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10.1017/mag.2019.126

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103.45 Improving elementary numerical integration using numerical differentiation

Introduction

The two most basic numerical methods of evaluating $I = \int_a^b f(x)\,dx$ rely on dividing the range (a, b) into n equal intervals of width $h = \dfrac{b-a}{n}$ and approximating each area by a trapezium or rectangle to provide the eponymous rules

$$T_n = h\left(\tfrac{1}{2}y_0 + y_1 + y_2 + \ldots + y_{n-1} + \tfrac{1}{2}y_n\right) \text{ and } R_n = h\left(y_{\frac{1}{2}} + y_{\frac{3}{2}} + \ldots + y_{n-\frac{3}{2}} + y_{n-\frac{1}{2}}\right)$$

where $y_k = f(a + kh)$.

The simplicity of these rules make them easy to program on a calculator or spreadsheet but if a high degree of accuracy is required n may need to be so large that accumulated rounding errors make this impossible. Something better is needed.

Approximating $f(x)$ to a quadratic which matches the values at the ends and middle of each interval produces the more accurate Simpson's Rule

$$S_{2n} = \tfrac{1}{6}h\left(y_0 + 4y_{\frac{1}{2}} + 2y_1 + 4y_{\frac{3}{2}} + 2y_2 + \ldots + 4y_{n-\frac{1}{2}} + y_n\right)$$

which reduces this problem.

Comparing basic methods.

Provided the integrand is 'well-behaved' so that it has continuous derivatives throughout the integration interval, the relative merits of these rules can be revealed by the Euler-Maclaurin sum formulae [1] which allow their errors to be expressed as a series of correction terms in the forms

$$\begin{aligned} E(T_n) = I - T_n &\approx -\frac{h^2}{12}\Delta_1 + \frac{h^4}{720}\Delta_3 - \frac{h^6}{30240}\Delta_5 + \ldots \\ \text{and } E(R_n) = I - R_n &\approx \left(\tfrac{1}{2}\right)\frac{h^2}{12}\Delta_1 - \left(\tfrac{7}{8}\right)\frac{h^4}{720}\Delta_3 + \left(\tfrac{31}{32}\right)\frac{h^6}{30240}\Delta_5 - \ldots \end{aligned} \tag{1}$$

where $\Delta_{2k-1} = f^{(2k-1)}(b) - f^{(2k-1)}(a)$ is the increase in the k th odd derivative over the integration interval.

For large n, the first non-zero term gives an estimate of the error. Hence if $\Delta_1 \neq 0$ and n is sufficiently large we can expect the trapezium rule to have about twice the error of the rectangle rule but with opposite sign. Combining them in the form $\frac{1}{3}(T_n + 2R_n)$ eliminates the first correction term

https://doi.org/10.1017/mag.2019.127 Published online by Cambridge University Press

and gives the S_{2n} of Simpson's rule. Hence from (1) its error is

$$E\left(S_{2n}\right) = I - S_{2n} \approx -\frac{h^4}{2880}\Delta_3 + \frac{h^6}{96768}\Delta_5 + \ldots .$$

As S_{2n} uses about twice as many ordinates as T_n or R_n direct comparison is unfair unless we replace h by $2h$ to find, for even n,

$$E\left(S_{\text{even } n}\right) = I - S_{\text{even } n} \approx -\frac{h^4}{180}\Delta_3 + \frac{h^6}{1512}\Delta_5 + \ldots . \tag{2}$$

If the integrand is easily differentiated, it is well worth calculating Δ_1 and using as an approximation either $I \approx T_n - \frac{1}{12}h^2\Delta_1$ or $I \approx R_n + \frac{1}{24}h^2\Delta_1$ with errors of about $\frac{h^4}{720}\Delta_3$ and $-\left(\frac{7}{8}\right)\frac{h^4}{720}\Delta_3$ respectively: both about a quarter that of S_n. The challenge here is to find methods of using some form of numerical differentiation to improve on the accuracy of T_n and R_n whilst retaining their simplicity.

Improving the trapezium rule.

If we are to avoid differentiation we need to find a numerical approximation to Δ_1 which may need the evaluation of extra ordinates near to those of y_0 and y_n.

