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The L_p Gauss image problem for C-pseudo-cones

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Abstract. In this article, we study the L_p Gauss image problem for C-pseudo-cones, where C is a pointed closed convex cone in $\mathbb{R}^n$ with vertex at the origin and having nonempty interior. We first establish the existence of solutions for a Borel measure supported on compact subsets, and then prove a general existence theorem for $p > 0$ via approximation. In addition, we prove a uniqueness theorem for $p < 0$ under appropriate assumptions.

1 Introduction

The well-known Minkowski problem in convex geometry originated from the seminal works of Minkowski [16, 17], and was developed by Aleksandrov [2, 3] and Fenchel–Jessen [6]. The classical Minkowski problem asks: Given a finite Borel measure on the unit sphere $\mathbb{S}^{n-1}$. Does there exist a convex body in $\mathbb{R}^n$ whose surface area measure equals the given measure? Here, a convex body denotes a compact convex subset with nonempty interior. Since the influential work [14], this problem has attracted increasing attention over the past decades, and many significant contributions can be found in the literatures (see, e.g., [7, 9–11, 15, 28]).

Böröczky et al. [4] introduced a significant generalization – the Gauss image problem – which concerns the existence of a convex body so that two given spherical Borel measures satisfy a prescribed relationship via the Gauss image map. For more detailed explanations, see the introduction in [4]. Böröczky and coauthors obtained the necessary and sufficient conditions for the existence of solutions, as well as their uniqueness. Subsequently, Chen et al. [5] proved the existence of smooth solutions, and Semenov [24] studied the existence under a weak assumption, where one measure is discrete and the other is absolutely continuous. For Musielak–Orlicz functions, a generalized Gauss image problem has been comprehensively analyzed in [8, 13].

In recent years, the Brunn–Minkowski theory for convex bodies [18] has been generalized to the noncompact setting, for instance, the Minkowski problems [12, 19, 20] and the Brunn–Minkowski inequality [26]. In this work, we advance this direction by studying the L_p Gauss image problem for unbounded closed convex sets. Before stating our main results, we recall necessary definitions.

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Let C denote a pointed closed convex cone in $\mathbb{R}^n$ with vertex at the origin and having nonempty interior, and let $C^\circ := \{x \in \mathbb{R}^n : x \cdot y \le 0 \text{ for all } y \in C\}$ be its dual cone, where $x \cdot y$ is the standard inner product of x and y. Denote $\Omega_C := \mathrm{int}C \cap \mathbb{S}^{n-1}$ and $\Omega_{C^\circ} := \mathrm{int}C^\circ \cap \mathbb{S}^{n-1}$, where $\mathrm{int}C$ (resp. $\mathrm{int}C^\circ$) denotes the interior of C (resp. C°).

Let K denote a nonempty closed convex set not containing the origin. If $tK \subset K$ for any $t \ge 1$, we say that K is a pseudo-cone. If $\{z \in \mathbb{R}^n : K + z \subset K\} = C$, or equivalently, C is the recession cone of K, then K is called a C-pseudo-cone.

Let λ be a nonzero finite measure on the Lebesgue-measurable subsets of $\mathrm{cl}\Omega_{C^\circ}$. The Gauss image measure $\lambda(K,\cdot)$ of λ, associated with a C-pseudo-cone K, is defined by [22] $\lambda(K,\eta) := \lambda(\boldsymbol{\alpha}_K(\eta))$, for any Borel set $\eta \subset \Omega_C$, where $\boldsymbol{\alpha}_K(\eta)$ denotes the radial Gauss image of η. Based on this, we prove a variational formula related to the Wulff shape and hence induce a new Borel measure, the L_p Gauss image measure, denoted as $\lambda_p(K,\cdot)$. Specifically, for $p \in \mathbb{R}$, the L_p Gauss image measure is defined by

$$\lambda_p(K,\cdot) = \rho_K^p \lambda(K,\cdot) \text{ on } \eta, \tag{1.1}$$

where ρ_K is the radial function of K. Moreover, we study the following problem.

The L_p Gauss image problem for C-pseudo-cones. *Let λ be a nonzero finite measure on the Lebesgue measurable subsets of $\mathrm{cl}\Omega_{C^\circ}$, and μ be a Borel measure on Ω_C. What are the necessary and sufficient conditions on λ and μ, such that there is a C-pseudo-cone K satisfying $\mu = \lambda_p(K,\cdot)$ on the Borel subsets of Ω_C?*

When λ is the spherical Lebesgue measure, L_p Gauss image problem becomes the L_p Alexandrov problem [1]. Recently, Xu et al. [25] and Schneider [21] studied the pseudo-cones. Schneider [22, 23] introduced the Gauss image problem for pseudo-cones and interpreted it from the measure transportation viewpoint. It is worth emphasizing that the L_p Gauss image problem for C-pseudo-cones is considerably more intricate than the case $p = 0$, necessitating the development of new techniques.

To address the problem, we first prove an existence result for $p > 0$.

