Proc. Camb. Phil. Soc. (1974), **75**, 295
PCPS 75–31
Printed in Great Britain

Waves on a shearing current: a uniformly valid asymptotic solution

By W. D. McKEE

Mathematics Department, Monash University, Clayton, Victoria 3168, *Australia*

(*Received* 12 *March* 1973)

Abstract. A uniformly valid asymptotic solution for waves on a shearing current is presented. This solution is more general than the previous WKB solutions because it is valid at and near a caustic. Far from the caustic line, the solution reduces to the WKB solution. It predicts that an incident wave will be reflected from a straight caustic with the same amplitude and a $\frac{1}{2}\pi$ phase shift.

1. *Introduction.* When waves propagate across a shearing current such as might be produced, for example, by a river flowing into a lake, they are refracted and undergo changes in amplitude. These processes have been studied by Longuet-Higgins and Stewart (7) and, in a more general setting, by Bretherton and Garrett (1). Experiments supporting the conclusions of Longuet-Higgins and Stewart have been performed by Hughes and Stewart (4).

If the fractional change in current speed over a distance of one wavelength is sufficiently small, the motion may be studied by means of the WKB approximation. However, it may happen that the waves are refracted so much that the crests become perpendicular to the current. The WKB approximation is not valid at such places which are known as caustics. This work provides a solution valid at and near the caustic as well as away from it. Analogous solutions for ordinary differential equations have been given by Langer in a series of papers culminating in Langer (6). This has been generalized to a system of linear partial differential equations by Ludwig (8). This paper applies a simplified form of Ludwig's 'ansatz' to the problem of waves on a shear flow. A similar treatment has been given by Chao (2) in the case of surface gravity waves on stationary water of variable depth. The predictions of this theory have been verified experimentally by Chao and Pierson (3).

The need for a corresponding solution in the case of waves on a shearing current was pointed out by Longuet-Higgins and Stewart (7). The theory presented here was originally worked out for the case treated by them, namely surface gravity waves on an ocean of infinite depth and constant density. However, it turned out to be no more difficult to include the effects of density stratification, vertical shear and one-dimensional bottom topography.

2. *The model.* We take the x and y axes horizontal and the z axis vertical with the free surface at $z = 0$ and the bottom at $z = -h(x)$. The fluid is stratified with density

https://doi.org/10.1017/S0305004100048490 Published online by Cambridge University Press

$\rho_0(z)$ and is moving with velocity $(0, V(x,z), 0)$. This configuration is an exact solution with the free surface undisturbed and the pressure hydrostatic. It is assumed to be stable. We consider wave-like perturbations to this basic state in which the velocity perturbation is $(u,v,w)\chi$, the pressure perturbation is $p\chi$, the density perturbation $\rho\chi$ and the free surface elevation is $\zeta\chi$ where $\chi = \exp i(ly - \omega t)$. The quantities u, v, w, ρ and p are all functions of x and z whereas ζ is a function of x only. We take l and ω both to be positive.

The linearized equations of motion, continuity and incompressibility are

$$\rho_0\{-i\omega u + ilVu\} = -p_x, \tag{1}$$

$$\rho_0\{-i\omega v + uV_x + ilVv + wV_z\} = -ilp, \tag{2}$$

$$\rho_0\{-i\omega w + ilVw\} = -p_z - \rho g, \tag{3}$$

$$u_x + ilv + w_z = 0, \tag{4}$$

$$-i\omega\rho + ilV\rho + w\rho_{0z} = 0. \tag{5}$$

From (1), (2), (3) and (5), we can express u, v and w in terms of p and its derivatives. Substituting these into (4) gives the following equation for the pressure perturbation

$$p_{xx} - l^2 p - \frac{2lV_x}{-\omega + lV}p_x + \rho_0(-\omega + lV)\left[\frac{(-\omega+lV)p_z}{\rho_0\{(-\omega+lV)^2 - N^2\}}\right]_z - \frac{lV_z(-\omega+lV)p_z}{(-\omega+lV)^2 - N^2} = 0, \tag{6}$$

where $N^2 = -(g\rho_{0z}/\rho_0)$ is the square of the Brunt–Väisälä frequency.

