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Anticritical graphs

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Abstract. The term 'critical graph' has been used most frequently to mean a graph such that the removal of any line decreases the chromatic number. This can be generalized to define a graph G to be 'critical (μ)' for an arbitrary invariant μ if, for each line e, $\mu(G-e) \neq \mu(G)$. Analogously we may consider any line e which joins two points not already adjacent in G and define G as 'anticritical (μ)' if $\mu(G+e) \neq \mu(G)$. Anticritical graphs are investigated here with respect to the following invariants: chromatic number, line-chromatic number, point-connectivity, line-connectivity, point-independence number, line-independence number, diameter, radius, and cyclability number.

Introduction and notation. Let μ denote an arbitrary parameter of a graph G. Then G is called *critical* with respect to μ, denoted *critical* (μ), if for every line e in it, $\mu(G-e) \neq \mu(G)$. From an opposite point of view, we call G *anticritical* (μ) if G is not a complete graph and if, for every pair of non-adjacent points in G, the line e joining them has the property $\mu(G+e) \neq \mu(G)$. In this paper we investigate anticritical graphs with respect to nine parameters: chromatic and line-chromatic numbers, point- and line-connectivity, point- and line-independence numbers, diameter, radius, and cyclability number. We use the terminology of (5) with slight changes. For completeness, the nine invariants are defined here.

The concepts of critical and anticritical graphs with respect to an arbitrary parameter serve to unify results which were previously isolated from each other. For example, the theorems of Mader (7) and Ore (8) stated below are now seen to be characterizations of two types of anticritical graphs.

The *chromatic number* $\chi(G)$ is the smallest number of colours which can be assigned to the points of G so that no two adjacent points have the same colour. The *connectivity* $\kappa(G)$ is the smallest number of points whose removal from G results in a disconnected or trivial graph. The *line-chromatic number* $\chi'(G)$ and the *line-connectivity* $\kappa'(G)$ are defined by changing 'point' to 'line' in the preceding sentences. The maximum number of independent (mutually non-adjacent) points is $\beta(G)$, and $\beta'(G)$ is the corresponding number for lines. These are called the *point-independence number* and the *line-independence number*, respectively.

To define the remaining three invariants, recall that $d(u, v)$, the *distance* between points u and v, is the length of a shortest path joining them. The *diameter* $d(G)$ of a graph G is $\max\{d(u, v): u, v \text{ in } G\}$. The eccentricity of a point u is $\max\{d(u, v): v \text{ in } G\}$. The *radius* $r(G)$ is the minimum eccentricity of the points. Finally, the *cyclability*

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number $\xi(G)$ is the largest k such that every set of k points of G are contained in a common cycle. As usual we write $\delta(G)$ and $\Delta(G)$ for the minimum and maximum degrees of the points in G. Note that if e is a line of $\bar{G}$, the complement of G, then e joins two points not adjacent in G.

The graphs which are anticritical with respect to β' and d were characterized by Mader (7) and Ore (8), respectively. We characterize graphs which are anticritical with respect to χ, κ, κ', β, and show that to do so for χ' would settle the Four Colour Conjecture. Furthermore we investigate anticritical graphs with respect to r and disprove a conjecture of Chvátal (2) on graphs which are anticritical with respect to ξ.

If G is anticritical (μ), then for any line e in $\bar{G}$, $\mu(G+e) > \mu(G)$ when μ denotes χ, χ', κ, κ', β' or ξ, whereas $\mu(G+e) < \mu(G)$ when μ denotes β, d or r. Furthermore, $\mu(G+e)$ equals $\mu(G)+1$ or $\mu(G)-1$ if μ denotes any of the invariants χ, χ', κ, κ', β or β'. Analogous statements hold for critical (μ) graphs.

We shall find, not surprisingly, that complete subgraphs play a major role in the construction of the various anticritical graphs.

Chromatic number (χ). Graphs which are critical (χ) were the first graphs called 'critical' in the literature. These were introduced by Dirac (3) who characterized them for $\chi \leqslant 3$. For $\chi \geqslant 4$ there is as yet no characterization. In contrast, we have a simple specification of the anticritical (χ) graphs. Note that $K(1, 1, \ldots, 1)$ is a complete graph.

