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A generalization of the Balescu–Lenard equation and the pair correlation function for a weakly non-ideal plasma

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Taking into account the first non-Markovian correction to the Balescu–Lenard equation, we derive an expression for the pair correlation function and a nonlinear kinetic equation valid for a weakly non-ideal polarized classical plasma. This equation allows for the description of the correlational energy evolution and exhibits global conservation of energy with dynamical polarization.

1. Introduction

The importance of polarization effects on plasmas in the kinetic regime has been recognized for a long time. A first attempt to include these effects in the linearized collision integral was made by Gasirovicz *et al.* (1956). Later, the nonlinear equation for a weakly coupled polarizable plasma was derived by Balescu (1960), with the help of Prigogine's diagram techniques (Prigogine 1963) and by Lenard (1960), who solved the Bogoliubov equation (Bogoliubov 1962) for the pair correlation function in the plasma approximation. These results were generalized to the quantum case by Konstantinov and Perel (1961) and by Wyeld and Pines (1962).

Kadanoff and Baym (1964) and Klimontovich (1975) have noted that the Balescu–Lenard (BL) equation takes into account the polarization of the system only in the collision integral, while the thermodynamics corresponds to an ideal gas – the dissipative and non-dissipative phenomena are not treated on an equal footing. They have shown that this discrepancy can be avoided if non-Markovian effects are taken into account. Klimontovich (1972) wrote the system of equations for the one-particle distribution function (df) and the pair correlations for the electric field and for the charge density, but he did not solve the equation for the pair correlation function and thus did not obtain a closed kinetic equation. On the other hand, Résibois (1963) and Dorfman and Cohen (1982) have formally derived the fully non-Markovian generalization of the BL equation, but their results are not easily tractable in practical cases. In particular, they did not give an explicit expression for the correlation energy.

In this paper, we solve, in the plasma approximation, the equation for the pair

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correlation function, considering the first non-Markovian correction. In this way, we obtain a nonlinear kinetic equation that generalizes the BL equation for a weakly non-ideal plasma. This equation, which includes the dynamical screening of the interaction potential, describes correctly the conservation of the total energy in a non-trivial way.

Let us mention that recently papers on non-Markovian nonlinear kinetic equations have been published. They treat the first density correction to the Uehling–Uhlenbeck equation (Morawetz and Roepke 1995) and the case of quantum Landau equation (Morawetz *et al.* 1994; Bonitz *et al.* 1996), which accounts for the potential energy at first order only. We propose a better kinetic equation that shows the global energy conservation in an approximation that, at equilibrium, is equivalent to the Debye–Hückel one.

A summary of these results has been published elsewhere (Belyi *et al.* 1996); in the present paper, their derivation is treated in more detail.

It is an especially great pleasure for us to have the opportunity of reporting on a generalization of the Balescu–Lenard equation in this special issue in honour of Professor Radu Balescu.

2. Truncation of the BBGKY hierarchy

We consider a spatially uniform weakly non-ideal multicomponent plasma. Out of equilibrium, its evolution is exactly described by the BBGKY hierarchy (Balescu 1975) for the reduced df or, equivalently, for the correlation functions. Since there are no mean fields in a uniform system, the first two equations are

$$\frac{\partial}{\partial t} f_a(\mathbf{p}_1, t) = \sum_b \int d\mathbf{r}_2\, d\mathbf{p}_2 \left[\frac{\partial}{\partial \mathbf{r}_1}\Phi_{ab}(|\,\mathbf{r}_1 - \mathbf{r}_2\,|)\right] \cdot \frac{\partial}{\partial \mathbf{p}_1} g_{ab}(\mathbf{p}_1, \mathbf{p}_2, |\,\mathbf{r}_1 - \mathbf{r}_2\,|, t), \quad (2.1)$$

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial}{\partial \mathbf{r}_1} + \mathbf{v}_2 \cdot \frac{\partial}{\partial \mathbf{r}_2}\right) g_{ab}(\mathbf{p}_1, \mathbf{p}_2, |\,\mathbf{r}_1 - \mathbf{r}_2\,|, t) \\
&\quad - \sum_c \int d\mathbf{r}_3\, d\mathbf{p}_3 \left\{ \frac{\partial}{\partial \mathbf{r}_1}\Phi_{ac}(|\,\mathbf{r}_1 - \mathbf{r}_3\,|) \cdot \frac{\partial}{\partial \mathbf{p}_1} f_a(\mathbf{p}_1, t) g_{bc}(\mathbf{p}_2, \mathbf{p}_3, |\,\mathbf{r}_2 - \mathbf{r}_3\,|, t) \right. \\
&\quad + \frac{\partial}{\partial \mathbf{r}_2}\Phi_{bc}(|\,\mathbf{r}_2 - \mathbf{r}_3\,|) \cdot \frac{\partial}{\partial \mathbf{p}_2} f_b(\mathbf{p}_2, t) g_{ac}(\mathbf{p}_1, \mathbf{p}_3, |\,\mathbf{r}_1 - \mathbf{r}_3\,|, t) \\
&\quad + \left[\frac{\partial}{\partial \mathbf{r}_1}\Phi_{ac}(|\,\mathbf{r}_1 - \mathbf{r}_3\,|) \cdot \frac{\partial}{\partial \mathbf{p}_1} + \frac{\partial}{\partial \mathbf{r}_2}\Phi_{bc}(|\,\mathbf{r}_2 - \mathbf{r}_3\,|) \cdot \frac{\partial}{\partial \mathbf{p}_2}\right] \\
&\quad \left. \times g_{abc}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, |\,\mathbf{r}_1 - \mathbf{r}_2\,|, |\,\mathbf{r}_2 - \mathbf{r}_3\,|, t) \right\} \\
&\quad - \left[\frac{\partial}{\partial \mathbf{r}_1}\Phi_{ab}(|\,\mathbf{r}_1 - \mathbf{r}_2\,|)\right] \cdot \left(\frac{\partial}{\partial \mathbf{p}_1} - \frac{\partial}{\partial \mathbf{p}_2}\right) g_{ab}(\mathbf{p}_1, \mathbf{p}_2, |\,\mathbf{r}_1 - \mathbf{r}_2\,|, t) \\
&= \left[\frac{\partial}{\partial \mathbf{r}_1}\Phi_{ab}(|\,\mathbf{r}_1 - \mathbf{r}_2\,|)\right] \cdot \left(\frac{\partial}{\partial \mathbf{p}_1} - \frac{\partial}{\partial \mathbf{p}_2}\right) f_a(\mathbf{p}_1, t) f_b(\mathbf{p}_2, t), \qquad (2.2)
\end{aligned}$$

where $f_a(\mathbf{p}_1, t)$ is the one-particle df for species a and $g_{ab}(\mathbf{p}_1, \mathbf{p}_2, |\,\mathbf{r}_1 - \mathbf{r}_2\,|, t)$ is the pair correlation function; $\mathbf{r}_1$, $\mathbf{p}_1$, $\mathbf{v}_1 = \mathbf{p}_1/m_a$ are respectively the position, the momentum and the velocity of particle 1, and $\Phi_{ab}(|\,\mathbf{r}_1 - \mathbf{r}_2\,|)$ is the interaction potential between particle 1 of species a and particle 2 of species b. In order to