The condition of constant pressure at the free surface gives, on linearization,

$$p_z = \frac{1}{g}[(-\omega + lV)^2 - N^2]p \quad \text{at} \quad z = 0. \tag{7}$$

The bottom boundary condition $w = -uh_x$ at $z = -h(x)$ becomes

$$(-\omega + lV)^2 p_z + [(-\omega + lV)^2 - N^2]h_x p_x = 0 \tag{8}$$

at $z = -h(x)$.

Although this is not essential, we will scale the variables as follows, denoting dimensionless quantities by asterisks:

$$h = \frac{g}{\omega^2}h^*,$$

$$z = \frac{g}{\omega^2}z^*,$$

$$(x,y) = L(x^*, y^*),$$

$$V = \frac{g}{\omega}V^*,$$

$$l = \frac{\omega^2}{g}l^*.$$

In these definitions, g is the acceleration due to gravity and L is a horizontal length scale associated with the basic flow $V(x,z)$. Defining

$$\beta = \frac{\omega^2 L}{g}, \quad \Omega(x,z) = -1 + lV, \quad \delta(x,z) = \rho_0(\Omega^2 - N^2/\omega^2)$$

https://doi.org/10.1017/S0305004100048490 Published online by Cambridge University Press

and dropping asterisks puts the governing equation into the form

$$(\Omega^2 - N^2/\omega^2)\left(\frac{p_x}{\Omega^2}\right)_x - \beta^2 l^2\left(\frac{\Omega^2 - N^2/\omega^2}{\Omega^2}\right)p + \beta^2\left(p_{zz} - p_z\frac{\delta_z}{\delta}\right) = 0, \tag{9}$$

with the boundary conditions

$$\beta^2\Omega^2 p_z + (\Omega^2 - N^2/\omega^2)\,h_x p_x = 0 \quad \text{at} \quad z = -h(x), \tag{10}$$

and

$$p_z = (\Omega^2 - N^2/\omega^2)\,p \quad \text{at} \quad z = 0. \tag{11}$$

We are interested in the form of the solutions for high frequencies, i.e. large β. The WKB approximation is obtained by seeking a solution of the form

$$p(x, z, \beta) = A(x, z, \beta)\,\phi(x, z)\exp i\beta S(x),$$

and assuming that $A(x, z, \beta)$ has the asymptotic expansion for large β

$$A(x, z, \beta) \sim A_0(x) + \sum_{m=1}^{\infty}\frac{A_m(x, z)}{(i\beta)^m}. \tag{12}$$

The function ϕ is a solution of local eigenvalue problem for wave propagation in water of uniform depth h moving with velocity V

$$\phi_{zz} - \phi_z\frac{\delta_z}{\delta} = n^2\left(\frac{\Omega^2 - N^2/\omega^2}{\Omega^2}\right)\phi, \tag{13}$$

$$\phi_z = 0 \quad \text{at} \quad z = -h, \tag{14}$$

$$\phi_z = (\Omega^2 - N^2/\omega^2)\,\phi \quad \text{at} \quad z = 0. \tag{15}$$

The separation constant is $n^2(x)$ and the eigenfunction is $\phi(x, z)$. For example, if N^2 and V_z are both zero and the depth infinite, we have $\phi = \exp[nz]$ and $n = \Omega^2$.

The postulated expansion (12) leads to the eikonal equation

$$(S_x)^2 = n^2 - l^2 \tag{16}$$

which indicates a transition from wave-like to exponential-like behaviour where $n^2(x) = l^2$. Such places are the caustics. The amplitudes A_m are found by solving the transport equations. The calculation is analogous to that given by Keller and Mow (5) for internal waves in still water of variable depth. The details will not be given here since the WKB solution is superseded by the uniformly valid asymptotic solution presented in section 3. The result is that the lowest order term A_0 satisfies the relation

$$A_0^2 S_x\int_{-h}^{0}\frac{\phi^2}{\rho_0\Omega^2}dz = \text{constant} \tag{17}$$

which indicates a singularity where $S_x = 0$, i.e. on the caustic. The wave is reflected as $p = B\phi\exp(-i\beta S)$ but the WKB method gives no way of determining B in terms of A. Before presenting a solution which overcomes these defects we point out that (16) and (17) are in agreement with the results of Longuet-Higgins and Stewart (7) and Bretherton and Garrett (1).

