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REMARKS ON ENTIRE SOLUTIONS FOR TWO FOURTH-ORDER ELLIPTIC PROBLEMS

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Abstract We are interested in entire solutions for the semilinear biharmonic equation $\Delta^2 u = f(u)$ in $\mathbb{R}^N$, where $f(u) = \mathrm{e}^u$ or $-u^{-p}$ $(p > 0)$. For the exponential case, we prove that for the polyharmonic problem $\Delta^{2m} u = \mathrm{e}^u$ with positive integer m, any classical entire solution verifies $\Delta^{2m-1} u < 0$; this completes the results of Dupaigne *et al.* (*Arch. Ration. Mech. Analysis* **208** (2013), 725–752) and Wei and Xu (*Math. Annalen* **313** (1999), 207–228). We also obtain a refined asymptotic expansion of the radial separatrix solution to $\Delta^2 u = \mathrm{e}^u$ in $\mathbb{R}^3$, which answers a question posed by Berchio *et al.* (*J. Diff. Eqns* **252** (2012), 2569–2616). For the negative power case, we show the non-existence of the classical entire solution for any $0 < p \leqslant 1$.

Keywords: biharmonic equation; entire solution; asymptotic behaviour; non-existence

2010 *Mathematics subject classification:* Primary 35J91; 35B08; 35B53; 35B40

1. Introduction

In this paper we are interested in entire solutions for two semilinear biharmonic equations

$$\Delta^2 u = \mathrm{e}^u \quad \text{in } \mathbb{R}^N \tag{1.1}$$

and

$$\Delta^2 u = -u^{-p} \quad \text{in } \mathbb{R}^N, \text{ where } p > 0. \tag{1.2}$$

Recently, the fourth-order equations have attracted the interest of many researchers. In particular, a lot of effort has been devoted to understanding the existence, multiplicity, stability and qualitative properties of solutions for $\Delta^2 u = f(u)$ with classical nonlinearities, like the polynomial growth $f(u) = u^p$, the exponential growth $f(u) = \mathrm{e}^u$, and the negative power situation $f(u) = -u^{-p}$. For the conformal dimension $N = 4$, (1.1) appears naturally in conformal geometry as the constant Q-curvature problem; the existence and asymptotic behaviour of solutions with finite total curvature, i.e. $\mathrm{e}^u \in L^1(\mathbb{R}^4)$, were studied in [**3**,**9**,**15**]. Entire radial solutions of (1.1) were also studied for $N \geqslant 5$ in [**1**]

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and the stability of these entire radial solutions were considered in [**2**, **6**]. In particular, it was proved [**2**] that (1.1) admits no entire radial solution if $N = 2$.

Recently, Farina informed us (personal communication, 2013) that a very general non-existence result was proved by Walter in 1957 (see [**12**]). In particular, Walter proved that no classical entire solution exists in $\mathbb{R}^2$ for the polyharmonic problem $\Delta^{2m}u = \mathrm{e}^u$ with any positive integer m. Here we give an alternative proof (see Corollary 2.1 and Remark 2.2). Indeed, we will make use of a general observation for entire solutions to $\Delta^{2m}u = \mathrm{e}^u$.

Theorem 1.1. *Let u be a classical solution of $\Delta^{2m}u = \mathrm{e}^u$ in $\mathbb{R}^N$ with $m \in \mathbb{N}^*$. Then $\Delta^{2m-1}u < 0$, i.e. $(-\Delta)^{2m-1}u > 0$ in $\mathbb{R}^N$.*

We note that similar results were obtained under additional conditions [**6**, **14**]. Dupaigne *et al.* [**6**] considered solutions to (1.1) that are stable outside a bounded domain. In [**14**] it was proved that $(-\Delta)^{m-1}u > 0$ for any classical entire solution of $(-\Delta)^m u = \mathrm{e}^u$ with $m \geqslant 2$ satisfying $u(x) = o(|x|^2)$ at infinity.