Theorem 1 *Let λ be a nonzero finite measure on the Lebesgue measurable subsets of $\mathrm{cl}\Omega_{C^\circ}$ which vanishes on Borel sets of Hausdorff dimension $(n-2)$. If μ is a finite Borel measure on Ω_C and $p > 0$, there exists a C-pseudo-cone K such that $\mu = \lambda_p(K,\cdot)$.*

Next, we consider the restricted C-pseudo-cones, in the sense that their outer unit normal vectors belong to Ω_{C°, and we establish a uniqueness characterization for the L_p Gauss image measure for $p < 0$ as follows.

Theorem 2 *Let $p < 0$ and K, L be two restricted C-pseudo-cones. If λ is strictly positive on open sets in Ω_{C° and $\lambda_p(K,\cdot) = \lambda_p(L,\cdot)$, then $K = L$.*

In this article, we first study a new optimization problem for $p \neq 0$, and we establish the existence of solutions for a Borel measure on compact subsets. Then, the general existence shown in Theorem 1 can be proved via a limiting argument. To characterize the uniqueness, we shall decompose Ω_C and employ a proof by contradiction to prove Theorem 2. The existence and uniqueness of solutions are nontrivial and have not been completely solved. We have established the existence of solutions for $p > 0$ and

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the uniqueness for $p < 0$. However, the methods employed here may not apply to the remaining cases, which will be investigated in future work.

This article is organized as follows. In Section 2, we recall some necessary notions and lemmas for the C-pseudo-cones. In Section 3, we prove a variational formula and introduce the L_p Gauss image measure as well as its weak convergence. In Section 4, Theorems 1 and 2 will be proved.

2 Preliminaries

In this section, we recall known results from the Brunn–Minkowski theory of pseudo-cones, for more details, we refer to [12, 19–22]. Throughout the article, the pseudo-cone C and its dual cone C° share analogous properties; thus we may interchange their roles. We use the same notations regardless of whether the sets are in C or C°.

2.1 Support function, radial function, and the copolar set

A pseudo-cone K with recession cone C is called a C-pseudo-cone. It is internal if vectors of Ω_{C° occur as outer unit normals of K only at points in $\mathrm{int}C$. We use $ps(C)$ and $psi(C)$ to denote the set of all C-pseudo-cones and that of all internal C-pseudo-cones, respectively.

Let $K \in ps(C)$. The support function $h_K : C^\circ \to \mathbb{R}$ of K is defined by

$$h_K(x) = \sup\{x \cdot y : y \in K\} \text{ for } x \in C^\circ.$$

Here, sup can be replaced by max if $x \in \mathrm{int}C^\circ$. As $h_K \le 0$, we also write $\tilde{h}_K = -h_K$. The radial function $\rho_K : \Omega_C \to \mathbb{R}$ of K is defined by

$$\rho_K(v) = \min\{\lambda \in \mathbb{R} : \lambda v \in K\} \text{ for } v \in \Omega_C.$$

The support and radial functions can, respectively, determine a pseudo-cone, and they are related by

$$\tilde{h}_{K^\circ}(v) = \frac{1}{\rho_K(v)} \text{ for } v \in \Omega_C, \tag{2.1}$$

where $K^\circ := \{x \in \mathbb{R}^n : x \cdot y \le -1 \text{ for all } y \in K\}$ denotes the copolar set of K. In fact, it follows from Lemma 4 of [21] that K° is a C°-pseudo-cone.

Let $\omega \subset C^\circ$ be nonempty and compact. A C-pseudo-cone K is C-determined by ω if $K = C \cap \bigcap_{u\in\omega} H_K^-(u)$, where $H_K^-(u)$ is the supporting halfspace of K with outer normal u. The set of all C-pseudo-cones which are C-determined by ω is denoted by $\mathcal{K}(C, \omega)$.

Let $\eta \subset \Omega_C$ be a nonempty compact subset and $f : \eta \to (0, \infty)$ be a continuous function. Then, one can define the Wulff shape $[f]$ associated with (C°, η, f) by

$$[f] := C^\circ \cap \bigcap_{v\in\eta} \{x \in \mathbb{R}^n : x \cdot v \le -f(v)\},$$

and one has $[f] \in \mathcal{K}(C^\circ, \eta)$.

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Analogously, one can define the convexification $\langle f\rangle$ associated with (C,η,f) by

$$\langle f\rangle = \bigcap\{K\in ps(C): f(v)v\in K \text{ for any } v\in\eta\}. \tag{2.2}$$

Clearly, $\langle f\rangle$ is a C-pseudo-cone and $\langle sf\rangle = s\langle f\rangle$ for $s>0$. Notice that $\langle f\rangle$ is contained in int$C$, thus we have $\langle f\rangle\in psi(C)$. It follows from definition (2.2) that

$$\rho_{\langle f\rangle}(v)\le f(v) \text{ on } \eta. \tag{2.3}$$

The C-pseudo-cone K is called C-defined by η if $K=\langle\rho_K|_\eta\rangle$, where $\rho_K|_\eta$ is the restriction of the function ρ_K to η. The set of all C-pseudo-cones that are C-defined by η is denoted by $\mathcal{K}^*(C,\eta)$. It turns out that $K\in\mathcal{K}^*(C,\eta)$ if and only if $K^\circ\in\mathcal{K}(C^\circ,\eta)$ (see page 8 of [22]).