3. *Uniformly valid asymptotic solutions.* In line with the 'uniform method' of Ludwig (8), we seek a solution of the form

$$p = \phi(x, z)\{F(x, z, \beta)\,A(-\beta^{\frac{2}{3}}\xi(x)) + i\beta^{-\frac{1}{3}}G(x, z, \beta)\,A'(-\beta^{\frac{2}{3}}\xi(x))\}, \tag{18}$$

https://doi.org/10.1017/S0305004100048490 Published online by Cambridge University Press

where ϕ is a solution of the eigenvalue problem given by (13), (14), (15) and F, G are assumed to have the asymptotic expansions for large β

$$F(x,z,\beta) \sim F_0(x) + \sum_{m=1}^{\infty} \frac{F_m(x,z)}{(i\beta)^m}; \quad G(x,z,\beta) \sim G_0(x) + \sum_{m=1}^{\infty} \frac{G_m(x,z)}{(i\beta)^m}. \tag{19}$$

In (18), $A(x)$ is one of the Airy functions Ai (x), Bi (x) or a linear combination of them. These are solutions of the equation $(d^2/dx^2)A(x) - xA(x) = 0$.

Substituting the postulated expansions into the governing equation and boundary conditions and equating coefficients of $(i\beta)^{2-m}$ and $(i\beta)^{\frac{5}{2}-m}$ where $m \geqslant 0$ gives

$$\xi\xi_x^2 = n^2 - l^2, \tag{20}$$

$$-\phi F_{mzz} - 2\phi_z F_{mz} + \phi F_{mz}\frac{\delta_z}{\delta} - \frac{2(\Omega^2 - N^2/\omega^2)\,\Omega_x}{\Omega^3}\{(\phi F_{m-2})_x + (\phi G_{m-1}\xi\xi_x)\}$$
$$+ \frac{(\Omega^2 - N^2/\omega^2)}{\Omega^2}\{(\phi F_{m-2})_{xx} + (\phi G_{m-1})_x \xi\xi_x + (\phi G_{m-1}\xi\xi_x)_x\} = 0, \tag{21}$$

and

$$-\phi G_{mzz} - 2\phi_z G_{mz} + \phi G_{mz}\frac{\delta_z}{\delta} - \frac{2(\Omega^2 - N^2/\omega^2)\,\Omega_x}{\Omega^3}\{(\phi G_{m-2})_x + \phi F_{m-1}\xi_x\}$$
$$+ \frac{\Omega^2 - N^2/\omega^2}{\Omega^2}\{(\phi G_{m-2})_{xx} + (\phi F_{m-1})_x \xi_x + (\phi F_{m-1}\xi_x)_x\} = 0, \tag{22}$$

subject to
$$F_{mz} = G_{mz} = 0 \quad \text{at} \quad z = 0, \tag{23}$$

and
$$\Omega^2\phi F_{mz} = (\Omega^2 - N^2/\omega^2)\,h_x[(F_{m-2}\phi)_x + \phi G_{m-1}\xi\xi_x], \tag{24}$$

$$\Omega^2\phi G_{mz} = (\Omega^2 - N^2/\omega^2)\,h_x[(G_{m-2}\phi)_x + \phi F_{m-1}\xi_x], \tag{25}$$

at $z = -h(x)$. In these expressions we have defined F_m and G_m to be identically zero for $m < 0$ and so (21), (22), (23), (24) and (25) hold for all $m \geqslant 0$. Notice that (22), (23) and (25) are identical with (21), (23), (24) if we interchange all the F_m and G_m and replace $\xi\xi_x$ by ξ_x.

Equation (20) is clearly analogous to the eikonal equation (16). For definiteness we take $x = 0$ to be the caustic and assume $n^2 > l^2$ for $x < 0$ and $n^2 < l^2$ for $x > 0$. We define

$$\left.\begin{aligned} \tfrac{2}{3}\xi^{\frac{3}{2}} &= -\int_0^x (n^2(t) - l^2)^{\frac{1}{2}}\,dt \quad (x < 0), \\ \tfrac{2}{3}(-\xi)^{\frac{3}{2}} &= \int_0^x (l^2 - n^2(t))^{\frac{1}{2}}\,dt \quad (x > 0). \end{aligned}\right\} \tag{26}$$

