Math. Struct. in Comp. Science (2003), *vol.* 13, *pp.* 259–320. © 2003 Cambridge University Press
DOI: 10.1017/S0960129502003924 Printed in the United Kingdom

Free iterative theories: a coalgebraic view

JIŘÍ ADÁMEK†, STEFAN MILIUS and JIŘÍ VELEBIL†

Institute of Theoretical Computer Science, Technical University, Braunschweig, Germany
Email: {adamek,milius,velebil}@iti.cs.tu-bs.de

Received 30 August 2001; revised 15 March 2002

Every finitary endofunctor of Set is proved to generate a free iterative theory in the sense of Elgot. This work is based on coalgebras, specifically on parametric corecursion, and the proof is presented for categories more general than just Set.

1. Introduction

Iterative algebraic theories were introduced by Calvin C. Elgot in Elgot (1975) as a concept serving the study of computation (on, say, Turing machines) at a level abstracting from the nature of external memory. The main example presented by Elgot is the theory of rational trees, that is, infinite trees that are solutions of systems of finitary iterative equations. Or, equivalently, that possess only finitely many subtrees. He and his coauthors later proved that this theory is a free iterative theory on a given (finitary) signature (Elgot *et al.* 1978).

The purpose of the present paper is to generalise Elgot's result from signatures (in other words, polynomial endofunctors of the category of sets) to finitary endofunctors of Set and some 'set-like' categories, for example, the category of posets. Using a very general Solution Theorem, developed in previous work, which shows by coalgebraic methods how iterative equations can be solved in categories, we prove that finitary endofunctors generate free iterative theories (in other words, finitary monads), called *rational monads*. We construct the rational monad in two steps:

(1) A rational monad of a *strongly finitary* endofunctor, that is, a finitary endofunctor 'preserving finiteness', is constructed in Section 4.

and

(2) A rational monad of a general finitary endofunctor is derived in Section 5.

We work with categories called strongly LFP, which we introduce in Section 2. And we assume that the given endofunctor preserves monomorphisms. For the category Set this last assumption can be omitted, as we show in Section 6. For all these cases the common formulation of a rational monad, R, is: R assigns to every object Y (of 'parameters') the union RY of all images of solutions of finitary flat equations with parameter Y.

In the rest of the introduction we explain the concepts mentioned above, and give further references.

† The first and third authors acknowledge the support of the Grant Agency of the Czech Republic under the Grant No. 201/02/0148.

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

1.1. *What is a rational tree?*

The algebra of finite and infinite Σ-labelled trees has, for every finitary signature Σ, the important property that iterative equational systems of a certain (liberal) type have unique solutions. And when we solve finitary flat iterative systems, we obtain precisely the so-called rational trees. We will now describe this in more detail.

For any set X of variables, we use

$$T_\Sigma X$$

to denote the algebra of all finite and infinite Σ-labelled trees with variables from X. That is, trees labelled so that a node with $n > 0$ children is labelled by an n-ary operation symbol (an element of Σ_n), and a leaf is labelled by a variable or a constant symbol (an element of $X + \Sigma_0$). The operations on $T_\Sigma X$ are given by tree-tupling. Now consider a system of iterative equations[†]

$$\begin{aligned} x_0 &\approx t_0(x_0, x_1, x_2, \ldots, y_0, y_1, y_2, \ldots) \\ x_1 &\approx t_1(x_0, x_1, x_2, \ldots, y_0, y_1, y_2, \ldots) \\ &\vdots \\ x_n &\approx t_n(x_0, x_1, x_2, \ldots, y_0, y_1, y_2, \ldots) \\ &\vdots \end{aligned} \tag{1}$$

where t_i are trees with leaves labelled by variables from $X = \{x_0, x_1, x_2, \ldots\}$ and parameters from $Y = \{y_0, y_1, y_2, \ldots\}$, that is,

$$t_i \in T_\Sigma(X + Y) \quad \text{for } i = 0, 1, 2, \ldots$$

Such a system is called *guarded* if none of the trees t_i is a variable from X. This condition alone guarantees that there exists a unique *solution* of (1), that is, a unique list ${x_i}^\dagger(y_0, y_1, y_2, \ldots)$ of trees in $T_\Sigma Y$ such that the expected identities

$$\begin{aligned} {x_0}^\dagger &= t_0({x_0}^\dagger, {x_1}^\dagger, {x_2}^\dagger, \ldots, y_0, y_1, y_2, \ldots) \\ {x_1}^\dagger &= t_1({x_0}^\dagger, {x_1}^\dagger, {x_2}^\dagger, \ldots, y_0, y_1, y_2, \ldots) \\ &\vdots \\ {x_n}^\dagger &= t_n({x_0}^\dagger, {x_1}^\dagger, {x_2}^\dagger, \ldots, y_0, y_1, y_2, \ldots) \\ &\vdots \end{aligned}$$

hold in $T_\Sigma Y$.

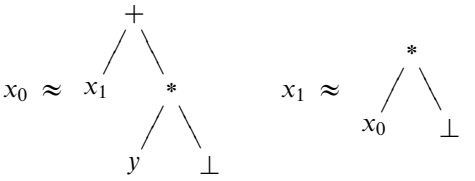
Theorem 1.1. Every guarded system of iterative equations has a unique solution.

This is a special case of a much more general Solution Theorem, which we mention in Section 1.2 below.

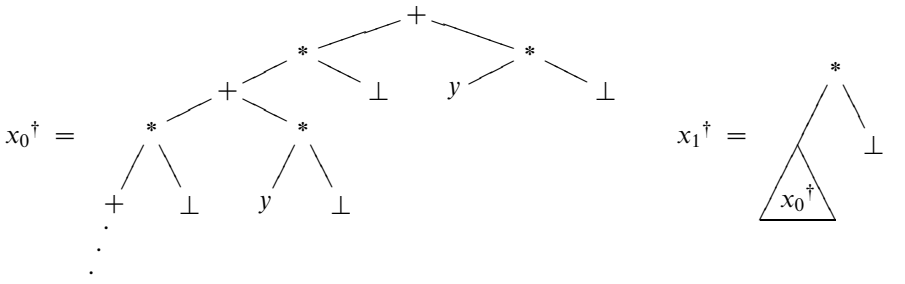
[†] We use $\approx$ to denote formal equations and $=$ to denote the identity of the two sides.

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

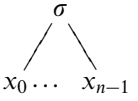
Example 1.2. Let Σ consist of binary operation symbols $+$ and $*$ and a constant symbol $\perp$. The following system of iterative equations



is guarded. The solution is given by the following trees in $T_\Sigma Y$:



A guarded equational system (1) is called *finitary* if it has only finitely many variables and the right-hand sides t_i are finite trees (such as those in Example 1.2). It is easy to see that every finitary system has a simple reduction to a finitary *flat* system, that is, one with finitely many variables, and with right-hand sides t_i that are either

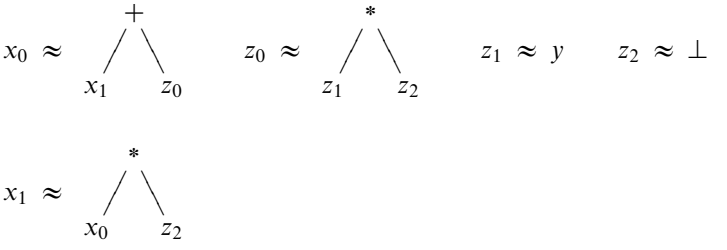
(a) flat trees  with $\sigma \in \Sigma_n$ and $x_0, \ldots, x_{n-1} \in X$

or

(b) single parameters from Y.

Observe that (a) includes single constant symbols $\sigma \in \Sigma_0$.

Example 1.3. A reduction of the system of Example 1.2 is obtained by introducing new variables z_0, z_1 and z_2 as follows:

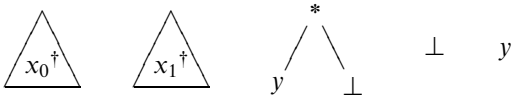


The unique solution of this reduced system is the original solution $x_0{}^\dagger$ and $x_1{}^\dagger$ together with the obvious new trees $z_i{}^\dagger$ for $i = 0, 1, 2$.