The convergence of C-pseudo-cones is defined via Hausdorff convergence of compact sections: write $K_i\to K_0$ if $K_i\cap t\mathbb{B}_2^n\to K_0\cap t\mathbb{B}_2^n$ for each $t\ge t_0$. Here, $\mathbb{B}_2^n$ is the standard unit ball in $\mathbb{R}^n$. It follows from (2.1) that

$$K_i\to K_0 \Leftrightarrow K_i^\circ\to K_0^\circ.$$

Let $\eta\subset\Omega_C$ be a nonempty compact subset and $\mathcal{C}^+(\eta)$ be the set of all positive continuous functions on η. If $f\in\mathcal{C}^+(\eta)$, then

$$[f] = \langle 1/f\rangle^\circ \text{ and } [f]^\circ = \langle 1/f\rangle. \tag{2.4}$$

Moreover, the C-pseudo-cones satisfy the continuity properties and selection theorem as follows.

Lemma 1 (cf. Lemma 2 of [22]) *If $\{f_i\}_{i\in\mathbb{N}_0}\subset\mathcal{C}^+(\eta)$, and $\{f_i\}_{i\in\mathbb{N}}$ converges uniformly on η to f_0 as $i\to\infty$, then $[f_i]\to[f_0]$ and $\langle f_i\rangle\to\langle f_0\rangle$ as $i\to\infty$.*

Lemma 2 (cf. Lemma 1 of [21]) *Every sequence of C-pseudo-cones whose distances from the origin are bounded and bounded away from* 0 *has a subsequence that converges to a C-pseudo-cone.*

2.2 Some maps for pseudo-cones

Huang et al. [9] introduced some critical maps for convex bodies, including the spherical image map, the reverse spherical image map, the radial map, the radial Gauss map, and the reverse radial Gauss map. In this section, we review these maps for pseudo-cones.

Let $K\in ps(C)$ with boundary ∂K. The spherical image of $\sigma\subset\partial K$ is defined by

$$\boldsymbol{v}_K(\sigma) = \{u\in\Omega_{C^\circ}: x\in H_K(u) \text{ for some } x\in\sigma\}.$$

The spherical image map $v_K(x):\partial K\backslash\sigma_K\to\Omega_{C^\circ}$ is defined by $v_K(x)=\boldsymbol{v}_K(\{x\})$, where σ_K is the set of all $x\in\partial K$ for which the set $\boldsymbol{v}_K(\{x\})$ contains more than a single element. The reverse spherical image of $\omega\subset\Omega_{C^\circ}$ is defined by

$$\boldsymbol{x}_K(\omega) = \{x\in\partial K: x\in H_K(u) \text{ for some } u\in\omega\},$$

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and the reverse spherical image map $x_K : \Omega_{C^\circ}\backslash\omega_K \to \partial K$ is defined by $x_K(u) = \boldsymbol{x}_K(\{u\})$, where ω_K is the set of all $u \in \Omega_{C^\circ}$ for which the set $\boldsymbol{x}_K(\{u\})$ contains more than one element.

The radial Gauss image of $\eta \subset \Omega_C$ is defined by

$$\boldsymbol{\alpha}_K(\eta) = \boldsymbol{\nu}_K(r_K(\eta)),$$

and the radial Gauss map $\alpha_K : \Omega_C\backslash\eta_K \to \Omega_{C^\circ}$ is defined by $\alpha_K = \nu_K \circ r_K$, where $r_K : \Omega_C \to \partial K$ is the radial map defined by $r_K(v) = \rho_K(v)v$ for $v \in \Omega_C$, and $\eta_K = r_K^{-1}(\sigma_K)$. Here, the radial map and the radial Gauss map are only defined on Ω_C, not on its closure.

For $K \in psi(C)$, the reverse radial Gauss image of $\omega \subset \Omega_{C^\circ}$ is defined by

$$\boldsymbol{\alpha}_K^*(\omega) = r_K^{-1}(\boldsymbol{x}_K(\omega)),$$

and the reverse radial Gauss map $\alpha_K^* : \Omega_{C^\circ}\backslash\omega_K \to \Omega_C$ is defined by $\alpha_K^* = r_K^{-1} \circ x_K$. Here, we have restricted this map to $psi(C)$ to ensure that for $u \in \Omega_{C^\circ}\backslash\omega_K$, the vector $r_K^{-1} \circ x_K(u)$ belongs to Ω_C. Two maps α_K and α_K^* are all continuous, and moreover,

$$\alpha_K^*(u) = \alpha_{K^\circ}(u), \text{ for any } u \in \Omega_{C^\circ}\backslash\omega_K. \tag{2.5}$$

Let $K \in ps(C)$, and λ be a nonzero finite measure on the Lebesgue measurable subsets of $\mathrm{cl}\Omega_{C^\circ}$ which vanishes on Borel sets of Hausdorff dimension $(n-2)$. Then,

$$\lambda(\omega_K) = 0. \tag{2.6}$$

3 The variational formula

Let λ be a nonzero finite measure on the Lebesgue measurable subsets of $\mathrm{cl}\Omega_{C^\circ}$ which vanishes on Borel sets of Hausdorff dimension $(n-2)$, unless specified otherwise. Then, for $K \in ps(C^\circ)$, we introduce the dual entropy functional associated with λ as follows:

$$E_\lambda(K) := \int_{\Omega_{C^\circ}} \log \rho_K(u) d\lambda(u). \tag{3.1}$$

When λ is the spherical Lebesgue measure, this definition is due to [12]. In the remaining part of this section, we use the dual entropy functional to prove a variational formula related to the Wulff shape, which induces the L_p Gauss image measure. We begin with two preliminary lemmas needed for the proof.

