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Robust transitivity and almost robust ergodicity

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Abstract. In this paper, we prove a relation between robust transitivity and robust ergodicity for conservative diffeomorphisms. In dimension two, robustly transitive diffeomorphisms are robustly ergodic. In higher dimensions, we work with partially hyperbolic diffeomorphisms with non-zero Lyapunov exponents. We define *almost robust ergodicity* and prove that in the three-dimensional case robustly transitive diffeomorphisms are approximated by almost robustly ergodic ones. In higher dimensions under the condition of the central Lyapunov exponents being of the same sign, we prove that robustly transitive diffeomorphisms are robustly ergodic.

1. *Introduction*

We address here the question of how the important concepts of topological transitivity and ergodicity (metric transitivity) are related. It is easy to verify that if Lebesgue measure, m, is ergodic or f preserves any ergodic probability measure which gives positive mass to open balls, then f is topologically transitive. That is, there exists a point $x \in M$ such that its forward orbit is dense or for any two open sets U and V there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$.

On the other hand, we know that the converse implication is not true; that is, transitivity is not enough to guarantee ergodicity. In fact, by an example of Furstenberg [**10**] we even know minimal diffeomorphisms which are not ergodic.

Our objective is to get robust ergodicity or *almost robust ergodicity* (see Definition 4) for volume preserving diffeomorphisms which are robustly transitive. Let us begin by defining these notions.

Let $\mathrm{Diff}_m^{1+\alpha}(M)$ denote the set of $C^{1+\alpha}$ volume preserving diffeomorphisms of M and $\mathrm{Diff}_m^{1+}(M) := \bigcup_{\alpha>0} \mathrm{Diff}_m^{1+\alpha}(M)$. From now on m always denotes the normalized Lebesgue measure.

Definition 1. We say that $f \in \mathrm{Diff}^1(M)$ is C^1-robustly transitive if there exists an open set in the C^1 topology $U \subset \mathrm{Diff}^1(M)$ containing f such that any g in U is also topologically transitive. We denote by $\mathcal{T}(M)$ the set of C^1 robustly transitive diffeomorphisms.

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Definition 2. We say that $f \in \mathrm{Diff}_m^{1+}(M)$ is C^1-robustly ergodic if there exists an open set in the C^1 topology $U \subset \mathrm{Diff}_m^1(M)$ containing f such that any g in $U \cap \mathrm{Diff}^{1+}(M)$ is also ergodic.

Observe that in Definition 2 we just take C^{1+} diffeomorphisms of a C^1 neighborhood of f. It is relevant to remember that even the ergodicity of volume preserving C^1 Anosov diffeomorphisms has not been proved in general. This is related to the fact that absolute continuity of stable and unstable foliations for the non-uniformly hyperbolic C^{1+} diffeomorphisms is a main ingredient to prove ergodicity in the majority of cases. We remark that a C^{1+} assumption on f is needed to get the absolute continuity of the invariant foliations. For an example of non-absolutely continuous foliation in the C^1 Anosov case, see [**2**].

In what follows we define *almost robust ergodicity* and would like to state that this definition seems to us reasonable and it is, perhaps, easier to check in relevant situations than robust ergodicity.

Definition 3. $f \in \mathrm{Diff}_m^{1+}(M)$ is called ϵ-ergodic if the Lebesgue measure m has an ergodic component C with $m(C) \geq 1 - \epsilon$.

Definition 4. $f \in \mathrm{Diff}_m^1(M)$ is called C^1 almost robustly ergodic, if for any $\epsilon > 0$ there is an open set $U_\epsilon \subset \mathrm{Diff}_m^1(M)$ containing f such that every $g \in U_\epsilon \cap \mathrm{Diff}_m^{1+}(M)$ is ϵ-ergodic.

By the above definition of almost robust ergodicity of f we require that the diffeomorphisms near enough to f have a 'greedy' enough ergodic component. This comes from the definition that if $f \in \mathrm{Diff}_m^{1+}(M)$ is almost robustly ergodic, then f is ergodic. We propose the following question for which we will give some partial answers in this paper.

