Proc. Camb. Phil. Soc. (1972), **71**, 457
PCPS 71–50
Printed in Great Britain

Complex bundles with two sections

By ELMER REES
University College of Swansea, Wales

(*Received* 7 *July* 1971)

Atiyah (2) defined the geometrical dimension of an element $x \in \tilde{K}(X)$ to be less than $k+1$ ($g\dim(x) \leqslant k$) if there is a k-dimensional bundle over X whose stable equivalence class is x. If ξ is an n-plane complex bundle over X, we say that it has r sections if there is an $(n-r)$-plane bundle η such that ξ is isomorphic to $\eta \oplus \epsilon^r$ where ϵ^r is the trivial r-plane bundle over X. If X has dimension $2n$ or less, then ξ has r sections if and only if $g\dim(\xi - n) \leqslant n-r$.

It is well known that if X has dimension $2n$ and the top Chern class $c_n(x)$ vanishes then $g\dim(x) \leqslant n-1$. The next Chern class $c_{n-1}(x)$ is the primary obstruction to the existence of a further section, in general there is also a secondary obstruction. Here we investigate conditions under which $c_{n-1}(x)$ is the only obstruction. The essential idea is contained in the proof of the following theorem.

Theorem 1. *Let X be a $2n$ dimensional complex such that there are no elements of order two in $H^{2n}(X)$ and the cohomotopy group $[X, S^{2n-1}]$ vanishes. If $x \in \tilde{K}(X)$ is such that $c_{n-1}(x) = 0$ and $c_n(x) = 0$ then $g\dim(x) \leqslant n-2$.*

When n is odd the same result appears as Theorem 3·5 of (8) without the condition on $[X, S^{2n-1}]$. We will also sketch a proof of this more general result by our method. When n is even, our theorem applies in particular for $X = \mathbb{C}P^n$ and we use it to solve a problem about $\mathbb{C}P^4$ raised by R. L. E. Schwarzenberger. We also give some examples to show the necessity of the conditions of the theorems.

Proof of Theorem 1. Let $\xi: X \to BU$ denote the classifying map for x. Because $c_n(x) = 0$, ξ factors through the inclusion $BU(n-1) \to BU$; for convenience we also denote a particular factorization by ξ. In fact when $[X, S^{n-1}] = 0$ this factorization is unique. The Thom complex $MU(n-1)$ is the cofibre of the inclusion mapping

$$BU(n-2) \to BU(n-1)$$

because this inclusion is up to homotopy the projection of the universal spherical fibration over $BU(n-1)$. Denote the cofibre map by p: $BU(n-1) \to MU(n-1)$. $MU(n-1)$ has cohomology in even dimensions only and in fact its integral cohomology is mapped monomorphically by p^*. $H^{2n-2}(MU(n-1)) = Z$ with generator the Thom class U; note that $p^*U = c_{n-1}$, the $(n-1)$st Chern class. $H^{2n}(MU(n-1)) = Z$ with generator V and $p^*V = c_1 . c_{n-1}$. Hence the $2n$-skeleton of $MU(n-1)$ is homotopy equivalent to a complex $S^{2n-2} \cup e^{2n}$ whose attaching map is η the non-zero element of $\pi_{2n-1}S^{2n-2}$ because when the coefficients are reduced mod 2 we have that

$$Sq^2 c_{n-1} = c_1 . c_{n-1}$$

by the Wu relations and so $Sq^2 U = V$ because p^* is a monomorphism on cohomology.

https://doi.org/10.1017/S0305004100050738 Published online by Cambridge University Press

LEMMA 2. *The composition*

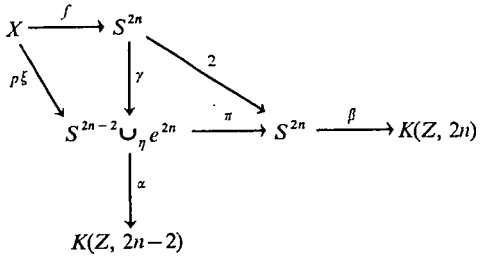
$$X \xrightarrow{\xi} BU(n-1) \xrightarrow{p} MU(n-1)$$

is inessential if $H^{2n}(X)$ *has no even torsion.*

Proof. Let $\alpha: MU(n-1) \to K(Z, 2n-2)$

be a map such that $\alpha^* \iota_{2n-2} = c_{n-1}$ then $\alpha_*: \pi_i MU(n-1) \to \pi_i K$ is an isomorphism for $i < 2n$.