THEOREM 1. *The anticritical* (χ) *graphs for* $\chi = m \geqslant 1$ *are precisely those complete* m*-partite graphs* $K(n_1, n_2, \ldots, n_m)$ *which are not complete.*

Proof. Clearly $\chi(K(n_1, n_2, \ldots, n_m)) = m$. If e is any line not present in $K(n_1, n_2, \ldots, n_m)$, then $K(n_1, n_2, \ldots, n_m)+e$ contains a complete subgraph K_{m+1} and therefore has chromatic number $m+1$. Assume conversely that G is an anticritical (χ) graph for $\chi = m$. Let $A_1, A_2, \ldots, A_m$ denote the colour classes of points of any m-colouring of G, with each $|A_i| = n_i \geqslant 1$. If $u \in A_i$, $v \in A_j$ and $i \neq j$, then u and v must be adjacent in G, for otherwise $G+uv$ has an m-colouring, a contradiction. Hence G is precisely the graph $K(n_1, n_2, \ldots, n_m)$.

Line-chromatic number (χ'). The critical (χ') graphs were first studied by Vizing (11). Obviously $\chi'(G) \geqslant \Delta(G)$. A graph is said to be of *class* ρ $(\rho \geqslant 1)$ when

$$\chi'(G) = \Delta(G) + \rho - 1.$$

Vizing's theorem (10) states that every graph is of class 1 or 2. If G is of class 1, has no isolated points and is critical (χ') for $\chi' = m$, then it is easy to see that G is the star $K(1, m)$.

The critical (χ') graphs are, however, not completely characterized. For the anticritical (χ') graphs we have the following negative result.

THEOREM 2. *There exists no anticritical* (χ') *graph of class* 2.

Proof. Suppose that $\chi'(G) = \Delta(G)+1$ and that G is anticritical (χ'). Clearly G is not a complete graph, so there exist nonadjacent points u and v of G. Then we have $\chi'(G+uv) = \chi'(G)+1 = \Delta(G)+2$. By Vizing's Theorem, $\chi'(G+uv) \leqslant \Delta(G+uv)+1$. So $G+uv$ is of class 2 and $\Delta(G+uv) = \Delta(G)+1$. At most two points of $G+uv$, namely

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u and v, can have degree $\Delta(G)+1$. By another theorem of Vizing (11), at least three points of a graph of class 2 have maximum degree. We have then obtained a contradiction, which proves the theorem.

A graph which is of class 1 and which is anticritical (Δ) is clearly also anticritical (χ'). The anticritical (Δ) graphs are simply the graphs in which the points of degree less than Δ are mutually adjacent. For example, all regular graphs are anticritical (Δ). It is well known ((5), p. 134), that the Four Colour Conjecture is equivalent to the statement that every cubic planar bridgeless graph is of class 1. Combining this with Theorem 2, we immediately obtain the next observation.

COROLLARY. *The Four Colour Conjecture is true if and only if every cubic planar bridgeless graph is anticritical* (χ').

There exist anticritical (χ') graphs which are not anticritical (Δ). For n odd,

$$\chi'(K_n) = n = \Delta(K_n)+1,$$

and if K_n is line-coloured in n colours, then every colour-class consists of $\frac{1}{2}(n-1)$ mutually independent lines. For n odd, let L_n be the graph obtained from K_n by deleting any $\frac{1}{2}(n-1)$ mutually independent lines. Then L_n has line-chromatic number $\chi'(L_n) = n-1 = \Delta(L_n)$. L_n is not anticritical (Δ). By an observation of Beineke and Wilson (1), Corollary 3, every graph obtained from K_n (n odd) by deleting any $\frac{1}{2}(n-3)$ lines is of class 2. So L_n is anticritical (χ').

Connectivity (κ). To construct these anticritical graphs, take three disjoint complete graphs of order m, s and t. Then form the join of K_m with the union $K_s \cup K_t$, denoted $K_m + (K_s \cup K_t)$, as defined in (5), p. 21. Clearly this graph can also be formed by taking K_{m+s} and K_{m+t} and identifying K_m within these two complete graphs.

THEOREM 3. *The anticritical* (κ) *graphs for* $\kappa = m$ *are the graphs* $K_m + (K_s \cup K_t)$.