For small h, the Taylor expansions,

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f^{(3)}(a) + \frac{h^4}{4!}f^{(4)}(a) + \ldots$$

and $$f(b-h) = f(b) - hf'(b) + \frac{h^2}{2!}f''(b) - \frac{h^3}{3!}f^{(3)}(b) + \frac{h^4}{4!}f^{(4)}(b) - \ldots$$

may be added to give

$$h\Delta_1 \approx f(a) - f(a+h) - f(b-h) + f(b) + \frac{h^2}{2!}\left(f''(a) + f''(b)\right) - \frac{h^3}{3!}\Delta_3 + \frac{h^4}{4!}\left(f^{(4)}(a) + f^{(4)}(b)\right). \tag{3}$$

This is a start but if we are to challenge $S_{\text{even } n}$ we need to eliminate the $h^2\left(f''(a) + f''(b)\right)$ term.

Re-writing (3) with $\frac{1}{2}h$ instead of h produces

$$\frac{h}{2}\Delta_1 \approx f(a) - f\left(a+\tfrac{h}{2}\right) - f\left(b-\tfrac{h}{2}\right) + f(b) + \frac{h^2}{4\times 2!}\left(f''(a) + f''(b)\right) - \frac{h^3}{8\times 3!}\Delta_3 + \frac{h^4}{16\times 4!}\left(f^{(4)}(a) + f^{(4)}(b)\right).$$

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Hence the combination $4 \times \frac{h}{2}\Delta_1 - h\Delta_1$ gives

$$h\Delta_1 \approx 3f(a) - 4f(a + \tfrac{h}{2}) + f(a + h) - 4f(b - \tfrac{h}{2}) + f(b - h) + 3f(b) + \frac{h^3}{12}\Delta_3 - \frac{h^4}{32}\left(f^{(4)}(a) + f^{(4)}(b)\right).$$

This does use the two extra ordinates, $y_{\frac{1}{2}}$ and $y_{n-\frac{1}{2}}$, but when substituted into (1) creates an improved trapezium rule which we shall denote by

$$T_n^{+2} = T_n - \tfrac{1}{12}h\left(3f(a) - 4f\left(a + \tfrac{h}{2}\right) + f(a + h) + f(b - h) - 4f\left(b - \tfrac{h}{2}\right) + 3f(b)\right)$$
$$= h\left(\tfrac{1}{4}y_0 + \tfrac{1}{3}y_{\frac{1}{2}} + \tfrac{11}{12}y_1 + y_2 + \ \ldots\ + y_{n-2} + \tfrac{11}{12}y_{n-1} + \tfrac{1}{3}y_{n-\frac{1}{2}} + \tfrac{1}{4}y_n\right),\ (n \geqslant 4)$$

with error, for large enough n,

$$E\left(T_n^{+2}\right) = I - T_n^{+2} \approx -\frac{h^4}{180}\Delta_3 + \frac{h^5}{12 \times 32}\left(f^{(4)}(a) + f^{(4)}(b)\right).$$

Comparing this with the correction terms shown in (2), T_n^{+2} appears to be about as accurate as $S_{\text{even } n}$. However as its second error term is proportional to h^5 rather than h^6 and two extra ordinates have been used it is really only nearly as good and then only when n is sufficiently large.

Improving the rectangle rule.

As introducing 2 extra ordinates, one at each end, was so successful with estimating Δ_1 for the trapezium rule this should also be tried with the rectangle rule. The obvious choices are $y_{-\frac{1}{2}}$ and $y_{n+\frac{1}{2}}$, provided the integrand remains 'well-behaved' in the extended interval $a - \frac{h}{2} \leqslant x \leqslant b + \frac{h}{2}$.