Question 1. Let $f \in \mathrm{Diff}_m^{1+}(M)$ be C^1 robustly transitive. May f be approximated by robustly ergodic diffeomorphisms or, even, is it true that any robustly transitive diffeomorphism is robustly ergodic?

Here, we recall the results of [**9**] (in three dimensions) and [**6**] (in higher dimensions) about robustly transitive diffeomorphisms. The authors show that for $f \in \mathcal{T}(M)$ the tangent bundle of M admits a dominated splitting $TM = E_1 \oplus E_2 \oplus \cdots \oplus E_k$.

An invariant decomposition $TM = E_1 \oplus E_2 \oplus \cdots \oplus E_k$ is *dominated* if the dimension of $E_i(x)$ is constant for $x \in M$ and there exist $0 < \lambda < 1$ and $C > 0$ such that

$$\|Df^n|E_i(x)\| \cdot \|Df^{-n}|E_{i+1}(f^n(x))\| \leq C\lambda^n \quad \text{for all } x \in M,\ n > 0.$$

Moreover, for robustly transitive diffeomorphisms the finest dominated splitting $E_1 \oplus E_2 \oplus \cdots \oplus E_k$ is *volume hyperbolic* (for finest splitting see [**6**]). That is for some $C > 0$ and $0 < \lambda < 1$

$$|\det(Df^{-n}|E_k(x))| \leq C\lambda^n,$$
$$|\det(Df^n|E_1(x))| \leq C\lambda^n.$$

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For surface diffeomorphisms, E_1 and E_k are one dimensional and consequently C^1 robust transitivity implies a global hyperbolic structure. This result was proved by Mañé [**12**]. On the other hand Anosov diffeomorphisms are C^1 robustly ergodic in the sense of Definition 2 (see [**3**]). So, there is a satisfactory positive answer to the above question in dimension two.

In the general case, for $f \in \mathcal{T}(M)$ the tangent bundle of the ambient manifold can be split in the following ways.

- $\mathcal{T}^{cs}$: the set of robustly transitive diffeomorphisms such that $TM = E^{cs} \oplus E^u$ where E^u is uniformly expanding.
- $\mathcal{T}^{cu}$: the tangent bundle has an invariant splitting $TM = E^s \oplus E^{cu}$ where E^s is uniformly contracting.
- $\mathcal{T}^d(M)$: the set of robustly transitive diffeomorphisms which have no hyperbolic invariant subbundle (see [**8**] and [**16**]).

In the three-dimensional case, robustly transitive diffeomorphisms may be non-Anosov, but at least one of the subbundles in the dominated splitting of their tangent bundle will be uniformly hyperbolic. A dominated splitting with at least one uniformly hyperbolic subbundle is called partially hyperbolic. If $\dim(M) = 3$ then $\mathcal{T}(M) = \mathcal{T}^{cu} \cup \mathcal{T}^{cs}$. In other words robustly transitive diffeomorphisms of a three-dimensional manifold are partially hyperbolic.

By Proposition 1, any diffeomorphism in $\mathcal{T}(M)$ for a three-dimensional manifold M has at least one positive and another negative Lyapunov exponent in a full Lebesgue measure set. For $f \in \mathcal{T}^{cs}(M) \cup \mathcal{T}^{cu}(M)$ we call the Lyapunov exponents corresponding to the subbundles E^{cs}(respectively E^{cu}) *central Lyapunov exponents.*

THEOREM 1. *Let $f \in \mathcal{T}^{cs} \cup \mathcal{T}^{cu}$ such that there exists a neighborhood $U \subset \mathrm{Diff}^1_m(M)$ containing f such that for every $g \in U$, all the central Lyapunov exponents have the same sign in a full Lebesgue measure set. Then f is C^1 robustly ergodic.*

For f in $\mathcal{T}^{cu} \cap \mathcal{T}^{cs}$ we have a splitting $E^s \oplus E^c \oplus E^u$ where $\dim(E^u)$, $\dim(E^s) > 0$. So, $\mathcal{T}^{cu} \cap \mathcal{T}^{cs}$ contains Anosov robustly transitive diffeomorphisms. In [**7**] the authors prove that for any C^2 diffeomorphism of a C^1 open and dense subset of $PH_m(M)$, the volume measure is ergodic where $PH_m(M)$ represents C^1 partially hyperbolic volume preserving diffeomorphisms with an invariant decomposition of the tangent bundle $TM = E^s \oplus E^c \oplus E^u$ such that $\dim(E^c) = 1$. In fact they use the results of Shub–Wilkinson [**15**] followed by Baraviera–Bonatti [**1**] about removing the zero Lyapunov exponents.