It is easy to compute that $\pi_{2n}(S^{2n-2} \cup_\eta e^{2n})$ is isomorphic to the integers for $n \geqslant 2$ and that by the standard method of constructing $K(Z, 2n-2)$ its $(2n+1)$-skeleton is homotopy equivalent to the cofibre of a generator γ of $\pi_{2n}(S^{2n-2} \cup_\eta e^{2n})$. Now consider the diagram.



The map f exists because the cohomology class represented by $\alpha p\xi$ is $c_{n-1}(x)$ and so $\alpha p\xi$ is null-homotopic. $\beta\pi p\xi$ represents $c_1 . c_{n-1}$ and is therefore also null homotopic. Hence $\pi p\xi$ is zero, i.e. $\pi\gamma f$ is zero. But as indicated in the diagram, $\pi\gamma$ is a map of degree two because γ is a generator of $\pi_{2n}(S^{2n-2} \cup_\eta e^{2n}) = Z$ and there is an exact sequence

$$\pi_{2n}(S^{2n-2} \cup_\eta e^{2n}) \to \pi_{2n} S^{2n} \to \pi_{2n} S^{2n-1} \to 0.$$

So $2f = 0$, but if $H^{2n}(X) \cong [X, S^{2n}]$ has no even torsion, f must be zero and therefore $p\xi = 0$, proving the lemma.

Let F_{n-1} be the fibre of the map $p: BU(n-1) \to MU(n-1)$, then by the lemma there is a lift of $\xi: X \to BU(n-1)$ to $\xi': X \to F_{n-1}$. The composite

$$BU(n-2) \to BU(n-1) \to MU(n-1)$$

is zero, choose a lift $BU(n-2) \to F_{n-1}$, an analysis of the cohomology of F_{n-1} (using the Serre spectral sequence of $BU(n-1) \to MU(n-1)$) shows that its $2n$-skeleton is homotopy equivalent to the $2n$-skeleton of a space $BU(n-2) \cup e^{2n-1}$. By the assumption that $[X, S^{2n-1}]$ vanishes and a standard Blakers–Massey argument, ξ' must lift to $BU(n-2)$ thus proving the theorem.

In this proof we have implicitly used a Blakers–Massey argument several times, the following is the actual result that is used.

LEMMA (Blakers–Massey). *Let* $A \subset X$ *have cofibre* p: $X \to X/A$ *with* A $(m-1)$*-connected and* X/A $(n-1)$*-connected, if* F_p *is the fibre of* p *and* i: $A \to F_p$ *the obvious map, then* i *is* $(m+n-3)$*-connected, i.e. any map* $Y \to F_p$ *factors through* i *as long as* dim $Y \leqslant m+n-2$.

This lemma can easily be deduced from (3).

We now give a reasonably detailed proof of Theorem 3·5 of (8) by our method.

https://doi.org/10.1017/S0305004100050738 Published online by Cambridge University Press

THEOREM 3. *Let X be a $2n$-dimensional complex with n odd and such that there are no elements of order two in $H^{2n}(X)$. If $x \in \tilde{K}(X)$ is such that $c_{n-1}(x) = 0$ and $c_n(x) = 0$ then* $g\dim(x) \leqslant n-2$.

Proof. When n is odd, $\pi_{2n-2}BU(n-2) = 0$ (see (6)) and so the $2n$-skeleton of F_{n-1} must be homotopy equivalent to that of $BU(n-2) \vee S^{2n-1}$. By the previous argument there are lifts $\xi: X \to BU(n-1)$ (now not unique) and $\xi': X \to F_{n-1}$, the only problem is that ξ' may be essential on S^{2n-1}. However, as the choice of ξ depends on $[X, S^{2n-1}]$ one would expect that one can change the lift ξ so that the composition

$$X \xrightarrow{\xi'} F_{n-1} \to S^{2n-1}$$

is zero and this is in fact what one does. It turns out that one can only do this when n is odd as we later show by an example.