Combining their Taylor expansions,

$$f\left(x + \tfrac{h}{2}\right) - f\left(x - \tfrac{h}{2}\right) \approx 2\,\frac{h}{2}f'(x) + 2\,\frac{h^3}{8 \times 3!}f^{(3)}(x) + 2\,\frac{h^5}{32 \times 5!}f^{(5)}(x) + \ \ldots.$$

Substituting $x = b$ then $x = a$ and subtracting then gives

$$h\Delta_1 \approx f\left(b + \tfrac{h}{2}\right) - f\left(b - \tfrac{h}{2}\right) - f\left(a + \tfrac{h}{2}\right) + f\left(a - \tfrac{h}{2}\right) - \frac{h^3}{24}\Delta_3 - \frac{h^5}{1920}\Delta_5. \quad (4)$$

Substituting into (1) produces an improved rectangle rule, using these 2 extra ordinates, which we shall call

$$R_n^{+2} = R_n + \tfrac{1}{24}h\left(f\left(a - \tfrac{h}{2}\right) - f\left(a + \tfrac{h}{2}\right) - f\left(b - \tfrac{h}{2}\right) + f\left(b + \tfrac{h}{2}\right)\right)$$
$$= h\left(\frac{1}{24}y_{-\frac{1}{2}} + \frac{23}{24}y_{\frac{1}{2}} + y_{\frac{3}{2}} + \ \ldots + y_{n-\frac{3}{2}} + \frac{23}{24}y_{n-\frac{1}{2}} + \frac{1}{24}y_{n+\frac{1}{2}}\right),\ (n \geqslant 3) \quad (5)$$

with error, for large n,

$$E\left(R_n^{+2}\right) = I - R_n^{+2} = -\frac{17}{5760}h^4\Delta_3 + \frac{1}{96768}h^6\Delta_5 - \ \ldots$$
$$\approx -\frac{17}{5760}h^4\Delta_3 \approx \frac{17}{32}E\left(S_n\right). \quad (6)$$

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Hence R_n^{+2} provides a method, using only 2 more ordinates and little more complicated than Simpson's rule, which almost halves its error and may be used with any positive n. However, the error is still much larger than the $\frac{h^4}{720}\Delta_3$ of $T_n - \frac{1}{12}h^2\Delta_1$ and $-\left(\frac{7}{8}\right)\frac{h^4}{720}\Delta_3$ of $R_n + \frac{1}{24}h^2\Delta_1$ which calculate the derivatives exactly. Could we do better?

Improving the Δ_1 estimate

We need to improve the accuracy of the numerical differentiation by using values of x, near a and b, but very close together. Define

$$\delta(\varepsilon) = \tfrac{1}{\varepsilon}\left(f\left(a - \tfrac{\varepsilon}{2}\right) - f\left(a + \tfrac{\varepsilon}{2}\right) - f\left(b - \tfrac{\varepsilon}{2}\right) + f\left(b + \tfrac{\varepsilon}{2}\right)\right).$$

Replacing h by ε in (4) shows that

$$\Delta_1 = \delta(\varepsilon) - \frac{\varepsilon^2}{24}\Delta_3 - \frac{\varepsilon^4}{1920}\Delta_5 - \ldots . \tag{7}$$

Substituting the approximation $\Delta_1 \approx \Delta_1' = \delta(\varepsilon)$ into (1) we find as first improvements to T_n and R_n:

$$T_n'(\varepsilon) = T_n - \tfrac{1}{12}h^2\Delta_1'$$

with error $E\left(T_n'(\varepsilon)\right) \approx \frac{h^4}{720}\left(1 + \frac{5\varepsilon^2}{2h^2}\right)\Delta_3 - \frac{h^6}{30240}\left(1 - \frac{21\varepsilon^4}{16h^4}\right)\Delta_5$

and

$$R_n'(\varepsilon) = R_n + \tfrac{1}{24}h^2\Delta_1'$$

with error $E\left(R_n'(\varepsilon)\right) \approx -\left(\frac{7}{8}\right)\frac{h^4}{720}\left(1 + \frac{10\varepsilon^2}{7h^2}\right)\Delta_3 + \left(\frac{31}{32}\right)\frac{h^6}{30}\left(1 - \frac{21\varepsilon^4}{31h^4}\right)\Delta_5.$

Notice that the complicated $\Delta_1' = \delta(\varepsilon)$ term is evaluated only once and therefore contributes a single extra term to the summations.

Some care needs to be taken over the choice of $\varepsilon = 10^{-k}$ (say) so that we balance the mathematical error in the approximation for Δ_1 of about $\varepsilon^2 = 10^{-2k}$ with the loss of about k significant figures in $\delta(\varepsilon)$ when subtracting almost equal numbers and dividing by a very small ε. With *Excel* normally giving 15 significant figures this numerical error would be about $10^{-(15-k)}$ so that a sensible choice would be $\varepsilon = 10^{-5}$.