Proof. Clearly $\kappa(K_m + (K_s \cup K_t)) = m$. Furthermore the set of m points which the complete subgraphs K_{m+s} and K_{m+t} have in common is the only set of m points whose removal results in a disconnected graph. From this it easily follows that the graph is anticritical (κ).

Assume conversely that G is anticritical (κ) for $\kappa = m$. Since G is neither complete nor $(m+1)$-connected, there exists a set S of m points of G such that $G-S$ is disconnected. Thus G contains two induced subgraphs G_1, G_2 such that $V(G_1) \cup V(G_2) = V(G)$, $V(G_1) \cap V(G_2) = S$ and $|V(G_i)| = n_i > m$ for $i = 1, 2$. If u and v are nonadjacent points of G_1 or of G_2 then $(G+uv)-S$ is disconnected, in contradiction to the anticriticality of G. So $G_i = K_{n_i}$ for $i = 1, 2$. Consequently $G = K_m + (K_s \cup K_t)$ where $n_1 - m = s$ and $n_2 - m = t$.

Line-connectivity (κ'). These anticritical graphs are easy to specify.

THEOREM 4. *The anticritical* (κ') *graphs for* $\kappa' = t$ *are obtained by adding any* t *lines to the disjoint union* $K_n \cup K_m$ *where* $n \geqslant t+1$ *and* $m = 1$, *or* $n, m \geqslant t+2$.

Proof. Since $\kappa'(K_p) = p-1$, it is easy to see that the graphs described in the theorem are anticritical (κ').

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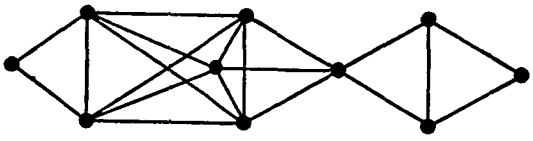


Fig. 1. An anticritical (d) graph.

Assume conversely that G is an anticritical (κ') graph for $\kappa' = t$. Then every point of G has degree at least t. Suppose first that G contains a point u_0 of degree t. If u, v are nonadjacent points different from u_0, then $\delta(G+uv) = \delta(G) = t$, so $\kappa'(G+uv) = t$, contradicting the anticriticality. So $G-u_0 = K_n$ where $n \geqslant t$. Since G is not complete, $n \geqslant t+1$. Suppose next that $\delta(G) \geqslant t+1$. Then G contains a separating set A of t lines such that $G-A = G_1 \cup G_2$, where G_1 and G_2 are disjoint. Every line of A joins a point of G_1 with a point of G_2, for otherwise $\kappa'(G) < t$. Because G is anticritical (κ'), $G_i = K_{n_i}$ for $i = 1, 2$. Now $n_i \geqslant 2$ for $i = 1, 2$ because $\delta(G) > t$. It only remains to show that $n_i \geqslant t+2$ for $i = 1, 2$. Assume therefore that $n_1 \leqslant t+1$. Then every point of G_1 is incident with at least $t+1-(n_1-1)$ lines of A because $\delta(G) \geqslant t+1$. So

$$|A| = t \geqslant n_1(t+2-n_1)$$

which implies that $n_1^2 - 2n_1 \geqslant t(n_1-1)$. Since $t \geqslant n_1 - 1$, this is a contradiction, and the theorem is proved.

Point-independence number (β) *and line-independence number* (β'). The join operation is most convenient in describing graphs anticritical with respect to these invariants.

THEOREM 5. *The anticritical* (β) *graphs for* $\beta = m$ *are the graphs* $\bar{K}_m + K_n$, *where* $n \geqslant 0$.

Proof. Clearly $\bar{K}_m + K_n$ is anticritical (β) for $\beta = m$. Conversely, let S be any set of m mutually independent points of a graph G. If e is any line of $\bar{G}$, then $G+e$ contains no set of m independent points, so e must join two points of S, and hence $G = \bar{K}_m + K_n$.

THEOREM 6 (*Mader* (7), p. 273). *The anticritical* (β') *graphs for* $\beta' = m$ *are the join graphs* $K_k + (K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_t})$ *where* $k \geqslant 0$, $t \geqslant k+2$, *every* n_i *is odd, and*

$$m = k + \sum_{i=1}^{t} \tfrac{1}{2}(n_i - 1).$$

Diameter (d) *and radius* (r). Anticritical (d) graphs were characterized by Ore.