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close this set of equations, we make the usual so-called plasma approximation: triple correlation functions as well as interactions between correlated particles are neglected. In this way, the last two terms on the left-hand side of (2.2) are dropped. In Fourier space, the closed set of equations then takes the form

$$\frac{\partial}{\partial t} f_a(\mathbf{p}_1, t) = J_a(\mathbf{p}_1, t) = -\frac{i}{8\pi^3} \sum_b \int d\mathbf{k}\, d\mathbf{p}_2\, \Phi_{ab}(\mathbf{k}) \mathbf{k} \cdot \frac{\partial}{\partial \mathbf{p}_1} g_{ab}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}, t), \tag{2.3}$$

$$\begin{aligned} &\left[\frac{\partial}{\partial t} + i\mathbf{k} \cdot (\mathbf{v}_1 - \mathbf{v}_2)\right] g_{ab}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}, t) \\ &\qquad - i\mathbf{k} \cdot \frac{\partial}{\partial \mathbf{p}_1} f_a(\mathbf{p}_1, t) \sum_c \Phi_{ac}(\mathbf{k}) \int d\mathbf{p}_3\, g_{bc}(\mathbf{p}_2, \mathbf{p}_3, \mathbf{k}, t) \\ &\qquad + i\mathbf{k} \cdot \frac{\partial}{\partial \mathbf{p}_2} f_b(\mathbf{p}_2, t) \sum_c \Phi_{bc}(\mathbf{k}) \int d\mathbf{p}_3\, g_{ac}(\mathbf{p}_1, \mathbf{p}_3, \mathbf{k}, t) \\ &\quad = i\Phi_{ab}(\mathbf{k}) \mathbf{k} \cdot \left(\frac{\partial}{\partial \mathbf{p}_1} - \frac{\partial}{\partial \mathbf{p}_2}\right) f_a(\mathbf{p}_1, t) f_b(\mathbf{p}_2, t), \end{aligned} \tag{2.4}$$

where

$$g_{ab}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}, t) = \int d\mathbf{r}\, \exp(-i\mathbf{k} \cdot \mathbf{r})\, g_{ab}(\mathbf{p}_1, \mathbf{p}_2, |\, \mathbf{r}\, |, t), \tag{2.5}$$

and

$$\Phi_{ab}(\mathbf{k}) = \frac{4\pi e_a e_b}{k^2} \tag{2.6}$$

is the Fourier transform of the Coulomb potential. We have also defined in (2.3) the collision integral $J_a(\mathbf{p}_1, t)$. To solve (2.4), it is generally assumed that the time dependence of the one-particle df can be neglected on the collision time scale. This results in the Markovian BL equation. To obtain a non-Markovian kinetic equation, one has to consider explicitly the time dependence of all the functions. Nevertheless, it is necessary to distinguish between fast and slow time scales.

In order to avoid unnecessary complications in this study, we introduce a compact notation. Let us define a scalar product

$$\underline{A} \cdot \underline{B} = \sum_{ab} \int d\mathbf{p}_1\, d\mathbf{p}_2\, A_{ab}(\mathbf{p}_1, \mathbf{p}_2) B_{ab}(\mathbf{p}_1, \mathbf{p}_2), \tag{2.7}$$

and rewrite (2.4) as

$$[\partial_t - \underline{\underline{L}} \cdot + \underline{\underline{\Gamma}}(t) \cdot] \underline{g}(t) = \underline{A}(t), \tag{2.8}$$

where

$$(\underline{\underline{L}})_{aba'b'}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_1', \mathbf{p}_2') = -i\mathbf{k} \cdot \left(\frac{\mathbf{p}_1}{m_a} - \frac{\mathbf{p}_2}{m_b}\right) \delta_{aa'} \delta_{bb'} \delta(\mathbf{p}_1 - \mathbf{p}_1') \delta(\mathbf{p}_2 - \mathbf{p}_2'), \tag{2.9}$$

$$\begin{aligned} (\underline{\underline{\Gamma}}(t))_{aba'b'}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_1', \mathbf{p}_2') &= -i\mathbf{k} \cdot \frac{\partial}{\partial \mathbf{p}_1} f_a(\mathbf{p}_1, t) \Phi_{aa'}(\mathbf{k}) \delta_{bb'} \delta(\mathbf{p}_2 - \mathbf{p}_2') \\ &\quad + i\mathbf{k} \cdot \frac{\partial}{\partial \mathbf{p}_2} f_b(\mathbf{p}_2, t) \Phi_{bb'}(\mathbf{k}) \delta_{aa'} \delta(\mathbf{p}_1 - \mathbf{p}_1'), \end{aligned} \tag{2.10}$$

$$\underline{A}_{ab}(\mathbf{p}_1, \mathbf{p}_2, t) = i\Phi_{ab}(\mathbf{k}) \mathbf{k} \cdot \left(\frac{\partial}{\partial \mathbf{p}_1} - \frac{\partial}{\partial \mathbf{p}_2}\right) f_a(\mathbf{p}_1, t) f_b(\mathbf{p}_2, t). \tag{2.11}$$

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The difficulty in solving (2.8) is that $\underline{\underline{\Gamma}}(t)$ depends on time. Therefore the solution will only be written as a series.

3. Green-function method and time scales

In this section, we shall formally solve (2.8) by the method of Green functions (Balescu 1963). This has already been done in some detail (Belyi *et al.* 1996), but we repeat it here for completeness.

When t is large enough, the initial correlations have decayed and the solution of (2.8) can be written as

$$\underline{g}(t) = \int_{-\infty}^{t} dt' \, \underline{\underline{R}}(t,t') \cdot \underline{A}(t'), \tag{3.1}$$

where the Green function $\underline{\underline{R}}(t,t')$ is the solution of the equation

$$[\partial_t - \underline{\underline{L}} \cdot + \underline{\underline{\Gamma}}(t) \cdot] \underline{\underline{R}}(t,t') = \delta(t-t') \underline{\underline{I}}, \tag{3.2}$$

with the causality condition $\underline{\underline{R}}(t,t') = 0$ when $t < t'$.