We would want to use only hundreds rather than thousands of intervals in our calculations of T_n and R_n in order to avoid accumulated rounding errors. This would mean $h \approx 10^{-3}$ at most so that $\frac{\varepsilon^2}{h^2} \approx 10^{-4} << 1$ and errors would then simplify to $E\left(T_n'\left(10^{-5}\right)\right) \approx \frac{h^4}{720}\Delta_3 - \frac{h^6}{30240}\Delta_5$ and

$$E\left(R_n'\left(10^{-5}\right)\right) \approx -\left(\tfrac{7}{8}\right)\frac{h^4}{720}\Delta_3 + \left(\tfrac{31}{32}\right)\frac{h^6}{30240}\Delta_5 \tag{8}$$

or about a quarter that of $S_{\text{even } n}$.

https://doi.org/10.1017/mag.2019.127 Published online by Cambridge University Press

Predicting the value of *n needed*

The usual way to decide the value of an integral by a numerical integration method is to increase n until successive results agree to the required degree of accuracy or at least the most consistent possible. How much better if we could know the value of n in advance?

Consider the original trapezium rule. For sufficiently large n, provided $\Delta_1 \neq 0$, we can expect $E(T_n) \approx -\frac{h^2}{12}\Delta_1$ so that the number of sub-intervals needed, $n = \frac{b-a}{h}$, for an error of size $\mathbf{E}$, is

$$n \approx (b-a)\sqrt{\frac{|\Delta_1'|}{12\mathbf{E}}}. \tag{9}$$

For the rectangle rule we simply replace the 12 by 24.

We shall use Macros in *Excel* like those in [2] to calculate T_n and R_n with another for $\delta(\varepsilon)$ to test these methods with the known integral $I_1 = \int_1^2 \frac{dx}{x} = \ln 2$. We find $\Delta_1' = \delta(10^{-5}) = 0.750000000049322$ which is correct to 10 significant figures with an error of about $0.5\varepsilon^2$. For $\mathbf{E} = 10^{-6}$ the prediction is $n = 250.00000001$ for T_n and $n = 176.78$ for R_n which are confirmed by the *Excel* results $E(T_{250}) = -9.99998 \times 10^{-7}$ and $E(T_{176}) = 1.009 \times 10^{-6}$ and $E(R_{177}) = 9.97 \times 10^{-7}$.

Accumulated rounding errors make an error as small as 10^{-12} impossible to achieve as (9) shows that n would need increasing by ×1000. (*Excel* calculations give $T_{250000} = -6.189 \times 10^{-12}$ and $R_{177000} = -4.718 \times 10^{-12}$).

Finding predictions for the 'improved' methods requires the estimation of Δ_3. This can be found by using 2ε as well as ε in (7) and solving

$$\Delta_1 = \delta(\varepsilon) - \frac{\varepsilon^2}{24}\Delta_3 - \frac{\varepsilon^4}{1920}\Delta_5 - \ldots = \delta(2\varepsilon) - \frac{4\varepsilon^2}{24}\Delta_3 - \frac{16\varepsilon^4}{1920}\Delta_5 - \ldots \tag{10}$$

to find

$$\Delta_3 = \frac{8}{\varepsilon^2}\left(\delta(2\varepsilon) - \delta(\varepsilon)\right) - \frac{\varepsilon^2}{16}\Delta_5 + \ldots . \tag{11}$$

Hence we may use $\Delta_3'' = \frac{8}{\varepsilon^2}\left(\delta(2\varepsilon) - \delta(\varepsilon)\right)$ as our approximation for Δ_3 with $\varepsilon = 10^{-k} = 10^{-3}$ as this balances the numerical error of about $10^{-(15-3k)}$ with the mathematical error of about 10^{-2k}. This gives $\Delta_3'' = 5.6250089$ which is correct to 5 significant figures with an error of about $9\varepsilon^2$.