It is worth mentioning that the corresponding result is no longer true for entire solutions to $(-\Delta)^m u = \mathrm{e}^u$ with odd m. In fact, Farina and Ferrero prove that when $m \geqslant 3$ is odd, there are infinitely many entire radial solutions of $(-\Delta)^{2m+1}u = \mathrm{e}^u$, $m \in \mathbb{N}$, such that $(-\Delta)^{2m}u$ changes sign (see [**7**, Lemma 6.8 and the proof of Lemma 5.4]; see also [**13**] for the equation $\Delta^{2m+1}u = \mathrm{e}^u$).

On the other hand, for $N \geqslant 3$ it is known that (1.1) admits infinitely many smooth radial solutions. These radial solutions are of either exactly quadratic growth or logarithmic growth at infinity for $N \geqslant 4$ (see [**1**, **2**]). For $N = 3$ it was proved [**2**] that the radial solution is of either exactly quadratic growth or it verifies $u(r) \leqslant -Cr$ at infinity for some $C > 0$. More precisely, let $u_{\alpha,\beta}$ be the unique radial solution of

$$\left.\begin{aligned} \Delta^2 u_{\alpha,\beta}(r) = \mathrm{e}^{u_{\alpha,\beta}(r)} \quad \text{for } r \in [0, R(\alpha,\beta)), \\ u_{\alpha,\beta}(0) = \alpha, \quad \Delta u_{\alpha,\beta}(0) = \beta, \quad u'_{\alpha,\beta}(0) = (\Delta u_{\alpha,\beta})'(0) = 0, \end{aligned}\right\} \tag{1.3}$$

where $[0, R(\alpha,\beta))$ denotes the maximal interval of existence. Noting that (1.3) is invariant under the scaling transformation

$$u_\lambda(x) = u(\lambda x) + 4\ln\lambda, \quad \lambda > 0,$$

we therefore need only to understand the case in which $\alpha = 0$. We will denote $u_{0,\beta}$ by u_β and $R(0,\beta)$ by $R(\beta)$ for simplicity. It was proved in [**1**, **2**] that any local solution to (1.3) satisfies

$$u_\beta(r) \geqslant \frac{\beta}{2N} r^2 \quad \text{for all } r \in [0, R(\beta)). \tag{1.4}$$

Furthermore, there exists $\beta_0 \in (-\infty, 0)$ such that the following hold.

(i) For $\beta < \beta_0$ we have $R(\beta) = +\infty$ and, in addition to (1.4), we have the upper bound

$$u_\beta(r) \leqslant -\frac{\beta_0 - \beta}{2N} r^2 \quad \text{for all } r \in [0, \infty).$$

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(ii) For $\beta = \beta_0$ the solution u_{β_0}, called the separatrix, verifies the following:

$$\begin{aligned} u_{\beta_0}(r) &\leqslant -Cr && \text{if } N = 3 \text{ and } r \text{ is large, with } C > 0; \\ u_{\beta_0}(r) &= -4\ln\left(1 + \frac{\mathrm{e}^{\alpha/2}}{8\sqrt{6}} r^2\right) && \text{for } N = 4; \\ \lim_{r\to\infty} [u_{\beta_0}(r) + 4\ln r] &= \ln[8(N-2)(N-4)] && \text{for } N \geqslant 5. \end{aligned}$$

(iii) For $\beta > \beta_0$, $R(\beta) < \infty$ and $\lim_{r \nearrow R(\beta)} u_\beta(r) = \infty$.

Berchio *et al*. left an open problem for the exact asymptotic behaviour of the separatrix u_{β_0} in dimension 3 (see [**2**]). The following result answers this issue.