When $\underline{\underline{\Gamma}}$ does not depend on time, it is easy to show that $\underline{\underline{R}}(t,t')$ can only depend on its time variables through the difference $t-t'$. In order to find the time dependence of $\underline{\underline{R}}$ in the general case, we write $\underline{\underline{\Gamma}}(t) = \underline{\underline{\Gamma}}(\mu t)$, where μ is a parameter that specifies the one-particle df time scale. Moreover, we consider $\underline{\underline{\Psi}}(\mu t) \equiv \underline{\underline{\Gamma}}(\mu t) - \underline{\underline{\Gamma}}(0)$ as a perturbation. Equation (3.2) is then written as

$$[\partial_t - \underline{\underline{\mathfrak{L}}} \cdot + \underline{\underline{\Psi}}(\mu t) \cdot] \underline{\underline{R}}(t,t') = \delta(t-t') \underline{\underline{I}}, \tag{3.3}$$

with $\underline{\underline{\mathfrak{L}}} = \underline{\underline{L}} - \underline{\underline{\Gamma}}(0)$.

At order zero, (3.3) becomes

$$(\partial_t - \underline{\underline{\mathfrak{L}}} \cdot) \underline{\underline{R_0}}(t-t') = \delta(t-t') \underline{\underline{I}}, \tag{3.4}$$

or

$$(\partial_t - \underline{\underline{\mathfrak{L}}} \cdot) \delta(t-t') = \underline{\underline{R_0^{-1}}}(t-t'), \tag{3.5}$$

where we have introduced the inverse Green function $\underline{\underline{R_0^{-1}}}(t-t')$ defined by

$$\int_{-\infty}^{\infty} dt' \, \underline{\underline{R_0}}(t-t') \cdot \underline{\underline{R_0^{-1}}}(t'-t'') = \delta(t-t'') \underline{\underline{I}}. \tag{3.6}$$

At order one, (3.3) is

$$(\partial_t - \underline{\underline{\mathfrak{L}}} \cdot) \underline{\underline{R_1}}(t,t') + \underline{\underline{\Psi}}(\mu t) \cdot \underline{\underline{R_0}}(t-t') = 0, \tag{3.7}$$

which becomes, using (3.5),

$$\underline{\underline{R_1}}(t,t') = -\int_{-\infty}^{\infty} dt_1 \, \underline{\underline{R_0}}(t-t_1) \cdot \underline{\underline{\Psi}}(\mu t_1) \cdot \underline{\underline{R_0}}(t_1-t'), \tag{3.8}$$

and, by a change of integration variable $t_1 \to t_1 + t'$,

$$\underline{\underline{R_1}}(t,t') = -\int_{-\infty}^{\infty} dt_1 \, \underline{\underline{R_0}}(t-t'-t_1) \cdot \underline{\underline{\Psi}}(\mu t_1 + \mu t') \cdot \underline{\underline{R_0}}(t_1) = \underline{\underline{R_1}}(t-t', \mu t'). \tag{3.9}$$

The procedure is easily iterated to any order in perturbation. We thus have in

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full generality

$$\underline{\underline{R}}(t,t') = \underline{\underline{R}}(t-t',\mu t'). \tag{3.10}$$

This means that the collision time and the relaxation time of the one-particle df have been disentangled. Introducing this result into (3.1) yields

$$\underline{g}(t) = \int_0^\infty d\tau\, \underline{\underline{R}}(\tau,\mu(t-\tau))\cdot\underline{A}(\mu(t-\tau)), \tag{3.11}$$

where we have taken into account the fact that $\underline{A}(t)$ varies on the time scale of the one-particle df. The expression (3.11), which is an exact result, is the starting point for the approximation scheme that we consider.

The Markovian limit of (3.11) is obtained when $t-\tau$ is replaced by t in the integrand, i.e. when the one-particle df varies slowly on the time scale τ of $\underline{\underline{R}}(\tau,\mu t)$. In this paper, we shall only treat the first non-trivial approximation, which leads to total-energy conservation, i.e. we consider the Markovian limit of (3.11) and its first non-Markovian correction. For this, we expand the integrand of (3.11) up to first order in τ:

$$\underline{g}(t) = \int_0^\infty d\tau\,(1-\tau\partial_t)\underline{\underline{R}}(\tau,\mu t)\cdot\underline{A}(\mu t). \tag{3.12}$$

The quantity of importance is thus $\underline{\underline{R}}(\tau,\mu t)$, i.e. the solution of (3.4), whose explicit expression will be given in the next section.

It is useful to introduce the resolvent $\underline{\underline{R}}(z,\mu t)$ as the Laplace transform of $\underline{\underline{R}}(\tau,\mu t)$:

$$\underline{\underline{R}}(z,\mu t) = \int_0^\infty d\tau\,\exp(iz\tau)\,\underline{\underline{R}}(\tau,\mu t), \qquad \operatorname{Im} z > 0. \tag{3.13}$$

Equation (3.12) then takes the form

$$\underline{g}(t) = \lim_{z\to i\varepsilon}\left(1+i\partial_t\frac{\partial}{\partial z}\right)\underline{\underline{R}}(z,t)\cdot\underline{A}(t) \tag{3.14}$$

(we put $\mu = 1$, since in the Laplace representation the quantities depend only on time through the one-particle df). As usual, the non-Markovianity is expressed by a derivative with respect to z. The extra dependence on time will be taken into account through (2.3).

4. Explicit results

As was pointed out in the previous section, we need the solution of (3.4), which is written explicitly with the help of (2.9) and (2.10) (since in this paper we only consider $\underline{\underline{R}}_0(t-t',\mu t)$, from now on we drop the subscript 0):

$$\begin{aligned}
&\left[\frac{\partial}{\partial t} + i\mathbf{k}\cdot(\mathbf{v}_1-\mathbf{v}_2)\right] R_{aba'b'}(\mathbf{k},t-t',\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_1',\mathbf{p}_2',\mu t)\\
&\quad -i\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{p}_1}f_a(\mathbf{p}_1,t)\sum_c\Phi_{ac}(\mathbf{k})\int d\mathbf{p}_3\, R_{cba'b'}(\mathbf{k},t-t',\mathbf{p}_3,\mathbf{p}_2,\mathbf{p}_1',\mathbf{p}_2',\mu t)\\
&\quad +i\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{p}_2}f_b(\mathbf{p}_2,t)\sum_c\Phi_{bc}(\mathbf{k})\int d\mathbf{p}_3\, R_{aca'b'}(\mathbf{k},t-t',\mathbf{p}_1,\mathbf{p}_3,\mathbf{p}_1',\mathbf{p}_2',\mu t)\\
&= \delta(t-t')\delta_{aa'}\delta_{bb'}\delta(\mathbf{p}_1-\mathbf{p}_1')\delta(\mathbf{p}_2-\mathbf{p}_2').
\end{aligned} \tag{4.1}$$