Therefore assuming the error is mainly due to the leading term in the expression for their error the predictions are given by:

$$n = (b-a)\sqrt[4]{\frac{|\Delta_3''|}{180\mathbf{E}}} \text{ for both } T_n^{+2} \text{ and } S_{\text{even}\,n};\ n = (b-a)\sqrt[4]{\frac{17|\Delta_3''|}{5760\mathbf{E}}} \text{ for } R_n^{+2}$$

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and if $\varepsilon^2 << h^2$

$$n = (b-a)\sqrt[4]{\frac{|\Delta_3''|}{720\mathbf{E}}} \text{ for } T_n'(10^{-5}); n = (b-a)\sqrt[4]{\frac{7|\Delta_3''|}{8\times 720\mathbf{E}}} \text{ for } R_n'(10^{-5}).$$

The results of applying these predictions to I_1 are shown in Table 1. Here *n*- and *n*+ are the nearest integers below and above *n* respectively. The predicted values of *n* are generally very good though there are slight overestimates for T_n^{+2} and an underestimate by 1 for R_n^{+2} when $\mathbf{E} = 10^{-12}$ $(R_{360}^{+2} = -9.90 \times 10^{-13})$. The results confirm that $S_{\text{even } n}$ is about as accurate as T_n^{+2} but is not as accurate as R_n^{+2} as more intervals are needed for the same error. Clearly $R_n'(10^{-5})$ is slightly better than $T_n'(10^{-5})$ with both improving significantly on $S_{\text{even } n}$.

	T_n^{+2}	S_{2n}	R_n^{+2}	$T_n'(10^{-5})$	$R_n'(10^{-5})$
$\mathbf{E = 10^{-6}}$					
n	13.30	6.65	11.35	9.40	9.09
***I*(*n*-)**	−9.35*E*−07	−1.48*E*−06	−1.13*E*−06	1.18*E*−06	−1.03*E*−06
***I*(*n*+)**	−7.03*E*−07	−8.03*E*−07	−8.00*E*−07	7.77*E*−07	−6.80*E*−07
$\mathbf{E = 10^{-9}}$					
n	74.77	37.38	63.83	52.87	51.13
***I*(*n*-)**	−1.01*E*−09	−1.04*E*−09	−1.05*E*−09	1.07*E*−09	−1.01*E*−09
***I*(*n*+)**	−9.61*E*−10	−9.36*E*−10	−9.90*E*−10	9.90*E*−10	−9.35*E*−10
$\mathbf{E = 10^{-12}}$					
n	420.45	210.22	358.95	297.30	287.54
***I*(*n*-)**	−9.9998*E*−13	−1.005*E*−12	−1.009*E*−12	1.010*E*−12	−1.010*E*−12
***I*(*n*+)**	−9.89*E*−13	−9.88*E*−13	−1.002*E*−12	9.94*E*−13	−9.94*E*−13

TABLE 1: A comparison of predicted *n* for the 'improved' methods applied to I_1 with three different errors, **E**.

Exceptional cases

An absurdly small value of $\Delta_1' = \delta(\varepsilon)$ indicates that $\Delta_1 = 0$ which would mean having to use the next correction term from (1), $\frac{h^4}{720}\Delta_3$ or $-\left(\frac{7}{8}\right)\frac{h^4}{720}\Delta_3$, when finding *n* with the trapezium and rectangle rules. Since $\Delta_3 = \frac{24}{\varepsilon^2}\delta(\varepsilon) - \frac{\varepsilon^2}{80}\Delta_5 - \ldots$ from (7), using $\Delta_3 \approx \Delta_3' = \frac{24}{\varepsilon^2}\delta(10^{-3})$, we should expect an error of **E** with T_n (and even less with R_n) if $n \approx (b-a)\sqrt[4]{\frac{|\Delta_3'|}{720\mathbf{E}}}$. Calculating the integral

$$\int_0^{\pi/2} x^2(1-\sin x)\,dx = \frac{\pi^3}{24} - \pi + 2 \approx 0.150335541422702,$$

we find $\Delta_1' = \delta(10^{-5}) \approx 6.43 \times 10^{-11}$ and $\Delta_3' \approx 15.4248$. Choosing $\mathrm{E} = 10^{-9}$, the prediction $n = 106.87$ proves correct as $E(T_{107}) = 9.9503 \times 10^{-10}$ and $E(R_{107}) = -8.7065 \times 10^{-10}$.

