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The Boltzmann relation and its validity in magnetized plasmas

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Abstract. This paper is concerned to give a definitive account of a physical situation of current practical interest by examining the plasma solution for a plasma in coaxial geometry with an applied axial magnetic field. It builds on earlier work concerned with plasma diamagnetism and concentrates on the parameters involved at low pressures and low collisionalities but can be extended to situations where the ions are magnetized.

1. Introduction

The plasma being of finite extent is necessarily non-uniform in density, and in the plasma approximation the ion radial speed v_i at both boundaries is the 'Bohm speed', but because of the geometry the ion velocity is directed inwards at the inner boundary and outwards at the outer boundary.

By analogy with a magnetized positive column as treated by Forrest and Franklin (1966), we expect to find a 'plasma balance equation', as Tonks and Langmuir (1929) called it, relating the volume generation rate Z, the plasma dimension $r_o - r_i$, and the ion sound speed c_s, where in this case it is $Z\int_{r_i}^{r_o} nr\,\mathrm{d}r/c_s(n_o r_o + n_i r_i) =$ a number of the order of 1, r_i being the radius of the inner cylinder and r_o the outer one, and n is the number density. The natural origin of co-ordinates in coaxial geometry is the axis $r = 0$, but the maximum of potential, which is at some point between r_i and r_o, that cannot be determined in advance for given r_i and r_o, and this creates a complication, which can be resolved in one of two ways, either choose $r = 0$ or $r = r_{\max}$, where V is taken to be zero.

We use the fluid approximation to describe the plasma because it is known that the free-fall power series expansion treatment given by Tonks and Langmuir (1929) for the positive column fails for quite modest magnetic fields, Forrest (1967), and in later work Tonks (1937) used the fluid approximation when treating magnetized plasmas. The existence of the magnetic field causes there to be azimuthal motion of both the electrons and ions but in opposite directions and not with a constant angular rate. This, in turn, can cause the *total* magnetic field acting on the plasma to vary with radius at sufficiently high applied magnetic field strengths.

This motion was examined by Forrest and Franklin (1968) in a positive column to explain the phenomenon of plasma diamagnetism commented on by Alfvén (1950).

We use the ion radial speed as the independent variable because it is well known (Franklin 1976) that this avoids any computational difficulty arising from the fact that the plasma equations are singular at the point(s), where the ion speed $v_i = c_s$.

2. The basic equations

We must take care over these since we are dealing with a non-uniform plasma in cylindrical geometry with volume generation. In the plasma approximation, the electron density n_e and the ion density n_i are equal to n, and the generation equations $\mathrm{div}(n_e v_e) = Z n_e$ and $\mathrm{div}(n_i v_i) = Z n_e$ require so-called ambipolarity, which in this case means that the radial components of velocity of both ions and electrons are equal, i.e. $v_{re} = v_{ri} = v_r$.

$$\text{Thus, } \mathrm{div}(nv_r) = Zn. \tag{1}$$

The electron and ion momentum equations need to reflect the fact that there is volume generation and this means that they have to be expressed in terms of dyadics giving –

$$m\mathrm{div}(n\mathbf{vv}) = -k_B T_e\,\mathrm{grad}(n) + en\,\mathrm{grad}(V) - en\,\mathbf{v}\times\mathbf{B} - nmv_e\mathbf{v} \tag{2}$$

$$M\mathrm{div}(n\mathbf{vv}) = -en\,\mathrm{grad}(V) + en\,\mathbf{v}\times\mathbf{B} - nMv_i\mathbf{v}, \tag{3}$$

where we can use $\mathrm{div}(n\mathbf{vv}) = \mathbf{v}\mathrm{div}(n\mathbf{v}) + n(\mathbf{v}.\mathrm{grad})\mathbf{v}$.

Separating out the radial (r) and azimuthal (θ) components, we arrive at four equations for $V, v_r, v_{e\theta}, v_{i\theta}$ and n, which combined with $\mathrm{d}(nv_r r)/r\mathrm{d}r = Zn$, give five first-order differential equations in the variables concerned.

We now make an apparent deviation by considering an unmagnetized collisionless plasma with volume generation in one dimension. The basic equations are $\mathrm{d}(nv)/\mathrm{d}x = Zn$, $Mv\mathrm{d}v/\mathrm{d}x + e\mathrm{d}V/\mathrm{d}x = -MvZ$ and $k_B T_e/n\,\mathrm{d}n/\mathrm{d}x + mv\mathrm{d}v/\mathrm{d}x - e\mathrm{d}V/\mathrm{d}x = -mvZ$, and the determinant of them regarded as three simultaneous