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The structure of this equation exhibits a separation of variables. Its solution can thus be factorized as

$$R_{aba'b'}(\mathbf{k}, t-t', \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}'_1, \mathbf{p}'_2, \mu t) = R_{aa'}(\mathbf{k}, t-t', \mathbf{p}_1, \mathbf{p}'_1, \mu t) R_{bb'}(\mathbf{k}, t-t', \mathbf{p}_2, \mathbf{p}'_2, \mathbf{k}, \mu t). \tag{4.2}$$

It can be verified that any factor on the right-hand side of (4.2) obeys the linearized Vlasov equation except for a trivial source term. The solution of this equation is easily obtained in the Laplace representation (Ichimaru 1973; Kalman 1975):

$$R_{aa'}(\omega + i0, \mathbf{k}, \mathbf{p}_1, \mathbf{p}'_1 t) = \frac{i}{\omega - \mathbf{k}\cdot\mathbf{v}_1 + i0}\left[\delta_{aa'}\delta(\mathbf{p}_1 - \mathbf{p}'_1) - \frac{1}{\omega - \mathbf{k}\cdot\mathbf{v}'_1 + i0}\frac{1}{\varepsilon(\omega, \mathbf{k}, t)}\Phi_{aa'}(\mathbf{k})\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{p}_1} f_a(\mathbf{p}_1, t)\right], \tag{4.3}$$

where, for future purposes, we have put $z = \omega + i0$, with ω a real frequency. Also in (4.3) there appears the dynamic dielectric function, defined as

$$\varepsilon(\omega, \mathbf{k}, t) = 1 + \sum_a \Phi_{aa}(\mathbf{k}) \int d\mathbf{p}\, \frac{1}{\omega - \mathbf{k}\cdot\mathbf{v}_1 + i0}\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{p}} f_a(\mathbf{p}, t). \tag{4.4}$$

The analytical properties of this function are well known, and have been discussed in the literature (Balescu 1963; Ichimaru 1973).

The complete resolvent is expressed as a convolution in frequency:

$$R_{ab,a'b'}(\omega + i0; k, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}'_1, \mathbf{p}'_2, t) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} R_{aa'}(\omega - \omega' + i0, \mathbf{k}, \mathbf{p}_1, \mathbf{p}'_1, t) R_{bb'}(\omega' + i0, -\mathbf{k}, \mathbf{p}_2, \mathbf{p}'_2, t), \tag{4.5}$$

The integration over ω' can be carried out in the following way. First, we remark that he first factor in the integrand, i.e. $R_{aa'}(\omega - \omega' + i0)$, is a regular function of ω' in the lower half of the complex plane S_-. Secondly, to evaluate the singularities in S_- of the second factor $R_{bb'}(\omega' + i0')$, which are not simple poles, we use the analytical properties of the integrand, as explained in Appendix A. The integration is then performed on a closed contour made by the real axis and a circle at infinity in S_-. The result is

$$\begin{aligned}
R_{ab,a'b'}(\omega + i0; k, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}'_1, \mathbf{p}'_2, t) = {} & \frac{i}{\omega + \mathbf{k}\cdot\mathbf{v}_2 - \mathbf{k}\cdot\mathbf{v}_1 + i0} \\
& \times \left[\delta_{aa'}\delta(\mathbf{p}_1 - \mathbf{p}'_1) - \frac{1}{\omega + \mathbf{k}\cdot\mathbf{v}_2 - \mathbf{k}\cdot\mathbf{v}'_1 + i0}\frac{\Phi_{aa'}(\mathbf{k})}{\varepsilon(\omega + \mathbf{k}\cdot\mathbf{v}_2, \mathbf{k})}\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{p}_1} f_a\right] \\
& \times \left[\delta_{bb'}\delta(\mathbf{p}_2 - \mathbf{p}'_2) + \frac{1}{\varepsilon(\mathbf{k}\cdot\mathbf{v}_2, \mathbf{k})}\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{p}_2} f_b \Phi_{bb'}(\mathbf{k}) \frac{1}{-\mathbf{k}\cdot\mathbf{v}_2 + \mathbf{k}\cdot\mathbf{v}'_2 - i0}\right] \\
& + \frac{i}{\omega + \mathbf{k}\cdot\mathbf{v}'_2 - \mathbf{k}\cdot\mathbf{v}_1 + i0}\frac{1}{-\mathbf{k}\cdot\mathbf{v}'_2 + \mathbf{k}\cdot\mathbf{v}_2 + i0}\frac{\Phi_{bb'}(\mathbf{k})}{\varepsilon^*(\mathbf{k}\cdot\mathbf{v}'_2, \mathbf{k})}\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{p}_2} f_b \\
& \times \left[\delta_{aa'}\delta(\mathbf{p}_1 - \mathbf{p}'_1) - \frac{1}{\omega + \mathbf{k}\cdot\mathbf{v}'_2 - \mathbf{k}\cdot\mathbf{v}'_1 + i0}\frac{\Phi_{aa'}(\mathbf{k})}{\varepsilon(\omega + \mathbf{k}\cdot\mathbf{v}'_2, \mathbf{k})}\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{p}_1} f_a\right] \\
& + \sum_c \int \mathbf{d}p_3 \frac{i}{\omega + \mathbf{k}\cdot\mathbf{v}_3 - \mathbf{k}\cdot\mathbf{v}_1 + i\Delta}\frac{\Phi_{bb'}(\mathbf{k})}{\mathbf{k}\cdot\mathbf{v}_3 - \mathbf{k}\cdot\mathbf{v}_2 - i0}\frac{\Phi_{cc}(\mathbf{k})}{\mathbf{k}\cdot\mathbf{v}_3 - \mathbf{k}\cdot\mathbf{v}'_2 + i0} \\
& \times \frac{1}{|\,\varepsilon(\mathbf{k}\cdot\mathbf{v}_3, \mathbf{k})\,|^2}\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{p}_3} f_c \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{p}_2} f_b \\
& \times \left[\delta_{aa'}\delta(\mathbf{p}_1 - \mathbf{p}'_1) - \frac{1}{\omega + \mathbf{k}\cdot\mathbf{v}_3 - \mathbf{k}\cdot\mathbf{v}'_1 + i0}\frac{\Phi_{aa'}(\mathbf{k})}{\varepsilon(\omega + \mathbf{k}\cdot\mathbf{v}_3, \mathbf{k})}\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{p}_1} f_a\right],
\end{aligned} \tag{4.6}$$