Definition 1.4. A tree in $T_\Sigma Y$ is called *rational* if it can be obtained by solving a finitary flat system of equations.

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

Thus, the trees $x_0{}^\dagger$ and $x_1{}^\dagger$ in Example 1.2 are examples of rational trees. Rational trees are fully characterised as those trees in $T_\Sigma Y$ that have, up to tree isomorphism, only finitely many subtrees (Elgot *et al.* 1978). For example, all subtrees of $x_0{}^\dagger$ in Example 1.2 are isomorphic to one of the following five (obtained by breadth-first search of the nodes of $x_0{}^\dagger$):



Now the subalgebra

$$R_\Sigma Y$$

of $T_\Sigma Y$ of all rational trees has a solution property 'almost as strong' as the algebra $T_\Sigma Y$ itself.

Theorem 1.5. Every finite guarded system of iterative equations with rational right-hand sides has a unique solution in $R_\Sigma Y$.

A direct proof of this theorem is not difficult, but we do not have to discuss it here because we are going to prove a much more general result, called the Rational Solution Theorem, see Corollary 5.8.

1.2. *What is a solution in general?*

In this subsection we recall briefly some results, which were obtained independently by Larry Moss (Moss 2001) and the present authors in collaboration with Peter Aczel (Aczel *et al.* 2002). The reader can find more details in the extended abstract Aczel *et al.* (2001), which has already been published.

We now generalise infinite trees, which are elements of the final coalgebra $T_\Sigma X$ of the polynomial endofunctor $H_\Sigma(_) + X$, to final coalgebras of $H(_) + X$ for an arbitrary endofunctor H. Recall that, given a signature Σ, the corresponding *polynomial functor*

$$H_\Sigma : \mathsf{Set} \longrightarrow \mathsf{Set}$$

defined by

$$H_\Sigma A = \Sigma_0 + \Sigma_1 \times A + \Sigma_2 \times A^2 + \cdots$$

has the property that H_Σ-algebras are just the classical universal algebras of signature Σ. A final H_Σ-coalgebra is well known to be the coalgebra $T_\Sigma \emptyset$ of all finite and infinite Σ-labelled trees without variables (Adámek and Koubek 1995). Now the functor $H_\Sigma(_) + X$ is also polynomial (for the signature obtained from Σ by adding a constant symbol for every variable in X), thus,

$$T_\Sigma X \text{ is a final coalgebra of } H_\Sigma(_) + X.$$

In our previous work (Aczel *et al.* 2002) we have introduced the concept of an *iteratable* endofunctor of Set (or, more generally, of any category with binary coproducts): it is an

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

endofunctor H such that $H(_)+X$ has a final coalgebra, for every X. We use the notation

$$TX$$

to denote a final coalgebra of $H(_)+X$, and the coalgebra-structure (which, by Lambek's Lemma (Lambek 1968), is an isomorphism between TX and $HTX+X$) is denoted by giving names to the coproduct injections of TX (as a coproduct of HTX and X) as follows:

$$\eta_X : X \longrightarrow TX \quad \text{('injection of variables')}$$

and

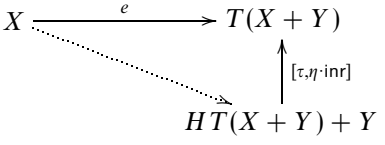
$$\tau_X : HTX \longrightarrow TX \quad (\text{'}TX \text{ becomes an } H\text{-algebra'}).$$

Recall that an endofunctor is called *finitary* if it preserves filtered colimits. Every finitary endofunctor is iteratable, see Example 2.11 of Aczel *et al.* (2002). The above concept of a guarded system of equations can be formalised by a function $e : X \longrightarrow T(X+Y)$, $x_i \mapsto t_i$. And since $T(X+Y) = \big(HT(X+Y)+Y\big)+X$ is a coproduct with injections $[\tau_{X+Y}, \eta_{X+Y} \cdot \mathsf{inr}] : HT(X+Y)+Y \longrightarrow T(X+Y)$ and $\eta_{X+Y} \cdot \mathsf{inl} : X \longrightarrow T(X+Y)$, to say that the right-hand sides t_i are never variables from X is precisely to say that e factors through $[\tau_{X+Y}, \eta_{X+Y} \cdot \mathsf{inr}]$. Hence, the notion of a guarded system (1) generalises as follows.

Definition 1.6. Let H be an iteratable endofunctor. We define a *guarded equation morphism* for H to be a morphism of the form

$$e : X \longrightarrow T(X+Y)$$

(X is the 'object of variables' and Y the 'object of parameters') that factors through the coproduct injection $[\tau_{X+Y}, \eta_{X+Y} \cdot \mathsf{inr}] : HT(X+Y)+Y \longrightarrow T(X+Y)$:



The H-algebras TX have a rich structure. First, substitution of trees in $T_\Sigma X$ for variables generalises to all iteratable endofunctors. Recall that given an interpretation of variables $x \in X$ as trees $s(x)$ over Y, that is, a function $s : X \longrightarrow T_\Sigma Y$, the corresponding substitution of trees from $T_\Sigma Y$ into (leaves of) trees of $T_\Sigma X$ is a homomorphism

$$\widehat{s} : T_\Sigma X \longrightarrow T_\Sigma Y$$

of Σ-algebras. Moreover, $\widehat{s}$ is the unique homomorphic extension of s. This can be generalised to all iteratable endofunctors.

Theorem 1.7 (Substitution Theorem). For every morphism $s : X \longrightarrow TY$ there exists a unique homomorphism $\widehat{s} : TX \longrightarrow TY$ of H-algebras extending s (that is, with $s = \widehat{s}\,\eta_X$).

The proof can be found in Moss (2001) or Aczel *et al.* (2001) (and a slightly improved version in Aczel *et al.* (2002)).

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

Corollary 1.8. $(T, \eta, \widehat{(_)})$ is a *Kleisli triple*, that is, the following three axioms are satisfied:

(i) $\widehat{\eta_X} = id_{TX}$, for every object X,
(ii) $\widehat{s}\,\eta_X = s$, for every morphism $s : X \longrightarrow TY$,
and
(iii) $\widehat{r}\,\widehat{s} = \widehat{\widehat{r}\,s}$, for every pair $s : X \longrightarrow TY$ and $r : Y \longrightarrow TZ$.

As shown in Manes (1976), axioms (i)–(iii) are equivalent to having a monad (T, η, μ), where $\mu_X = \mathrm{i}\widehat{d_{TX}} : TTX \longrightarrow TX$. Observe that μ_X is a homomorphism of H-algebras (since every $\widehat{s}$ is). However, this monad is usually not finitary; thus, it cannot be identified with an algebraic theory in the sense of Lawvere.

Example 1.9. For every signature Σ we have a monad (T_Σ, η, μ), where $T_\Sigma X$ is the algebra of all finite and infinite Σ-labelled trees above, η is the insertion of variables (as one-node trees), and μ is the usual substitution of trees into trees. This monad is seldom finitary. (In fact, it is finitary iff all operations of Σ are either unary or nullary.)

Next, we can introduce solutions for equation morphisms $e : X \longrightarrow T(X + Y)$ by mimicking the case of trees as follows. A solution of e is a morphism $e^\dagger : X \longrightarrow TY$. It has the property that when the following 'substitution' morphism

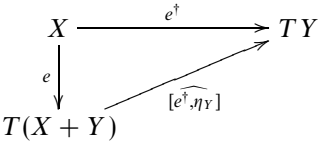
$$s = [e^\dagger, \eta_Y] : X + Y \longrightarrow TY$$

is considered (that is, every variable x_i is substituted by the tree $e^\dagger(x_i) = {x_i}^\dagger \in TY$, while parameters are unchanged), the composite

$$X \xrightarrow{\ e\ } T(X + Y) \xrightarrow{\ \widehat{s}\ } TY$$

(corresponding to performing the substitution s on all variables of the right-hand sides of (1)) is equal to $e^\dagger$.

Definition 1.10. We define a *solution* of an equation morphism $e : X \longrightarrow T(X + Y)$ to be a morphism $e^\dagger : X \longrightarrow TY$ such that the triangle



commutes.

