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A mixing completely scrambled system exists

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Abstract. We prove that there exists a topologically mixing homeomorphism which is completely scrambled. We also prove that, for any integer $n \geq 1$, there is a continuum of topological dimension n supporting a transitive completely scrambled homeomorphism and an n-dimensional compactum supporting a weakly mixing completely scrambled homeomorphism. This solves a 15-year-old open problem.

1. *Introduction*

The notion of a Li–Yorke pair was introduced for the first time by Li and Yorke in their seminal paper [**14**] from 1975. Recall that $x, y \in X$ form a *Li–Yorke pair* for a continuous map $T: X \to X$ on a compact metric space (X, d) if

$$\liminf_{n\to\infty} d(T^n(x), T^n(y)) = 0 \quad \text{and} \quad \limsup_{n\to\infty} d(T^n(x), T^n(y)) > 0,$$

and a set $D \subset X$ is *scrambled* when each pair of its distinct points is Li–Yorke. A natural question which arose immediately was about the possible size of a scrambled set. While it was already known that scrambled sets can be uncountable, it was soon realized that, when $X = [0, 1]$, a scrambled set is never residual [**9**] but can have full Lebesgue measure [**6**, **16**] (see also [**15**]; for more detailed discussion on the size of scrambled sets, see survey paper [**4**] and the references therein). Clearly, the most extreme situation is when the whole of X is a scrambled set. In such a case, we say that (X, T) is *completely scrambled*. The question of whether a completely scrambled system can exist for a compact X was open for a while, and a positive answer was brought by the paper [**13**] by Huang and Ye. They proved that completely scrambled sets exist on the Cantor set, and then used a nice extension method to show that, for every $n \geq 1$, there exists a compact set (in

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fact, a continuum) X together with a completely scrambled homeomorphism $T: X \to X$. Unfortunately, their examples are not transitive, and hence they wondered whether these examples can be improved to obtain a transitive map. It was also known that a completely scrambled transitive, or even completely scrambled and weakly mixing, homeomorphism exists (see [**1**] or its extension in [**8**]) for a closed subset X in $[0, 1]^{\mathbb{N}}$ of unknown dimension (most likely infinite). Therefore the following question has remained open since the publication of [**13**] in 2001.

Question 1.1. Does there exist, for every $n \geq 1$, a continuum X of dimension n supporting a completely scrambled transitive homeomorphism?

In §4.2, we will show that the answer to the above question is positive. We will also prove the following theorem which extends the results of [**8**].

THEOREM 1.2. *There exists a mixing homeomorphism on the Cantor set which is completely scrambled.*

Note that the above result cannot be obtained by a direct modification of the approach in [**8**] because uniformly rigid systems (a core property in the construction in [**8**]) are never mixing. In our approach, we will use a technique of Shimomura [**18**], which is based on the construction of an inverse limit of graph covers introduced by Akin, Glasner and Weiss in [**3**]. While, in many aspects, our construction in the proof of Theorem 1.2 is inspired by [**18**], our proofs are different and somehow shorter.

Using Theorem 1.2 and a class of examples discovered by Glasner and Maon [**10**], we prove the following theorem.

THEOREM 1.3. *For every $n \geq 1$ there exists a compact set Y of topological dimension n supporting a completely scrambled and weakly mixing homeomorphism.*

2. *Preliminaries*

A *dynamical system* (X, T) is a pair consisting of a compact metric space (X, d) and a continuous map $T: X \to X$. By *dimension*, we always mean topological dimension, and a *Cantor set* is any zero-dimensional compact metric space without isolated points: that is, any space homeomorphic with the standard Cantor set in the real line. A *continuum* is a compact connected metric space.

2.1. *Transitivity and chaos.* For each $x \in X$, we define its (positive) *orbit* by $O_T^+(x) = \{T^n(x) : n \geq 0\}$ and its *ω-limit set* by $\omega_T(x) = \bigcap_{n\geq 0} \overline{O_T^+(T^n(x))}$. A set $Z \subset X$ is *invariant* if $T(Z) \subset Z$, and Z is a subsystem if it is closed and invariant, which, in other words, means that (Z, T) is a dynamical system. A point $x \in X$ is *recurrent* if $x \in \omega_T(x)$ and *minimal* if it is recurrent and $\omega_T(x) = \omega_T(y)$ for every $y \in \omega_T(x)$. An invariant set Z is *minimal* if $Z = \omega_T(x)$ for some $x \in X$.

A pair of points $x, y \in X$ is:

(i) *proximal* if $\liminf_{n\to\infty} d(T^n(x), T^n(y)) = 0$;

(ii) *asymptotic* if $\limsup_{n\to\infty} d(T^n(x), T^n(y)) = 0$; and

(iii) *Li–Yorke* if it is proximal but not asymptotic.

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A set $D \subset X$ is *scrambled* if any two distinct $x, y \in D$ are a Li–Yorke pair. If there exists an uncountable scrambled set, then we say that (X, T) is *chaotic* in the sense of Li and Yorke. If the whole X is a scrambled set, then (X, T) is *completely scrambled*.

A dynamical system (X, T) is:

(i) *transitive* if, for any non-empty open sets U, V, there is $n > 0$ such that $T^n(U) \cap V \neq \emptyset$;

(ii) *weakly mixing* if the product system $(X \times X, T \times T)$ is transitive;

(iii) *mixing* if, for any two open sets U, V, there is N such that $T^n(U) \cap V \neq \emptyset$ for all $n \geq N$; and

(iv) *uniformly rigid* if $\liminf_{n\to\infty} \rho(T^n, \mathrm{id}_X) = 0$, where ρ is the metric of uniform convergence, that is, $\rho(S, R) = \sup_{x\in X} d(S(x), R(x))$.

