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Inequalities on partial traces of positive semidefinite block matrices

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Abstract. Inequalities on partial traces of positive semidefinite matrices are studied. Extensions of several existing inequalities on the determinant of partial traces are then obtained. Particularly, we improve a determinantal inequality given by Lin [Canad. Math. Bull. **59**(2016)].

1 Introduction

Let $\mathbf{M}_n$ and $\mathbf{M}_m(\mathbf{M}_n)$ be the set of all $n \times n$ complex matrices and the set of all $m \times m$ block matrices with each block in $\mathbf{M}_n$, respectively. For $A, B \in \mathbf{M}_n$, we write $A \geq B$ ($A > B$) if $A - B$ is positive semidefinite (positive definite). In particular, $A \geq 0$ ($A > 0$) if A is positive semidefinite (positive definite). The Kronecker product of $A = [a_{i,j}]_{i,j=1}^m \in \mathbf{M}_m$ and $B \in \mathbf{M}_n$ is denoted by $A \otimes B = [a_{i,j}B]_{i,j=1}^m \in \mathbf{M}_m(\mathbf{M}_n)$. The usual trace of A is denoted by $\mathrm{tr}A$.

Two partial traces of $A = [A_{i,j}]_{i,j=1}^m \in \mathbf{M}_m(\mathbf{M}_n)$ are defined by

$$\mathrm{tr}_1 A = \sum_{i=1}^m A_{i,i} \in \mathbf{M}_n \qquad \text{and} \qquad \mathrm{tr}_2 A = \left[\mathrm{tr}A_{i,j}\right]_{i,j=1}^m \in \mathbf{M}_m.$$

Partial traces are introduced with various applications in quantum information theory; see [13, 14]. In particular, they are closely connected to the subadditivity property of entropies; see [2, 3]. Besides, they are special matrix operations which are interesting in their own right. In fact, they have been studied in matrix theory without under a specific name, for example [9, 8, 15]. Recent studies on matrix inequalities involving partial traces can be found in [4, 6, 7, 11].

A density matrix is a positive semidefinite matrix with trace equal to 1. Lin [11] showed that for every density matrix $A \in \mathbf{M}_m(\mathbf{M}_n)$,

$$1 + \det A \geq \det(\mathrm{tr}_1 A)^m + \det(\mathrm{tr}_2 A)^n. \tag{1.1}$$

This inequality was inspired by an inequality of Schatten p-norms on partial traces which was given by Audenaert [2]. Fiedler and Markham [9] showed that for positive semidefinite $A \in \mathbf{M}_m(\mathbf{M}_n)$,

$$\left(\frac{\det(\mathrm{tr}_2 A)}{n}\right)^n \geq \det A. \tag{1.2}$$

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Analogous inequality on $\mathrm{tr}_1 A$ was given by Lin [12] which states that for positive semidefinite $A \in \mathbf{M}_m(\mathbf{M}_n)$,

$$\left(\frac{\det(\mathrm{tr}_1 A)}{m}\right)^m \geq \det A. \tag{1.3}$$

In fact, both inequalities (1.2) and (1.3) can be improved to

$$\left(\frac{\det(\mathrm{tr}_2 A)}{n^m}\right)^n \geq \det A \qquad \text{and} \qquad \left(\frac{\det(\mathrm{tr}_1 A)}{m^n}\right)^m \geq \det A \tag{1.4}$$

by applying the log-concavity of the determinant over the cone of positive semidefinite matrices; see [6, 12]. The main goal of the paper is to give an improvement on inequality (1.1). Moreover, we will present a conceptual and unifying proof of both inequalities in (1.4).

2 Determinantal inequalities of partial traces

We first recall the definition of completely positive maps. A linear map $\Phi : \mathbf{M}_n \to \mathbf{M}_k$ is called positive if $\Phi(A) \geq 0$ for all $A \geq 0$. A positive map $\Phi : \mathbf{M}_n \to \mathbf{M}_k$ is called m-positive if for every $A \in \mathbf{M}_m(\mathbf{M}_n)$,

$$A = [A_{i,j}]_{i,j=1}^m \geq 0 \qquad \Longrightarrow \qquad [\Phi(A_{i,j})]_{i,j=1}^m \geq 0. \tag{2.1}$$

It is said to be completely positive if (2.1) holds for all integers $m \geq 1$. A completely positive map $\Phi : \mathbf{M}_n \to \mathbf{M}_k$ is called unital if $\Phi(I_n) = I_k$, and it is called trace preserving if $\mathrm{tr}\,(\Phi(A)) = \mathrm{tr}\, A$ for all $A \geq 0$. We refer the reader to [4, Chapter 3] for basic properties of completely positive maps. It is known that $\mathrm{tr}_1(\cdot)$ and $\mathrm{tr}_2(\cdot)$ are trace preserving completely positive maps on $\mathbf{M}_m(\mathbf{M}_n)$. We have the following result.