In Theorem 1, we assume that the Lyapunov exponents in the center-stable (center-unstable) direction have the same sign. If this *compatibility* condition does not hold, we may have a mixed behavior. That is, a direction corresponding to the positive Lyapunov exponent does not dominate the direction corresponding to the negative exponent.

In Theorem 2, we deal with this mixed behavior in the three-dimensional case. For a three-dimensional manifold M, using a result of [**5**] we prove that C^{1+} volume preserving diffeomorphisms near to a generic diffeomorphism in $\mathrm{Diff}^1_m(M) \cap \mathcal{T}^{cu}(M) \cap \mathcal{T}^{cs}(M)$ have a big ergodic component in the measure theoretic sense.

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THEOREM 2. *Let $f \in \mathcal{T}^{cs}(M) \cup \mathcal{T}^{cu}(M)$ where M is three dimensional and there exists $U \subset \mathrm{Diff}^1_m(M)$ containing f such that for any $g \in U$ the central Lyapunov exponents are non-zero in a full Lebesgue measure set. Then, f is approximated by C^1 almost robustly ergodic diffeomorphisms. In fact the almost robustly ergodic diffeomorphisms are C^1 residual in a neighborhood of f.*

Another interesting question for robustly ergodic diffeomorphisms is the following.

Question 2. Is it true that any C^1 robustly ergodic conservative diffeomorphism admits a dominated splitting?

For more than three dimensions, in [**8**] the authors gave the first example of robustly transitive diffeomorphisms of $\mathbb{T}^4$ which do not have any hyperbolic direction, and in [**16**] the present author showed the robust ergodicity of such diffeomorphisms. In fact, the robust ergodicity of a class of diffeomorphisms of $\mathbb{T}^n$ which do not have any hyperbolic direction is verified.

2. *Preliminary results*

Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism of a compact d-dimensional manifold that preserves volume m. Oseledets theorem [**13**] states that for m-almost every point there are d numbers $\lambda_i, i = 1, \ldots, d$ which are Lyapunov exponents and some of them may coincide. So, identifying the equal Lyapunov exponents we get $\hat{\lambda}_1(x) > \cdots > \hat{\lambda}_{k(x)}(x)$ and

$$T_x M = E^1_x \oplus \cdots \oplus E^{k(x)}_x$$

such that

$$\lim_{n\to\infty} \frac{1}{n} \log \|Df^n(x)(v_j)\| = \hat{\lambda}_j(x) \quad \text{for all } v_j \in E^j_x \backslash \{0\}.$$

The $\hat{\lambda}_j$ are Lyapunov exponents which depend measurably on x. Let us define $\Lambda_i(f, x) = \lambda_1(x) + \cdots + \lambda_i(x)$ and $\Lambda_i(f) = \lambda_1(f) + \cdots + \lambda_i(f)$ where $\lambda_j(f) = \int_M \lambda_j(x)\, dm$. By semi-continuity arguments it is easy to show that the set of continuity points of $\Lambda : f \to (\Lambda_1(f), \ldots, \Lambda_{d-1}(f))$ contains a C^1 residual subset $\mathcal{R} \subset \mathrm{Diff}^1_m(M)$. In [**5**], the authors show that for all $f \in \mathcal{R}$, the Oseledet splitting is dominated or trivial.