The 'transport equations' (21), (22) can be readily solved by integration to yield

$$G_m(x,z) = G_m(x,-h) + J_{m-1}(x,z),$$
$$F_m(x,z) = F_m(x,-h) + I_{m-1}(x,z),$$

where
$$G_m(x,-h)\,(\psi\xi\xi_x)^{\frac{1}{2}} = \tfrac{1}{2}\int C_{m-1}(\psi\xi\xi_x)^{\frac{1}{2}}\,dx + \gamma_m,$$
$$F_m(x,-h)\,(\psi\xi_x)^{\frac{1}{2}} = \tfrac{1}{2}\int B_{m-1}(\psi\xi_x)^{\frac{1}{2}}\,dx + \alpha_m.$$

https://doi.org/10.1017/S0305004100048490 Published online by Cambridge University Press

In these expressions γ_m and α_m are constants and

$$\psi = \int_{-h}^{0} \frac{\phi^2}{\rho_0 \Omega^2} dz, \tag{27}$$

$$J_{m-1} = -\int_{-h}^{z} \frac{\delta}{\phi^2} \int_{z'}^{0} \frac{\phi}{\rho_0 \Omega^2} \Big\{ (\phi G_{m-2})_{xx} + (\phi G_{m-1})_x \xi_x + (\phi F_{m-1} \xi_x)_x - \frac{2\Omega_x}{\Omega} \begin{bmatrix} (\phi G_{m-2})_x \\ + (\phi F_{m-1} \xi_x) \end{bmatrix} \Big\} dz'' \, dz', \tag{28}$$

$$B_{m-1} = -\int_{-h}^{0} \frac{\phi}{\rho_0 \Omega^2} \Big\{ (\phi I_{m-1})_x \xi_x + (\phi I_{m-1} \xi_x)_x - \frac{2\Omega_x}{\Omega} (\phi I_{m-1} \xi_x) + (\phi G_{m-1})_{xx} - \frac{2\Omega_x}{\Omega} (\phi G_{m-1})_x \Big\} dz - \frac{\phi(-h) h_x [\phi(-h) G_{m-1}(-h)]_x}{\rho_0(-h) \Omega^2(-h)}. \tag{29}$$

I_m is given by an expression analogous to (28) with F's and G's interchanged and ξ_x replacing $\xi\xi_x$. Likewise C_{m-1} is obtained from (29) with the same changes and J_{m-1} replacing I_{m-1}.

Interest of course centres on the lowest order terms F_0, G_0 which are independent of depth and given by

$$F_0(x) = \frac{\alpha_0}{(\psi \xi_x)^{\frac{1}{2}}},$$

$$G_0(x) = \frac{\gamma_0}{(\psi \xi \xi_x)^{\frac{1}{2}}}.$$

We are particularly interested in the case where the flow field contains a caustic. (Otherwise we would use the simpler WKB approximation. This is equivalent to replacing the Airy functions by their asymptotic expansions for large argument.) As described above, we take $x = 0$ to be the caustic line. Then $\xi > 0$ for $x < 0$ and $\xi < 0$ for $x > 0$. Since the function $\mathrm{Bi}(X)$ is exponentially large for large negative X the function $A(-\beta^{\frac{2}{3}}\xi)$ in (18) must be taken to be $\mathrm{Ai}(-\beta^{\frac{2}{3}}\xi(x))$. We assume that the function $(n^2 - l^2)$ has a simple zero at $x = 0$. Then ξ is proportional to $-x$ for small x. To avoid singular behaviour at the caustic we must put $\gamma_0 = 0$. The higher order terms do not vanish however. The leading term in the asymptotic expansion is therefore given by

$$p \sim \alpha_0 \phi (\psi \xi_x)^{-\frac{1}{2}} \mathrm{Ai}(-\beta^{\frac{2}{3}}\xi(x)),$$

where ϕ is defined by (13), (14), (15); ξ by (26) and ψ by (27). Away from the caustic, the Airy function may be replaced by its asymptotic expansion for large argument to give

$$p \sim -\frac{i\alpha_0 \phi \beta^{-\frac{1}{6}}}{2(\pi\psi)^{\frac{1}{2}}(n^2 - l^2)^{\frac{1}{4}}} \left\{ \begin{array}{l} \exp i \left(\beta \int_x^0 (n^2(t) - l^2)^{\frac{1}{2}} dt + \frac{1}{4}\pi \right) \\ + \exp i \left(-\beta \int_x^0 (n^2(t) - l^2)^{\frac{1}{2}} dt + \frac{3}{4}\pi \right) \end{array} \right\}$$

for $x < 0$. This consists of an incident and a reflected wave which have the same amplitude but differ in phase by $\frac{1}{2}\pi$. Both these waves are solutions of the WKB

https://doi.org/10.1017/S0305004100048490 Published online by Cambridge University Press

equations. This behaviour differs from reflection from a rigid vertical wall where there is no phase change (the appropriate boundary condition being $p_x = 0$ at $x = 0$).