Let $T : X \to X$ be a homeomorphism and let Z be a compact metric space obtained from $X \times [0, 1]$ by identifying pairs $(x, 1)$ with $(T(x), 0)$. The *suspension flow over* (X, T) is the flow $\langle \Phi_t \rangle_{t\in\mathbb{R}}$ defined on Z by the formula

$$\Phi_t(s, x) = (\{s + t\}, T^{\lfloor s+t \rfloor}(x)),$$

where $\lfloor r \rfloor$ and $\{r\}$ denote the integer and fractional parts of a real number $r \in \mathbb{R}$, respectively. Clearly, if (X, T) is transitive, then Y is a continuum.

A continuous map $\pi : X \to Y$ is a *factor* map between dynamical systems (X, T) and (Y, S) if π is surjective and $\pi \circ T = S \circ \pi$.

2.2. *Graph covers.* By a *graph* we mean a pair $G = (V, E)$ of finite sets, where $E \subset V \times V$. By a *vertex* in G we mean any element of V, and elements of E are called the *edges*. The graphs we consider are always *edge surjective*, that is, for every $v \in V$, there are $u, w \in V$ such that $(u, v), (v, w) \in E$. In other words, each vertex has an incoming and outgoing edge. A map $\phi : V_1 \to V_2$ is a *homomorphism* between graphs (V_1, E_1), (V_2, E_2) if, for every $(u, v) \in E_1$, $(\phi(u), \phi(v)) \in E_2$; to emphasize that ϕ is a graph homomorphism, we write $\phi : (V_1, E_1) \to (V_2, E_2)$. For simplicity, for edge $e = (u, v) \in E_1$, we use the notation $\phi(e) = (\phi(u), \phi(v))$. This extends onto paths $e_1 \dots e_n$ on (V_1, E_1) by the standard rule $\phi(e_1 \dots e_n) = \phi(e_1) \dots \phi(e_n)$. A homomorphism ϕ is *bidirectional* if $(u, v), (u, v') \in E_1$ implies that $\phi(v) = \phi(v')$ and $(w, u), (w', u) \in E_1$ implies that $\phi(w) = \phi(w')$. If ϕ is a bidirectional map between edge-surjective graphs, then we call it a *bd-cover*. Let $\mathcal{G} = \langle \phi_i \rangle_{i=0}^{\infty}$ denote a sequence of bd-covers $\phi_i : (V_{i+1}, E_{i+1}) \to (V_i, E_i)$, and let

$$V_{\mathcal{G}} = \varprojlim (V_i, \phi_i) = \left\{ x \in \prod_{i=0}^{\infty} V_i : \phi_i(x_{i+1}) = x_i \text{ for all } i \geq 0 \right\}$$

be the inverse limit defined by $\mathcal{G}$. As usual, let $\phi_{m,n} = \phi_n \circ \phi_{n+1} \circ \cdots \circ \phi_{m-1}$ and denote the projection from $V_{\mathcal{G}}$ onto V_n by $\phi_{\infty,n}$. Set

$$E_{\mathcal{G}} = \{(x, y) \in V_{\mathcal{G}} \times V_{\mathcal{G}} : (x_i, y_i) \in E_i \text{ for each } i = 1, 2, \dots\}.$$

As usual, V_i is endowed with discrete topology and $\mathbb{X} = \prod_{i=0}^{\infty} V_i$ is endowed with product topology. This topology is compatible with the metric given by $d(x, y) = 0$ when $x = y$

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and $d(x, y) = 2^{-k}$ when $x \neq y$ and $k = \min\{i : x_i \neq y_i\}$. Then $V_{\mathcal{G}}$ is a closed subset of $\mathbb{X}$ and we consider it with topology (and metric) induced from $\mathbb{X}$.

By a cycle on graph G we mean any path starting and ending in the same vertex. If $c_1, \ldots, c_n$ are cycles starting in the same vertex v, then we use $a_1c_1 + \cdots + a_nc_n$ to denote the cycle at v obtained by passing a_1 times cycle c_1, then a_2 times cycle c_2, etc. The length of a path η (i.e., the number of edges on it) is denoted by $|\eta|$. By $V(\eta)$, we denote the set of vertices on path η. The following important fact is given in [**17**, Lemma 3.5], with injectivity of $E_{\mathcal{G}}$ relying on the fact that the sequence is bidirectional.

LEMMA 2.1. *Let $\mathcal{G} = \langle \phi_i \rangle$ be a sequence of bd-covers $\phi_i : (V_{i+1}, E_{i+1}) \to (V_i, E_i)$. Then $V_{\mathcal{G}}$ is a zero-dimensional compact metric space and the relation $E_{\mathcal{G}}$ defines a homeomorphism.*

3. *A special homeomorphism*

3.1. *Construction of homeomorphism $(V_{\mathcal{G}}, T_{\mathcal{G}})$.* For every integer $n > 0$, we define a special vertex $v_{n,0} \in V_n$ and a special edge $e_{n,0} = (v_{n,0}, v_{n,0}) \in E_n$, and we put $G_0 = (\{v_{0,0}\}, \{e_{0,0}\})$. Next we will specify other vertices in V_n and accompanying edges, so that a graph G_n is defined. Our aim is to construct a special sequence of bd-covers. In particular, we put $\phi_n(v_{n+1,0}) = v_{n,0}$. We embed in each V_n exactly n additional cycles $c_{n,1}, \ldots, c_{n,n}$ (of appropriate length, which will be determined later), such that each cycle starts in $v_{n,0}$ and all the other vertices are pairwise distinct. For $n = 1$, we include in V_1 a cycle $c_{1,1}$ of length 10 and define $\phi_1(c_{1,1}) = 10e_{0,0}$.