THEOREM 3. [**5**] *For any compact manifold M there exists a C^1 residual subset $\mathcal{R}$ of $\mathrm{Diff}^1_m(M)$ such that for every $f \in \mathcal{R}$ the Oseledet splitting is dominated or else trivial, at almost every point.*

Let us recall some facts from linear algebra. Given a vector space V and a non-negative integer p, let $\wedge^p(V)$ be the pth exterior power of V. It is generated by the p-vectors of the form $v_1 \wedge \cdots \wedge v_p$ with $v_i \in V$. We take the norm $\|\cdot\|$ in $\wedge^p(V)$ such that $\|v_1 \wedge \cdots \wedge v_p\|$ is the p-dimensional volume of the parallelepiped with edges $v_1, \ldots, v_p$. A linear map $L : V \to W$ induces a natural linear map from $\wedge^p(V)$ to $\wedge^p(V)$ such that

$$\wedge^p(L)(v_1 \wedge \cdots \wedge v_p) = L(v_1) \wedge \cdots \wedge L(v_p).$$

For almost every $x \in M$, we have

$$\frac{1}{n} \log \|\wedge^p(Df^n_x)\| \to \Lambda_p(f, x) \quad \text{as } n \to \infty.$$

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PROPOSITION 1. *Let $f \in \mathcal{T}(M)$, then almost every $x \in M$ has at least one positive and one negative Lyapunov exponent.*

Proof. We just recall that for any $f \in \mathcal{T}$ the tangent bundle has a dominated splitting $TM = E_1 \oplus \cdots \oplus E_k$ such that Df uniformly expands the volume form of E_k and contracts in the E_1 direction. Let $\dim(E_k) = p$, then by the above discussions we conclude that for almost every $x \in M$,

$$\Lambda_p(f, x) = \lim_{n\to\infty} \frac{1}{n} \log \|\wedge^p (Df_x^n)\| > 0.$$

This means that at least one Lyapunov exponent is positive. The same argument detects a negative Lyapunov exponent in the E_1 direction. □

2.1. *Ergodic components and local ergodicity.* Let f be a C^{1+} diffeomorphism preserving the Lebesgue measure m. If all the Lyapunov exponents of f are non-zero on a full Lebesgue measure subset, then there are a countable number of invariant sets C_i with $m(C_i) > 0$ and $f|C_i$ ergodic (see [**14**]). Each C_i is called an ergodic component.

We are going to prove the ergodicity of the Lebesgue measure by finding open sets (mod 0) in the ergodic components. This kind of approach to prove ergodicity was introduced by Pesin [**14**]. We say that f is locally ergodic if all ergodic components are open, neglecting a zero measure set.

Observe that for transitive diffeomorphisms, to prove ergodicity it is enough to prove that any ergodic component contains an open set (mod 0) and then use the transitivity as follows.

If C_1, C_2 are two ergodic components with $U_1 \subset C_1$, $U_2 \subset C_2$ (mod 0), that is $m(U_i \backslash C_i) = 0$, then by topological transitivity there exists $n > 0$ such that $f^n(U_1) \cap U_2 \neq \emptyset$. By the invariance of C_i and the fact that f is a diffeomorphism and preserves the Lebesgue measure m, it comes out that $C_1 \cap C_2 \neq \emptyset$ and $C_1 = C_2$.

In Theorem 2 we show that diffeomorphisms near enough to generic diffeomorphisms of Theorem 3 have large ergodic components in the measure sense. We would just like to mention that Dolgopyat, Hu and Pesin constructed an example of diffeomorphism with infinitely many open ergodic components (Addendum to [**4**]). It is obvious that such examples cannot be topologically transitive because open invariant sets are dense in the transitive case.

3. *Proof of Theorem 1*

Let $f \in \mathcal{T}^{cs}$ be as in Theorem 1 (the $\mathcal{T}^{cu}$ case can be proved in a similar way) and $U \in \mathrm{Diff}^1_m(M)$ the open set of transitive diffeomorphisms with non-zero Lyapunov exponents in a full Lebesgue measure set. By the hypothesis about the sign of Lyapunov exponents and the fact that at least one of the Lyapunov exponents in the E^{cs} subbundle is negative, all the Lyapunov exponents corresponding to E^{cs} are negative.