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equations in $\mathrm{d}v/\mathrm{d}x$, $\mathrm{d}(eV)/\mathrm{d}x$ and $1/n\ \mathrm{d}n/\mathrm{d}x$ is readily shown to be $(M+m)v^2 - k_B T_e$. Where this is zero, the equations become singular, and thus, the Bohm speed *taking electron inertia into account* c_s* is

$$(kT_e/(M+m))^{1/2}.$$

We later examine departures from the Boltzmann relation for electrons implied by the title, which from (2) applies strictly only when m, $\boldsymbol{v}_e$ and $\mathbf{B}$ are all zero, and so we must retain the electron mass in all equations because the effect of a magnetic field *in the absence of collisions*, either for momentum transfer or for ionization, is to 'tie' electrons to a particular field line and thus in response to an electric field they behave as if they are infinitely massive. Thus, to cover the full parameter range, we must be able to treat the electrons as having an effective mass, ranging from $m_{\mathrm{eff}} = m$ to $m_{\mathrm{eff}} = \text{infinity}$.

We normalize the equations from (2) and (3) separating out radial and azimuthal components of velocity and introduce $\varphi = -eV/k_B T_e$, $V_r = v_r/c_s^*$, $V_{e\theta} = v_{e\theta}/c_s^*$, $V_{i\theta} = v_{i\theta}/c_s^*$, $R = rZ/c_s^*$, $N = n/n_{\max}$, $\omega_{ce} = eB/m$, $\omega_{ci} = eB/M$, $\delta_e = \nu_e/Z$, $\delta_i = \nu_i/Z$, $\beta_e = \omega_{ce}/Z$, $\beta_i = \omega_{ci}/Z$ and $\mu = m/M$ to give

$$\mathrm{d}V_{\theta i}/\mathrm{d}R = -\beta_i - (1+\delta_i)V_{\theta i}/V_r, \tag{4}$$

$$\mathrm{d}V_{\theta e}/\mathrm{d}R = \beta_e - (1+\delta_e)V_{\theta e}/V_r \tag{5}$$

$$V_r \mathrm{d}V_r/\mathrm{d}R = \mathrm{d}\phi/\mathrm{d}R + V_{\theta i}\beta_i - V_r(1+\delta_i), \tag{6}$$

$$\mu V_r \mathrm{d}V_r/\mathrm{d}R = -\mathrm{d}\phi/\mathrm{d}R - 1/N\,\mathrm{d}N/\mathrm{d}R - \mu V_{\theta e}\beta_e - \mu V_r(1+\delta_e). \tag{7}$$

The first two terms on the right-hand side of (7), on their own, correspond to the Boltzmann relation, but (7) has to be folded with (1) and (6), where (1) has become

$$\mathrm{d}(NV_r R)/\mathrm{d}R = NR. \tag{8}$$

Our choice for the normalization of the spatial coordinate relates back to the plasma balance equation and does not 'favour' any of the radii involved. They are determined by the values of R where $V_r = \pm 1$, in terms of Zr/c_s^* being a number of the order of one with $R = R_{\max}$ at the plasma peak and having a negative value $-\lambda_1$, where $V_r = -1$ and a positive value λ_2, where $V_r = +1$. These two values give the ratio of r_i to r_o by a simple transformation.

We can obtain a local value for the deviation from the Boltzmann relation in terms of an integral from the radius, where $V_r = 0$, and this has consequences for the statement made by Allen (2008), and discussed by Ahedo (2009) that at the plasma boundary the Boltzmann relation in its differential form $-\mathrm{d}N/N\mathrm{d}\varphi = -1$, is satisfied there. $\mathrm{d}N/N\mathrm{d}R$ and $\mathrm{d}\varphi/\mathrm{d}R$ go to infinity in the same manner mathematically, but it has no physical content.

Elsewhere, there is a progressive departure until the boundaries are reached at $V_r = +1$, and -1. The question as to whether it is appropriate to suggest that the Boltzmann relation holds in the differential form, $\mathrm{d}N/N = -\mathrm{d}\varphi$, we leave open for the moment, but return to it later.

In order to test our treatment, we have carried out computations over a range of parameters, but in the next section, we concentrate on the matter of deviation from the Boltzmann relation because it has often been used as an *ab initio* assumption when it cannot be justified in terms of the range of physical parameters under consideration.

3. Deviations from the Boltzmann relation

From the above expressions for the spatial variation of potential and density, it is possible to give a local expression for the departure from the Boltzmann relation by taking their difference and showing that it increases towards the plasma boundary. This can be expressed as $-N = \exp(-\phi) \times Q(R)$, where $Q(R)$ is a positive quantity less than 1, but varying with R given by $-Q(R) = \exp[-\mu\{\int_{R_{\max}}^{R}(V_r\delta_e + \beta_e(V_{e\theta})\}\mathrm{d}R]$. Only electron terms enter here because we are comparing the modification of the Boltzmann relation *for electrons* in the presence of a magnetic field with its expression in the absence of a magnetic field and the limit $m/M \to 0$ validates our result.