Proposition 2.1 *Let $A \in \mathbf{M}_m(\mathbf{M}_n)$ be positive semidefinite. Then,*

$$\det(\mathrm{tr}_1 A) \leq \left(\frac{\mathrm{tr} A}{n}\right)^n \qquad \textit{and} \qquad \det(\mathrm{tr}_2 A) \leq \left(\frac{\mathrm{tr} A}{m}\right)^m.$$

Proof As $\mathrm{tr}_1(\cdot)$ is trace preserving, we have $\mathrm{tr}(\mathrm{tr}_1 A) = \mathrm{tr} A$. Let λ_j, $j = 1, \ldots, n$, be the eigenvalues of $\mathrm{tr}_1 A$. Then, by the AM–GM inequality, we have

$$\frac{\mathrm{tr} A}{n} = \frac{\lambda_1 + \cdots + \lambda_n}{n} \geq \sqrt[n]{\lambda_1 \cdots \lambda_n} = \sqrt[n]{\det(\mathrm{tr}_1 A)}.$$

The second inequality on $\det(\mathrm{tr}_2 A)$ can be shown similarly. ∎

As a consequence of Proposition 2.1, we have the following inequalities for density matrices.

Corollary 2.2 *Let $A \in \mathbf{M}_m(\mathbf{M}_n)$ be a density matrix. Then*

$$\det(\mathrm{tr}_1 A) \leq \left(\frac{1}{n}\right)^n \qquad \textit{and} \qquad \det(\mathrm{tr}_2 A) \leq \left(\frac{1}{m}\right)^m. \tag{2.2}$$

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Remark 2.3 Let $A \in \mathbf{M}_m(\mathbf{M}_n)$ be a density matrix. When $n = 1$, we have $\mathrm{tr}_1 A = A$ and $\mathrm{tr}_2 A = 1$. Similarly, for $m = 1$, we see that $\mathrm{tr}_2 A = A$ and $\mathrm{tr}_1 A = 1$. Hence, if $n = 1$ or $m = 1$, then, inequality (1.1) is equivalent to (2.2). Now for $m, n \geq 2$, Corollary 2.2 implies that

$$1 > \left(\frac{1}{n}\right)^n + \left(\frac{1}{m}\right)^m \geq \det(\mathrm{tr}_1 A) + \det(\mathrm{tr}_2 A) \geq \det(\mathrm{tr}_1 A)^m + \det(\mathrm{tr}_2 A)^n.$$

Therefore, Corollary 2.2 improves determinantal inequality (1.1) on the partial traces of density matrices.

For positive semidefinite $A = [A_{i,j}]_{i,j=1}^m \in \mathbf{M}_m(\mathbf{M}_n)$, it is known that $[\det A_{i,j}]_{i,j=1}^m \in \mathbf{M}_m$ is also positive semidefinite; see [1]. By applying Proposition 2.1 on $[\det A_{i,j}]_{i,j=1}^m$, we have the following result.

Corollary 2.4 *Let $A = [A_{i,j}]_{i,j=1}^m \in \mathbf{M}_m(\mathbf{M}_n)$ be positive semidefinite. Then,*

$$\det\left([\det A_{i,j}]_{i,j=1}^m\right) \leq \left(\frac{\sum_{i=1}^m \det A_{i,i}}{m}\right)^m.$$

Remark 2.5 Let $A = [A_{i,j}]_{i,j=1}^m \in \mathbf{M}_m(\mathbf{M}_n)$ be positive semidefinite. Choi [5, Theorem 2] showed that

$$\det A \leq \left(\frac{\sum_{i=1}^m \det A_{i,i}}{m}\right)^m.$$

As

$$\det A \leq \det\left([\det A_{i,j}]_{i,j=1}^m\right),$$

(see [15]), Corollary 2.4 is an improvement of Choi's result.

Let $A, B \in \mathbf{M}_n$ be positive semidefinite with eigenvalues $\lambda_1(A) \geq \cdots \geq \lambda_n(A) \geq 0$ and $\lambda_1(B) \geq \cdots \geq \lambda_n(B) \geq 0$, respectively. We write $A \prec B$ if the vector $\big(\lambda_1(A), \ldots, \lambda_n(A)\big)$ is majorized by the vector $\big(\lambda_1(B), \ldots, \lambda_n(B)\big)$, that is,

$$\sum_{i=1}^k \lambda_i(A) \leq \sum_{i=1}^k \lambda_i(B), \qquad k = 1, \ldots, n,$$

with equality holds for $k = n$.