As m is an invariant measure with non-zero Lyapunov exponents, it has a countable number of ergodic components. By the continuity of the invariant foliations of partially hyperbolic diffeomorphisms in [**11**], the unstable foliation of $f \in U$ varies continuously.

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Using Pesin theory for $f \in \mathrm{Diff}_m^{1+}(M)$, there is a full Lebesgue measure subset $\mathcal{R}$ such that $x \in \mathcal{R}$ is regular in the following sense:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \phi(f^i(x)) = \lim_{n\to-\infty} \frac{1}{n}\sum_{i=0}^{n-1} \phi(f^i(x)) \quad \text{for any } \phi \in C^0(M).$$

For any $x \in \mathcal{R}$, there is a local stable manifold $W^s_{\mathrm{loc}}(x)$ corresponding to the Lyapunov exponents in the E^{cs} direction. $W^s(x)$ depends measurably on x, moreover the stable foliation is absolutely continuous. For any ergodic component C, there is $x \in C$ such that $W^s_{\mathrm{loc}}(x)$ is almost (with respect to Lebesgue measure of $W^s(x)$) contained in $C \cap \mathcal{R}$. This means that there exists $C_x \subset W^s_\epsilon(x) \cap C \cap \mathcal{R}$, for some $\epsilon > 0$, such that $m_s(W^s_\epsilon(x) \backslash C_x) = 0$ where m_s is the Lebesgue measure induced on the stable manifold.

Now we saturate C_x by unstable leaves and let $U_x = \bigcup_{y\in W^s_\epsilon(x)} W^u(y)$. By the continuity of unstable foliation, U_x will contain an open set. On the other hand, for any $y \in C_x$ and $z \in W^u(y)$ we have

$$\begin{aligned}\lim_{n\to-\infty} \frac{1}{n}\sum_{i=0}^{n-1} \phi(f^i(z)) &= \lim_{n\to-\infty} \frac{1}{n}\sum_{i=0}^{n-1} \phi(f^i(y)) \\ &= \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \phi(f^i(y)) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \phi(f^i(x)).\end{aligned}$$

Now by the absolute continuity of W^u (unstable foliation) we conclude that $\bigcup_{y\in C_x} W^u(y)$ has full measure in U_x and, finally, as regular points have full measure, the ergodic component C contains a total Lebesgue measure subset of the open set U_x.

Now by the argument of local ergodicity in §2, transitivity of f implies the uniqueness of ergodic components of positive Lebesgue measure and consequently ergodicity of m. □

We mention that the absolute continuity of W^u by itself is not enough to get an open set (mod 0) in the above argument, but if we have some control from below on the size of local unstable manifolds of $y \in W^s_{\mathrm{loc}}(x)$ then the same argument works.

4. *Proof of Theorem 2*

Let $f \in \mathcal{T}^{cs}$ and U be as in Theorem 2. The $\mathcal{T}^{cu}$ case is similar. The difference between Theorems 2 and 1 appears when in some ergodic component with positive Lebesgue measure, a direction corresponding to a positive Lyapunov exponent does not dominate the direction corresponding to the negative one.

More precisely, let $TM = E^u \oplus E^{cs}$ and $\lambda_1(x) > \lambda_2(x) > \lambda_3(x)$ represent the Lyapunov exponents and $E_1(x) \oplus E_2(x) \oplus E_3(x)$ be the Oseledets splitting. It may happen that for an ergodic component C with $m(C) > 0, \lambda_2 > 0$ and E^{cs} cannot be split into one-dimensional dominating subbundles over C.

We show that the measure of such badly behaved sets is small enough for diffeomorphisms near to a C^1 generic diffeomorphism.

Take any $g \in \mathcal{R} \cap U$, where $\mathcal{R}$ is the residual subset in Theorem 3. Let Λ and Λ_i be as defined in §2. As g is a continuity point of Λ, for small $\delta_0 > 0$ there exists $\mathcal{O}$, a C^1 neighborhood of g in $\mathrm{Diff}_m^1(M)$, such that for all $f, f_1 \in \mathcal{O}$, $|\Lambda_i(f_1) - \Lambda_i(f)| < \delta_0$.