The following result is called Parametric Corecursion in Moss (2001) and Solution Theorem in Aczel *et al.* (2001); see also a much improved version of the proof in Aczel *et al.* (2002).

Theorem 1.11 (Solution Theorem). Given an iteratable endofunctor H, every guarded equation morphism has a unique solution.

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

1.3. *What is a rational monad?*

Consider a finitary endofunctor H of Set (that is, H preserves filtered colimits). The following generalises the above finitary flat systems of equations for $H = H_\Sigma$.

Definition 1.12. We define a *finitary flat equation morphism* for H to be a morphism of the following form

$$e : X \longrightarrow HX + Y \quad (X \text{ finite}).$$

In brief, finitary flat equation morphisms are just the finite coalgebras of the endofunctor $H(_) + Y$. (If $H = H_\Sigma$, then a finitary flat equation morphism

$$e : X \longrightarrow \coprod_{n\in\omega} \Sigma_n \times X^n + Y \quad (X \text{ finite})$$

is precisely the concept as introduced above.) Every finitary flat equation morphism gives rise to a guarded equation morphism using $\tau_X : HTX \longrightarrow TX$ above as follows:

$$X \xrightarrow{\ e\ } HX + Y \xrightarrow{\ \tau_X H\eta_X + Y\ } TX + Y \xrightarrow{\ [T\mathsf{inl}, \eta_{X+Y}\cdot \mathsf{inr}]\ } T(X + Y). \tag{2}$$

By a harmless abuse of notation, we use $e^\dagger : X \longrightarrow TY$ to denote the unique solution of (2). We have proved in Aczel *et al.* (2002) that for flat equation morphisms we have

$$\text{solution} = \text{corecursion}.$$

That is, $e^\dagger$ is the unique homomorphism of the coalgebra X into the final coalgebra TY.

Mimicking the above definition of $R_\Sigma Y$ as the algebra of all solutions of finitary flat equation systems (for $H = H_\Sigma$), we introduce below an H-algebra

$$RY = \bigcup \mathrm{im}(e^\dagger)$$

as a union of images of all solutions of finitary flat equation morphisms with the parameter set Y (and an arbitrary finite set X of variables). In Adámek *et al.* (2002) we show that this definition of RY can be formulated equivalently using all finitary equation morphisms, not just the flat ones.

These H-algebras RY, which are subalgebras of TY, also have a rich structure; we are going to prove the following.

Rational Substitution Theorem: Every morphism $s : X \longrightarrow RY$ has a unique extension into a homomorphism $\widetilde{s} : RX \longrightarrow RY$ of H-algebras (see Theorem 5.14).

Rational Solution Theorem: Every guarded equation morphism $e : X \longrightarrow R(X + Y)$ where X is finite has a unique solution $e^\ddagger : X \longrightarrow RY$ (see Corollary 5.8).

As a corollary of the Rational Substitution Theorem, we conclude immediately that R is also a monad on Set, this time a finitary monad (or, equivalently, Lawvere's algebraic theory (Manes 1976)). We call this monad the *rational monad* generated by the given finitary endofunctor H.

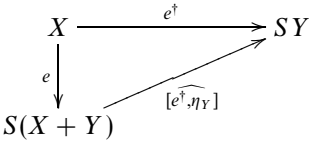
https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

1.4. *A rational monad is a free iterative monad*

C. C. Elgot has studied algebraic theories (in other words, finitary monads on Set) with the property that certain iterative equations have unique solutions; he called such theories iterative (Elgot 1975). From Section 1.2 above, it is quite natural to call, for a given monad (S, η, μ), any morphism

$$e : X \longrightarrow S(X + Y)$$

an *equation morphism* (for S). Recall that every monad defines substitution: given $s : X \longrightarrow SY$ we have the corresponding homomorphism $\widehat{s} = \mu_Y \cdot Ss : SX \longrightarrow SY$ of free S-algebras. Now, a morphism $e^\dagger : X \longrightarrow SY$ will be called a *solution* of e if the triangle



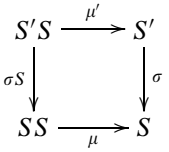
commutes. But how can we express the property of e being guarded? (Or *ideal*, which is a slightly stricter concept that Elgot used: for trees this means that the right-hand sides are neither variables, nor single parameters. But this difference is inessential as we prove below – see Remark 4.31.) For that purpose, Elgot introduced the concept of an ideal algebraic theory. Translated into the language of monads, it yields the following definition (see Aczel *et al.* (2002) for a very simple proof that our formulation is equivalent to Elgot's):

Definition 1.13. A monad (S, η, μ) is called *ideal* if:

(i) $S = S' + Id$ and $\eta : Id \longrightarrow S$ is the right-hand injection (notation: $\sigma : S' \longrightarrow S$ for the left-hand one)

and

(ii) μ restricts to $\mu' : S'S \longrightarrow S'$, that is, there is a natural transformation μ' such that the square



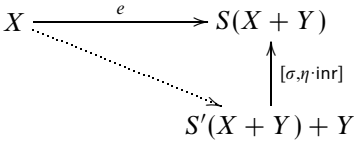
commutes.

This definition applies to all categories with finite coproducts such that coproduct injections are monomorphic.

Example 1.14. The monad T defined by Corollary 1.8 is ideal: here $T = HT + Id$ with injections τ (=σ) and η. Also, the monad R defined by the Rational Substitution Theorem is ideal, as we prove below.

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

For ideal monads (S, η, μ) we call an equation morphism $e : X \longrightarrow S(X + Y)$ *guarded* if it factors through $[\sigma_{X+Y}, \eta_{X+Y}\,\mathsf{inr}] : S'(X + Y) + Y \longrightarrow S(X + Y)$:



Definition 1.15. (Elgot 1975) A finitary monad S in Set is called *iterative* if it is ideal and every guarded equation morphism $e : X \longrightarrow S(X + Y)$ with X and Y finite has a unique solution.

The restriction to finite sets Y, in comparison to the Rational Solution Theorem above, is inessential because the rational monad is finitary. Thus, results using finitely presentable parameter objects Y extend easily to results on arbitrary Y.

The main result of the present paper is the fact that every finitary endofunctor of Set generates a free iterative monad, *viz*., the rational monad. For polynomial endofunctors this is the main result of Elgot *et al.* (1978), but is proved in a completely different manner.

This situation is analogous to the main result in Aczel *et al.* (2002) concerning *completely iterative monads*, as introduced in Elgot *et al.* (1978): they are defined as above, except that the requirement that X and Y be finite is dropped. We have proved that every iteratable endofunctor generates a free completely iterative monad, *viz*., the above monad T.

1.5. *Beyond the category of sets*

The concept of an ideal monad is immediately extended to any category with finite coproducts. And we can also extend Definition 1.15 naturally, as follows.

Definition 1.16. A monad S is called *iterative* if it is ideal and every guarded equation morphism $e : X \longrightarrow S(X + Y)$ with X and Y finitely presentable (see Section 2.2 below) has a unique solution.

However, for the main result of our paper, stating that every finitary endofunctor generates a free iterative monad, we need to make rather strong additional assumptions on the category we work with. This contrasts to the situation of Aczel *et al.* (2002), where the result that every iteratable functor generates a free completely iterative monad is proved for all categories having finite coproducts with monomorphic injections. In the present paper we will assume that the given category is strongly LFP, that is, it has to be extensive and locally finitely presentable with finitely presentable objects closed under strong quotients and have finite hom-sets for finitely presentable objects. We introduce these concepts in Section 2.

Examples 1.17. The category Set^S (for S-sorted operations) is strongly LFP. Also, the category of posets (for order-enriched operations) is strongly LFP.

The only additional assumption on the finitary endofunctor is that it preserves monomorphisms, an assumption not needed for set functors, as we show in Section 6.

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

In the next section we recall everything about locally finitely presentable categories that we need later. Even for readers only interested in $\mathscr{A} = \mathsf{Set}$, this section brings information about finitary monads that will be used further.

2. Finitary monads

2.1. *Finitary Kleisli triples in* Set.