We have an action map

$$a: S^{2n-1} \vee BU(n-1) \to BU(n-1),$$

this is the action of the fibre S^{2n-1} on the total space $BU(n-1)$ and exists in the range of dimensions in which we are interested, we do not pursue the details of how this can be made precise as we are only reproving Thomas' theorem. If $\alpha \in [X, S^{2n-1}]$ we can choose a new lift $\xi^\alpha: X \to BU(n-1)$ which is $a(\alpha, \xi)$.

Consider the diagram

$$\begin{array}{ccc} S^{2n-1} \vee BU(n-1) & \xrightarrow{a} & BU(n-1) \\ \uparrow & & \uparrow \\ S^{2n-1} \vee F_{n-1} & \xrightarrow{b} & F_{n-1} \end{array}$$

where b is track addition on the S^{2n-1} factor. The diagram can be completed as shown because $\pi_{2n-1}(MU(n-1)) = 0$. We also need the following diagram

$$\begin{array}{ccc} S^{2n-1} \vee F_{n-1} & \xrightarrow{b} & F_{n-1} \\ \downarrow & & \downarrow \\ S^{2n-1} \vee S^{2n-1} & \longrightarrow & S^{2n-1} \end{array}$$

This commutes from the very definition of b. When n is even, b cannot be chosen to make both these diagrams commute and this is the only place where the argument breaks down when n is even.

Suppose now that we have lifts $\xi: X \to BU(n-1)$ and $\xi': X \to F_{n-1}$, let α denote the composition $X \longrightarrow F_{n-1} \to S^{2n-1}$, the lift $b(-\alpha, \xi'): X \to F_{n-1}$ clearly projects to 0 in $[X, S^{2n-1}]$ and so this lifts to a map $X \to BU(n-2)$ whose stable class is $\xi: X \to BU$, thus proving the theorem.

COROLLARY 4. *If x is an element of $\tilde{K}(\mathbb{C}P^n)$ with $c_{n-1}(x) = 0$ and $c_n(x) = 0$ then* $g\dim(x) \leqslant n-2$.

Proof. When n is odd, this follows from Theorem 3 and when n is even it follows from Thereom 1 because then

$$[\mathbb{C}P^n, S^{2n-1}] = [S^{2n-2} \cup_\eta e^{2n}, S^{2n-1}] = 0.$$

https://doi.org/10.1017/S0305004100050738 Published online by Cambridge University Press

It is very tempting to conjecture (but probably far too optimistic!) that if

$$x \in \tilde{K}(\mathbb{C}P^n)$$

is such that $c_{k+1}(x) = 0, c_{k+2}(x) = 0, \ldots, c_n(x) = 0$ then $g\dim(x) \leqslant k$. It is easy to prove this when $k = 1, n-1$ and this corollary gives it for $k = n-2$.

Some examples. 1. Consider an even dimensional real projective space P^{2n}, then $\tilde{K}(P^{2n}) = Z_{2^n}$ generated by $\eta = (L-1) \otimes \mathbb{C}$ where L is the canonical bundle on P^{2n}. Choose r such that

$$\binom{r}{n} \equiv 0 \bmod 2 \quad \text{and} \quad \binom{r}{n-1} \equiv 2 \bmod 4$$

(such an r exists if and only if n is not a power of two). Then $c_n(r\eta) = 0$ and $c_{n-1}(r\eta) = 0$, however when one calculates the γ operations of K-theory (2) one sees that

$$\gamma^{n-1}(r\eta) = \binom{r}{n-1}\eta^{n-1} = \binom{r}{n-1} 2^{n-2}\eta \neq 0$$

hence $g\dim(r\eta) \geqslant n-1$. Hence we have found examples of a bundle x over a $2n$-dimensional complex X with $c_{n-1}(x) = 0$, $c_n(x) = 0$ but $g\dim(x) \geqslant n-1$, $H^{2n}(X)$ has two torsion and when n is odd $[X, S^{2n-1}] \neq 0$ but $[X, S^{2n-1}] = 0$ when n is even.