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As the Lyapunov exponents of $f \in U$ are non-zero, by [**14**] m has a countable number of ergodic components C_n with $m(C_n) > 0$ and $m\big(\bigcup C_n\big) = 1$.

Now let $f \in \mathcal{O} \cap \mathrm{Diff}_m^{1+}(M)$; we show that f has a big ergodic component in the measure theoretic sense. So, let us define two kinds of ergodic components for f.

(1) $\mathcal{C}_{\mathrm{good}}$: Union of ergodic components C_i such that one of the following happens:
- $\lambda_2(x) < 0$ for $x \in C_i$;
- $E^{cs} = E_2 \oplus E_3$ is a dominated splitting on C_i and $\lambda_2(x) > 0$.

(2) $\mathcal{C}_{\mathrm{bad}}$: Complement of $\mathcal{C}_{\mathrm{good}}$.

By the arguments in the previous section in the proof of Theorem 1 we deduce that all $C_i \subset \mathcal{C}_{\mathrm{good}}$ contain the open set (mod 0), because of the following.

- If $\lambda_2(x) < 0$ then we are in the setting of Theorem 1. First we use the absolute continuity of local stable manifolds (two dimensional) corresponding to λ_2 and λ_3 and then the continuity of unstable foliation tangent to E^u.
- If $\lambda_2(x) = \lambda > 0$ and E_2 dominates E_3, then on the corresponding ergodic component C_i, the tangent bundle has the following invariant dominated decomposition: $T_{C_i}M = E^{cu} \oplus E^{cs}$ with $\dim(E^{cs}) = 1$ and $E^{cu} = E^u \oplus E^2$ where E^2 corresponds to λ_2. Now as f is conservative, it is easy to show that the above dominated splitting has the volume hyperbolicity property $|\det Df^n \mid E^{cs}(x)| \leq C\lambda^n$ for some $C > 0$, $\lambda < 1$ for any $x \in C_i$ (see [**6**] for the volume hyperbolicity of dominated splitting in the conservative case). As E^{cs} is one dimensional, the volume hyperbolicity is equivalent to uniform hyperbolicity. Now apply the same argument as Theorem 1 using the absolute continuity of the local unstable manifolds for the points in C_i and the existence of large stable manifolds which is a result of uniform hyperbolicity of E^{cs}.

Consequently, by topological transitivity, there is just one ergodic component in $\mathcal{C}_{\mathrm{good}}$.

For the bad components we prove that they occupy a small region in the measure theoretic sense. First, we prove an easy lemma about ergodic components. Any ergodic component C is an invariant set and m_C, the normalized restriction of the Lebesgue measure on C, is an ergodic measure for the restriction of f on C.

LEMMA 1. *There exists $C' \subseteq C$ such that $m_C(C \backslash C') = 0$ and any $x \in C'$ has a dense orbit in C.*

Proof. As $\mathrm{Supp}(m_C)$ has full measure in C and is dense in it, we substitute C by $\mathrm{Supp}(m_C)$.

Remember that by the ergodicity of m_C, the basin of m_C has full measure. The basin of m_C consists of points z such that $(1/n)\sum_{i=0}^{n-1} \delta_{f^i(z)}$ converges to m_C in the weak* topology. As m_C has a full support, the orbit of the points in the basin will be dense in C. Now it is enough to take $C' =$ basin of m_C and the claim is proved. □

By an m-dominated splitting of $E_2 \oplus E_3$ along the orbit of a point x we require that for all $n \in \mathbb{Z}$:

$$\frac{\|Df^m_{f^n(x)} \mid E_3\|}{\|Df^m_{f^n(x)} \mid E_2\|} \leq \frac{1}{2}.$$

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We say that an invariant set C admits an m-dominated splitting if there is an m-dominated splitting for all orbits in C. Observe that an m-dominated splitting extends to the closure of an invariant subset.

Let $\Gamma(f, m)$ denote the subset of points such that $E_2 \oplus E_3$ does not admit an m-dominated splitting and let $\Gamma(f, \infty) = \bigcap_{m\in\mathbb{N}} \Gamma(f, m)$. For the ergodic components $C \subseteq \mathcal{C}_{\text{bad}}$, $\lambda_2 > 0$ and $E_2 \oplus E_3$ does not admit an m-dominated splitting over C for any $m \in \mathbb{N}$.