We assume that the length of each $c_{n,i}$ is appropriate so that the following bd-covers are well defined. For $2 \leq i \leq n$, we put

$$\phi_n(c_{n+1,i}) = e_{n,0} + 2c_{n,i} + \cdots + 2c_{n,n} + e_{n,0}.$$

We also put $\phi_n(c_{n+1,n+1}) = (n+2)^2(\sum_{i=1}^{n} |c_{n,i}|)e_{n,0}$ and, finally,

$$\begin{aligned}\phi_n(c_{n+1,1}) = {} & e_{n,0} + 2c_{n,1} + 2e_{n,0} + 2c_{n,1} + \cdots \\ & + k_n e_{n,0} + 2c_{n,1} + e_{n,0} + 2c_{n,2} + \cdots + 2c_{n,n} + e_{n,0},\end{aligned}$$

where $k_n = 2(|e_{n,0}| + \sum_{i=1}^{n} |c_{n,i}|)$. Since the first and the last edges in each cycle $c_{n+1,i}$ are sent by ϕ_n onto $e_{n,0}$, each ϕ_n is a bd-cover.

Definition 3.1. Let $\mathcal{G} = \langle \phi_i \rangle_{i=0}^{\infty}$ be the sequence defined above. We denote by $T_{\mathcal{G}} : V_{\mathcal{G}} \to V_{\mathcal{G}}$ the homeomorphism induced by $E_{\mathcal{G}}$ in view of Lemma 2.1.

3.2. *Properties of $(V_{\mathcal{G}}, T_{\mathcal{G}})$.* In this section, we prove a few auxiliary lemmas about properties of the system $(V_{\mathcal{G}}, T_{\mathcal{G}})$ constructed in §3.1. They will be used later in the proof of Theorem 1.2. For simplicity of notation, we denote $X = V_{\mathcal{G}}$ and $T = T_{\mathcal{G}}$.

Denote $p = (v_{0,0}, v_{1,0}, v_{2,0}, \ldots)$. It is not hard to see that $p \in V_{\mathcal{G}}$ and $(p, p) \in E_{\mathcal{G}}$. In particular, p is a fixed point of T. We easily obtain the following lemma.

LEMMA 3.2. *The point p is the only periodic point of (X, T).*

Proof. Note that the length of each cycle $c_{n,i}$ increases with n and each such cycle has pairwise distinct vertexes. Hence, if z is a periodic point, then, for sufficiently large n,

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it may not contain any vertex from $c_{n,i}$, and so the only choice is $z_n = v_{n,0}$ for each n. Indeed, $z = p$. □

LEMMA 3.3. *For every $x \in X$, $p \in \omega_T(x)$.*

Proof. Fix any integers $n > 0$ and $m > 0$ and let $y = T^n(x)$. If $T^n(x)_m = v_{m,0}$, then $T^n(x)_i = v_{i,0}$ for all $i \leq m$, and so $d(x, p) \leq 2^{-m}$. In the case $T^n(x)_m \neq v_{m,0}$, vertex $T^n(x)_m$ belongs to some cycle $c_{m,j}$ and then, in a finite number of iterations (say $l > 0$), $T^{n+l}(x)_m$ reaches the terminal vertex of the cycle $c_{m,j}$, which is $v_{m,0}$. This shows that, for every n, there is $s \geq n$ such that $T^s(x)_m = v_{m,0}$, and so $d(T^s(x), p) \leq 2^{-m}$. Therefore $d(\omega_T(x), p) \leq 2^{-m}$ and, since m can be arbitrarily large and $\omega_T(x)$ is closed, $p \in \omega_T(x)$. □

COROLLARY 3.4. *The dynamical system (X, T) is proximal.*

Proof. By Lemma 3.3, the fixed point p belongs to every ω-limit set, and hence $\{p\}$ is the unique minimal subsystem in X. But it is well known (e.g., see [**2**]) that it equivalently means that (X, T) is proximal. □

LEMMA 3.5. *The dynamical system (X, T) is topologically mixing.*

Proof. By the definition of product topology, it is enough to show that, for any m and any vertices $u, v \in V_m$, there is $N > 0$ such that, for every $n \geq N$, there is $x \in X$ such that $x_m = u$ and $T^n(x)_m = v$. Observe that $\phi_m(c_{m+1,1})$ contains all vertices from V_m. So it is enough to show that there is $N > 0$ such that, for every $n \geq 0$, there is a point $x \in X$ such that path induced by $T^i(x)_{m+1}$ on (V_{m+1}, E_{m+1}) contains $c_{m+1,1} + \eta + c_{m+1,1}$ as a subpath, with length of intermediate path $|\eta| = n$. Note that each $c_{j,1}$ is an inner element of the path $\phi_j(c_{j+1,1})$, that is, a subpath not containing the first or last vertex, and hence, for any j, it is not hard to find $x \in V_{\mathcal{G}}$ such that $T^{i-1}(x)_j$ is exactly the ith vertex of $c_{j,1}$ for each $0 < i \leq |c_{j,1}|$.