This expression corresponds to an integral relation given by Zimmermann et al. (2010), but is local rather than being integrated spatially, and so it is the more 'dramatic'.

4. The influence of the plasma motion on the magnetic field

The externally applied magnetic field is of magnitude B and it causes motion of the charged particles such that an azimuthal current, in turn, generates a magnetic field given by $-\mathrm{d}B/\mathrm{d}r = \mu_0 ne(v_{e\theta} - v_{i\theta})$. Converting to normalized variables requires one to introduce a value for the maximum plasma density n_0. When that is done, one obtains a local magnetic field $B_L = B[1 + \int_{R\max}^{R}\omega_{pe}^2/Z\,\omega_{ce}c_s^2/c^2(V_{e\theta} - V_{i\theta})\mathrm{d}R]$, so the effect of the plasma motion is to increase the magnetic field near the plasma boundary where it reaches its maximum value, and it is the greater the plasma density and the magnetic field. $B_L/B - 1$ is positive definite, but the coefficient $\omega_{pe}^2/\omega_{ce}Zc_s^2/c^2$ though dimensionless cannot be transformed to anything simpler.

This confirms the fact that a bounded magnetized plasma is diamagnetic, the matter that Alfvén raised, and which was confirmed for the low pressure positive column by Forrest and Franklin (1968), both theoretically and experimentally. Using results given by them, it is possible to show that the magnitude of $v_{e\theta}$ is typically

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Table 1. A selection of computed results for low pressure plasmas.

1 ω_{ce}/Z	2 ω_{ci}/Z	3 ν_e/Z	4 Zr_i/c_s^*	5 $Zr_{\max}/c_s^*$	6 Zr_o/c_s^*	7 r_o/r_i	Boltz$_o$	Comment
1000	1	1	0.044	1.0	1.13	25.4	1.91	Varying B at low pressure
500	0.5	1	0.177	1.0	1.25	7.06	1.45	
25	0.025	1	0.576	1.0	1.70	2.96	1.01	
1	0.001	1	0.577	1.0	1.71	2.95	1.00	
25	0.025	100	0.600	1.0	1.61	2.68	1.00	Varying pressure
25	0.025	10	0.579	1.0	1.70	2.94	1.00	
25	0.025	1	0.576	1.0	1.70	2.95	1.01	
25	0.025	0.1	0.575	1.0	1.70	2.96	1.01	
25	0.025	1	4.45	5.0	5.58	1.25	1.01	Varying the plasma peak
25	0.025	1	1.51	2.0	2.63	1.74	1.01	
25	0.025	1	0.576	1.0	1.70	2.95	1.00	
25	0.025	1	0.193	0.5	1.19	6.17	1.00	

Column 1 gives a measure of the magnetization of electrons and column 2 that of ions.
Column 3 gives the collisionality of the plasma going from free fall to collisional.
Columns 4–6 give the normalized boundaries and the maximum, and as column 5 varies so does the shape of the plasma.
Column 7 gives the 'aspect ratio' of the discharge.
Column 8 labelled Boltz$_o$ is the value of $(N\exp(\varphi))_o$, and thus, is a measure of the deviation from the Boltzmann relation at r_o. The values at the inner radius r_i vary similarly.
Groups of rows indicate that parameter is being varied as described in the **Comment**. This means that some rows are repeated.

four orders larger than $v_{i\theta}$. For some time, this diamagnetic effect has been used as a standard diagnostic technique in fusion plasmas where electrostatic probes confront severe problems (see Wesson 1987).

5. An extended discussion of the two fluid momentum equations

Here, we examine these in more detail in the presence of a magnetic field. Taking this field to be uniform, there is general agreement that for the case of a non-uniform plasma – and all bounded plasmas are non-uniform in density – there is ambipolar motion of both electrons and ions along the density gradient and that there is motion of both types of particle in opposite directions about the axial direction. Since the particles have speeds varying in the radial direction, the azimuthal motion of the fluids is not one with constant angular speed, and so the introduction of forces of the form $v_{i\theta}^2/r$ as in Zimmermann et al. (2010) cannot be justified, the forces acting on the fluids causing the azimuthal motion are given *directly* by the eBv_r/m and eBv_r/M terms.

Thus, the terms involving $v_{i\theta}^2$ given in Allen (2008) and used by Zimmermann et al. were not found above. If the $v_{i\theta}^2$ terms were valid then the authors should have correspondingly carried through terms in $v_{e\theta}^2$, but those terms were not included apparently on the grounds of the relative smallness of the electron mass compared to the ion mass and they were explicitly dropped by Sternberg et al. (2008). But $v_{e\theta} \gg v_{i\theta}$, and so with the knowledge that $M^2/m^2 > v_{e\theta}^2/v_{i\theta}^2 > M^2v_i^2/m^2v_e^2$, as given in Franklin (1976), the $v_{e\theta}^2$ terms, if valid, should be included, but would be orders of magnitude larger than the $v_{i\theta}^2$ ones.