We claim that there exits a full m_C measure subset of C which we also denote by C such that $C \subseteq \Gamma(f, \infty)$. To prove the claim observe that as an m-dominated splitting passes to the closure of sets, for any point $x \in C'$, there is no m-dominated splitting along the orbit of x where C' is the same as in Lemma 1. So, there is no m-dominated splitting of $E_2 \oplus E_3$ along the orbit of x. This proves that $\mathcal{C}_{\text{bad}} \subseteq \Gamma(f, \infty)$ (mod 0).

Let

$$J(f) = \int_{\Gamma(f,\infty)} \frac{\lambda_2 - \lambda_3}{2}\, dm(x).$$

We apply the following proposition [**5**, Proposition 4.17].

PROPOSITION 2. *Given any $\delta > 0$ and $\epsilon > 0$, there exists a diffeomorphism f_1, ϵ-near to f such that*

$$\int_M \Lambda_2(f_1, x)\, dm(x) < \int_M \Lambda_2(f, x)\, dm(x) - J(f) + \delta.$$

By the above proposition we deduce that if the measure of bad components is not small enough then after perturbing f a little, the average of $\lambda_1 + \lambda_2$ drastically drops.

As $\mathcal{C}_{\text{bad}} \subseteq \Gamma(f, \infty)$ and on $\mathcal{C}_{\text{bad}}$, $\lambda_2(x) > 0$ by the above proposition it turns out that

$$\begin{aligned}\Lambda_2(f) - \Lambda_2(f_1) &\geq \frac{1}{2}\int_{\mathcal{C}_{\text{bad}}} (\lambda_2 - \lambda_3)(f)\, dm - \delta \geq \frac{1}{2}\int_{\mathcal{C}_{\text{bad}}} -\lambda_3(f)\, dm - \delta \\ &\geq m(\mathcal{C}_{\text{bad}}) \inf_{x\in\mathcal{C}_{\text{bad}}} \frac{-\lambda_3(f, x)}{2} - \delta.\end{aligned}$$

Now we observe that the infimum above is bounded away from zero uniformly in $\mathcal{O}$.

LEMMA 2. *There is some $\alpha < 1$, depending only on g such that $\lambda_3(x) \leq \log(\alpha)$ for all $x \in \mathcal{C}_{\text{bad}}$.*

Proof. As g is volume hyperbolic and partially hyperbolic $TM = E^u \oplus E^{cs}$, so $\det(Df \mid E^{cs}(x)) < \alpha < 1$ for all $x \in M$. We can take α uniform in a C^1 neighborhood of g, just because $g \to E^{cs}(x, g)$ varies continuously in the C^1 topology between partially hyperbolic diffeomorphisms. Take $x \in \mathcal{C}_{\text{bad}}$ then

$$\lambda_2(x) + \lambda_3(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \det(Df \mid E^{cs}(f^i(x))) \leq \log(\alpha)$$

and as $\lambda_2(x) > 0$ we have $\lambda_3(x) < \log(\alpha)$. □

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So by the above lemma

$$\inf_{x\in\mathcal{C}_{\text{bad}}} \frac{-\lambda_3(f,x)}{2} \geq \frac{-\log(\alpha)}{2}$$

and

$$\delta_0 \geq \Lambda_2(f) - \Lambda_2(f_1) \geq m(\mathcal{C}_{\text{bad}})\frac{-\log(\alpha)}{2} - \delta.$$

We then get

$$m(\mathcal{C}_{\text{bad}}) \leq \frac{\delta+\delta_0}{-\log(\alpha)}.$$

Taking δ_0 and δ small, for $f \in \mathcal{O} \cap \text{Diff}_m^{1+}(M)$, $m(\mathcal{C}_{\text{bad}})$ is small enough and so $m(\mathcal{C}_{\text{good}})$ is large enough. This implies that g is almost robustly ergodic.

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