We claim that:

(i) for each $j \geq 1$, the cycle $\phi_{m+j,m}(c_{m+j,1})$ contains cycle $c_{m,1} + \eta_i + c_{m,1}$ with intermediate path η_i of length $|\eta_i| = i$, where $i = 0, \ldots, k_{m+j-1}$;

(ii) $je_{m,0} + c_{m,1}$ is a prefix of $\phi_{m+j,m}(c_{m+j,1})$; and

(iii) there is α with $|\alpha| \leq k_{m+j-1} - j$ such that $c_{m,1} + \alpha$ is a suffix of $\phi_{m+j,m}(c_{m+j,1})$.

The claim is clear for $j = 1$ since $c_{m,1} + ke_{m,0} + c_{m,1}$ is a subpath of $\phi_{m+1,m}(c_{m+1,1}) = \phi_m(c_{m+1,1})$ for $k = 0, \ldots, k_m$, $\phi_m(c_{m+1,1})$ which starts with $e_{m,0} + c_{m,1}$ and ends with path

$$c_{m,1} + e_{m,0} + 2c_{m,2} + \cdots + 2c_{m,m} + e_{m,0},$$

whose length is exactly $k_m - |c_{m,1}| \leq k_m - m$.

Next assume that the claim holds for some $j \geq 0$ and consider

$$\xi = \phi_{m+j+1,m}(c_{m+j+1,1}) = \phi_{m+j,m}(\phi_{m+j+1}(c_{m+j+1,1})).$$

This cycle contains a subpath

$$\phi_{m+j,m}(c_{m+j,1}) + \eta + \phi_{m+j,m}(c_{m+j,1}) \tag{1}$$

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for each path η of lengths $|\eta| = 0, 1, \ldots, k_{m+j}$. By assumption (3.2), each $\phi_{m+j,m}(c_{m+j,1})$ contains a copy of $c_{m,1} + \alpha + c_{m,1}$ with any $|\alpha| = 0, \ldots, k_{m+j-1}$.

On the other hand, by assumptions (3.2) and (3.2) and (1), there is a path α of length $|\alpha| \le k_{m+j-1}$ such that we can find in ξ a path $c_{m,1} + \gamma + c_{m,1}$ with γ of length $|\gamma| = |\alpha| + |\eta|$. The claim is proved, and so the result follows. □

As the last element, we will show that (X, T) does not contain an asymptotic pair. The following notion of degree was used in [**18**].

Definition 3.6. For each $v \in V_n$, we define its degree by

$$\deg(v) = \begin{cases} +\infty & \text{if } v = v_{n,0}, \\ i & \text{if } v \in V(c_{n,i})\backslash\{v_{n,0}\}, \end{cases}$$

and next, for $x \in X$, we put $\deg(x) = \min_i \deg(x_i)$.

It is not hard to see that, for each $x \in X$ and each $n < m$, $\deg(x_n) \ge \deg(x_m) \ge 1$. Clearly, if $\deg(x) = +\infty$, then $x = p$. It is also clear that if $\deg(x) = i < +\infty$, then there is N such that $x_n \in V(c_{n,i})\backslash\{v_{n,0}\}$ for all $n \ge N$.

Remark 3.7. Note that $\phi_n(c_{n+1}, i)$ starts and ends with edge $e_{n,0}$ and hence, if $x_n \in V(c_{n,i})\backslash\{v_{n,0}\}$ for $n \ge N$, then $T(x)_n \in V(c_{n,i})\backslash\{v_{n,0}\}$ for $n > N$.

Remark 3.7 immediately proves the following lemma which is an analogue of [**18**, Lemma 3.11].

LEMMA 3.8. *For each $x \in X$, $\deg(x) = \deg(T(x))$.*

The following observation will simplify further considerations.

LEMMA 3.9. *Suppose that $\deg(x) = i < \infty$ and fix any $n > i$. Then, for every m, $\min_{j \ge m} \deg(T^j(x)_n) \le i + 1$.*

Proof. First note that if $x_j \in V(c_{j,i})\backslash\{v_{j,0}\}$, then also $x_{j+1} \in V(c_{j+1,i})\backslash\{v_{j+1,0}\}$ and so, by Remark 3.7, we must have $T(x)_{j+1} \in V(c_{j+1,i})\backslash\{v_{j+1,0}\}$. As a consequence, by induction, we easily obtain that $T^m(x)_{j+m} \in V(c_{j+m,i})\backslash\{v_{j+m,0}\}$. But the occurrence of $c_{j+m,i}$ in $\phi_{j+m}(c_{j+m+1,i})$ is eventually followed by $c_{j+m,i+1}$, and clearly $c_{j+m,i+1}$ contains a subpath η such that $\phi_{j+m,n}(\eta) = c_{n,i+1}$. This immediately implies that there is an integer $r \ge 0$ such that $T^{m+r}(x)_n \in V(c_{n,i+1})$. □

COROLLARY 3.10. *If $x, y \in X$ are an asymptotic pair, then $|\deg(x) - \deg(y)| \le 1$. In particular, there is no point $x \ne p$ asymptotic with p.*