Theorem 1.2. *Let β_0 be defined as above and let $N = 3$. Then we have, as $r \to \infty$, $u_{\beta_0}(r) = \alpha_1 r + \alpha_2 + \alpha_3 r^{-1} + O(\mathrm{e}^{-cr})$, where $c > 0$ and*

$$\alpha_1 = \frac{-1}{8\pi} \int_{\mathbb{R}^3} \mathrm{e}^{u_{\beta_0}}\, \mathrm{d}x, \qquad \alpha_2 = \frac{1}{8\pi} \int_{\mathbb{R}^3} |x| \mathrm{e}^{u_{\beta_0}}\, \mathrm{d}x, \qquad \alpha_3 = \frac{-1}{24\pi} \int_{\mathbb{R}^3} |x|^2 \mathrm{e}^{u_{\beta_0}}\, \mathrm{d}x.$$

The second part of this paper is devoted to the consideration of the classical solutions of (1.2). Recently, the radial solutions to (1.2) were studied [**5**], and some Liouville-type results were obtained for stable entire solutions of (1.2) [**8**]. We remark that all of these results concern the negative exponent $-p$ with $p > 1$, and it seems curious to us that no study existed for entire solutions of (1.2) with $p \leqslant 1$. Here we prove that no such entire solution could exist if $p \in (0, 1]$. That is, we have the following theorem.

Theorem 1.3. *If $0 < p \leqslant 1$, then (1.2) admits no entire smooth solution.*

In fact, our proof is inspired by the work of Choi and Xu in [**4**], where the above result was established for $N = 3$.

2. Proof of Theorem 1.1

In this section we prove Theorem 1.1. In the following, for a given function f, we write

$$\bar{f}(r) = -\int_{\partial\mathbb{B}_r(0)} f\, \mathrm{d}\sigma = \frac{1}{|\partial\mathbb{B}_r(0)|} \int_{\partial\mathbb{B}_r(0)} f\, \mathrm{d}\sigma \quad \forall r > 0,$$

where $|\partial\mathbb{B}_r(0)|$ denotes the volume of the sphere. Furthermore, we will consider $\Delta^{2m} u = \mathrm{e}^u$ as a system:

$$v_1 := u,\ v_{k+1} := \Delta v_k \text{ for } 1 \leqslant k \leqslant 2m-1 \text{ so that } \Delta v_{2m} = \mathrm{e}^u \text{ in } \mathbb{R}^N. \tag{2.1}$$

Proof of Theorem 1.1. First we show that $v_{2m} = \Delta^{2m-1} u \leqslant 0$. If this is not the case, there is a point $x_0 \in R^N$ such that $v_{2m}(x_0) > 0$. Up to a translation, we may assume that $x_0 = 0$. Therefore, with v_k given by (2.1), the $\overline{v_k}(r)$ satisfy

$$\Delta\overline{v_k} = \overline{v_{k+1}} \quad \text{for } 1 \leqslant k \leqslant 2m-1, \qquad \Delta\overline{v_{2m}} = \overline{\mathrm{e}^u} \geqslant \mathrm{e}^{\bar{u}} \quad \text{in } \mathbb{R}^N. \tag{2.2}$$

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We remark that $\Delta\overline{v_{2m}} = r^{1-N}(r^{N-1}\overline{v_{2m}}')' = \overline{\mathrm{e}^u} > 0$, so $\overline{v_{2m}}$ is increasing with respect to the radius r. There holds $\Delta\overline{v_{2m-1}} \geqslant \overline{v_{2m}}(0) > 0$. Integrating this, we obtain

$$v_{2m-1}(r) \geqslant v_{2m-1}(0) + \frac{\overline{v_{2m}}(0)}{2N} r^2.$$

Hence, $v_{2m-1}(r) \to \infty$ as $r \to \infty$. By iteration, we see that $\bar{u}(r) = \bar{v}_1(r) \to \infty$ as $r \to \infty$. Now, let $r = \mathrm{e}^t$, $w(t) = \bar{u}(\mathrm{e}^t)$, so that a direct calculation yields