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where for brevity we have dropped the arguments of the one-particle dfs: $f_a \equiv f_a(\mathbf{p}_1, t)$, etc. Finally, as shown by (3.14), to obtain the pair correlation function, we need the quantity

$$\underline{H}(\omega, t) \equiv \underline{\underline{R}}(\omega + i0, t) \cdot \underline{A}(t), \tag{4.7}$$

which takes the explicit form

$$\begin{aligned} H_{ab}(\omega, \mathbf{k}, \mathbf{p}_1, \mathbf{p}_2, t) = &-\frac{1}{\omega + \mathbf{k}\cdot\mathbf{v}_2 - \mathbf{k}\cdot\mathbf{v}_1 + i0} \sum_c \int d\mathbf{p}_3 \left\{ \delta_{bc}\delta(\mathbf{p}_2 - \mathbf{p}_3) \right. \\ &\left. + \left[\frac{1}{\omega + \mathbf{k}\cdot\mathbf{v}_3 - \mathbf{k}\cdot\mathbf{v}_1 + i0} + \frac{1}{\mathbf{k}\cdot\mathbf{v}_2 - \mathbf{k}\cdot\mathbf{v}_3 + i0} \right] \frac{\Phi_{cc}(\mathbf{k})}{\varepsilon^*(\mathbf{k}\cdot\mathbf{v}_3, \mathbf{k})} \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{p}_2} f_b \right\} \\ &\times \left\{ \left[f_c - \frac{Q(\mathbf{k}, \mathbf{k}\cdot\mathbf{v}_3)}{\varepsilon(\mathbf{k}\cdot\mathbf{v}_3, \mathbf{k})} \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{p}_3} f_c \right] \frac{\Phi_{ab}(\mathbf{k})}{\varepsilon(\omega + \mathbf{k}\cdot\mathbf{v}_3, \mathbf{k})} \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{p}_1} f_a \right. \\ &\left. - \left[f_a - \frac{Q(\mathbf{k}, \omega + \mathbf{k}\cdot\mathbf{v}_3)}{\varepsilon(\omega + \mathbf{k}\cdot\mathbf{v}_3, \mathbf{k})} \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{p}_1} f_a \right] \frac{\Phi_{ab}(\mathbf{k})}{\varepsilon(\mathbf{k}\cdot\mathbf{v}_3, \mathbf{k})} \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{p}_3} f_c \right\}, \end{aligned} \tag{4.8}$$

where

$$Q(\mathbf{k}, \omega) = \sum_a \Phi_{aa}(\mathbf{k}) \int d\mathbf{p}_1 \frac{1}{\omega - \mathbf{k}\cdot\mathbf{v}_1 + i0} f_a(\mathbf{p}_1, t). \tag{4.9}$$

Let us point out that our method of integration over the frequency has broken an important symmetry property of the pair correlation function. By closing the contour in S_+, we would obtain, instead of (4.8), $H_{ba}(\omega, -\mathbf{k}, \mathbf{p}_2, \mathbf{p}_1, t)$. The half sum of the two expressions guarantees that, at any order of approximation, we have

$$g_{ab}(\mathbf{k}, \mathbf{p}_1, \mathbf{p}_2, t) = g_{ba}(-\mathbf{k}, \mathbf{p}_2, \mathbf{p}_1, t) \tag{4.10}$$

To obtain the expression for the pair correlation function is now straightforward. In the first approximation (Balescu and Taylor 1961; Balescu 1963), we take the limit as $\omega \to 0$ of (4.8):

$$\begin{aligned} g^0_{ab}(\mathbf{k}, \mathbf{p}_1, \mathbf{p}_2, t) &= \lim_{\omega\to 0} H_{ab}\left(\omega, \mathbf{k}, \mathbf{p}_1, \mathbf{p}_2, t\right) \\ &= -\frac{\Phi_{ab}(\mathbf{k})}{\mathbf{k}\cdot\mathbf{v}_2 - \mathbf{k}\cdot\mathbf{v}_1 + i0} \left[\frac{1}{\varepsilon(\mathbf{k}\cdot\mathbf{v}_2, \mathbf{k})} \mathbf{k}\cdot\left(\frac{\partial}{\partial \mathbf{p}_1} - \frac{\partial}{\partial \mathbf{p}_2} \right) f_a f_b \right. \\ &\quad + \sum_c \Phi_{cc}(\mathbf{k}) \int d\mathbf{p}_3 \left(\frac{1}{\mathbf{k}\cdot\mathbf{v}_3 - \mathbf{k}\cdot\mathbf{v}_1 + i0} + \frac{1}{-\mathbf{k}\cdot\mathbf{v}_3 + \mathbf{k}\cdot\mathbf{v}_2 + i0} \right) \\ &\quad \left. \times \frac{1}{|\varepsilon(\mathbf{k}\cdot\mathbf{v}_3, \mathbf{k})|^2} \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{p}_2} f_b \left(f_c \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{p}_1} f_a - f_a \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{p}_3} f_c \right) \right]. \end{aligned} \tag{4.11}$$

This expression can be simplified further if the last term is transformed by using a property of the dielectric function that is demonstrated in Appendix B:

$$1 - \frac{1}{\varepsilon(\mathbf{k}\cdot\mathbf{v}_1, \mathbf{k})} = \sum_b \int d\mathbf{p}_2 \frac{\Phi_{bb}(\mathbf{k})}{|\,\varepsilon(\mathbf{k}\cdot\mathbf{v}_2, \mathbf{k})\,|^2} \frac{1}{\mathbf{k}\cdot\mathbf{v}_1 - \mathbf{k}\cdot\mathbf{v}_2 + i0} \mathbf{k}_2\cdot\frac{\partial}{\partial \mathbf{p}_2} f_b(\mathbf{p}_2, t). \tag{4.12}$$

Then

$$\begin{aligned} g^0_{ab}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}, t) = &-\frac{\Phi_{ab}(\mathbf{k})}{\mathbf{k}\cdot\mathbf{v}_2 - \mathbf{k}\cdot\mathbf{v}_1 + i0} \left[\frac{1}{\varepsilon(\mathbf{k}\cdot\mathbf{v}_2, \mathbf{k})} f_b(\mathbf{p}_2, t) \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{p}_1} f_a(\mathbf{p}_1, t) \right. \\ &\left. - \frac{1}{\varepsilon^*(\mathbf{k}\cdot\mathbf{v}_1, \mathbf{k})} f_a(\mathbf{p}_1, t) \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{p}_2} f_b(\mathbf{p}_2, t) \right] \end{aligned}$$

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$$+\sum_c \int \mathbf{d}p_3 \, \frac{1}{|\,\varepsilon(\mathbf{k}\cdot\mathbf{v}_3,\mathbf{k})\,|^2} \frac{\Phi_{ab}(\mathbf{k})}{\mathbf{k}\cdot\mathbf{v}_3 - \mathbf{k}\cdot\mathbf{v}_1 + i0} \frac{\Phi_{cc}(\mathbf{k})}{\mathbf{k}\cdot\mathbf{v}_3 - \mathbf{k}\cdot\mathbf{v}_2 - i0}$$
$$\times f_c(\mathbf{p}_3)\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{p}_1} f_a(\mathbf{p}_1,t)\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{p}_2} f_b(\mathbf{p}_2,t). \tag{4.13}$$

On the other hand, g^1_{ab} is obtained from (3.14), (4.7) and (4.8) as

$$g^1_{ab}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k},t) = \frac{i}{2}\frac{\partial}{\partial t}\frac{\partial}{\partial\omega}[H_{ab}(\omega,\mathbf{k},\mathbf{p}_1,\mathbf{p}_2,t) + H_{ba}(\omega,-\mathbf{k},\mathbf{p}_2,\mathbf{p}_1,t)]|_{\omega=0}. \tag{4.14}$$

A more explicit expression can be found elsewhere (Belyi *et al.* 1996). We shall not reproduce it here, because it is not needed in the following.