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Similarly, calculating $\int_0^1 \frac{4}{1+x^2} = \pi$ using $T_n'(10^{-5})$ finds $\Delta_3'' \approx 1.776 \times 10^{-6}$ showing that $\Delta_3 = 0$. We therefore use the next error terms from (6) and (8) giving $n = (b-a)\sqrt[6]{\frac{|\Delta_5|}{96768\mathbf{E}}}$ for R_n^{+2} (and S_{2n}) and $n = (b-a)\sqrt[6]{\frac{|\Delta_5|}{30240\mathbf{E}}}$ for $T_n'(10^{-3})$ and $R_n'(10^{-3})$. Equation (11) provides the approximation $\Delta_5 \approx \Delta_5'' = \frac{128}{\varepsilon^5}(\delta(2\varepsilon) - \delta(\varepsilon)) \approx 60.00247$ using $\varepsilon = 10^{-15/7} \approx 0.0072$. Using $\mathbf{E} = 10^{-12}$ we correctly find

$$E(R_{30}^{+2}) \approx E(S_{60}) \approx E(R_{36}'(10^{-5})) \approx -E(T_{36}'(10^{-5})) \approx 9 \times 10^{-13}.$$

Higher order methods

Simpson's rule and our improvements $T_n(\varepsilon)$, $R_n(\varepsilon)$, $T_n'(\varepsilon)$ and $R_n'(\varepsilon)$ all have errors proportional to h^4 and are referred to as being of $O(h^4)$. Faster convergence will be achieved if we can achieve methods of $O(h^6)$ or $O(h^8)$. As well as using the approximation (11) for Δ_3 to predict n, it may also be substituted into (7) to find a better approximation to Δ_1 of $\Delta_1'' = \frac{4\delta(\varepsilon) - \delta(2\varepsilon)}{3}$ which can then be used to produce the $O(h^6)$ rules defined by

$$T_n''(\varepsilon) = T_n - \frac{1}{12}h^2\Delta_1'' + \frac{1}{720}h^4\Delta_3'' \text{ and } R_n''(\varepsilon) = R_n + \frac{1}{24}h^2\Delta_1'' - \frac{7}{8}\frac{1}{720}h^4\Delta_3''$$

which the reader might like to pursue.

However, should we be prepared to calculate both T_n' and R_n' a simpler method is available which combine these as in $S_{2n} = \frac{1}{3}(T_n + 2R_n)$. Using only the first error term in (8) the ratio of the errors expected with T_n' and R_n' are of opposite sign in the ratio $K = \left|\frac{E(T_n')}{E(R_n')}\right| \approx \frac{8}{7}$. Therefore defining $S_{2n}'(10^{-5}) = \frac{1}{15}(8R_n'(10^{-5}) + 7T_n'(10^{-5}))$ we eliminate this term when $\frac{\varepsilon^2}{h^2} << 1$ and have an approximately $O(h^6)$ method with

$$E(S_{2n}'(\varepsilon)) \approx \frac{1}{20} \times \frac{1}{30240}h^6\Delta_5. \tag{12}$$

Similarly $S_{2n}''(10^{-3}) = \frac{32R_n''(10^{-3}) + 31T_n''(10^{-3})}{63}$ would even produce an approximately $O(h^8)$ method.

For maximum benefit from S_{2n}' we need to be able to predict the number of intervals needed for a specified error. This needs an estimate of Δ_5 which may be found by substituting Δ_3 from (11) into (10) with 2ε replaced by 3ε on the right-hand side to find

$$\Delta_5 \approx \Delta_5''' = \frac{16}{\varepsilon^4}(5\delta(\varepsilon) - 8\delta(2\varepsilon) + 3\delta(\varepsilon))$$

using $10^{-15/7} \approx 0.0072$ as the appropriate value for ε.

https://doi.org/10.1017/mag.2019.127 Published online by Cambridge University Press

Determining an unknown integral

Consider the integral $I_2 = \int_0^{\pi} \frac{2^x \sin x}{1 + x}\, dx$ which is a combination of common 'well-behaved' functions. A quick calculation of S_2 shows that $I_2 \approx 2.42$. With the 15 significant figures possible with *Excel* we can only expect to determine the exact value of I_2 within one figure in the last place so that we should choose $\mathbf{E} = 10^{-14}$ for maximum accuracy. To see which methods might produce such accuracy we first find the appropriate Δ terms:

$$\Delta_1' = -3.13081743302895\left(\varepsilon = 10^{-5}\right), \Delta_3'' = -1.82856792818598\left(\varepsilon = 10^{-3}\right)$$

$$\text{and } \Delta_5''' = -47.7397169705215\left(\varepsilon = 0.0072\right).$$