THEOREM 7 (*Ore* (8), *Theorem* 2·1). *A graph* G *is anticritical* (d) *for* $d = m$ *if and only if its points can be partitioned into* $m+1$ *mutually disjoint non-empty classes*

$$A_1, A_2, \ldots, A_{m+1}$$

such that $|A_1| = |A_{m+1}| = 1$, *and* $u \in A_i$, $v \in A_j$ *are adjacent if and only if* $|i-j| \leqslant 1$.

Thus the anticritical (d) graphs are described structurally; Fig. 1 shows an example with $m = 6$. While there seems to be no simple characterization of the critical (d) graphs, in contrast, the critical (r) graphs are easy to specify.

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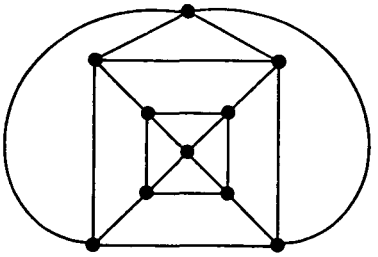


Fig. 2. Another admissible graph.

THEOREM 8. *A graph G is critical (r) if and only if G is a tree.*

Proof. Clearly every tree is critical (r). Suppose conversely that G is critical (r) for $r = m$. Let u be a point of eccentricity m. Then G has a spanning tree T such that the distances $d(u, v)$ in T equal the distances $d(u, v)$ in G for all points v of G (see König (6), p. 60). So $r(T) = r(G) = m$. Since G is critical (r), $G = T$.

We have been unable to characterize anticritical (r) graphs but present a partial result. It is well known (see (5), p. 421) that $r(G) \leqslant d(G) \leqslant 2r(G)$. For anticritical (r) graphs this can be improved a little. If in a graph G the points u and v have distance $d(u, v) \geqslant 2r(G) - 1 > 1$, then it is easy to show that $r(G + uv) = r(G)$. Thus if G is anticritical (r), then $r(G) \leqslant d(G) \leqslant 2r(G) - 2$. We shall now demonstrate that these bounds cannot be improved. A cycle of even length provides a case of an anticritical (r) graph having the same radius and diameter. To realize the other bound, for $m \geqslant 3$, let H_m denote the graph consisting of the $3m - 1$ points

$$u_1, u_2, \ldots, u_{2m-1}, v_m, v_{m+1}, \ldots, v_{2m-1}$$

and the lines $u_i u_{i+1} (1 \leqslant i \leqslant 2m-2)$, $v_i v_{i+1} (m \leqslant i \leqslant 2m-2)$, $u_i v_i (m \leqslant i \leqslant 2m-1)$ and $u_{m-1} v_m$. Then H_m is anticritical (r) for $r = m$, and $d(H_m) = 2m - 2$.

The anticritical (r) graphs for $r = 2$ are simply the graphs G such that each component of $\bar{G}$ is a star $K_{1,s}$. We shall now describe a class of anticritical (r) graphs for $r = 3$. For $r > 3$, these graphs have not been characterized.

We shall say that a graph G is *admissible* (for want of a more descriptive adjective) if its points can be labelled $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n$, where u_i and v_i are non-adjacent for $i = 1, 2, \ldots, n$, and for each i and each point w, $w \neq u_i$, $w \neq v_i$, w is adjacent to precisely one of the points u_i, v_i. If G is admissible, then a new admissible graph is produced from G by adding two new points u_{n+1}, v_{n+1} and for each $i = 1$ to n, either the two lines $u_{n+1} u_i$, $v_{n+1} v_i$ or the two lines $u_{n+1} v_i$, $v_{n+1} u_i$. Furthermore every admissible graph with $2n + 2$ points can be obtained in this manner from an admissible graph with $2n$ points. The smallest admissible graphs are $2K_1$ and $2K_2$, while C_6, the cycle of length six, is the smallest connected admissible graph. Also, the graphs $2K_n$ and $K_{n,n} - nK_2$ (the graph obtained from $K_{n,n}$ by deleting n independent lines) are admissible graphs. Fig. 2 shows another example.