To state the first lemma of this section, we let $\eta \subset \Omega_C$ be a nonempty compact subset and $f : \eta \to \mathbb{R}$ be continuous. Let $\delta > 0$ be small enough and for $t \in (-\delta, \delta)$. We define a continuous function $h_t : \eta \to (0, \infty)$ by

$$\log h_t(v) = \log h_0(v) + tf(v) + o(t, v), \text{ for any } v \in \eta, \tag{3.2}$$

where $o(t, \cdot) : \eta \to \mathbb{R}$ is continuous, and $\lim_{t\to 0} o(t, \cdot)/t = 0$ uniformly on η.

By adapting the arguments in [9] and interchanging the roles of C and C° in Lemma 5.4 of [12], we can prove the following lemma.

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Lemma 3 *Let $\eta \subset \Omega_C$ be a nonempty compact subset and $f : \eta \to \mathbb{R}$ be continuous. If $[h_t]$ is the Wulff shape associated with (C°, η, h_t), where h_t is given by (3.2), then for almost all $u \in \Omega_{C^\circ}$,*

$$\lim_{t\to 0} \frac{\log \rho_{[h_t]}(u) - \log \rho_{[h_0]}(u)}{t} = f(\alpha_{[h_0]}(u)).$$

Moreover, there exist two constants $\delta_0 > 0$ and $M > 0$ such that

$$|\log \rho_{[h_t]}(u) - \log \rho_{[h_0]}(u)| \le M|t|,$$

for all $u \in \Omega_{C^\circ}$ and $t \in (-\delta_0, \delta_0)$.

Furthermore, we apply Lemma 3 to obtain the following result.

Lemma 4 *Let $\eta \subset \Omega_C$ be a nonempty compact subset and $f : \eta \to \mathbb{R}$ be continuous. If $[h_t]$ is the Wulff shape associated with (C°, η, h_t), where h_t is given by (3.2), then for $t \in (-\delta, \delta)$, we have*

$$\frac{d}{dt} E_\lambda([h_t])\Big|_{t=0} = \int_\eta f(v) d\lambda([h_0]^\circ, v). \tag{3.3}$$

Proof Note that $\langle 1/h_0 \rangle \in psi(C)$. It follows from (2.4) and (2.5) that

$$\alpha_{[h_0]}(u) = \alpha_{\langle 1/h_0\rangle^\circ}(u) = \alpha^*_{\langle 1/h_0\rangle}(u) = \alpha^*_{[h_0]^\circ}(u) \text{ for any } u \in \Omega_{C^\circ} \backslash \omega_{\langle 1/h_0\rangle}.$$

Then, by definition (3.1), Lemma 3, the dominated convergence theorem, and (2.6), we have

$$\begin{aligned}
\frac{d}{dt} E_\lambda([h_t])\Big|_{t=0} &= \lim_{t\to 0} \int_{\Omega_{C^\circ}} \frac{\log \rho_{[h_t]}(u) - \log \rho_{[h_0]}(u)}{t} d\lambda(u) \\
&= \int_{\Omega_{C^\circ}} f(\alpha_{[h_0]}(u)) d\lambda(u) \\
&= \int_{\Omega_{C^\circ}} f(\alpha^*_{[h_0]^\circ}(u)) d\lambda(u).
\end{aligned}$$

Consequently, the desired (3.3) immediately follows from Lemma 4 of [22]. Here, we have extended the function f to Ω_C by taking $f = 0$ outside η. ∎

We are now in the position to prove a variational formula related to the Wulff shape.

Lemma 5 *Let $\eta \subset \Omega_C$ be a nonempty compact subset and $K, L \in ps(C)$, and let $[h_{t,p}]$ be the Wulff shape associated with $(C^\circ, \eta, h_{t,p})$. If $p \neq 0$ and $h_{t,p}(v) = (\bar{h}^p_{K^\circ}(v) + t\bar{h}^p_{L^\circ}(v))^{1/p}$ for any $v \in \eta$, then*

$$\frac{d}{dt} E_\lambda([h_{t,p}])\Big|_{t=0} = \frac{1}{p} \int_\eta \rho_L^{-p}(v) \rho_K^p(v) d\lambda(\langle \rho_K|_\eta \rangle, v). \tag{3.4}$$

If $p = 0$ and $h_{t,0}(v) = \bar{h}_{K^\circ}(v)\bar{h}^t_{L^\circ}(v)$ for any $v \in \eta$, then

$$\frac{d}{dt} E_\lambda([h_{t,0}])\Big|_{t=0} = -\int_\eta \log \rho_L(v) d\lambda(\langle \rho_K|_\eta \rangle, v). \tag{3.5}$$