Proof. Suppose that $x \ne y$ and they are asymptotic. Then, without loss of generality, we may assume that $x \ne p$ and so $\deg(x) = i < \infty$. Fix any $n > i$. Since x, y are asymptotic, there exists m such that $T^j(x)_n = T^j(y)_n$ for all $j \ge m$. By Lemma 3.9, we obtain that $T^j(y)_n \le i + 1$ for some j and so $\deg(T^j(y)) \le i + 1$, which, by Lemma 3.8, gives $\deg(y) \le i + 1$. In particular, $\deg(y) < +\infty$ and hence also $y \ne p$. Therefore we can change the role of x and y obtain by a symmetrical argument that $\deg(x) \le \deg(y) + 1$. □

LEMMA 3.11. *If $\deg(x) = i$ and $\deg(y) = i + 1$ for some $i \in \mathbb{N}$, then the pair x, y is not asymptotic.*

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Proof. Assume to the contrary that $\deg(y) = \deg(x) + 1$, but x and y are asymptotic, and denote $i = \deg(x)$. For each $m, s \geq 0$ and $z \in \{x, y\}$, write $z_m^s := \phi_{\infty,m}(T^s(z))$. Note that, by Lemma 3.8, $y_{i+1}^j \notin V(c_{i+1,i})$ for every $j \geq 0$ and so, if x, y are asymptotic, then $x_{i+1}^j \notin V(c_{i+1,i})$ for all j sufficiently large. Therefore, without loss of generality, we may assume that $x_{i+1}^j \notin V(c_{i+1,i})$ for all $j \geq 0$. Fix any $n > i+1$ sufficiently large to have $\phi_{\infty,k}(x) \in V(c_{k,i})$ and $\phi_{\infty,k}(y) \in V(c_{k,i+1})$ for all $k \geq n-1$. We also assume that n is large enough to have $x_{i+1}^j = y_{i+1}^j$ for all $j \geq n$.

We can present a path induced on the graph G_n by the orbit of x by

$$\langle x_n^j \rangle_{j=0}^{\infty} = \eta + 2c_{n,i+1} + \cdots,$$

where η is a path which does not contain complete cycle $c_{n,i+1}$ as a subpath. Clearly, η cannot contain complete cycle $c_{n,i}$ because, otherwise, $x_{i+1}^j \in V(c_{i+1,i})$ for some $j \geq 0$, which is a contradiction. Therefore either $|\eta| \leq |c_{n,i+1}|$ or

$$\phi_{n-1}(\eta) = \eta' + 2c_{n-1,i+1} + \cdots + 2c_{n-1,n-1} + e_{n-1,0},$$

where η' is a suffix of $c_{n-1,i}$, and hence, in this case, we obtain $|\eta| \leq |\eta'| + |c_{n,i+1}|$. Repeating the above arguments inductively, and keeping in mind that $\phi_{\infty,i+1}(x) \notin V(c_{i+1,i})$,

$$|\eta| \leq \sum_{r=1}^{n} |c_{r,i+1}| \leq |c_{n,i+1}| + n|c_{n-1,i+1}| \leq |c_{n,i+1}| + |c_{n,n}| - n. \tag{2}$$

Since $\phi_{\infty,n-1}(x) \in V(c_{n-1,i})$, also $|\eta| \geq |c_{n,i+1}|$.

On the other hand, we must have $y_{i+1}^j \in V(c_{i+1,i+1})$ for infinitely many iterates j, as, otherwise, it is again easy to see that x, y are not asymptotic. Hence we may assume that n is sufficiently large to ensure that

$$\langle y_n^j \rangle_{j=0}^{\infty} = \xi + c_{n,i+1} + 2c_{n,i+2} + \cdots + 2c_{n,n} + \cdots,$$

where ξ is a suffix of $c_{n,i+1}$. Let $j > 0$ be the smallest integer such that $y_n^j \in \bigcup_{s=i+2}^{n} V(c_{n,s})$. Since $\phi_{\infty,n-1}(y) \in V(c_{n-1,i+1})$, $|\xi| > |c_{n,n}|$ and hence

$$n \leq |\eta| < j \leq 2|c_{n,i+1}| \leq |\eta| + |c_{n,i+1}|.$$

Roughly speaking, it means that x_{n-1}^j belongs to the first occurrence of $c_{n,i+1}$ on the path $\eta + 2c_{n,i+1} + \cdots$. By the choice of j, $y_{i+1}^j \notin V(c_{i+1,i+1})$ and $x_{i+1}^j = y_{i+1}^j$. Then calculations similar to (2) yield that there is $r < 2|c_{n,i+2}|$ such that x_n^{j+r} is the first vertex of $c_{n,i+1}$. This implies that $x_{i+1}^{j+r+n} \in V(c_{i+1,i+1})$ while $r + n \leq 2|c_{n,i+2}| + |c_{n,n}|$, and hence we still have $y_n^{j+r+n} \in \bigcup_{s=i+2}^{n} V(c_{n,s})$ and, as a consequence, $y_{i+1}^{j+r+n} \notin V(c_{i+1,i+1})$. This shows that x, y are not asymptotic, which is a contradiction. □

It remains to consider the case when x and y have distinct finite degrees.

LEMMA 3.12. *Let* $\deg(x) = \deg(y) = i$ *for some* $i \in \mathbb{N}$ *and assume that there are* $n > i+2$ *and* $k \geq 1$ *such that*

$$\langle \phi_{\infty,n}(T^j(x)) \rangle_{j=k}^{\infty} = \langle \phi_{\infty,n}(T^j(y)) \rangle_{j=k}^{\infty} = c_{n,i} + \cdots.$$

Then $\langle \phi_{\infty,n}(T^j(x)) \rangle_{j=0}^{\infty} = \langle \phi_{\infty,n}(T^j(y)) \rangle_{j=0}^{\infty}$.