Finally, from (2.3) and the above results, and using (4.4), we obtain the collision integral as

$$J_a(\mathbf{p}_1,t) = J^0_a(\mathbf{p}_1) + J^1_a(\mathbf{p}_1,t)$$
$$= -\frac{i}{8\pi^3}\sum_b \int d\mathbf{k}\, d\mathbf{p}_2\, \Phi_{ab}(\mathbf{k})\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{p}_1}[g^0_{ab}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k},t) + g^1_{ab}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k},t)] \tag{4.15}$$
$$= -\sum_b \int \frac{d\mathbf{k}}{(2\pi)^3} d\mathbf{p}_2 \Phi^2_{ab}(\mathbf{k}) i\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{p}_1}\left(1 + i\frac{\partial}{\partial t}\frac{\partial}{\partial\omega}\right)\frac{1}{\omega + \mathbf{k}\cdot\mathbf{v}_2 - \mathbf{k}\cdot\mathbf{v}_1 + i0}\frac{1}{\varepsilon^*(\mathbf{k}\cdot\mathbf{v}_2,\mathbf{k})}$$
$$\times\left\{\frac{1}{\varepsilon(\omega+\mathbf{k}\cdot\mathbf{v}_2,\mathbf{k})}\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{p}_1} f_a\left[f_b - \frac{Q(\mathbf{k},\mathbf{k}\cdot\mathbf{v}_2)}{\varepsilon(\mathbf{k}\cdot\mathbf{v}_2,\mathbf{k})}\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{p}_2} f_b\right]\right.$$
$$\left. -\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{p}_2} f_b \frac{1}{\varepsilon(\mathbf{k}\cdot\mathbf{v}_2,\mathbf{k})}\left[f_a - \frac{Q(\mathbf{k},\omega+\mathbf{k}\cdot\mathbf{v}_2)}{\varepsilon(\omega+\mathbf{k}\cdot\mathbf{v}_2,\mathbf{k})}\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{p}_1} f_a\right]\right\}\Bigg|_{\omega=0}. \tag{4.16}$$

The first contribution immediately gives the BL collision integral (Balescu 1963):

$$J^0_a(\mathbf{p}_1,t) = -\sum_b \int \frac{d\mathbf{k}}{(2\pi)^3}\, d\mathbf{p}_2\, \Phi^2_{ab}(\mathbf{k})\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{p}_1}\delta(\mathbf{k}\cdot\mathbf{v}_2 - \mathbf{k}\cdot\mathbf{v}_1)$$
$$\times\frac{1}{|\,\varepsilon(\mathbf{k}\cdot\mathbf{v}_2,\mathbf{k})\,|^2}\mathbf{k}\cdot\left(\frac{\partial}{\partial\mathbf{p}_1} - \frac{\partial}{\partial\mathbf{p}_2}\right) f_a f_b, \tag{4.17}$$

where we have used the relation

$$\frac{1}{x+i0} = -i\pi\delta_+(\mathbf{x}) = -i\pi\delta(\mathbf{x}) + \frac{P}{x}, \tag{4.18}$$

with P indicating the Cauchy principal part.

5. Properties of the kinetic equation

The most important properties that a kinetic equation must possess are the conservation laws. We now show how the kinetic equation

$$\frac{\partial}{\partial t} f_a(\mathbf{p}_1,t) = J^0_a(\mathbf{p}_1,t) + J^1_a(\mathbf{p}_1,t), \tag{5.1}$$

where the collision integrals $J^0_a(\mathbf{p}_1,t)$ and $J^1_a(\mathbf{p}_1,t)$ are given by (4.16) and (4.17), fulfills these conditions.

The conservation of the number of particles of each species is obtained by integrating (5.1) over $\mathbf{p}_1$:

$$\frac{\partial}{\partial t}\int d\mathbf{p}_1\, f_a(\mathbf{p}_1,t) = \int d\mathbf{p}_1\,[J^0_a(\mathbf{p}_1,t) + J^1_a(\mathbf{p}_1,t)] = 0. \tag{5.2}$$

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The second equality is straightforward because $J_a^0(\mathbf{p}_1, t)$ and $J_a^1(\mathbf{p}_1, t)$ begin with a derivative with respect to $\mathbf{p}_1$.

Momentum conservation is given by

$$\frac{\partial}{\partial t}\int d\mathbf{p}_1\, \mathbf{p}_1 f_a(\mathbf{p}_1, t) = \int d\mathbf{p}_1\, \mathbf{p}_1 [J_a^0(\mathbf{p}_1, t) + J_a^1(\mathbf{p}_1, t)]. \tag{5.3}$$

After an integration by parts, the right-hand side of (5.3) becomes

$$\frac{i}{8\pi^3}\sum_{ab}\int d\mathbf{k}\, d\mathbf{p}_1\, d\mathbf{p}_2\, \Phi_{ab}(\mathbf{k})\mathbf{k}[g_{ab}^0(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}, t) + g_{ab}^1(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}, t)], \tag{5.4}$$

where we have used the expressions (4.15) and the collision integrals. Owing to the symmetry property (4.10), the expression (5.4) vanishes.