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Proof We first prove the case of $p \neq 0$. As $h_{t,p}(v) = (\bar{h}^p_{K^\circ}(v) + t\bar{h}^p_{L^\circ}(v))^{1/p}$ for any $v \in \eta$, it follows that

$$\log h_{t,p} = \log \bar{h}_{K^\circ} + \frac{1}{p}\left(\frac{\bar{h}_{L^\circ}}{\bar{h}_{K^\circ}}\right)^p t + o_p(t,\cdot),$$

where $o_p(t,\cdot) : \eta \to \mathbb{R}$ is continuous, and $\lim_{t\to 0} o_p(t,\cdot)/t = 0$ uniformly on η. Then, from Lemma 4 and (2.1), and noting that $[\bar{h}_{K^\circ}]^\circ = [1/\rho_K]^\circ = \langle \rho_K|_\eta \rangle$, we obtain the desired (3.4) immediately.

Analogously, to deal with the case of $p = 0$, we note that

$$\log h_{t,0} = \log \bar{h}_{K^\circ} + t \log \bar{h}_{L^\circ} \text{ on } \eta.$$

Then, by the arguments similar to those for the case of $p \neq 0$, we can easily verify (3.5). ∎

For a C-pseudo-cone K, the variational formula (3.4) induces an important geometric measure $\rho^p_K \lambda(K,\cdot)$. We call it the L_p Gauss image measure and denote it by $\lambda_p(K,\cdot)$. Since the Gauss image measure $\lambda(K,\cdot)$ introduced by Schneider [22] is a Borel measure, and the radial function ρ_K is positive and continuous, the L_p Gauss image measure $\lambda_p(K,\cdot)$ remains a Borel measure. In the following, we further prove its weak convergence.

Lemma 6 *Let $\eta \subset \Omega_C$ be a nonempty compact subset, $K_i \in psi(C)$ for $i \in \mathbb{N}_0$ and $p \neq 0$. If $K_i \to K_0$ as $i \to \infty$, then $\lambda_p(K_i,\cdot)$ converges to $\lambda_p(K_0,\cdot)$ weakly on η.*

Proof As $K_i \to K_0$ as $i \to \infty$, it follows that ρ_{K_i} converges to ρ_{K_0} uniformly on η. Note that ρ_{K_0} is positive, and ρ_{K_i} are uniformly bounded on η. Thus, we conclude that $\rho^p_{K_i}$ converges to $\rho^p_{K_0}$ uniformly, and $g\rho^p_{K_i}$ converges to $g\rho^p_{K_0}$ uniformly as well, where g denotes a continuous function on η.

Furthermore, it follows from Lemma 5 of [22] that restriction of the measure $\lambda(K_i,\cdot)$ to η is weakly convergent. That is, $\lambda(K_i,\cdot)\llcorner_\eta$ converges to $\lambda(K_0,\cdot)\llcorner_\eta$ weakly as $i \to \infty$. Therefore, we have

$$\lim_{i\to\infty} \int_\eta g(v)\rho^p_{K_i}(v)d\lambda(K_i,v) = \int_\eta g(v)\rho^p_{K_0}(v)d\lambda(K_0,v).$$

Equivalently,

$$\lim_{i\to\infty} \int_\eta g(v)d\lambda_p(K_i,v) = \int_\eta g(v)d\lambda_p(K_0,v),$$

and the desired weak convergence follows. ∎

4 Proofs of the main results

This section is dedicated to proving our existence and uniqueness theorems. To complete the proof of the former, we first recall a necessary lemma in [22], and we establish an existence result under the additional assumption that measure μ is

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defined on the nonempty compact set. We shall begin with stating the following lemma.

Lemma 7 (cf. Lemma 9 of [22]) *Let λ be a nonzero finite measure on the Lebesgue measurable subsets of $\mathrm{cl}\Omega_{C^\circ}$ which vanishes on Borel sets of Hausdorff dimension $(n-2)$. Let $\eta \subset \Omega_C$ be a nonempty compact subset, $g : \eta \to \mathbb{R}$ be continuous, and $K \in ps(C)$. If $\langle f_t \rangle$ is the convexification associated with (C, η, f_t), where*

$$\log f_t(v) = \log \rho_K(v) + tg(v) \text{ for } v \in \eta,$$

then

$$\frac{d}{dt}\int_{\Omega_{C^\circ}} \log \rho_{\langle f_t\rangle^\circ}(u)d\lambda(u)\Big|_{t=0} = -\int_\eta g(v)d\lambda(\langle f_0\rangle, v). \tag{4.1}$$

To prove the existence of solutions for a finite Borel measure μ on the nonempty compact subset, we study a critical optimization problem that differs from the one considered in [22]. Then, we use Lemma 7 to obtain the following existence result.