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Proof. For each $m, s \geq 0$ and $z \in \{x, y\}$, write $z_m^s := \phi_{\infty,m}(T^s(z))$. Then sequence $\langle z_m^s \rangle_{s=0}^{\infty}$ is an infinite path on G_m. By assumption, there exist finite paths η_n, ξ_n, $|\eta_n| = |\xi_n|$ and an infinite path γ_n on G_n such that

$$\langle x_n^s \rangle_{s=0}^{\infty} = \eta_n + c_{n,i} + \gamma_n, \quad \langle y_n^s \rangle_{s=0}^{\infty} = \xi_n + c_{n,i} + \gamma_n. \tag{3}$$

We claim that either $\eta_n = \xi_n$ or, for every $m > n$, there exist finite paths η_m, ξ_m, $|\eta_m| = |\xi_m|$ such that:

(1) $\langle x_m^s \rangle_{s=0}^{\infty} = \eta_m + c_{m,i} + \cdots$, $\langle y_m^s \rangle_{s=0}^{\infty} = \xi_m + c_{m,i} + \cdots$;
(2) $|\eta_m| + |c_{m,i}| > |\eta_{m-1}| + |c_{m-1,i}|$; and
(3) $|\eta_m| < |\eta_{m-1}|$.

Let us assume that $\eta_n \neq \xi_n$. This implies, in particular, that $x_m^0 \neq y_m^0$ for every $m > n$. Observe that the cycle $c_{n,i}$ in $\langle x_n^s \rangle_{s=0}^{\infty}$ must be contained in an image of (possibly a part of) cycle $c_{n+1,i}$ in $\langle x_n^s \rangle_{s=0}^{\infty}$ (the same is true for $\langle y_n^s \rangle_{s=0}^{\infty}$). Picture these images written one over another as in (4).

$$\begin{aligned} \langle x_n^s \rangle_{s=0}^{\infty} &= \overbrace{\cdots\cdots + c_{n,i} + \cdots\cdots}^{c_{n+1,i}}\cdots \\ \langle y_n^s \rangle_{s=0}^{\infty} &= \cdots\underbrace{\cdots\cdots + c_{n,i} + \cdots\cdots}_{c_{n+1,i}}\cdots \end{aligned} \tag{4}$$

If these images are shifted, then, moving to the right from $c_{n,i}$, the first vertex of $c_{n,i+1}$ must appear before the trace of $\phi_n(c_{n+1,i})$ finishes in each of these sequences, and it will appear earlier in $\langle y_n^s \rangle_{s=0}^{\infty}$ than in $\langle x_n^s \rangle_{s=0}^{\infty}$ or *vice-versa* (the first case occurs if the situation is as in (4)). It leads to a contradiction with the definition of γ_n. On the other hand, the whole of $c_{n+1,i}$ must appear in $\langle x_{n+1}^s \rangle_{s=0}^{\infty}$ because $x_{n+1}^0 \neq y_{n+1}^0$. This proves the claim for $m = n + 1$.

For the proof of the case $m > n + 1$, we can repeat previous observations. However, now the relative shift in images of $c_{m,i}$ in paths

$$\begin{aligned} \langle x_{m-1}^s \rangle_{s=0}^{\infty} &= \overbrace{\cdots\cdots + c_{m-1,i} + \cdots\cdots}^{c_{m,i}}\cdots \\ \langle y_{m-1}^s \rangle_{s=0}^{\infty} &= \cdots\underbrace{\cdots\cdots + c_{m-1,i} + \cdots\cdots}_{c_{m,i}}\cdots \end{aligned} \tag{5}$$

does not lead to an immediate contradiction because we cannot use γ_n directly in this case. By the symmetry of the argument, we may assume that the relative shift is as in (5). As before, moving to the right from $c_{m-1,i}$ in both paths on G_{m-1}, the first occurrence of a vertex of $c_{m-1,i+1}$ will appear earlier in $\langle x_{m-1}^s \rangle_{s=0}^{\infty}$ than in $\langle y_{m-1}^s \rangle_{s=0}^{\infty}$. Note that the trace of $\phi_{m-1,n}(c_{m-1,i+1})$ in $\langle x_n^s \rangle_{s=0}^{\infty}$ contains a copy of cycle $c_{n,i+1}$ in γ_n, and hence $c_{n,i+1}$ will also appear in $\langle y_n^s \rangle_{s=0}^{\infty}$. But we know that it cannot be obtained as an image of $c_{m-1,i+1}$ (it is too early a position to see a vertex of $c_{m-1,i+1}$), and hence the only possibility is that it is a part of image $\phi_{m-1,n}(c_{m-1,i})$. But then at level $m-2$ we will see one of the two paths given by

$$\begin{aligned} \langle y_{m-2}^s \rangle_{s=0}^{\infty} &= \overbrace{\cdots\cdots + c_{m-2,m-2} + e_{m-2,0} + c_{m-2,i+1} + \cdots\cdots}^{c_{m,i}}\cdots \\ \langle x_{m-2}^s \rangle_{s=0}^{\infty} &= \cdots\underbrace{\cdots\cdots\cdots\cdots + c_{m-2,i} + c_{m-2,i+1} + \cdots\cdots}_{c_{m,i}}\cdots \end{aligned} \tag{6}$$

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or

$$\begin{aligned}\langle x^s_{m-2}\rangle^\infty_{s=0} &= \overbrace{\cdots\cdots + c_{m-2,i+1} + c_{m-2,i+1} + \cdots\cdots\cdots}^{c_{m,i}}\\ \langle y^s_{m-2}\rangle^\infty_{s=0} &= \cdots\cdots + \underbrace{c_{m-2,i+1} + c_{m-2,i+2} + \cdots\cdots\cdots}_{c_{m,i}}\end{aligned} \tag{7}$$

Note that, in the case given by (6), if we move left from $c_{m-2,i+1}$ then in $\langle y^s_{m-2}\rangle^\infty_{s=0}$, we will reach the first position r (within $c_{m-2,i}$) in which $\phi_{m-2,n}(x^r_{m-2})$ points to the last vertex of $c_{n,i}$ while $x^r_n \notin V(c_{n,i})$. But the position r occurs after the start of γ_n, which is a contradiction.