$$\mathrm{e}^{4mt}\mathrm{e}^{w(t)} = \mathrm{e}^{4mt}\mathrm{e}^{\bar{u}(r)} \leqslant \mathrm{e}^{4mt}\Delta^{2m}\bar{u}(r) = w^{(4m)}(t) + \sum_{i=1}^{4m-1} c_i w^{(i)}(t), \tag{2.3}$$

where the c_i are some constants depending only on N and i. Hereafter, $g^{(i)}$ denotes the ith derivative of a function g. Since $\lim_{t\to\infty} w(t) = \infty$, there exists T_1 such that

$$\mathrm{e}^{4mt}\mathrm{e}^{w(t)} \geqslant w^2(t) \quad \text{for all } t \geqslant T_1.$$

We now apply the test function method developed by [**11**]. More precisely, we can choose a non-negative function $\phi_0 \in C_0^\infty[0,\infty)$ satisfying $\phi_0 > 0$ in $[0,2)$:

$$\phi_0(\tau) = \begin{cases} 1 & \text{for } \tau \in [0,1], \\ 0 & \text{for } \tau \geqslant 2, \end{cases} \qquad \text{and} \qquad \int_0^2 \frac{|\phi_0^{(i)}(\tau)|^2}{\phi_0(\tau)}\,\mathrm{d}\tau := A_i < \infty \quad \forall i \in \mathbb{N}.$$

Letting $T > T_1$, multiplying (2.3) by $\phi(t) = \phi_0((t-T_1)/(T-T_1))$ and integrating by parts, we obtain

$$\int_{T_1}^{\infty} \left[\phi^{(4m)}(t) + \sum_{i=1}^{4m-1} (-1)^i c_i \phi^{(i)}(t)\right] w(t)\,\mathrm{d}t \geqslant \int_{T_1}^{\infty} w^2(t)\phi(t)\,\mathrm{d}t - C. \tag{2.4}$$

By Young's inequality, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$w(t)\phi^{(i)}(t) \leqslant \varepsilon w^2(t)\phi(t) + C_\varepsilon \frac{|\phi^{(i)}(t)|^2}{\phi(t)} \quad \forall t \in [T_1, 2T - T_1).$$

Then, provided that ε is chosen sufficiently small, (2.4) yields

$$\begin{aligned} C' \sum_{i=1}^{4m} A_i (T-T_1)^{1-2i} = C' \sum_{i=1}^{i=4m} \int_{T_1}^{2T-T_1} \frac{|\phi^{(i)}(t)|^2}{\phi(t)}\,\mathrm{d}t &\geqslant \int_{T_1}^{2T-T_1} w^2(t)\phi(t)\,\mathrm{d}t - C'' \\ &\geqslant \int_{T_1}^{T} w^2(t)\,\mathrm{d}t - C'', \end{aligned}$$

with fixed constants $C', C'' > 0$. Letting $T \to \infty$, we observe a contradiction with $w(t) \to \infty$. So we have $v_{2m} \leqslant 0$ in $\mathbb{R}^N$.

Now suppose that there exists $x_0 \in \mathbb{R}^N$ verifying $v_{2m}(x_0) = 0$. Then x_0 is a maximum of v_{2m}, and hence $\Delta v_{2m}(x_0) \leqslant 0$, which is impossible as $\Delta v_{2m} = \mathrm{e}^u$, so $\Delta^{2m-1}u = v_{2m} < 0$ in $\mathbb{R}^N$. □

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As an immediate consequence of Theorem 1.1, we can make the following claim.