Recall the concept of a *Kleisli triple* $(T, \eta, \widehat{(_)})$ on a category $\mathscr{A}$, which is equivalent to that of a monad on $\mathscr{A}$ (Manes 1976). It consists of:

(1) A function assigning to every object X in $\mathscr{A}$ an object TX and a morphism $\eta_X : X \longrightarrow TX$.

and

(2) A function assigning to every morphism $s : X \longrightarrow TY$ a morphism $\widehat{s} : TX \longrightarrow TY$ so that the axioms (i)–(iii) of Corollary 1.8, hold.

A *finitary Kleisli triple in* Set consists, analogously, of two functions: one assigns to every finite set X a set TX and a morphism $\eta_X : X \longrightarrow TX$, and the other one assigns to every morphism $s : X \longrightarrow TY$ with X and Y finite a morphism $\widehat{s} : TX \longrightarrow TY$ so that (i)–(iii) of corollary hold. In Adámek *et al.* (2002) we proved that there exists a unique *finitary monad* on Set, that is, a monad (T, η, μ) such that T preserves filtered colimits, generating that triple in the expected sense:

(a) For every finite set X, the given $\eta_X : X \longrightarrow TX$ is the component of the unit $\eta : Id \longrightarrow T$.

and

(b) For every $s : X \longrightarrow TY$ with X and Y finite, the usual 'substitution formula'

$$\widehat{s} = \mu_Y \cdot Ts : TX \longrightarrow TY$$

holds.

In other words, a finitary Kleisli triple can be extended to a Kleisli triple corresponding to a finitary monad, and vice versa.

Let us also recall from Manes (1976) that finitary monads on Set are precisely the same as Lawvere's algebraic theories: given a finitary monad (T, η, μ), we can consider its theory consisting of natural numbers as objects, and functions $n \longrightarrow Tk$ as morphisms from n to k. The composite of

$$n \xrightarrow{f} Tk \quad \text{and} \quad k \xrightarrow{g} Tl$$

is

$$n \xrightarrow{f} Tk \xrightarrow{Tg} TTl \xrightarrow{\mu_l} Tl$$

Conversely, every Lawvere's theory represents a finitary variety $\mathscr{V}$ whose monad (describing $\mathscr{V}$-free algebras) is finitary.

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

2.2. *Locally finitely presentable (LFP) categories*

We generalise the concept of a finite set in Set to that of a *finitely presentable object* A in a category $\mathscr{A}$: it is an object such that the hom-functor $\mathscr{A}(A, _) : \mathscr{A} \longrightarrow \mathsf{Set}$ preserves filtered colimits.

Definition 2.1. (Gabriel and Ulmer 1971). A category is called LFP if it is cocomplete and has a set of finitely presentable objects whose closure under filtered colimits is the whole category.

Remark 2.2. It follows that every LFP category $\mathscr{A}$ has, up to isomorphism, only a set of finitely presentable objects. Moreover, if

$$\mathscr{A}_{fp}$$

denotes a set of representatives of finitely presentable objects (considered as a full subcategory), then for every object X of $\mathscr{A}$ we have:

(i) The comma-category $\mathscr{A}_{fp}/X$ (consisting of all morphisms $a : A \longrightarrow X$ with A in $\mathscr{A}_{fp}$) is filtered.

and

(ii) X is a canonical colimit of the diagram

$$D_X : \mathscr{A}_{fp}/X \longrightarrow \mathscr{A}$$

given by

$$D_X(a : A \longrightarrow X) = A.$$

Consequently,

(iii) $\mathscr{A}$ is a free cocompletion of $\mathscr{A}_{fp}$ under filtered colimits. That is, every functor $F : \mathscr{A}_{fp} \longrightarrow \mathscr{B}$, where $\mathscr{B}$ has filtered colimits, has an extension, unique up to natural isomorphism, to a functor $F' : \mathscr{A} \longrightarrow \mathscr{B}$ preserving filtered colimits.

See Adámek and Rosický (1994).

Examples 2.3.

(i) Set is LFP, choose Set_{fp} to be the full subcategory of natural numbers.
(ii) The category Pos of posets and monotone maps is LFP, here Pos_{fp} is a full subcategory representing all finite (in other words, finitely presentable) posets.
(iii) The category Gra of graphs (sets with a binary relation) and graph homomorphisms is LFP with Gra_{fp} representing all finite graphs.
(iv) Every variety $\mathscr{V}$ of (finitary, many-sorted) algebras is LFP. An algebra V in $\mathscr{V}$ is a finitely presentable object iff it can be presented by finitely many generators and finitely many equations.

Definition 2.4. We define a *finitary Kleisli triple* on an LFP category $\mathscr{A}$ to be a triple $(T, \eta, \widehat{(_)})$ where T is a function assigning to every finitely presentable object X an object TX, with a morphism $\eta_X : X \longrightarrow TX$, and $\widehat{(_)}$ is a function assigning to every morphism

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

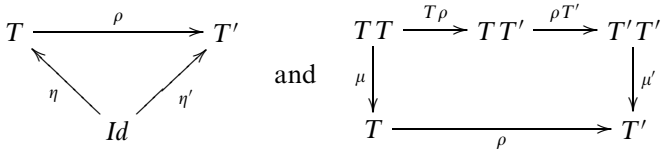
$s : X \longrightarrow TY$ with X and Y finitely presentable a morphism $\widehat{s} : TX \longrightarrow TY$ so that:

(i) $\widehat{\eta_X} = id_{TX}$, for all X in $\mathscr{A}_{fp}$;
(ii) $\widehat{s}\,\eta_X = s$, for all $s : X \longrightarrow TY$, (X, Y in $\mathscr{A}_{fp}$);
and
(iii) $\widehat{r}\,\widehat{s} = \widehat{\widehat{r}\,s}$, for all $s : X \longrightarrow TY$ and $r : Y \longrightarrow TZ$ (X, Y, Z in $\mathscr{A}_{fp}$).

Example 2.5. Every finitary monad (T, η, μ) on an LFP category $\mathscr{A}$ generates a finitary Kleisli triple with $\widehat{s} = \mu_Y \cdot Ts$ for all $s : X \longrightarrow TY$ with X, Y in $\mathscr{A}_{fp}$.

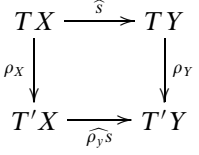
Remark 2.6. It is proved in Adámek *et al.* (2002) that finitary Kleisli triples on an LFP category $\mathscr{A}$ correspond precisely to finitary monads on $\mathscr{A}$. In fact, the correspondence extends to the level of morphisms of Kleisli triples and monads.

Recall that, given monads (T, η, μ) and (T', η', μ') on a category, a *monad homomorphism* is a natural transformation $\rho : T \longrightarrow T'$ such that the diagrams



commute. If the components of ρ are monomorphisms, then (T, η, μ) is called a *submonad* of (T', η', μ').

Expressed by Kleisli triples, a homomorphism is given by assigning, to every object X, a morphism $\rho_X : TX \longrightarrow T'X$ such that (a) $\rho_X \cdot \eta_X = \eta'_X$ and, (b) given $s : X \longrightarrow TY$, the square



commutes (where we use $\widehat{(_)}$ for both Kleisli triples). When both triples are finitary, we restrict to X and Y finitely presentable in the above definition.

The precise formulation of the correspondence mentioned above is given by the following proposition (Adámek *et al.* 2002).

Proposition 2.7. Let $\mathscr{A}$ be an LFP category. The category of finitary Kleisli triples on $\mathscr{A}$ and their homomorphisms is equivalent to the category of finitary monads on $\mathscr{A}$ and their homomorphisms.

Corollary 2.8. The category of finitary monads on $\mathscr{A}$ is coreflective in the category of all monads on $\mathscr{A}$. A coreflection of a monad T is the finitary monad generating the same finitary Kleisli triple as T.

Remark 2.9. From Corollary 2.8 it follows trivially that for a finitary monad R and an arbitrary monad T, given any morphism of the corresponding finitary Kleisli triples, there is a unique extension to a monad homomorphism from R to T.

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

2.3. *Strongly LFP categories*

In the subsequent sections we have several additional requirements on the LFP category we work with, which we now summarise in an (admittedly *ad hoc*) definition. The main point is that Set has all these properties.