Lemma 8 *Let λ be a nonzero finite measure on the Lebesgue measurable subsets of $\mathrm{cl}\Omega_{C^\circ}$ which vanishes on Borel sets of Hausdorff dimension $(n-2)$. If μ is a finite Borel measure on the nonempty compact subset $\eta \subset \Omega_C$ and $p \neq 0$, there exists a C-pseudo-cone $K \in \mathcal{K}^*(C, \eta)$ such that $\mu = \lambda_p(K, \cdot)$.*

Proof Let $|\lambda| = \lambda(\Omega_{C^\circ})$. Recall that $\mathcal{C}^+(\eta)$ denotes the set of all positive continuous functions on η. Then, for $p \neq 0$, we construct a functional $\Phi_{\lambda,\mu,p}$ on $\mathcal{C}^+(\eta)$ as follows:

$$\Phi_{\lambda,\mu,p}(f) = -\frac{1}{p}\log\left(\frac{1}{\mu(\eta)}\int_\eta f^{-p}(v)d\mu(v)\right) + \frac{1}{|\lambda|}\int_{\Omega_{C^\circ}} \log \rho_{\langle f\rangle^\circ}(u)d\lambda(u),$$

where $f \in \mathcal{C}^+(\eta)$. Obviously, Lemma 1 shows that functional $\Phi_{\lambda,\mu,p}$ is continuous on $\mathcal{C}^+(\eta)$. Moreover, we verify $\Phi_{\lambda,\mu,p}(sf) = \Phi_{\lambda,\mu,p}(f)$ for $s > 0$. That is, functional $\Phi_{\lambda,\mu,p}$ is homogeneous of degree 0. Next, we consider a minimization problem

$$\inf\left\{\Phi_{\lambda,\mu,p}(f) : f \in \mathcal{C}^+(\eta)\right\}, \tag{4.2}$$

and we prove that problem (4.2) is indeed solvable.

For any $f \in \mathcal{C}^+(\eta)$, it follows from (2.3) that $\rho_{\langle f\rangle} \le f$ on η, and $\langle \rho_{\langle f\rangle}|_\eta\rangle = \langle f\rangle \in psi(C)$. Then, we obtain $\Phi_{\lambda,\mu,p}(\rho_{\langle f\rangle}|_\eta) \le \Phi_{\lambda,\mu,p}(f)$. Now, we focus on the set of all radial functions of pseudo-cones in $psi(C)$. Thus, the problem (4.2) can be equivalently transformed into the problem

$$\inf\left\{\Phi_{\lambda,\mu,p}(\rho_K|_\eta) : K \in psi(C)\right\}. \tag{4.3}$$

In fact, we can verify that $K \in psi(C)$ is a solution to the problem (4.3) if and only if $\rho_K|_\eta$ is a solution to the problem (4.2).

In the following, we further have the following Claim.

Claim. The infimum in (4.3) can be attained by some C-pseudo-cone in $psi(C)$.

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To verify the Claim, we take a minimizing sequence $\{K_i\}$ of pseudo-cones in $psi(C)$, such that

$$\lim_{i\to+\infty}\Phi_{\lambda,\mu,p}(\rho_{K_i}|_\eta)=\inf\left\{\Phi_{\lambda,\mu,p}(\rho_K|_\eta):K\in psi(C)\right\}.$$

Since the functional $\Phi_{\lambda,\mu,p}$ is homogeneous of degree 0, we assume that each K_i has distance 1 from the origin. Then, by Lemma 2, we conclude that sequence $\{K_i\}$ has a subsequence, still denoted by $\{K_i\}$, convergent to some C-pseudo-cone K_0 as $i\to\infty$. Consequently, from the continuity of functional $\Phi_{\lambda,\mu,p}$, we can verify the assertion of Claim.

To continue with the proof of Lemma 8, we take a sufficiently small $\delta>0$. For each $t\in(-\delta,\delta)$, we define a function ρ_t on η by $\rho_t:=\rho_{K_0}e^{tg}$, where $g\in\mathcal{C}^+(\eta)$.

For any $s\in(-1,1)$, we have the inequality $|e^s-1-s|\le es^2$. Taking $s=-ptg(v)$, then for any $|t|<\min\{\frac{1}{|p|\max\{g(v):v\in\eta\}},\delta\}$, we have

$$\left|\frac{e^{-ptg(v)}-1}{t}+pg(v)\right|\le ep^2|t|g^2(v),\ \text{for } v\in\eta.$$

Furthermore, we conclude that, as $t\to0$,

$$\frac{\rho_t^{-p}-\rho_0^{-p}}{t}=\rho_{K_0}^{-p}\frac{e^{-ptg}-1}{t}\to-\rho_{K_0}^{-p}pg,\ \text{uniformly on } \eta.$$

Then, we have the following calculation:

$$\begin{aligned}
&\frac{d}{dt}\log\left(\frac{1}{\mu(\eta)}\int_\eta\rho_t^{-p}(v)d\mu(v)\right)\Big|_{t=0}\\
=&\frac{1}{\int_\eta\rho_{K_0}^{-p}(v)d\mu(v)}\frac{d}{dt}\int_\eta\rho_t^{-p}(v)d\mu(v)\Big|_{t=0}\\
=&\frac{1}{\int_\eta\rho_{K_0}^{-p}(v)d\mu(v)}\lim_{t\to0}\int_\eta\frac{\rho_t^{-p}(v)-\rho_0^{-p}(v)}{t}d\mu(v)\Big|_{t=0}\\
=&-\frac{p}{\int_\eta\rho_{K_0}^{-p}(v)d\mu(v)}\int_\eta\rho_{K_0}^{-p}(v)g(v)d\mu(v).
\end{aligned}\tag{4.4}$$