Proposition 2.6 *Let $A \in \mathbf{M}_m(\mathbf{M}_n)$ be positive semidefinite. Then,*

$$I_m \otimes \frac{tr_1 A}{m} \prec A \qquad \textit{and} \qquad \frac{tr_2 A}{n} \otimes I_n \prec A.$$

Proof It is straightforward to check that $X \mapsto I_m \otimes \frac{\mathrm{tr}_1 X}{m}$ and $X \mapsto \frac{\mathrm{tr}_2 X}{n} \otimes I_n$ are unital trace preserving completely positive maps of $\mathbf{M}_m(\mathbf{M}_n)$. The result follows by the well-known Uhlmanns theorem; see [16]. ∎

We have the following consequence of Proposition 2.6.

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Theorem 2.7 *Let $A \in \mathbf{M}_m(\mathbf{M}_n)$ be positive semidefinite matrix. Then,*

$$\left(\frac{\det(\mathrm{tr}_1 A)}{m^n}\right)^m \geq \det A \quad \textit{and} \quad \left(\frac{\det(\mathrm{tr}_2 A)}{n^m}\right)^n \geq \det A.$$

Proof Note that if A, B are positive semidefinite and $A \prec B$, then $\det B \leq \det A$; see [17, p. 346]. Hence by Proposition 2.6, we have

$$\left(\frac{\det(\mathrm{tr}_1 A)}{m^n}\right)^m = \det\left(\frac{\mathrm{tr}_1 A}{m}\right)^m = \det\left(I_m \otimes \frac{\mathrm{tr}_1 A}{m}\right) \geq \det A,$$

and

$$\left(\frac{\det(\mathrm{tr}_2 A)}{n^m}\right)^n = \det\left(\frac{\mathrm{tr}_2 A}{n}\right)^n = \det\left(\frac{\mathrm{tr}_2 A}{n} \otimes I_n\right) \geq \det A. \qquad \blacksquare$$

Remark 2.8 In general, the two quantities $\left(\frac{\det(\mathrm{tr}_1 A)}{m^n}\right)^m$ and $\left(\frac{\det(\mathrm{tr}_2 A)}{n^m}\right)^n$ are not comparable. For example, let

$$A = \begin{pmatrix} I_2 & I_2 \\ I_2 & I_2 \end{pmatrix} \quad \text{and} \quad \tilde{A} = \mathrm{diag}(1, 0, 1, 0).$$

Then,

$$\det(\mathrm{tr}_2 A) = \det\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 0, \quad \det(\mathrm{tr}_1 A) = \det(2I_2) = 4,$$

and

$$\det\left(\mathrm{tr}_2 \tilde{A}\right) = \det I_2 = 1, \quad \det\left(\mathrm{tr}_1 \tilde{A}\right) = \det\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

Remark 2.9 Lin [12] showed that for positive semidefinite $A \in \mathbf{M}_m(\mathbf{M}_n)$,

$$\det(I_n + \mathrm{tr}_1 A)^m \geq \det(I_{mn} + mA) \quad \text{and} \quad \det(I_m + \mathrm{tr}_2 A)^n \geq \det(I_{mn} + nA). \tag{2.3}$$

In fact, both inequalities become immediate by applying Theorem 2.7 for matrices $I_{mn} + mA$ and $I_{mn} + nA$, respectively.

The following result can be regarded as a complement to (2.3).

Theorem 2.10 *Let $A \in \mathbf{M}_m(\mathbf{M}_n)$ be positive definite. Then,*

$$\frac{\det(I_{mn} + mA)}{m^{nm} \det A} \geq \frac{\det(I_n + \mathrm{tr}_1 A)^m}{\det(\mathrm{tr}_1 A)^m} \quad \textit{and} \quad \frac{\det(I_{mn} + nA)}{n^{mn} \det A} \geq \frac{\det(I_m + \mathrm{tr}_2 A)^n}{\det(\mathrm{tr}_2 A)^n}.$$

Proof By [10, Theorem 2.3(b)], for every unital trace preserving completely positive map $\Phi : \mathbf{M}_N \to \mathbf{M}_N$ and positive definite $A \in \mathbf{M}_N$,

$$\det(I_N + A^{-1}) \geq \det(I_N + \Phi(A)^{-1}).$$

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Equivalently, we have

$$\frac{\det(I_N + A)}{\det A} \geq \frac{\det(I_N + \Phi(A))}{\det \Phi(A)}.$$

Hence, by considering the map $X \mapsto I_m \otimes \frac{\mathrm{tr}_1 X}{m}$ with $X = mA$, we have

$$\frac{\det(I_{mn} + mA)}{m^{mn} \det A} = \frac{\det(I_{mn} + mA)}{\det(mA)} \geq \frac{\det(I_{mn} + I_m \otimes \mathrm{tr}_1 A)}{\det(I_m \otimes \mathrm{tr}_1 A)} = \left(\frac{\det(I_n + \mathrm{tr}_1 A)}{\det(\mathrm{tr}_1 A)}\right)^m.$$

The second inequality on $\mathrm{tr}_2 A$ can be shown similarly by considering the map $X \mapsto \frac{\mathrm{tr}_2 X}{n} \otimes I_n$ with $X = nA$. ∎

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