Corollary 2.1. *For any $m \in \mathbb{N}^*$ the equation $\Delta^{2m} u = \mathrm{e}^u$ admits no classical entire solution in $\mathbb{R}^2$.*

Proof. We suppose by contradiction that u is a smooth function verifying $\Delta^{2m} u = \mathrm{e}^u$ in $\mathbb{R}^2$. Using Theorem 1.1, $v := \Delta^{2m-1} u < 0$ in $\mathbb{R}^2$. Moreover,

$$\bar{v}'(r) = \frac{1}{2\pi r} \int_{\mathbb{B}_r(0)} \Delta \bar{v} \, \mathrm{d}x = \frac{1}{2\pi r} \int_{\mathbb{B}_r(0)} \Delta^{2m} u \, \mathrm{d}x = \frac{1}{2\pi r} \int_{\mathbb{B}_r(0)} \mathrm{e}^u \, \mathrm{d}x \geqslant \frac{C}{r} \quad \forall r \geqslant 1,$$

where C is a positive constant. Hence,

$$\bar{v}(r) - \bar{v}(1) = \int_1^r \bar{v}'(r) \, \mathrm{d}r \geqslant C \ln r \quad \forall r \geqslant 1.$$

This contradicts the fact that $\bar{v}(r) < 0$ if we let r tend to ∞, so we are done. □

Remark 2.2. By adapting a similar approach, the results of Theorem 1.1 and Corollary 2.1 hold true for the equation $\Delta^{2m} u = f(u)$ with general convex positive nonlinearity f verifying

$$\liminf_{t \to \infty} f(t) t^{-1-\mu} > 0 \quad \text{for some } \mu > 0. \tag{2.5}$$

We should mention that Walter [**12**] proved the non-existence of smooth entire solutions to $\Delta^{2m} u = f(u)$ in $\mathbb{R}^2$ for any $m \in \mathbb{N}^*$ and any positive function f satisfying (2.5), without the convexity assumption.

3. Proof of Theorem 1.2

We use here the notation for radial solutions from the introduction, and also the results (i)–(iii) cited there, given by [**1**, **2**]. Recall that u_β is the unique radial solution of

$$\Delta^2 u_\beta = \mathrm{e}^{u_\beta}, \quad \Delta u_\beta(0) = \beta, \quad u_\beta(0) = u_\beta'(0) = (\Delta u_\beta)'(0) = 0, \tag{3.1}$$

and the solution exists globally if and only if $\beta \leqslant \beta_0$. First, we show the following characterization of the separatrix solution u_{β_0}.

Proposition 3.1. *For any $\beta \leqslant \beta_0$, $\lim_{r \to \infty} \Delta u_\beta(r) \leqslant 0$ and $\lim_{r \to \infty} \Delta u_\beta(r) = 0$ if and only if $\beta = \beta_0$.*

Proof. For any solution u of (1.1),

$$\frac{\mathrm{d}\Delta u(r)}{\mathrm{d}r} = r^{N-1} \int_0^r s^{1-N} \mathrm{e}^u \, \mathrm{d}s > 0.$$

According to Theorem 1.1, $\lim_{r \to \infty} \Delta u_\beta(r) = \sigma \leqslant 0$ exists. For $\beta < \beta_0$ we see that $\sigma < 0$ since $u_\beta \leqslant -Cr^2$ by (i), and $\sigma = 0$ implies readily that $u_\beta(r) = o(r^2)$ at ∞.

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Similarly, we easily obtain $\lim_{r\to\infty}\Delta u_{\beta_0}=0$ for $N\geqslant 4$ by (ii). Consider now u_{β_0} when $N=3$. In fact, we will prove that if $\sigma<0$, then $\beta<\beta_0$.

For $N=3$, (1.3) reads

$$(r^4u'''(r))'=r^4\mathrm{e}^u \quad \forall r>0. \tag{3.2}$$

Integrating over $[0,r]$, we see that for all $r\geqslant 1$,

$$r^4u'''(r)=\int_0^r s^4\mathrm{e}^{u(s)}\,\mathrm{d}s\leqslant\int_0^\infty s^4\mathrm{e}^{u(s)}\,\mathrm{d}s<\infty.$$