In the case given by (7), moving to the right from $c_{m-2,i+1}$ in $\langle x^s_{m-2}\rangle^\infty_{s=0}$ will eventually reach the first position r (within $c_{m-2,i+1}$) in which $\phi_{m-2,n}$ sends to the first vertex of $c_{n,i+1}$, while still $y^r_n \in V(c_{n,i+1})$. This is again a contradiction. The claim is proved.

To complete the proof, note that condition (3) of the claim cannot be satisfied for infinitely many m, and hence the only possibility is that $\eta_n = \xi_n$ and, consequently, $\langle x^s_n\rangle^\infty_{s=0} = \langle y^s_n\rangle^\infty_{s=0}$. □

LEMMA 3.13. *Let* $\deg(x) = \deg(y) = i$ *for some* $i \in \mathbb{N}$. *If the pair* $x, y \in X$ *is asymptotic, then* $x = y$.

Proof. Let us choose an asymptotic pair $x, y \in X$, $x \neq y$ for which $\deg(x) = \deg(y) = i$ for some $i \in \mathbb{N}$. There exists an integer $N > i + 2$ for which $\deg(x_n) = \deg(y_n) = i$ for all $n \geq N$. We may also assume that $x_n \neq y_n$ for all $n \geq N$. As before, for each m, $s \geq 0$ and $z \in \{x, y\}$, write $z^s_m := \phi_{\infty,m}(T^s(z))$. Since the pair x, y is asymptotic, there exists $K > 0$ such that $x^k_N = y^k_N$ for all $k \geq K$. If the sequence $\langle x^k_N\rangle^\infty_{k=0}$ contains infinitely many copies of $c_{N,i}$, then we may apply Lemma 3.12 to obtain that $x^k_N = y^k_N$ for every $k \geq 0$, and, in particular, that $x_N = y_N$, which is a contradiction. Hence the only possibility is that $\langle x^k_N\rangle^\infty_{k=0}$ contains finitely many copies of $c_{N,i}$. Note that each complete cycle $c_{n,i}$ appearing in $\langle x^k_n\rangle^\infty_{k=0}$ induces at least two copies of cycle $c_{n-1,i}$ in $\langle x^k_{n-1}\rangle^\infty_{k=0}$, and hence, without loss of generality, we may assume that, for all $n \geq N$, complete cycle $c_{n,i}$ does not appear in either $\langle x^k_n\rangle^\infty_{k=0}$ or in $\langle y^k_n\rangle^\infty_{k=0}$. Note that $x_n, y_n \in c_{n,i}$ for all $n \geq N$ and, for each n, there exist suffixes η_n and ξ_n of $c_{n,i}$, starting with vertices x_n and y_n, respectively, which are a prefixes of paths $\langle x^k_n\rangle^\infty_{k=0}$ and $\langle y^k_n\rangle^\infty_{k=0}$. First observe that, by the definition of $c_{n,i}$ and ϕ_n,

$$\lim_{n\to\infty} |\eta_n| = \lim_{n\to\infty} |\xi_n| = +\infty.$$

We may also assume that $|\eta_n| < |\xi_n|$ for infinitely many n. If $\limsup_{n\to\infty} ||\eta_n| - |\xi_n|| = +\infty$, then, for each M, there are $n > N$ and $j > M$ such that $x^j_N = \phi_{n,N}(x^j_n) = y_N \in V(c_{N,i})$ or $y^j_N = \phi_{n,N}(y^j_n) = x_N \in V(c_{N,i})$, which is a contradiction. We obtain that the sequence $\langle ||\eta_n| - |\xi_n||\rangle^\infty_{n=N}$ is bounded, and therefore there exists an integer $m \in \mathbb{Z}$ such that $T^m(x) = y$. If $m = 0$, then the proof is complete, so, without loss of generality, we assume that $m \neq 0$, which implies that x is a periodic point. But $\deg(x) < \infty$, so $x \neq p$ contradicts Lemma 3.3. The proof is complete. □

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4. *Applications*

4.1. *Proof of Theorem 1.2.* Consider the dynamical system $(X, T)(=(V_{\mathcal{G}}, T_{\mathcal{G}}))$ constructed in §3.1 and denote by p its unique (fixed) point with $\deg(p) = +\infty$. By Corollary 3.4, (X, T) is proximal and, by Lemma 3.5, it is topologically mixing. Take any $x, y \in X$ which are asymptotic. By Corollary 3.10 and Lemma 3.11, we either have $x = y = p$ or there is $i \in \mathbb{N}$ such that $\deg(x) = \deg(y) = i$. By Lemma 3.13, we obtain that also, in the second case, $x = y$. This shows that when $x \neq y$, they are proximal but not asymptotic. Indeed, (X, T) is completely scrambled.