Recall that a category with coproducts and pullbacks is said to be *extensive* if every commutative diagram

$$\begin{array}{ccccc} A_0 & \xrightarrow{a_0} & A & \xleftarrow{a_1} & A_1 \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_1} \\ X_0 & \xrightarrow[\mathsf{inl}]{} & X_0 + X_1 & \xleftarrow[\mathsf{inr}]{} & X_1 \end{array}$$

comprises a pair of pullback squares iff the top row is a coproduct diagram. Equivalently, coproducts are disjoint and universal (Carboni *et al.* 1993).

In particular, extensive categories have the property that

coproduct injections are monomorphisms,

and

a coproduct of two monomorphisms is monomorphic.

Recall that an epimorphism e is called *strong* iff it has the diagonal fill-in property with respect to all monomorphisms m (that is, given morphisms u, v with $ve = mu$, there exists d with $u = de$ and $v = md$). *Strong quotients* are quotients carried by strong epimorphisms.

Remark 2.10. Every LFP category has a factorisation of morphisms as strong epimorphisms followed by monomorphisms (Adámek and Rosický 1994, 1.61).

Definition 2.11. A category is called *strongly LFP* if it is LFP and extensive, hom-sets of finitely presentable objects are finite, and a strong quotient of a finitely presentable object is finitely presentable.

Examples 2.12.

(i) Set is strongly LFP.
(ii) Pos and Gra are strongly LFP.
(iii) Every category of relational structures (of any finitary relational signature) is strongly LFP: the finitely presentable objects are those with finitely many elements, and with all relations but finitely many empty.

2.4. *Finitary endofunctors of an LFP category*

It follows from Remark 2.2(iii) above that the category $\mathsf{Fin}[\mathscr{A}, \mathscr{A}]$ of finitary endofunctors of an LFP category $\mathscr{A}$ is equivalent to the functor category $[\mathscr{A}_{fp}, \mathscr{A}]$. From this result it follows that $\mathsf{Fin}[\mathscr{A}, \mathscr{A}]$ is itself an LFP category. We now describe its finitely presentable objects.

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

For any pair A, B of finitely presentable objects of on LFP category $\mathscr{A}$ we use $A \bullet B : \mathscr{A}_{fp} \longrightarrow \mathscr{A}$ to denote the following functor:

$$A \bullet B : X \mapsto \coprod_{\mathscr{A}_{fp}(A,X)} B. \tag{3}$$

These functors were called *step functors* in Adámek (1997) and were proved to be finitely presentable in $[\mathscr{A}_{fp}, \mathscr{A}]$ (see Lemma 1 there).

Proposition 2.13. Every functor F in $[\mathscr{A}_{fp}, \mathscr{A}]$ is a filtered colimit of functors that are finite colimits of step functors.

Proof. See the proof of Theorem 3 in Adámek (1997). □

Definition 2.14. A finitary functor $F : \mathscr{A} \longrightarrow \mathscr{A}$ is called *strongly finitary* if it preserves finitely presentable objects.

Proposition 2.15. For an LFP category $\mathscr{A}$ the following are equivalent:

(i) The category $\mathscr{A}_{fp}$ has finite hom-sets.
(ii) Every finitary endofunctor of $\mathscr{A}$ is a filtered colimit of strongly finitary endofunctors.

Proof. (i) $\Rightarrow$ (ii): Since the category of finitary endofunctors of $\mathscr{A}$ is equivalent to $[\mathscr{A}_{fp}, \mathscr{A}]$, it suffices to show that every functor $F : \mathscr{A}_{fp} \longrightarrow \mathscr{A}$ is a filtered colimit of functors that preserve finitely presentable objects.

By (i), every functor $A \bullet B$ is strongly finitary and a finite colimit of strongly finitary functors is strongly finitary. Now use Proposition 2.13.

(ii) $\Rightarrow$ (i): Suppose that A and B are finitely presentable objects in $\mathscr{A}$ with $\mathscr{A}_{fp}(A, B)$ infinite. We show that the object

$$(A \bullet A)B = \coprod_{\mathscr{A}_{fp}(A,B)} A \tag{4}$$

is not finitely presentable in $\mathscr{A}$, and therefore the functor $A \bullet A$ is not strongly finitary. Since $A \bullet A$ is finitely presentable, it cannot be expressed as a filtered colimit of strongly finitary functors: in fact, $A \bullet A$ would be a retract of one of them, but retracts of strongly finitary functors are strongly finitary.

We prove that $\coprod_J A$ is not finitely presentable for any infinite set J. In fact, $\coprod_J A$ is a filtered colimit of $\coprod_I A$ for all finite sets $I \subseteq J$. If $\coprod_J A$ were finitely presentable, some of the colimit maps, that is, some canonical map

$$c_I : \coprod_I A \longrightarrow \coprod_J A \qquad (I \text{ finite})$$

would be a split epimorphism (because the identity of $\coprod_J A$ would factor through c_I, by definition of finite presentability). However, c_I is not an epimorphism: choose distinct morphisms $f, g \in \mathscr{A}_{fp}(A, B)$ and let $\overline{f}, \overline{g} : \coprod_J A \longrightarrow B$ be morphisms whose components are equal for all indices in I, but whose j-components (for some $j \in J \setminus I$) are f and g, respectively. Then $\overline{f} \neq \overline{g}$ but $\overline{f} c_I = \overline{g} c_I$. □

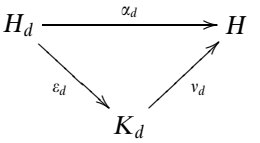
https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

Corollary 2.16. Suppose that $\mathscr{A}$ is a strongly LFP category. Every finitary endofunctor H preserving monomorphisms can be expressed as a filtered colimit of strongly finitary functors preserving monomorphisms.

Proof. Use Proposition 2.15 and express H as a filtered colimit

$$H = \operatorname*{colim}_d H_d$$

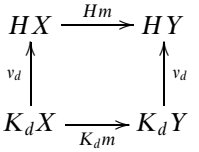
of strongly finitary functors and use $\alpha_d : H_d \longrightarrow H$ to denote a colimit cocone. The category $\mathsf{Fin}[\mathscr{A}, \mathscr{A}] \simeq [\mathscr{A}_{fp}, \mathscr{A}]$ of finitary endofunctors of $\mathscr{A}$ is LFP, therefore, it has (StrongEpi, Mono)-factorisations of morphisms, see Remark 2.10. Moreover, both strong epimorphisms and monomorphisms are defined component-wise in any functor category. We can factor each $\alpha_d : H_d \longrightarrow H$ in $\mathsf{Fin}[\mathscr{A}, \mathscr{A}]$ as a strong epimorphism followed by a monomorphism:



Since each ε_d is pointwise a strong epimorphism, every functor K_d is strongly finitary: given a finitely presentable object X, we have K_dX is a strong quotient of a finitely presentable object H_dX (we know that H_d is strongly finitary), thus, K_dX is finitely presentable.

It is easy to see that the cocone $(v_d : K_d \longrightarrow H)$ is a filtered colimit, this follows, once again from the fact that $\varepsilon_d : H_d \longrightarrow K_d$ are (strong) epimorphisms.

It remains to show that every functor K_d preserves monomorphisms. Let $m : X \longrightarrow Y$ be a monomorphism. From commutativity of the square



it follows that $(v_d)_Y \cdot K_dm$ is a monomorphism, thus, K_dm is a monomorphism. □

3. Some properties of solutions

Assumptions 3.1. We use H to denote a finitary, monomorphism preserving endofunctor of a strongly LFP category $\mathscr{A}$.

Notice that every finitary functor is iteratable, which implies (as in the case $\mathscr{A} = \mathsf{Set}$ mentioned in Section 1.2 above) that final coalgebras

$$TX$$

of $H(_) + X$ form a 'completely iterative' monad T (Aczel *et al.* 2001). As in Section 1.2, T is a coproduct $T = HT + Id$ with injections

$$\tau : HT \longrightarrow T \quad \text{and} \quad \eta : Id \longrightarrow T.$$

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

Notation 3.2. Put

$$\tau^* \equiv H \xrightarrow{H\eta} HT \xrightarrow{\tau} T.$$

Observe that τ, η and τ^* have monomorphic components (because $\mathscr{A}$ has monomorphic coproduct injections, and H preserves monomorphisms).