As has already been mentioned, the kinetic energy E^K is not conserved by the kinetic equation (5.1), since it can be transformed into potential energy. To see this property, let us write

$$\frac{\partial}{\partial t}E^k \equiv \frac{\partial}{\partial t}\int d\mathbf{p}_1\, \frac{p_1^2}{2m_a} f_a(\mathbf{p}_1, t) = \sum_a \int \frac{p_1^2}{2m_a}[J_a^0(t, \mathbf{p}_1) + J_a^1(t, \mathbf{p}_1)]\, d\mathbf{p}_1. \tag{5.5}$$

Using (4.15) and integrating by parts, we obtain, with the help of the symmetry property (4.10),

$$\frac{\partial}{\partial t}E^k \equiv \frac{i}{16\pi^3}\sum_{ab}\int d\mathbf{k}\, d\mathbf{p}_1\, d\mathbf{p}_2\, \Phi_{ab}(\mathbf{k})(\mathbf{k}\cdot\mathbf{v}_1 - \mathbf{k}\cdot\mathbf{v}_2) \times[g_{ab}^0(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}, t) + g_{ab}^1(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}, t)]. \tag{5.6}$$

A simple inspection of (4.13) shows that $g_{ab}^0(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}, t)$ does not contribute to the right-hand side of (5.6). (It is indeed a property of the BL collision integral (4.17) that it conserves kinetic energy.) In order to evaluate the contribution of $g_{ab}^1(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}, t)$, we first remark that

$$\frac{\mathbf{k}\cdot\mathbf{v}_1 - \mathbf{k}\cdot\mathbf{v}_2}{\omega - \mathbf{k}\cdot\mathbf{v}_1 + \mathbf{k}\cdot\mathbf{v}_2 + i0} = \frac{\omega}{\omega - \mathbf{k}\cdot\mathbf{v}_1 + \mathbf{k}\cdot\mathbf{v}_2 + i0} - 1.$$

We then express $g_{ab}^1(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}, t)$ by (4.14) with (4.8), to obtain

$$\begin{aligned}
\frac{\partial}{\partial t}E^k = {} & \frac{1}{16\pi^3}\frac{\partial}{\partial t}\lim_{\omega\to 0}\frac{\partial}{\partial\omega}\sum_{ab}\int d\mathbf{k}\, d\mathbf{p}_1 d\mathbf{p}_2 \Phi_{ab}^2(\mathbf{k}) \\
& \times\Bigg\{\frac{\omega}{\mathbf{k}\cdot\mathbf{v}_2 - \mathbf{k}\cdot\mathbf{v}_1 + i0}\Bigg[\frac{1}{\varepsilon(\mathbf{k}\cdot\mathbf{v}_2, \mathbf{k})}\mathbf{k}\cdot\left(\frac{\partial}{\partial\mathbf{p}_1} - \frac{\partial}{\partial\mathbf{p}_2}\right) f_a(\mathbf{p}_1, t) f_b(\mathbf{p}_2, t) \\
& + \sum_c \int \mathbf{d}p_3 \frac{\Phi_{cc}(\mathbf{k})}{|\,\varepsilon(\mathbf{k}\cdot\mathbf{v}_3, \mathbf{k})\,|^2}\left(\frac{1}{\mathbf{k}\cdot\mathbf{v}_3 - \mathbf{k}\cdot\mathbf{v}_1 + i0} + \frac{1}{\mathbf{k}\cdot\mathbf{v}_2 - \mathbf{k}\cdot\mathbf{v}_3 + i0}\right) \\
& \times\left(f_c(\mathbf{p}_3, t)\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{p}_1} f_a(\mathbf{p}_1, t)\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{p}_2} f_b(\mathbf{p}_2, t)\right. \\
& \left. - f_a(\mathbf{p}_1, t)\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{p}_3} f_c(\mathbf{p}_3, t)\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{p}_2} f_b(\mathbf{p}_2, t)\right)\Bigg] \\
& + \frac{1}{\varepsilon(\mathbf{k}\cdot\mathbf{v}_2, \mathbf{k})} f_a(\mathbf{p}_1, t)\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{p}_2} f_b(\mathbf{p}_2, t)
\end{aligned}$$

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$$-\sum_c \int \mathbf{d}p_3 \frac{\Phi_{cc}(\mathbf{k})}{|\,\varepsilon(\mathbf{k}\cdot\mathbf{v}_3,\mathbf{k})\,|^2} \frac{1}{\mathbf{k}\cdot\mathbf{v}_2 - \mathbf{k}\cdot\mathbf{v}_3 + i0}$$
$$\times f_a(\mathbf{p}_1,t)\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{p}_3} f_c(\mathbf{p}_3,t)\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{p}_2} f_b(\mathbf{p}_2,t)\Big\}, \tag{5.7}$$

where we have not written the terms that vanish on integration over $\mathbf{p}_1$ or $\mathbf{p}_2$ nor the ω-dependent coefficient of the factor ω, since $\lim_{\omega\to 0}(\partial/\partial\omega)\omega f(\omega) = f(0)$. The differentiation with respect to ω is then trivial. By comparison with (4.11), we may finally write

$$\frac{\partial}{\partial t}E^k = -\frac{1}{16\pi^3}\frac{\partial}{\partial t}\sum_{ab}\int d\mathbf{k}\, dp_1\, \mathbf{d}p_2\, \Phi_{ab}(\mathbf{k}) g^0_{ab}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k},t) = -\frac{\partial}{\partial t}E^P, \tag{5.8}$$

where E^P is the potential energy, defined as

$$E^P = \frac{1}{2}\sum_{ab}\int d\mathbf{r}\, dp_1\, \mathbf{d}p_2\, \Phi_{ab}(\mathbf{r}) g^0_{ab}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{r},t)$$
$$= \frac{1}{32\pi^3}\sum_{ab}\int d\mathbf{k}\, d\mathbf{p}_1\, d\mathbf{p}_2 \frac{\Phi^2_{ab}(\mathbf{k})}{\mathbf{k}\cdot\mathbf{v}_1 - \mathbf{k}\cdot\mathbf{v}_2}\left[\frac{1}{|\,\varepsilon(\mathbf{k}\cdot\mathbf{v}_1),\mathbf{k})\,|^2} + \frac{1}{|\,\varepsilon(\mathbf{k}\cdot\mathbf{v}_2),\mathbf{k})\,|^2}\right]$$
$$\times\left(f_b\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{p}_1}f_a - f_a\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{p}_2}f_b\right). \tag{5.9}$$

Equation (5.8) shows that, in the approximation considered, the total energy $E^K + E^P$ is conserved. By construction, E^P is of first order in the plasma parameter; thus, at equilibrium, it is equal to the Debye–Hückel energy (Balescu 1963, 1975; Ichimaru 1973).

It is easy to verify that $J^0_a(\mathbf{p}_1,t)+J^1_a(\mathbf{p}_1,t)$ vanishes at equilibrium. This is indeed a property of the BL collision integral $J^0_a(\mathbf{p}_1,t)$. On the other hand, the time derivative before the expression (4.14) for $g^1_{ab}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k},t)$ ensures that it vanishes when the one-particle df does not depend on time. As a consequence, $J^1_a(\mathbf{p}_1,t)$ is zero for any stationary state.

6. Conclusions

We have derived a nonlinear kinetic equation for a spatially uniform multicomponent polarizable plasma that exhibits conservation of particle number, total momentum and energy up to first order in the plasma parameter. It is thus a generalization of the BL kinetic equation to weakly non-ideal plasmas whose equilibrium state is correctly described in the Debye–Hückel approximation.