Beyond the caustic, the motion decays rapidly being given by

$$p \sim \frac{\alpha_0 \phi \beta^{-\frac{1}{6}}}{2(\pi\psi)^{\frac{1}{2}}(l^2 - n^2)^{\frac{1}{4}}} \exp\left\{-\beta \int_0^x (l^2 - n^2(t))^{\frac{1}{2}}\, dt\right\}.$$

This is also a solution of the WKB equations. The uniform asymptotic solution thus provides a smooth transition from the wave region to the 'shadow' region beyond the caustic.

4. *Discussion.* The uniformly valid asymptotic solution presented in section 3 is entirely analogous to the corresponding solutions for other physical problems, e.g. Chao (2). The waves considered here reflect in the usual way from a straight caustic, i.e. with incident and reflected amplitudes equal and with a phase shift of $\frac{1}{2}\pi$. Beyond the caustic the wave amplitude decays exponentially. The maximum wave amplitude occurs not at the caustic but just before it. Like the WKB solution, this solution does not predict any partial reflexions as the waves advance into a region of varying current or depth.

The solution was obtained by linearizing about the basic shear flow $V(x, z)$. Since the resulting wave amplitudes turn out to be bounded, this linearization is consistent provided the wave amplitude is sufficiently small.

As a numerical indication of the circumstances under which the present solution would have to be used, we consider waves of length π metres in still water of great depth approaching a current $V(x)$ with their crests at 45° to the current. If the phase speed in the y-direction has the same sign as V, the waves will be reflected where the current speed reaches just over half a metre per second. Waves of the same wavelength approaching with their crests at 60° to the current would not penetrate so far and would be reflected where the current speed reached 0·18 metres per second.

The solution given in this paper is strictly only applicable if the current speed beyond the caustic is everywhere too large for wave propagation in the x-direction. For example, if the current speed drops to zero a long way beyond the caustic, wave propagation will again be possible and a second caustic will exist separating the region of exponential amplitude behaviour from this new wave region. In this case, a small amount of wave energy would be transmitted beyond this system of two caustics and the amplitude of the reflected wave would be slightly less than that of the incident wave. The analysis of this situation would presumably involve parabolic cylinder functions rather than Airy functions.

REFERENCES

(1) Bretherton, F. P. and Garrett, C. J. R. Wavetrains in inhomogeneous moving media. *Proc. Roy. Soc. Ser. A* **302** (1969), 529.

(2) Chao, Y. Y. An asymptotic evaluation of the wave field near a smooth caustic. *J. Geophys. Res.* **76** (1971), 7401.

(3) Chao, Y. Y. and Pierson, W. J. Experimental studies of the refraction of uniform wave trains and transient wave groups near a straight caustic. *J. Geophys. Res.* **77** (1972), 4545.

https://doi.org/10.1017/S0305004100048490 Published online by Cambridge University Press

(4) HUGHES, B. A. and STEWART, R. W. Interaction between gravity waves and a shear flow. *J. Fluid Mech.* **10** (1961), 385.

(5) KELLER, J. B. and MOW, V. C. Internal wave propagation in an inhomogeneous fluid of non-uniform depth. *J. Fluid Mech.* **38** (1969), 365.

(6) LANGER, R. E. The asymptotic solutions of ordinary linear differential equations of the second order, with special reference to a turning point. *Trans. Amer. Math. Soc.* **34** (1949), 461.

(7) LONGUET-HIGGINS, M. S. and STEWART, R. W. The changes in amplitude of short gravity waves on steady non-uniform currents. *J. Fluid Mech.* **10** (1961), 529.

(8) LUDWIG, D. Uniform asymptotic expansions at a caustic. *Comm. Pure Appl. Math.* **19** (1966), 215.

https://doi.org/10.1017/S0305004100048490 Published online by Cambridge University Press