The concept of a flat system of equations as introduced in the Introduction (see Sections 1.1 and 1.3) immediately generalises as follows.

Definition 3.3. We define a *flat equation morphism* to be a morphism

$$e : X \longrightarrow HX + Y,$$

that is, a coalgebra of $H(_) + Y$. It is called *finitary* if X is finitely presentable.

Remark 3.4. Every flat equation morphism $e : X \longrightarrow HX + Y$ yields a guarded equation morphism in the sense of Section 1.3 by composing with the following monomorphism

$$m_{X,Y} \equiv HX + Y \xrightarrow{\tau^* + Y} TX + Y \xrightarrow{[T\mathsf{inl}, \eta_{X+Y} \cdot \mathsf{inr}]} T(X + Y).$$

We also write, whenever there is no danger of confusion,

$$e^\dagger : X \longrightarrow TY$$

for the unique solution of the corresponding guarded equation. Thus $e^\dagger$ is the unique morphism such that the square

$$\begin{array}{ccc} X & \xrightarrow{e^\dagger} & TY \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle \mu_Y} \\ HX + Y & & \\ {\scriptstyle m_{X,Y}}\downarrow & & \\ T(X + Y) & \xrightarrow[T[e^\dagger, \eta]]{} & T^2 Y \end{array}$$

commutes. The following proposition was proved in Aczel *et al.* (2002).

Proposition 3.5 (Solution is Corecursion). For every flat equation morphism $e : X \longrightarrow HX + Y$, the (unique) solution $e^\dagger : X \longrightarrow TY$ is precisely the unique homomorphism of the coalgebra $e : X \longrightarrow HX + Y$ into the final coalgebra TY of $H(_) + Y$.

Remark 3.6. We also need a simple result concerning general iterative monads, see Definitions 1.13 and 1.16.

Let S be an iterative monad. An equation morphism $e : X \longrightarrow S(X + Y)$ that factors through σ_{X+Y} is, of course, guarded. Moreover, the unique solution $e^\dagger : X \longrightarrow SY$ factors through σ_Y : if $e = \sigma_{X+Y} \cdot e'$, then

$$e^\dagger = \mu_Y \cdot S[e^\dagger, \eta_Y] \cdot \sigma_{X+Y} \cdot e' = \sigma_Y \cdot (\mu'_Y \cdot S'[e^\dagger, \eta_Y] \cdot e').$$

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

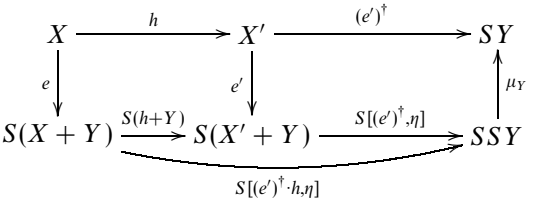
Lemma 3.7. Let S be an iterative monad and let $e : X \longrightarrow S(X + Y)$ and $e' : X' \longrightarrow S(X' + Y)$ be guarded equation morphisms. For every morphism $h : X \longrightarrow X'$ such that the square

$$\begin{array}{ccc} X & \xrightarrow{e} & S(X+Y) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle S(h+Y)} \\ X' & \xrightarrow[e']{} & S(X'+Y) \end{array}$$

commutes, we have

$$e^\dagger = (e')^\dagger \cdot h.$$

Proof. The diagram



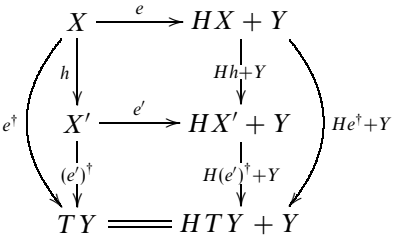
obviously commutes. □

Remark 3.8.

(i) Returning to flat equation morphisms, the corresponding statement here is a direct corollary of Proposition 3.5: given flat equation morphisms $e : X \longrightarrow HX + Y$ and $e' : X' \longrightarrow HX' + Y$, we have, for every homomorphism $h : X \longrightarrow X'$ of those coalgebras (of $H(_) + Y$), that

$$e^\dagger = (e')^\dagger \cdot h.$$

(ii) In particular, when both $e^\dagger$ and $(e')^\dagger$ happen to be monomorphisms (that is, subobjects of TY), then the existence of a homomorphism $h : e \longrightarrow e'$ implies $e^\dagger \subseteq (e')^\dagger$. The converse is also true: if $e^\dagger \subseteq (e')^\dagger$, then the unique $h : X \longrightarrow X'$ with $e^\dagger = (e')^\dagger h$ is a homomorphism. In fact, consider the following diagram:



The outer and lower squares commute by Proposition 3.5. Hence the upper square commutes when extended by $H(e')^\dagger + Y$, which is a monomorphism since H preserves monomorphisms (by assumption) and monomorphisms are closed under coproducts. Thus $h : e \longrightarrow e'$ is a homomorphism, as desired.

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

4. Rational monad – strongly finitary case

Assumptions 4.1. Throughout this section, $\mathscr{A}$ denotes a strongly LFP category, and H an endofunctor that preserves monomorphisms and is *strongly finitary*, that is, not only preserves filtered colimits but also finite presentability (if X is a finitely presentable object, then so is HX).

In the next section we extend the results to all finitary endofunctors preserving monomorphisms.

Recall from Remark 2.10 that the category $\mathscr{A}$ has (StrongEpi, Mono)-factorisations of morphisms. We use $\mathrm{im}(f)$ to denote the monomorphic part of the factorisation of f.

The following definition generalises the definition of the algebra $R_\Sigma Y$ of all rational trees as solutions of systems of finitary flat equations, see Section 1.1.

Definition 4.2. For every finitely presentable object Y of $\mathscr{A}$, we define

$$RY = \bigcup \mathrm{im}(e^\dagger) \qquad (e \text{ a finitary flat equation morphism})$$

that is, we define an object RY together with a monomorphism

$$\varepsilon_Y : RY \longrightarrow TY$$

to be the subobject of TY that is the union of images of all solutions $e^\dagger : X \longrightarrow TY$ of finitary flat equation morphisms $e : X \longrightarrow HX + Y$ (X finitely presentable).

Remark 4.3. Explicitly, for every finitary flat equation morphism $e : X \longrightarrow HX + Y$, we factor $e^\dagger : X \longrightarrow TY$ as a strong epimorphism $k : X \longrightarrow X'$ followed by a monomorphism $m : X' \longrightarrow TY$. Then ε_Y is the union of all those monomorphisms m. Union here means just the supremum in the lattice of subobjects, but this is actually a colimit, as we see below.

Observe that each of the images m is, itself, a solution of a finitary flat equation morphism $e' : X' \longrightarrow HX' + Y$. In fact, since finitely presentable objects are closed under strong quotients, X' is finitely presentable, and, since Hm is a monomorphism (by assumption on H), so is $Hm + Y$ (since coproducts are extensive, and thus monomorphisms are stable under finite coproducts, see Section 2.3). Thus we can use the diagonal fill-in property to find a coalgebra structure e' on X' such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{e} & HX + Y \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle Hk+Y} \\ X' & \xrightarrow{e'} & HX' + Y \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle Hm+Y} \\ TY & = & HTY + Y \end{array}$$

commutes. Now m is a coalgebra homomorphism, and it follows from Proposition 3.5 that $m = (e')^\dagger$.

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

Notation 4.4. For every finitely presentable object Y of $\mathscr{A}$, we use

$$\mathsf{EQ}_Y$$

to denote the category of all finitary flat equations with the parameter object Y. That is:

objects are all finitary flat equation morphisms $e : X \longrightarrow HX + Y$ (X in $\mathscr{A}_{fp}$)

and

morphisms are coalgebra homomorphisms with respect to $H(_) + Y$:

$$\begin{array}{ccc} X & \xrightarrow{e} & HX + Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle Hh+Y} \\ X' & \xrightarrow[e']{} & HX' + Y \end{array}$$

The category EQ_Y comes with a forgetful functor into the underlying category $\mathscr{A}$. We denote it by

$$\mathrm{Eq}_Y : \mathsf{EQ}_Y \longrightarrow \mathscr{A}, \qquad (X \xrightarrow{e} HX + Y) \mapsto X.$$

Remark 4.5. Recall that a full subcategory $\mathscr{D}_0$ of a filtered category $\mathscr{D}$ is *cofinal* if every object of $\mathscr{D}$ has a morphism into an object of $\mathscr{D}_0$; it follows that (in every category) colimits of $\mathscr{D}$-diagrams coincide with colimits of the restricted $\mathscr{D}_0$-diagrams.