It follows from Lemma 7 that $\Phi_{\lambda,\mu,p}(\rho_t)$ is differentiable at $t=0$. As $\rho_{K_0}|_\eta$ is a solution to the problem (4.2), we obtain that

$$\frac{d}{dt}\Phi_{\lambda,\mu,p}(\rho_t)\Big|_{t=0}=0.$$

This, together with (4.1) and (4.4), gives that

$$\frac{1}{\int_\eta\rho_{K_0}^{-p}(v)d\mu(v)}\int_\eta\rho_{K_0}^{-p}(v)g(v)d\mu(v)-\frac{1}{|\lambda|}\int_\eta g(v)d\lambda(\langle\rho_{K_0}|_\eta\rangle,v)=0.\tag{4.5}$$

Then, from (4.5) and the Riesz representation theorem, we employ (1.1) to obtain

$$\mu=\frac{\int_\eta\rho_{K_0}^{-p}(v)d\mu(v)}{|\lambda|}\lambda_p(\langle\rho_{K_0}|_\eta\rangle,\cdot).\tag{4.6}$$

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Since the right-hand side of (4.6) is invariant under any dilations of K_0, we can rescale K_0 to obtain the desired equality. ∎

Lemma 8 is an essential step toward the general existence theorem. We now remove the assumption of compactness condition, and then we adopt the approximation method to prove our existence theorem.

Proof of Theorem 1 Let μ be a Borel measure on Ω_C, and let $\{\eta_j\}_{j\in\mathbb{N}}$ be a sequence of compact sets in Ω_C such that

$$\mu(\eta_1) > 0,\ \eta_j \subset \eta_{j+1} \text{ for } j \in \mathbb{N},\ \bigcup_{j\in\mathbb{N}} \eta_j = \Omega_C.$$

Define $\mu_j = \mu\llcorner_{\eta j}$; that is, $\mu_j(\beta) = \mu(\beta \cap \eta_j)$ for any Borel set β in Ω_C. Thus, μ_j is a finite Borel measure on η_j. Then, for each j, the proof of Lemma 8 shows that there exists a C-pseudo-cone $K_j \in \mathcal{K}^*(C, \eta_j)$ with $\mu_j = c_j\lambda_p(K_j, \cdot)$ on η_j, where

$$c_j = \frac{1}{|\lambda|}\int_{\eta_j} \rho_{K_j}^{-p} d\mu_j = \frac{1}{|\lambda|}\int_{\Omega_c} \rho_{K_j}^{-p}\chi_{\eta_j} d\mu,$$

where χ_{η_j} is the characteristic function of η_j.

As $\lambda(K_j, \cdot)$ is positively homogeneous of degree 0, we assume that K_j has distance 1 from the origin. Then, we apply Lemma 2 again to obtain that sequence $\{K_j\}$ has a subsequence, still denoted by $\{K_j\}$, converging to C-pseudo-cone K as $j \to \infty$. Moreover, we obtain that $\rho_{K_j}^{-p} \le 1$ due to $p > 0$. It follows that

$$c_j \to \frac{1}{|\lambda|}\int_{\Omega_c} \rho_K^{-p} d\mu =: c > 0 \text{ as } j \to \infty.$$

Fix $k \in \mathbb{N}$, then for any Borel set $\beta \subseteq \eta_k$ and $j \ge k$, $\mu(\beta) = \mu_j(\beta) = c_j\lambda_p(K_j, \beta)$, and the restrictions to η_k satisfy

$$\frac{1}{c_j}\mu\llcorner_{\eta_k} = \lambda_p(K_j, \cdot)\llcorner_{\eta_k}.$$

Taking $j \to \infty$ and applying Lemma 6 on η_k, we conclude that $\mu(\beta) = c\lambda_p(K, \beta)$ for $\beta \subseteq \eta_k$. As $k \in \mathbb{N}$ is arbitrary and $\bigcup_{k\in\mathbb{N}} \eta_k = \Omega_C$, we have $\mu = c\lambda_p(K, \cdot)$ on Ω_C. Finally, by rescaling K, we can obtain the desired equality. ∎

Recall that Böröczky et al. [4] adopted the method in [27] to prove a uniqueness lemma, and Schneider [22] has recently carried over that proof to pseudo-cones. In the following, we prove that uniqueness remains valid in the L_p setting for $p < 0$. Our proof technique follows from [22].

Proof of Theorem 2 Let $p < 0$. Suppose $\lambda_p(K, \cdot) = \lambda_p(L, \cdot)$, but $K \ne L$. This implies that there exists some $v_0 \in \Omega_C$ so that $r_K(v_0)$ and $r_L(v_0)$ are regular points, and the support hyperplane of K at $r_K(v_0)$ and that of L at $r_L(v_0)$ have distinct normal vectors. Without loss of generality, we assume that $r_K(v_0)$ is closer to the origin than $r_L(v_0)$, then we rescale K and denote by $K' = cK$, where $c =: r_L(v_0)/r_K(v_0) \ge 1$.