6. Extension into the sheath

This is a natural requirement to treat the problem fully, and the most obvious way would be to examine solutions involving two parameters of small magnitude, namely the ratio of the Debye length λ_{De} to the plasma size L and the electron cyclotron radius ρ_{ce} to the plasma size. Using matched asymptotic approximations is complicated with two *small* parameters, and probably, the best approach would be to carry out computations and then analyze them in the spirit of Franklin and Snell (2000, 2001), but we have not attempted to carry through such work.

However, we have carried out computations in plane geometry for *finite* values of λ_{De}/L and examined the behaviour of $\mathrm{d}N/N\mathrm{d}\varphi$ throughout the plasma sheath in the absence of a magnetic field and find that its departure from -1 is greatest in the plasma-sheath boundary region. This is to be expected because of the disjunction between the plasma solution and the sheath solution in that region, given that the Boltzmann relation is satisfied by definition at the plasma centre, and is well satisfied in the sheath near the wall.

7. Results and discussion

We have carried through computations for a wide range of parameters – corresponding to the physical variables $m, M, Z, T_e, B, \nu_e, \nu_i$, – and verified that in the case where the inner radius is zero, we obtain the results given by Forrest and Franklin (1968) for the positive column, equally for large values of both r_i and r_o, we showed that the results approached those obtained for plane geometry.

Separately, we examined the plasma balance equation and found the relation between volume generation rate

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and wall loss rate, showing that the eigenvalue had limiting values that corresponded similarly.

When the electrons were strongly magnetized, there was radial asymmetry with the plasma being driven towards the outer wall.

When the ions were magnetized, the departure from Boltzmann was greater and broke down due to instability from MHD modes.

Accordingly, we give as a rather comprehensive table (see Table 1) results for various parameters keeping some of the variables constant, on which the general features are intuitively not strongly dependent. Thus, we have taken m/M constant at 0.0001, $v_e = 100v_i$, but our table allows variation of other quantities to show the general trends as one particular variable is given three or more significantly different values and it is presented in such a way that one can see the trend and draw one's own conclusions for any particular practical situation.

To give this information over the range of parameters chosen would have required 36 rows with several repetitions – hence, the choices made. Giving figures with the spatial distributions would have required more space in terms of diagrams than could reasonably have been justified in order to reach the conclusions stated – hence, the approach adopted in Table 1.

To relate Z to T_e in the simplest of discharge models as described in Franklin (1976) over the pressure range would require one to designate a specific gas, and so we have left T_e as an implicit variable. But the general features for electropositive gases would remain much the same, whatever the gas.

Conclusions

(1) Interesting effects occur in a magnetized plasma in coaxial geometry;

(2) To examine these over a full range of parameters, it is necessary to include the electron mass at all stages;

(3) The Boltzmann relation as an *a priori* condition should be used with caution and be fully justified from the start when treating any problem involving a real plasma, that is one of finite extent, with volume ionization and particle collisions, but even the more so when there is an applied magnetic field of such magnitude as to affect the charged particle motion.

References

Ahedo, E. 2009 *Phys. Plasmas* **16**, 113503.

Alfvén, H. 1950 *Cosmical Electrodynamics*. Oxford: Clarendon, p. 58.

Allen, J. E. 2008 *Contrib. Plasma Phys.* **48**, 400.

Forrest, J. R. 1967 *D Phil thesis*, Oxford University.

Forrest, J. R. and Franklin, R. N. 1966 *Brit. J. Appl. Phys.* 17, 1061.

Forrest, J. R. and Franklin, R. N. 1968 *Proc. Roy. Soc.* A **305**, 251.

Franklin, R. N. 1976 *Plasma Phenomena in Gas Discharges*. Oxford: Clarendon.

Franklin, R. N. and Snell, J. 2000 *Phys. Plasmas* **7**, 3077; 2001 *Phys. Plasmas* **8**, 643.

Sternberg, N. et al. 2008 In: *Proc. 35th EPS Conf. Plasma Phys, Hersonissos*, ECA, Vol. 32.

Tonks, L. and Langmuir, I. 1929 *Phys. Rev.* **34**, 876.

Tonks, L. 1937 *Phys. Rev.* **51**, 744.

Wesson, J. 1987 *Tokomaks*. Oxford: Clarendon.

Zimmermann, T. M. G., Coppins, M. and Allen, J. E. 2010 *Phys. Plasmas* **17**, 022301.

https://doi.org/10.1017/S0022377811000298 Published online by Cambridge University Press