2. Consider the universal example E_1 given by Thomas on page 1197 of (9), it is the universal example for the problem we are considering and is described there by its Postnikov resolution. If X is the $(2n-1)$-skeleton of E_1 (with $n = 2s+2$ in Thomas' notation) it is clear that the universal bundle on X has $g\dim \geqslant n-1$, $H^{2n}(X)$ is zero and so has no elements of order 2. This example shows that when n is even the cohomotopy condition that we give is necessary.

An example that indicates that many more conditions are necessary to obtain a result for the existence of three sections is

3. If $n = 2^j + 2$, $(j > 0)$ there is an $x \in \tilde{K}(P^{2n-1})$ with $c_i(x) = 0$ for all i but $g\dim(x) \geqslant n-2$.

Proof. $\tilde{K}(P^{2n-1}) = Z_{2^{n-1}}$ generated by η. Let $x = 2^{j+1}\eta$, then $c_i(x) = 0$ for all i and $x \neq 0$ if $j+1 < n-1$ i.e. $j > 0$. However

$$\gamma^{n-2}(x) = \binom{2^{j+1}}{2^j} . 2^{n-3}\eta \quad \text{and} \quad \binom{2^{j+1}}{2^j} \equiv 2 \bmod 4$$

hence $\gamma^{n-2}(x) \neq 0$ and so $g\dim(x) \geqslant n-2$. In fact $g\dim(x) = n-2$ because an even multiple of η has a quaternionic structure and for such bundles the vanishing of the top Chern class already ensures two sections.

An application. Using the Atiyah–Singer index theorem, R. L. E. Schwarzenberger showed in (5) that if ξ is a 2-plane bundle over $\mathbb{C}P^4$ with Chern classes c_1, c_2 then

$$c_2(c_2 + 1 - 3c_1 - 2c_1^2) \equiv 0 \bmod 12.$$

He has asked the converse question: given a pair of integers c_1, c_2 satisfying this congruence, is there a 2-plane bundle over $\mathbb{C}P^4$ with these Chern classes? Here we answer

https://doi.org/10.1017/S0305004100050738 Published online by Cambridge University Press

his question positively. Eric Brown has also studied this question extensively using Postnikov systems and I am very grateful to him for arousing my interest in the problem.

By Corollary 4, it is enough to find a stable bundle over $\mathbb{C}P^4$ with the required Chern classes. Recently Alan Thomas (10) has calculated which Chern classes can be realized by stable bundles over $\mathbb{C}P^n$–this would allow us to read off the result. However for completeness we give an *ad hoc* proof for $\mathbb{C}P^4$, following the method given in (7) for constructing suitable bundles over $\mathbb{C}P^3$.

PROPOSITION 5. *For every pair of integers a, b such that*

$$b(b+1-3a-2a^2) \equiv 0 \bmod 12$$

there is an element $x \in \tilde{K}(\mathbb{C}P^4)$ whose Chern classes are $c_1 = a$, $c_2 = b$, $c_3 = 0$, $c_4 = 0$.

We need a preliminary result.

LEMMA 6. *There are bundles whose only non-zero Chern classes are as follows*

(*a*) *θ over S^8 with $c_4\theta = 6$.*

(*b*) *ϕ over $X = S^6 \cup_\eta e^8$ with $c_3\phi = 2$, $c_4\phi = 3$.*

(*c*) *ψ over $HP^2 = S^4 \cup_\nu e^8$ with $c_2\psi = -1$, $c_4\psi = 0$.*

Proof. (*a*) is a well known corollary of Bott's periodicity theorem and is given in (4).

(*b*) Let h_3, h_4 generate H^6X, H^8X respectively. There are many ways of seeing (e.g. (1)) that there is a $\xi \in \tilde{K}(X)$ with

$$\mathrm{ch}\,(\xi) = h_3 + \tfrac{1}{2}h_4.$$

But
$$3!\,\mathrm{ch}_3(\xi) = 3c_3(\xi)$$

and
$$4!\,\mathrm{ch}_4(\xi) = -4c_4(\xi)$$

by Newton's formula ((5), p. 92) (note that all the cross-terms vanish). Hence

$$c_3(\xi) = 2h_3 \quad \text{and} \quad c_4(\xi) = -3h_4.$$

Let $q: X \to X \vee S^8$ be the map that pinches the boundary of a small disc in e^8 to a point. Then the bundle ϕ whose classifying map is

$$X \xrightarrow{q} X \vee S^8 \xrightarrow{\xi \vee \theta} BU$$

has Chern classes $c_3(\phi) = 2h_3$ and $c_4(\phi) = 3h_4$.