Here we used the fact that $u(r)\leqslant -Cr$ for large r. Thus, $u'''(r)<Cr^{-4}$ for $r\geqslant 1$. Suppose now that $\sigma=\lim_{r\to\infty}\Delta u(r)<0$ for some entire solution u of (3.1) with $N=3$. As

$$u'(r)=r^{-2}\int_0^r s^2\Delta u(s)\,\mathrm{d}s,$$

we then have

$$u(r)\sim\frac{\sigma}{6}r^2,\quad u'(r)\sim\frac{\sigma}{3}r,\quad u''(r)\sim\frac{\sigma}{3}\quad\text{when } r\to\infty.$$

Consider now the function $\tilde{u}$ defined by

$$\tilde{u}(r)=-\varepsilon r^2+\ln(1+r)-b,$$

where

$$\varepsilon>0,\quad b\geqslant\ln\Big(\max_{\mathbb{R}_+}\psi\Big)\quad\text{with } \psi(r):=\frac{r(1+r)^5}{2(r+4)}\mathrm{e}^{-\varepsilon r^2}\text{ in }\mathbb{R}_+.$$

Direct computation shows that $\tilde{u}$ is a supersolution of (3.2) in $\mathbb{R}^3$, and

$$\tilde{u}'(r)=-2\varepsilon r+\frac{1}{r+1},\qquad \tilde{u}''(r)=-2\varepsilon-\frac{1}{(r+1)^2},\qquad \tilde{u}'''(r)=\frac{2}{(r+1)^3}.$$

Hence, if we fix $\varepsilon\in(0,-\sigma/6)$ and some large enough r_0, there holds $u^{(i)}(r_0)<\tilde{u}^{(i)}(r_0)$ for $0\leqslant i\leqslant 3$. By continuous dependence on initial data, there is $\beta_1>\beta=-\Delta u(0)$ such that $u_{\beta_1}^{(i)}(r_0)<\tilde{u}^{(i)}(r_0)$ for $0\leqslant i\leqslant 3$. We claim then that

$$u_{\beta_1}(r)<\tilde{u}(r)\quad\text{for all } r\geqslant r_0. \tag{3.3}$$

If this is not the case, then

$$r_1=\sup\{s>r_0 \text{ such that } u_{\beta_1}(r)<\tilde{u}(r) \text{ in } [r_0,s]\}<\infty.$$

By (3.2), we have $(r^4u'''_{\beta_1}(r))'<(r^4\tilde{u}'''(r))'$ in $[r_0,r_1)$, and successive integrations yield that $u'_{\beta_1}<\tilde{u}'$ on $[r_0,r_1)$, and hence $u_{\beta_1}(r_1)<\tilde{u}(r_1)$. This contradicts the definition of r_1, so the claim (3.3) holds true. By point (iii), u_{β_1} is then defined for all $r\geqslant 0$, which means that $\beta_1\leqslant\beta_0$, so $\beta<\beta_0$. □

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Proof of Theorem 1.2. To simplify the presentation, we erase the index β_0 and denote u_{β_0} by u. Recall that $u \leqslant -Cr$ for some $C > 0$ by (ii). Let $v = -\Delta u$. Then we have

$$v(r) = \beta_0 - \int_0^r s^{-2} \int_0^s t^2 \mathrm{e}^{u(t)} \,\mathrm{d}t \,\mathrm{d}s \quad \forall r > 0.$$

Applying Proposition 3.1, as $\lim_{r\to\infty} v(r) = 0$ we obtain

$$\begin{aligned} v(r) = \int_r^\infty s^{-2} \int_0^s t^2 \mathrm{e}^{u(t)} \,\mathrm{d}t \,\mathrm{d}s &= \frac{1}{r} \int_0^r t^2 \mathrm{e}^{u(t)} \,\mathrm{d}t + \int_r^\infty t \mathrm{e}^{u(t)} \,\mathrm{d}t \\ &= \frac{1}{4\pi r} \int_{\mathbb{R}^3} \mathrm{e}^u \,\mathrm{d}x - \frac{1}{r} \int_r^\infty t^2 \mathrm{e}^u \,\mathrm{d}t + \int_r^\infty t \mathrm{e}^u \,\mathrm{d}t. \end{aligned}$$