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We now define a disjoint decomposition $\Omega_C = \eta' \cup \eta_0 \cup \eta$ as follows:

$$\eta' = \{v \in \Omega_C : \rho_{K'}(v) < \rho_L(v)\},$$
$$\eta_0 = \{v \in \Omega_C : \rho_{K'}(v) = \rho_L(v)\},$$
$$\eta = \{v \in \Omega_C : \rho_{K'}(v) > \rho_L(v)\}.$$

If $v \in \eta'$ and H_L is a support hyperplane of L at $r_L(v)$, then pseudo-cone K' has a parallel support hyperplane $H_{K'}$ at some $r_{K'}(v')$ with $v' \in \eta'$. Moreover, we have

$$\boldsymbol{\alpha}_L(\eta') \subseteq \boldsymbol{\alpha}_{K'}(\eta') = \boldsymbol{\alpha}_K(\eta'). \tag{4.7}$$

Note that $\eta' \cup \eta_0$ and $\eta \cup \eta_0$ are closed sets. It follows that $\boldsymbol{\alpha}_L(\eta' \cup \eta_0)$ and $\boldsymbol{\alpha}_{K'}(\eta \cup \eta_0)$ are two closed subsets in Ω_{C°. Consequently, $\Omega_{C^\circ}\backslash\boldsymbol{\alpha}_L(\eta' \cup \eta_0)$ and $\Omega_{C^\circ}\backslash\boldsymbol{\alpha}_{K'}(\eta \cup \eta_0)$ are open sets, and moreover, we can easily verify that

$$\Omega_{C^\circ}\backslash\boldsymbol{\alpha}_{K'}(\eta \cup \eta_0) \subset \boldsymbol{\alpha}_{K'}(\eta') \text{ and } (\Omega_{C^\circ}\backslash\boldsymbol{\alpha}_L(\eta' \cup \eta_0)) \cap \boldsymbol{\alpha}_L(\eta') = \varnothing.$$

Taking an open set $\beta = (\Omega_{C^\circ}\backslash\boldsymbol{\alpha}_{K'}(\eta \cup \eta_0)) \cap (\Omega_{C^\circ}\backslash\boldsymbol{\alpha}_L(\eta' \cup \eta_0))$, we shall prove that

$$v_1 := (\alpha_K(v_0) + \alpha_L(v_0))/|\alpha_K(v_0) + \alpha_L(v_0)| \in \beta.$$

We first verify that $v_1 \notin \boldsymbol{\alpha}_{K'}(\eta \cup \eta_0)$. Let $H_{K'}(v_1)$ be a support hyperplane of K' with unit normal v_1. Then, for $r_{K'}(v) \in H_{K'}(v_1)$, there exist a $x' \in P$ and $s > 0$ so that $r_{K'}(v) = x' + sv_1$, where P is a hyperplane passing through $x_0 := r_{K'}(v_0)$ with v_1 as its normal vector. Since

$$x_0 \cdot \alpha_{K'}(v_0) = h_{K'}(\alpha_{K'}(v_0)) \geq x' \cdot \alpha_{K'}(v_0) + sv_1 \cdot \alpha_{K'}(v_0),$$

and $v_1 \cdot \alpha_{K'}(v_0) > 0$, we obtain that $x_0 \cdot \alpha_{K'}(v_0) > x' \cdot \alpha_{K'}(v_0)$. Then, from $x' \cdot v_1 = x_0 \cdot v_1$, we conclude $x' \cdot \alpha_L(v_0) > x_0 \cdot \alpha_L(v_0)$. It follows from $v_1 \cdot \alpha_L(v_0) > 0$ that

$$r_{K'}(v) \cdot \alpha_L(v_0) > x' \cdot \alpha_L(v_0) > x_0 \cdot \alpha_L(v_0) = h_L(\alpha_L(v_0)).$$

This shows that $r_{K'}(v) \notin L$. Thus, we obtain $v \in \eta'$, and $v_1 \notin \boldsymbol{\alpha}_{K'}(\eta \cup \eta_0)$.

Similarly, we can verify that $v_1 \notin \boldsymbol{\alpha}_L(\eta' \cup \eta_0)$. Therefore, β is a nonempty open set in Ω_{C°.

As $\beta \subset \boldsymbol{\alpha}_{K'}(\eta')$ and $\beta \cap \boldsymbol{\alpha}_L(\eta') = \varnothing$, by (4.7), we obtain $\boldsymbol{\alpha}_L(\eta') = \boldsymbol{\alpha}_L(\eta')\backslash\beta \subseteq \boldsymbol{\alpha}_{K'}(\eta')\backslash\beta$. Consequently, $\lambda(\boldsymbol{\alpha}_L(\eta')) < \lambda(\boldsymbol{\alpha}_{K'}(\eta'))$ due to that $\lambda(\beta) > 0$, and

$$\lambda_p(L, \eta') = \int_{\eta'} \rho_L^p(v) d\lambda(\alpha_L(v)) < \int_{\eta'} \rho_{K'}^p(v) d\lambda(\alpha_{K'}(v))$$
$$\leq \int_{\eta'} \rho_K^p(v) d\lambda(\alpha_K(v)) = \lambda_p(K, \eta').$$

This contradicts our assumption; hence, $K = L$. ∎

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