(*c*) We take ψ to be the complex bundle underlying the canonical bundle on the quaternionic projective plane, HP^2.

Proof of Proposition 5. We will construct the element x by pulling a suitable bundle back along a diagonal map

$$\mathbb{C}P^4 \to \mathbb{C}P^4 \times HP^2 \times X \times S^8.$$

This map is the diagonal of the following:

(1) the identity map,

(2) the map obtained by composing the inclusion $\mathbb{C}P^4 \to \mathbb{C}P^5$ with the fibration $\mathbb{C}P^5 \to HP^2$ that has fibre S^2,

https://doi.org/10.1017/S0305004100050738 Published online by Cambridge University Press

(3) the map that pinches $\mathbb{C}P^2$ to a point, and

(4) the map that pinches $\mathbb{C}P^3$ to a point.

Let λ be the canonical bundle over $\mathbb{C}P^4$ whose Chern class is x, then the total Chern class of the pullback of the external direct sum

$$\lambda^a - b\psi + c\phi + d\theta$$

along this diagonal map is

$$(1+ax)(1-x^2)^{-b}(1+2x^3+3x^4)^c(1+6x^4)^d$$
$$= 1+ax+bx^2+(ab+2c)x^3+(2ac+\tfrac{1}{2}b(b+1)+3c+6d)x^4.$$

We want $$c_3 = ab+2c = 0$$

and $$c_4 = 2ac+\tfrac{1}{2}b(b+1)+3c+6d = 0.$$

Hence we must have $ab = -2c$ (so we must have $ab \equiv 0 \bmod 2$) and

$$\begin{aligned} -12d &= 6c+b^2+b+4ac \\ &= -3ab+b^2+b-2a^2b \\ &= b(b+1-3a-2a^2). \end{aligned}$$

Hence if a, b are such that $ab \equiv 0 \bmod 2$ and

$$b(b+1-3a-2a^2) \equiv 0 \bmod 12$$

we can find c, d so that $\quad c_1 = a, \quad c_2 = b, \quad c_3 = 0, \quad c_4 = 0$

(note that $ab \equiv 0 \bmod 2$ is a consequence of the mod 12 congruence, and is the necessary and sufficient condition for the existence of a bundle on $\mathbb{C}P^3$ with Chern classes a and b).

It is of interest to know which Chern classes can be realized by algebraic $\mathbb{C}^2$-bundles over $\mathbb{C}P^4$. The referee pointed out that recently Horrocks and Mumford have constructed such a bundle with total Chern class $1+5x+10x^2$.

REFERENCES

(1) ADAMS, J. F. On the groups $J(X)-IV$, *Topology* **5** (1966), 21–71.

(2) ATIYAH, M. F. Immersions and embeddings of manifolds. *Topology* **1** (1961), 125–132.

(3) BLAKERS, A. L. and MASSEY, W. S. The homotopy groups of a triad II. *Ann. of Math.* **55** (1952), 192–201.

(4) BOTT, R. The space of loops on a Lie group. *Michigan Math. J.* **5** (1958), 35–61.

(5) HIRZEBRUCH, F. *Topological methods in algebraic geometry*, translated with new appendix by R. L. E. Schwarzenberger (Springer Verlag, 1966).

(6) KERVAIRE, M. A. Some nonstable homotopy groups of Lie groups. *Illinois J. Math.* **4** (1960), 161–169.

(7) REES, E. Embeddings of real projective space. *Topology* **10** (1971), 309–312.

(8) THOMAS, E. Postnikov invariants and higher order cohomology operations. *Ann. of Math.* (1967)**85**, 184–217.

(9) THOMAS, E. Real and complex vector fields on manifolds. *J. Math. Mech.* **16** (1967), 1183–1206.

(10) THOMAS, A. Almost complex structures on complex projective spaces (to appear).

https://doi.org/10.1017/S0305004100050738 Published online by Cambridge University Press