Therefore,

$$(r^2 u'(r))' = ar + r \int_r^\infty t^2 \mathrm{e}^u \,\mathrm{d}t - r^2 \int_r^\infty t \mathrm{e}^u \,\mathrm{d}t, \quad \text{where } a = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \mathrm{e}^u \,\mathrm{d}x. \tag{3.4}$$

Integrating (3.4), we obtain

$$u(r) = \frac{ar}{2} + \frac{1}{2} \int_0^r t^3 \mathrm{e}^u \,\mathrm{d}t - \frac{1}{6r} \int_0^r t^4 \mathrm{e}^u \,\mathrm{d}t + \frac{r^2}{6} \int_r^\infty t \mathrm{e}^u \,\mathrm{d}t.$$

Then it is easy to get the claimed expansion for u. □

4. Proof of Theorem 1.3

The proof of Theorem 1.3 is based on the following lemma.

Lemma 4.1. *If u is a smooth solution of (1.2), then $\Delta u > 0$ in $\mathbb{R}^N$.*

Indeed, this lemma is an immediate consequence of the following result.

Lemma 4.2. *If u is a C^4 lower-bounded function verifying that $\Delta^2 u < 0$ in $\mathbb{R}^N$, then $\Delta u > 0$ is in $\mathbb{R}^N$.*

Proof. First we show by contradiction that $\Delta u \geqslant 0$. Suppose that there is $x_0 \in \mathbb{R}^N$ verifying $\Delta u(x_0) < 0$. By translation, we can assume that $x_0 = 0$. Let $w = \Delta u$. Then $\Delta \bar{u} = \bar{w}$ and $\Delta \bar{w} = \overline{\Delta^2 u} < 0$, where $\bar{u}$ and $\bar{w}$ are the average over the sphere for u and w, respectively. Consequently, $\bar{w}'(r) \leqslant 0$, and hence $\bar{w}(r) \leqslant \bar{w}(0) = \Delta u(0) < 0$. Therefore, $\Delta \bar{u} \leqslant \bar{w}(0)$ in $\mathbb{R}^N$, which yields

$$\bar{u}(r) \leqslant \bar{u}(0) + \frac{\bar{w}(0)}{2N} r^2.$$

We obtain that $\bar{u}(r)$ goes to $-\infty$ as r tends to ∞, which is impossible since u is lower bounded. So $\Delta u \geqslant 0$ in $\mathbb{R}^N$. Now, if there is $x_1 \in \mathbb{R}^N$ such that $\Delta u(x_1) = 0$, then x_1 is a minimum point of Δu and $\Delta^2 u(x_1) \geqslant 0$, which contradicts the hypothesis, so the proof is complete. □

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From the above proof, as $\bar{w} \leqslant w(0)$, we immediately have the following corollary.

Corollary 4.3. *If u is a C^4 lower-bounded solution in $\mathbb{R}^N$ verifying $\Delta^2 u < 0$ in $\mathbb{R}^N$, then there exists $C > 0$ such that $\bar{u}(r) \leqslant C(1 + r^2)$ for any $r \geqslant 0$.*

Proof of Theorem 1.3. For $N = 1$, we have $u'' > 0$ from Lemma 4.1 and $u^{(4)} < 0$. However, except being a constant, no function can be concave and lower bounded on $\mathbb{R}$, so we obtain the non-existence of an entire solution for $u^{(4)} = u^{-p}$ in $\mathbb{R}$ for any $p > 0$. For $N = 2$ the superharmonic function Δu is bounded from below by Lemma 4.1, so it must be constant, and again it cannot verify (1.2), so we are done.