Proposition 4.6. For every finitely presentable object Y, Eq_Y is a small, filtered diagram, and RY can be defined as a colimit

$$RY = \operatorname{colim} \mathrm{Eq}_Y.$$

Proof. First, notice that EQ_Y is the full subcategory of $\mathsf{Coalg}(H(_) + Y)$ of those coalgebras with carrier from $\mathscr{A}_{fp}$. Since $\mathscr{A}_{fp}$ is small and for each object X of $\mathscr{A}$ there is only a set of morphisms $e : X \longrightarrow HX + Y$, it follows that EQ_Y is a small category. Moreover, observe that EQ_Y is filtered because it is finitely cocomplete: in fact, the category of coalgebras has colimits formed on the level of $\mathscr{A}$, and since finite colimits of finitely presentable objects are finitely presentable, it follows that EQ_Y is closed under finite colimits in $\mathsf{Coalg}(H(_) + Y)$. Remark 4.3 shows that the filtered diagram Eq_Y has a cofinal subdiagram formed by all finitary flat equation morphisms with monomorphic solutions. (In fact, for every object e of EQ_Y we have constructed a morphism $k : e \longrightarrow e'$ in EQ_Y with e' having monomorphic solution m.)

Now we have defined RY as the union of all $(e')^\dagger = m : X' \longrightarrow TY$, and since this union is filtered, it is a colimit of the corresponding filtered diagram (whose morphisms are all existing subobject inclusions, or, equivalently, all morphisms of EQ_Y, see Remark 3.8(ii)); this is true in every LFP category, see Adámek and Rosický (1994, 1.63). That diagram is cofinal in Eq_Y, whence Eq_Y has the same colimit, that is, $RY = \operatorname{colim} \mathrm{Eq}_Y$. □

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

Notation 4.7.

(i) We use

$$e^\sharp : X \longrightarrow RY \quad (\text{for all } e : X \longrightarrow HX + Y \text{ in } \mathsf{EQ}_Y)$$

to denote the colimit cocone of Eq_Y.

(ii) Observe that for any finitely presentable object Y of $\mathscr{A}$, $\mathsf{inr} : Y \longrightarrow HX + Y$ is an object of EQ_Y. We use

$$\eta^R_Y : Y \longrightarrow RY$$

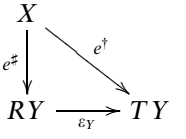
to denote the colimit morphism $\mathsf{inr}^\sharp$.

Example 4.8. Let $H = H_\Sigma$ be a polynomial endofunctor of Set. Then RY is the algebra of all rational trees over Y, and for every finitary flat equation system $e : X \longrightarrow H_\Sigma X + Y$ the colimit morphism $e^\sharp : X \longrightarrow RY$ is a codomain restriction of the solution $e^\dagger : X \longrightarrow TY$.

Remark 4.9. Notice that for every finitely presentable object Y of $\mathscr{A}$ the monomorphism

$$\varepsilon_Y : RY \longrightarrow TY$$

of Definition 4.2 makes the triangles



commutative for all $e : X \longrightarrow HX + Y$ in EQ_Y. This is obvious if $e^\dagger$ is monomorphic; otherwise use the fact established in the proof of Proposition 4.6 that the finitary flat equation morphisms with monomorphic solutions form a cofinal subcategory of EQ_Y.

This makes ε_Y uniquely determined by the equations $\varepsilon_Y e^\sharp = e^\dagger$.

Proposition 4.10. **($RY = HRY + Y$.)** For every finitely presentable object Y of $\mathscr{A}$ the diagram Eq_Y has the colimit

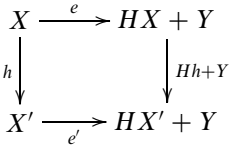
$$\operatorname{colim} \mathrm{Eq}_Y = HRY + Y$$

with the following colimit cocone

$$X \xrightarrow{e} HX + Y \xrightarrow{He^\sharp + Y} HRY + Y \quad (\text{for } e : X \longrightarrow HX + Y \text{ in } \mathsf{EQ}_Y)$$

Proof.

(i) The morphisms $\delta_e = (He^\sharp + Y) \cdot e$ form a cocone of Eq_Y, that is, for every morphism



of EQ_Y we have

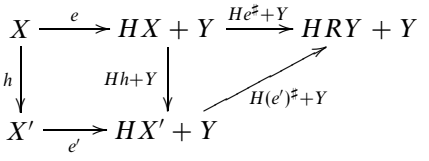
$$\delta_e = \delta_{e'} \cdot h.$$

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

Indeed, since h is a morphism, we know that

$$e^\sharp = (e')^\sharp \cdot h.$$

Therefore, the diagram



commutes.

(ii) In order to complete the proof, we shall show that the unique morphism

$$i : RY \longrightarrow HRY + Y \quad \text{with } i \cdot e^\sharp = \delta_e \text{ (for } e \text{ in } \mathsf{EQ}_Y)$$

is an isomorphism. Define $j : HRY + Y \longrightarrow RY$ as follows. Since H preserves filtered colimits, we have

$$HRY = \operatorname{colim} H\mathrm{Eq}_Y,$$

and therefore

$$HRY + Y = \operatorname{colim}(H\mathrm{Eq}_Y + Y)$$

with colimit cocone $He^\sharp + Y$ (e in EQ_Y). Define for every

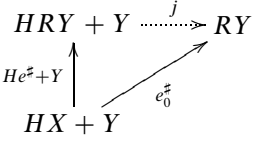
$$e : X \longrightarrow HX + Y \quad \text{in } \mathsf{EQ}_Y$$

a new member

$$e_0 \equiv\ HX + Y \xrightarrow{He+Y} H(HX + Y) + Y$$

of EQ_Y. Note that $HX + Y$ lies in $\mathscr{A}_{fp}$ since H is strongly finitary and $\mathscr{A}_{fp}$ is closed under coproducts.

Define a morphism $j : HRY + Y \longrightarrow RY$ by the commutativity of the following triangles:



for all e in EQ_Y.

The morphism j is well-defined since $e_0^\sharp$ (for e in EQ_Y) form a cocone. In fact, for every morphism $h : e \longrightarrow e'$ in EQ_Y, it is easy to show that $Hh + Y : e_0 \longrightarrow e_0'$ is also a morphism of EQ_Y. But then

$$e_0^\sharp = (e_0')^\sharp \cdot (Hh + Y),$$

as desired.

(ii a) Proof of $j \cdot i = id$: It is our task to show that

$$jie^\sharp = e^\sharp \quad \text{for all } e \text{ in } \mathsf{EQ}_Y.$$

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

For this observe that e is a morphism in EQ_Y from e to e_0. Therefore

$$j \cdot i \cdot e^\sharp = j \cdot (He^\sharp + Y) \cdot e = e_0^\sharp \cdot e = e^\sharp.$$

(ii b) Proof of $i \cdot j = id$: Here we must show that

$$i \cdot j \cdot (He^\sharp + Y) = He^\sharp + Y \quad \text{for all } e \text{ in } \mathsf{EQ}_Y.$$

But this is again easily seen:

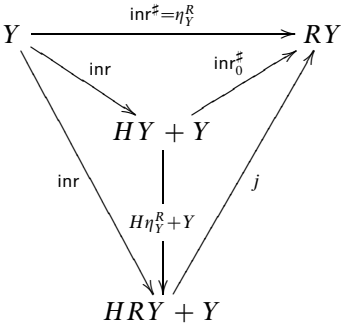
$$i \cdot j \cdot (He^\sharp + Y) = i \cdot e_0^\sharp = (He_0^\sharp + Y) \cdot (He + Y) = He^\sharp + Y$$

where the last step again uses $e : e \longrightarrow e_0$ in EQ_Y. □

Remark 4.11. We have established that

$$RY = HRY + Y \quad (Y \text{ in } \mathscr{A}_{fp})$$

with coproduct injection given by $j \cdot \mathsf{inl}$ and $j \cdot \mathsf{inr}$. Note that $j \cdot \mathsf{inl} = \eta_Y^R : Y \longrightarrow RY$. Indeed, the diagram

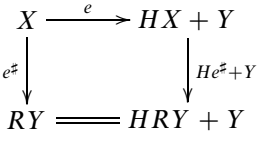


commutes.