This equation, which can be used for instance to study temperature relaxation when different species have different temperatures does not, however, have the same universality as the BL equation. Indeed, an H theorem is missing. This should ensure that the system goes monotonically to equilibrium and that the transport coefficients are positive-definite. We can only say that the dominant approximation gives the correct behaviour of the system; we cannot say anything about the corrections except that they must be small.

On the other hand, to obtain our equation, we have closed the BBGKY hierarchy in the so-called plasma approximation (first order in the plasma parameter), but we considered non-Markovian effects. Our closure procedure excludes Markovian terms of second order in the plasma parameter, while the non-Markovian terms are clearly

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of second order. In fact, the non-Markovian terms introduce an important new qualitative effect, namely the conservation of the total energy, while the neglected Markovian terms give rise to small quantitative corrections to the relaxation time of the one-particle df. If necessary, these corrections could be added to our equation without qualitative change.

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Appendix A

In the expression (4.5) for $R_{ab,a'b'}(\omega+i0;k,\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_1',\mathbf{p}_2',t)$, there are some ambiguous terms of the form

$$I=\int d\omega'\,\frac{1}{\omega'-\mathbf{k}\cdot\mathbf{v}_2+i0}\,\frac{1}{\omega'-\mathbf{k}\cdot\mathbf{v}_2'+i0}F(\omega'), \tag{A 1}$$

which are transformed as

$$\begin{aligned} I&=\int d\omega'\,\frac{1}{\omega'-\mathbf{k}\cdot\mathbf{v}_2+i0}\left(\frac{1}{\omega'-\mathbf{k}\cdot\mathbf{v}_2'+i0}-\frac{1}{\omega'-\mathbf{k}\cdot\mathbf{v}_2'-i0}+\frac{1}{\omega'-\mathbf{k}\cdot\mathbf{v}_2'-i0}\right)F(\omega')\\ &=\int d\omega'\,\frac{1}{\omega'-\mathbf{k}\cdot\mathbf{v}_2+i0}\left(2\,\mathrm{Im}\,\frac{1}{\omega'-\mathbf{k}\cdot\mathbf{v}_2'+i0}+\frac{1}{\omega'-\mathbf{k}\cdot\mathbf{v}_2'-i0}\right)F(\omega'). \end{aligned} \tag{A 2}$$

The second term in the large parentheses on the right-hand side of (A 2) is regular in S_-, and thus the integral is evaluated by the residue of the pole at $\omega'=\mathbf{k}\cdot\mathbf{v}_2-i0$ when the contour is closed by a circle at infinity in S_-. On the other hand

$$\mathrm{Im}\,\frac{1}{\omega'-\mathbf{k}\cdot\mathbf{v}_2'+i0}=-\pi\delta(\omega'-\mathbf{k}\cdot\mathbf{v}_2')$$

(cf. (4.18)). The delta function is then used to compute the integral corresponding to the first term in the large parentheses.

Similarly, since the singularities in S_- of $1/\varepsilon(\omega',\mathbf{k})$ are unknown (Balescu 1963; Ichimaru 1973), we take

$$\begin{aligned} \frac{1}{\varepsilon(\omega',\mathbf{k})}&=\frac{1}{\varepsilon^*(\omega',\mathbf{k})}+\frac{1}{\varepsilon(\omega',\mathbf{k})}-\frac{1}{\varepsilon^*(\omega',\mathbf{k})}\\ &=\frac{1}{\varepsilon^*(\omega',\mathbf{k})}+2i\,\mathrm{Im}\,\frac{1}{\varepsilon(\omega',\mathbf{k})}\\ &=\frac{1}{\varepsilon^*(\omega',\mathbf{k})}+2\pi i\sum_a\frac{\Phi_{aa}(\mathbf{k})}{|\,\varepsilon(\omega',\mathbf{k})\,|^2}\int d\mathbf{p}_1\delta(\omega'-\mathbf{k}\cdot\mathbf{v}_1)\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{p}_1}f_a(\mathbf{p}_1,t), \end{aligned} \tag{A 3}$$

where we have used the definition of the dielectric function (4.4). For a stable plasma, the function $1/\varepsilon^*(\omega',\mathbf{k})$ has no singularities in S_-, and in the second term on the right-hand side of (A 3) the delta function is used for the integration over the variable ω'.

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Appendix B

Starting from the Kramers–Kronig relation for $1/\varepsilon(\omega,\mathbf{k})$ (Ichimaru 1973),

$$\mathrm{Re}\,\frac{1}{\varepsilon(\omega,\mathbf{k})}-1=\int\frac{d\omega'}{\pi}\frac{P}{\omega'-\omega}\,\mathrm{Im}\,\frac{1}{\varepsilon(\omega',\mathbf{k})}, \tag{B 1}$$

where P indicating the Cauchy principal part, one can verify the formula

$$\frac{1}{\varepsilon(\omega,\mathbf{k})}=1-\int\frac{d\omega'}{\pi}\frac{1}{\omega-\omega'+i0}\,\mathrm{Im}\,\frac{1}{\varepsilon(\omega',\mathbf{k})} \tag{B 2}$$

by taking the real and imaginary parts of each side. Using the property (4.18) the real part of (B 2) gives (B 1), while the imaginary part transforms as

$$\begin{aligned}\mathrm{Im}\frac{1}{\varepsilon(\omega,\mathbf{k})}&=-\int\frac{d\omega'}{\pi}\,\mathrm{Im}\,\frac{1}{\omega-\omega'+i0}\,\mathrm{Im}\,\frac{1}{\varepsilon(\omega',\mathbf{k})}\\&=\int\frac{d\omega'}{\pi}\pi\delta(\omega-\omega')\,\mathrm{Im}\,\frac{1}{\varepsilon(\omega',\mathbf{k})}=\mathrm{Im}\,\frac{1}{\varepsilon(\omega,\mathbf{k})}.\end{aligned}$$

On the other hand, (B 2) transforms as

$$\begin{aligned}\frac{1}{\varepsilon(\omega,\mathbf{k})}-1&=\int\frac{d\omega'}{\pi}\frac{1}{\omega-\omega'+i0}\frac{\mathrm{Im}\varepsilon(\omega',\mathbf{k})}{\mid\varepsilon(\omega',\mathbf{k})\mid^2}\\&=-\int d\omega'\,\frac{1}{\omega-\omega'+i0}\frac{1}{\mid\varepsilon(\omega',\mathbf{k})\mid^2}\sum_c\Phi_{cc}(\mathbf{k})\\&\quad\times\int\mathbf{d}p_3\,\delta(\omega'-\mathbf{k}\cdot\mathbf{v}_3)\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{p}_3}f_c(\mathbf{p}_3,t),\end{aligned}$$

where we have used the definition (4.4) of the dielectric function. On replacing ω by $\mathbf{k}\cdot\mathbf{v}_1$, we finally obtain (4.12)

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