THEOREM 9. *If G is admissible then either $G = 2K_n$ or G is anticritical (r) for $r = 3$.*

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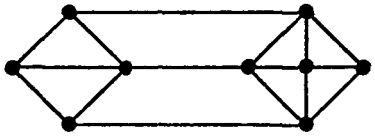


Fig. 3. An anticritical (r) graph for $r = 3$.

Proof. The admissible graph G with $2n$ points is regular of degree $n-1$, so if G is disconnected, then $G = 2K_n$. We shall therefore consider the case where G is connected. In this case, clearly $d(u_i, v_i) = 3$ for $i = 1, 2, \ldots, n$, so $d(G) \geqslant r(G) \geqslant 3$. We now show that the eccentricity of every point is 3. We need only consider point u_1. Let w be any point, $w \neq v_1$, such that $d(u_1, w) \geqslant 3$. Then w and v_1 are adjacent. We shall show that a point u, $u \neq w$, $u \neq v_1$, is adjacent to w if and only if it is adjacent to v_1. If u is adjacent to w then u is not adjacent to u_1 since $d(u_1, w) \geqslant 3$. But then u is adjacent to v_1. Suppose conversely that u is adjacent to v_1. Then $u \in \{u_j, v_j\}$ for some j. Write u' for the point $\{u_j, v_j\} - \{u\}$. Since v_1 is adjacent to u, point v_1 is not adjacent to u'. But then u' is adjacent to u_1. Since $d(w, u_1) \geqslant 3$, w is not adjacent to u'. But then w is adjacent to u. Thus we have shown that u is adjacent to w if and only if u is adjacent to v_1, so $d(u_1, w) = d(u_1, v_1) = 3$. From this it follows that $d(G) = r(G) = 3$. Let $e = u_1 v$ be any line not present in G. We can show that in $G + e$, u_1 has eccentricity 2. Since u_1 is not adjacent to v in G, either $v = v_1$ or v is adjacent to v_1. From the arguments above, it follows that every point at distance 3 from u_1 is either adjacent to v or is v. So $G + e$ has radius 2 and we have proved that G is anticritical (r) for $r = 3$.

If in an admissible graph with at least 6 points every point u is replaced by a complete graph $K(u)$ such that a point of $K(u)$ and a point of $K(v)$ are adjacent if and only if u and v are adjacent in G, then the resulting graph has diameter 3 and is anticritical (r) for $r = 3$. There exist, however, graphs G which are anticritical (r) with

$$r(G) = d(G) = 3$$

and which cannot be described as above. Fig. 3 shows such a graph.

Cyclability number (ξ). Dirac (4) proved that the cyclability number is at least its connectivity: $\xi(G) \geqslant \kappa(G)$. Watkins and Mesner (12) showed that equality holds in the case $\kappa(G) \geqslant 3$ if and only if G contains a set S of $\kappa(G)$ points such that $G - S$ has at least $\kappa(G) + 1$ components. From this we deduce the following characterization.

THEOREM 10. *A graph G with $\kappa(G) = k \geqslant 3$ is anticritical (ξ) for $\xi = k$ if and only if*

$$G = K_k + (K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_s}),$$

where $s = k + 1$.

Watkins and Mesner also characterized the graphs G with $\xi(G) < 3$. From their results one can easily characterize the anticritical (ξ) graphs for $\xi = 1$ and 2. Note the similarity of the graphs of Theorems 10 and 6.

A graph G with p points is *hamiltonian* if $\xi(G) = p$ and is *hypohamiltonian* if

$$\xi(G) = p - 1.$$

Chvátal's Conjecture (2). Every anticritical (ξ) graph is maximal nonhamiltonian.