The predictions show that basic trapezium and rectangle rules need over 1.6 million and 1.1 million intervals so they will certainly fail. Simpson's rule needs $2n = 3154$ which is too large to avoid rounding errors as is the $n = 2693$ needed for R_n^{+2}. Predictions for $T_n'\left(10^{-5}\right)$ and $R_n'\left(10^{-5}\right)$ reduce n to 2231 and 2157 respectively which are also too large.

We are left with $S_{2n}'\left(10^{-3}\right)$. The error is given by (12) so that $n \approx (b-a)\sqrt[6]{\frac{|\Delta_5'''|}{604800\mathbf{E}}} = 140.18$ which is small enough to avoid serious rounding errors so we can claim $I_2 = S_{280}'\left(10^{-5}\right) = 2.45115490457673$ with an error of approximately 10^{-14}. To check this we find other estimates using values near $n = 140$. These are shown in Table 2.

Slight rounding errors produce fluctuations of $\pm 10^{-14}$ and suggest the final digit should be 2 rather than 3.

n	$S_{2n}'\left(10^{-3}\right)$
130	2.45115490457673
140	2.45115490457673
150	2.45115490457672
160	2.45115490457671
170	2.45115490457673
180	2.45115490457673

TABLE 2: Calculations of $S_{2n}'\left(10^{-3}\right)$ for I_2 near $n = 140$ showing effect of rounding errors on the final digit

Using this we can then verify the accuracy of the other predictions for I_2 using the 'improved' rules. For example an error of less than 10^{-6} is correctly achieved by $R_{27}^{+2} = 2.451153917$ and an error of less than 10^{-9} by $T_{126}'\left(10^{-5}\right) = 2.451154905558$.

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Conclusions

Clearly these methods, incorporating numerical differentiation, are effective in reducing the number of ordinates to be calculated and thereby reducing or eliminating the effects of cumulative rounding errors.

The relative simplicity of R_n^{+2} given by (5) and its superiority over T_n, R_n, T_n^{+2} and $S_{\text{even } n}$ does have some appeal particularly for use on a programmable calculator. On a spreadsheet, however, by writing simple Macros to calculate T_n, R_n and $\delta(\varepsilon)$ the best method involves finding both $T_n'(\varepsilon)$ and $R_n'(\varepsilon)$ before combining them into $S_{2n}'(\varepsilon)$.

The choice of ε depends on the rule employed and the calculating device to be used. Finding Δ_{2p-1} working to m significant figures, in order to balance the mathematical error of the approximate differentiation with the numerical error of inexact arithmetic, we need $\varepsilon = 10^{-k}$ where k satisfies $m \approx (2p + 1)k$.

The ability to calculate the Δ_{2p-1} terms allows for an accurate prediction of the number of intervals needed to achieve a specified error. This means a single calculation of the integral rather than a search for consistency with increasing n. This could be particularly useful in accurate calculation of functions which are only defined as a definite integral.

Finally, it must be stressed, that these methods are dependent on the integrand being 'well behaved'. If not, they are unlikely to offer improvement on the basic methods of T_n or R_n with their combination as described in [2].

References

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10.1017/mag.2019.127

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103.46 A four-hundred year calendar

The relationships between days and dates, in the form of calendars, tables, schedules, etc., figure prominently in our management of affairs, both domestically and nationally. A new version of these relationships, valid from the introduction of the Gregorian calendar in 1752, is presented in this Note in the form of a nomogram, the use of which is exemplified.

The objective in constructing the nomogram (Figure 1) was to develop a means whereby, given the date in the conventional form of date, month and year (e.g., 18th June 1815), the corresponding day of the week could be found. The starting point for the writer's interest in the topic was a short paper by Lewis Carroll, published in the journal *Nature* [1].

https://doi.org/10.1017/mag.2019.127 Published online by Cambridge University Press