From now on, consider $N \geqslant 3$. We claim that if u is a smooth solution of (1.2), then

$$\text{there exists } C > 0 \text{ such that } \bar{u}(r) \geqslant Cr^{4/(p+1)} \quad \forall r > 0. \tag{4.1}$$

In fact, $\bar{w}$ is decreasing, where $w = \Delta u$, and $\bar{u}$ is increasing as $\bar{w} > 0$ by Lemma 4.1. Using $\Delta \bar{u} = \bar{w}$, we have, by the monotonicity of $\bar{w}$, that

$$\bar{u}(r) \geqslant u(0) + \frac{\bar{w}(r)}{2N} r^2. \tag{4.2}$$

On the other hand, by Jensen's inequality,

$$f(r) := -\Delta \bar{w}(r) = \overline{u^{-p}}(r) \geqslant \bar{u}^{-p}(r) > 0.$$

For any $s \geqslant r > 0$,

$$\bar{w}'(s) = -s^{1-N} \int_0^s t^{N-1} f(t)\,\mathrm{d}t \leqslant -s^{1-N} \int_0^r t^{N-1} f(t)\,\mathrm{d}t,$$

so we obtain, using the monotonicity of $\bar{u}$,

$$\begin{aligned} \bar{w}(r) \geqslant \bar{w}(2r) + \int_r^{2r} s^{1-N} \int_0^r t^{N-1} f(t)\,\mathrm{d}t\,\mathrm{d}s &\geqslant \bar{w}(2r) + Cr^{2-N} \int_0^r t^{N-1} f(t)\,\mathrm{d}t \\ &\geqslant Cr^{2-N} \int_0^r t^{N-1} \bar{u}^{-p}(t)\,\mathrm{d}t \\ &\geqslant Cr^2 \bar{u}^{-p}(r). \end{aligned} \tag{4.3}$$

Inserting (4.3) into (4.2), we have

$$\bar{u}(r) \geqslant u(0) + Cr^4 \bar{u}^{-p}(r) \geqslant Cr^4 \bar{u}^{-p}(r).$$

Hence, (4.1) follows.

Combining (4.1) and Corollary 4.3, if u is a classical solution of (1.2), necessarily there holds $p \geqslant 1$. Finally, we will exclude the $p = 1$ case. Let u be a smooth entire solution to $\Delta^2 u = -u^{-1}$. Then $\bar{u}$ is a subsolution to the following equation:

$$\Delta^2 U(r) + U^{-1}(r) = 0, \quad U(0) = u(0), \quad U''(0) = \bar{u}''(0), \quad U'(0) = U'''(0) = 0. \tag{4.4}$$

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Consider

$$Z(r) = u(0) + \frac{\bar{u}''(0)}{2} r^2.$$

Obviously, Z is biharmonic and a supersolution of (4.4). A comparison principle (see [**10**, Lemma 3.2]) ensures that $Z \geqslant \bar{u}$, and there is a solution U to (4.4) satisfying $\bar{u} \leqslant U \leqslant Z$.

By Lemma 4.1, $W := \Delta U > 0$, so U is increasing. As $\Delta W = -U^{-1} < 0$, W is decreasing and $W(r) \geqslant Cr^2 U^{-1}(r)$ (see, for example, (4.3)). By Corollary 4.3, $\lim_{r\to\infty} W(r) = \alpha > 0$. Therefore, $\lim_{r\to\infty}(U/r^2) = \alpha/2N$ and

$$\lim_{r\to\infty} rW'(r) = -\lim_{r\to\infty} r^{2-N} \int_0^r \frac{t^{N-1}}{U(t)}\,\mathrm{d}t = -\lim_{r\to\infty} \frac{r^2}{(N-2)U(r)} = -\frac{2N}{(N-2)\alpha} < 0.$$

This implies that $W(r) < 0$ for r large enough, which contradicts $W > 0$. □

Remark 4.4. The proof of Theorem 1.3 also means that no solution exists in dimension 1 or 2 for any $p > 0$. It answers Question 2 of McKenna and Reichel in [**10**].

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