4.2. *Solution to Question 1.1.* Let (X, T) be a transitive completely scrambled homeomorphism acting on a compact metric space X of topological dimension $n \geq 0$. We will show how to extend it to a system (Y, S) with connected Y of topological dimension $n + 1$. Note that the case of (X, T) for $n = 0$ is provided by Theorem 1.2, and hence, by induction using the method described below, we can construct it for any $n > 0$. This will solve Question 1.1.

Let $\langle \Phi_t \rangle_{t \in \mathbb{R}}$ be the suspension flow of (X, T) acting on a corresponding space Z obtained as a respective quotient of $X \times [0, 1]$. Since (X, T) is transitive, it is not hard to verify that there is an irrational $s > 0$ such that the homeomorphism $R = \Phi_s : Z \to Z$ is transitive (in fact, there is a residual set of parameters $t \in \mathbb{R}$ for which Φ_t is transitive; e.g., see the first part of the proof of [**7**, Proposition 4]). Since there is no asymptotic pair in (X, T), it is also clear that (Z, R) does not contain asymptotic pairs. Let p be a fixed point which is the unique minimal set for $(X.T)$. Then it is not hard to see that the set $M = \omega_R(p, 0)$ is a minimal set for (Z, R), homeomorphic to the unit circle. Now fix any $(x, r) \in Z$. Since (p, x) are proximal in (X, T), for every $m > 0$ and open neighbourhood V of p, there is $i > 0$ such that $T^{i+j}(x) \in V$ for all $j = 0, \ldots, m$. But m can be arbitrarily large, so, for every neighbourhood U of $(p, 0)$, there exists $n > 0$ such that $R^n(x, q) \in U$ and, in particular, $(p, 0) \in \omega_R(x, r)$. This proves that M is the unique minimal set for (Z, R). Let Y be obtained from Z by collapsing M to a point. This induces in a natural way a factor (Y, S) with a factor map $\pi : (Z, R) \to (Y, S)$. Denote by q the unique fixed point of (Y, S) defined by $\{q\} = \pi(M)$. It is obvious that there is no other minimal subset in (Y, S), and hence (Y, S) is proximal (e.g., see [**2**]). Additionally, observe that there is no non-trivial asymptotic pair in (Y, S). Since there is no asymptotic pair in (Z, R) consisting of distinct points, the only possibility could be that there is $y \in Y$, $y \neq q$, asymptotic to q. But then there exists $(z, r) \in Z$ such that $\omega_R(z, t) = \omega_R(p, 0)$, which, in turn, implies that $\omega_T(z) = p$, which is a contradiction.

We have just proved that (Y, S) is proximal without non-trivial asymptotic pairs, which, equivalently, means that (X, S) is completely scrambled. It follows directly from the construction, that Y is of topological dimension $n + 1$. We have proved the following theorem.

THEOREM 4.1. *For every $n \geq 1$, there exists a continuum X of dimension n supporting a completely scrambled transitive homeomorphism.*

4.3. *Proof of Theorem 1.3.* First, fix any $n \geq 2$. Apply Theorem 1.2 to obtain a mixing completely scrambled homeomorphism (X, T), where X is a Cantor set (strictly speaking,

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it is the dynamical system $(V_{\mathcal{G}}, T_{\mathcal{G}})$ constructed in §3.1). On the other hand, Glasner and Maon, inspired by an earlier work of Glasner and Weiss [**11**], provided, in 1989, a method of construction of minimal, weakly mixing and uniformly rigid homeomorphisms [**10**, Proposition 6.5]. As a particular example of an application of this technique, we obtain a uniformly rigid, weakly mixing and minimal dynamical system (Z, R), where $Z = \mathbb{T}^n$ is n-dimensional torus. Denote by p the unique fixed point of (X, T). Let (Y, S) be a factor of $(X \times Z, T \times R)$ obtained by collapsing minimal set $M = \{p\} \times Z$ to a point $q \in Y$, and let $\pi : X \times Z, T \times R) \to (Y, S)$ be a corresponding factor map. Assume that $x, y \in Y$ are distinct points forming an asymptotic pair and take any $u, v \in Z$, $s, t \in X$ such that $\pi(s, u) = x$ and $\pi(t, v) = y$. If $s \neq p$, then there is an open set $V \supset M$ such that $(T \times R)^n(s, u) \notin V$ for infinitely many $n \geq 0$. Let us fix one such n and any open set $U \supset M$ such that $\overline{U} \subset V$. Take any $\lambda > 0$ such that $\operatorname{dist}(X \setminus V, \overline{U}) > 0$ and $\limsup_{i \to \infty} d(T^i(s), T^i(t)) > \lambda$. Since we may assume that π outside U does not decrease distances, it is not hard to see that $d(S^n(x), S^n(y)) > \lambda/2$, which is impossible. But then the only remaining possibility is that $s = t = p$, which is also impossible because $x \neq y$. This shows that distinct points in Y are never asymptotic. But it is also clear that $\{q\}$ is the unique minimal subset in Y, which shows that (Y, S) is proximal. Finally, (Y, S) is weakly mixing as a factor of a weakly mixing dynamical system $(X \times Z, T \times R)$. We have just proved Theorem 1.3 for $n \geq 2$.

Finally, to prove Theorem 1.3 for $n = 1$, it suffices to use Handel's minimal homeomorphism on the pseudo-circle [**12**], since this homeomorphism is weakly mixing and uniformly rigid [**5**].

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