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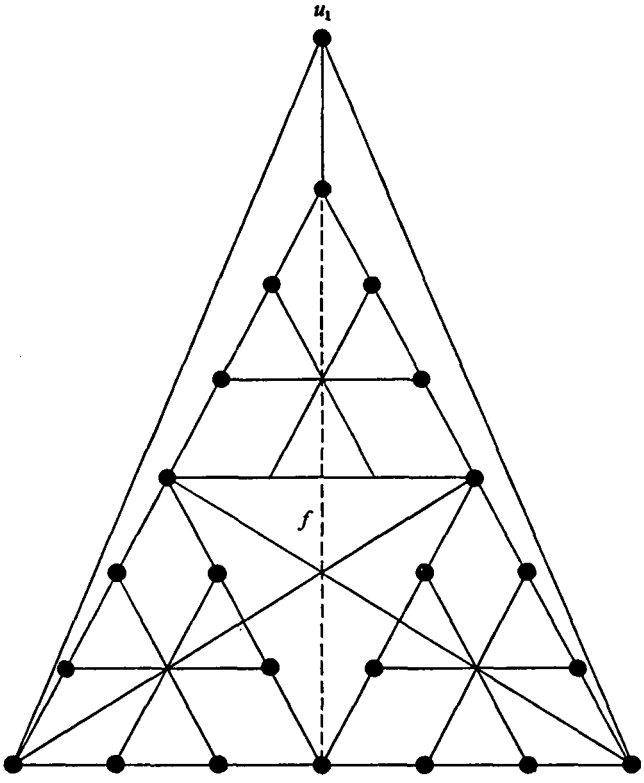


Fig. 4. A counter-example to Chvátal's Conjecture. The line f is dashed.

By Theorem 10, the conjecture is true for anticritical (ξ) graphs with $\xi(G) = \kappa(G)$. Also, the conjecture holds for graphs G with $\xi(G) < 3$, by (12), Theorem 2. For hypohamiltonian graphs the conjecture is obviously true, but the following theorem shows that it is not true for graphs with p points and cyclability number $p-2$.

THEOREM 11. *There exists a hypohamiltonian graph G_0 containing a line f such that $\xi(G_0-f) = \xi(G_0)-1$ and for every line e not present in G_0, G_0-f+e is hamiltonian.*

Remark. Clearly G_0-f is an anticritical (ξ) graph which is not maximal non-hamiltonian.

Proof of Theorem 11. Let P denote the Petersen graph and let $P' = P-f$, where f is any line in P. It is well known that P is hypohamiltonian. Let u_1, u_2, u_3, u_4 denote the points which are adjacent to one of the two points of f. Then $P'-u_i$ is nonhamiltonian for $i = 1, 2, 3, 4$ and $P'-w$ is hamiltonian for $w \neq u_1, u_2, u_3, u_4$. Replace each of the points u_2, u_3, u_4 by any point-deleted subgraph of P. These are all isomorphic and one is shown in (9), Figure 1. The resulting graph is denoted G_1; then G_0 is defined as G_1+f and is shown in Fig. 4. By (9), Lemma 1, G_1 is nonhamiltonian, G_1-u_1 is nonhamiltonian, and G_1-w is hamiltonian for every point $w \neq u_1$ of G_1. Again by (9), Lemma 1, G_0 is hypohamiltonian, since G_0 is obtained from a hypohamiltonian graph (namely P) by

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replacing each of three independent points by a point-deleted subgraph of P. In order to complete the proof we only need to show that $G_0 - f + e$ is hamiltonian for every line e not present in G_0.

Since P is hypohamiltonian, $\xi(P) = 9$. For every line e not present in P,

$$\xi(P+e) = 10,$$

and for every line e in P, $\xi(P-e) = 5$. So P is both anticritical (ξ) and critical (ξ)! This is a special case of the following.

THEOREM 12. *There exist infinitely many hypohamiltonian graphs which are both critical* (ξ) *and anticritical* (ξ).

Proof. For $k \geqslant 2$, let J_k be the graph consisting of the points $u_0, v_1, v_2, v_3, w_1, w_2, \ldots, w_{3k}$ and the lines

$$w_i w_{i+1} \quad (1 \leqslant i \leqslant 3k-1), \quad w_1 w_{3k}, \quad v_1 w_{3i+1} \quad (0 \leqslant i \leqslant k-1),$$

$$v_2 w_{3i+2} \quad (0 \leqslant i \leqslant k-1), \quad v_3 w_{3i+3} \quad (0 \leqslant i \leqslant k-1), \quad u_0 v_1, u_0 v_2, \quad \text{and} \quad u_0 v_3.$$

In particular, J_2 is isomorphic to P. The graphs J_k were discovered by V. Van Diest and J. Doyen who showed that they are hypohamiltonian. Since every line e of J_k is incident with a point of degree 3, $J_k - e$ is not hypohamiltonian, so J_k is critical (ξ). Moreover one can verify that the addition of any new line to J_k results in a hamiltonian graph, so J_k is also anticritical (ξ).

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