Notation 4.12. From now on we shall use for any finitely presentable object Y the following notation for the coproduct injections:

$$\rho_Y : HRY \longrightarrow RY \quad \text{and} \quad \eta_Y^R : Y \longrightarrow RY.$$

Corollary 4.13. Let Y be a finitely presentable object. For every finitary flat equation morphism $e : X \longrightarrow HX + Y$, the diagram



commutes.

Proof. In fact, to show that

$$e^\sharp = \left[\rho_Y, \eta_Y^R\right] \cdot (He^\sharp + Y) \cdot e,$$

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

just use the fact that $[\rho_Y, \eta_Y^R] = j$ and that (since $e : e \longrightarrow e_0$ is a morphism in EQ_Y), $e^\sharp = e_0^\sharp \cdot e$. Thus,

$$e^\sharp = e_0^\sharp \cdot e = \left[\rho_Y, \eta_Y^R\right] \cdot (He^\sharp + Y) \cdot e$$

□

Remark 4.14.

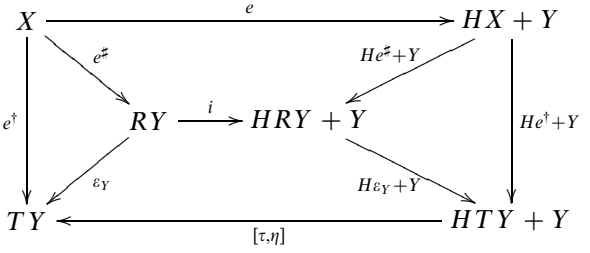
(i) For any finitely presentable object Y, we have

$$\varepsilon_Y = H\varepsilon_Y + Y.$$

More precisely, if $i : RY \longrightarrow HRY + Y$ denotes the isomorphism of Proposition 4.10, then its composite with $H\varepsilon_Y + Y : HRY + Y \longrightarrow HTY + Y = TY$ is equal to ε_Y. This follows from the equalities

$$\varepsilon_Y \cdot e^\sharp = (H\varepsilon_Y + Y) \cdot i \cdot e^\sharp \quad (e \text{ in } \mathsf{EQ}_Y).$$

In fact, consider the following diagrams:



The upper square commutes by definition of i, the two outer triangles by Remark 4.9. The outer square commutes since the solution $e^\dagger$ of e is given by corecursion, see Proposition 3.5. Thus, the lower square commutes, too.

(ii) Consequently, ε_Y is a homomorphism of H-algebras:

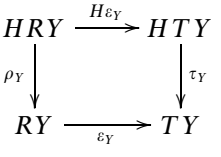
$$\varepsilon_Y \cdot \rho_Y = \tau_Y \cdot H\varepsilon_Y$$

with

$$\varepsilon_Y \cdot \eta_Y^R = \eta_Y. \tag{5}$$

In fact, $\varepsilon_Y \cdot \eta_Y^R = \mathsf{inr}^\dagger$ for $\mathsf{inr} : Y \longrightarrow HY + Y$, and it is easy to verify that $\mathsf{inr}^\dagger = \eta_Y$.

Corollary 4.15. For every finitely presentable object Y, the square



is a pullback.

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

Proof. In fact, since $\mathscr{A}$ is extensive, for every morphism $f : A \longrightarrow B$ the squares

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \mathrm{inl}}\downarrow & & \downarrow{\scriptstyle \mathrm{inl}} \\ A + Y & \xrightarrow[f+Y]{} & B + Y \end{array}$$

are pullbacks. Apply this to $f = H\varepsilon_Y$. □

Theorem 4.16 (Rational Substitution Theorem). For every morphism

$$s : X \longrightarrow RY \quad (X,\ Y \text{ finitely presentable}),$$

there exists a unique extension to a homomorphism

$$\widetilde{s} : RX \longrightarrow RY$$

of H-algebras (that is, a unique homomorphism with $s = \widetilde{s} \cdot \eta^R_Y$).

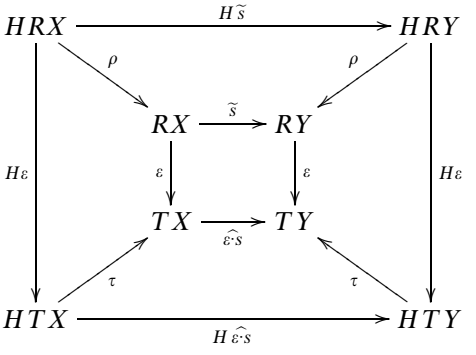
Proof. (I) **Existence:** We are going to find a morphism $\widetilde{s}$ such that the square

$$\begin{array}{ccc} RX & \xrightarrow{\widetilde{s}} & RY \\ {\scriptstyle \varepsilon_X}\downarrow & & \downarrow{\scriptstyle \varepsilon_Y} \\ TX & \xrightarrow[\widehat{\varepsilon_Y \cdot s}]{} & TY \end{array} \tag{6}$$

commutes. It follows from Remark 4.14(ii) and Theorem 1.7 that

$$\varepsilon_Y \cdot \widetilde{s} \cdot \eta^R_Y = \widehat{\varepsilon_Y \cdot s} \cdot \eta_X = \varepsilon_Y \cdot s,$$

which, since ε_Y is a monomorphism by definition, proves that $\widetilde{s}$ extends s. It also follows that $\widetilde{s}$ is a homomorphism of H-algebras: to conclude $\widetilde{s} \cdot \rho_X = \rho_Y \cdot H\,\widetilde{s}$, use the fact that ε_Y is a monomorphism and that the diagram



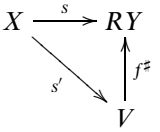
commutes (because the inner square is the above square (6), the outer one is its image and $\widehat{\varepsilon_Y \cdot s}$, ε_X and ε_Y are homomorphisms of H-algebras).

In order to define $\widetilde{s}$, use the fact that $RY = \operatorname{colim} \mathrm{Eq}_Y$ is a filtered colimit and X is finitely presentable. Thus, there exists an object

$$f : V \longrightarrow HV + Y \quad \text{of } \mathsf{EQ}_Y$$

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

such that s factors through its colimit morphism:



This allows us to define the desired morphism $\widetilde{s} : \operatorname{colim} \mathrm{Eq}_X \longrightarrow RY$ by providing a 'suitable' cocone of the diagram Eq_X as follows: Every object of EQ_X, say, $e : Z \longrightarrow HZ + X$, will be turned into an object $\overline{e}$ of EQ_Y by adding V (the domain of f) as new variables: $\overline{e} : Z + V \longrightarrow H(Z + V) + Y$. In more detail, for every object

$$e : Z \longrightarrow HZ + X \quad \text{of } \mathsf{EQ}_X$$

we use

$$\overline{e} : Z + V \longrightarrow H(Z + V) + Y$$

to denote the object of EQ_Y with the following components:

$$\begin{aligned} \overline{e} \cdot \mathsf{inl} \;\equiv\; & Z \xrightarrow{e} HZ + X \xrightarrow{HZ + fs'} HZ + HV + Y \xrightarrow{[H\mathsf{inl}, H\mathsf{inr}] + Y} H(Z + V) + Y \qquad (7) \\ \overline{e} \cdot \mathsf{inr} \;\equiv\; & V \xrightarrow{f} HV + Y \xrightarrow{H\mathsf{inr} + Y} H(Z + V) + Y\,. \end{aligned}$$

We shall show below that the morphisms

$$Z \xrightarrow{\mathsf{inl}} Z + V \xrightarrow{\overline{e}^\sharp} RY \quad (e \text{ in } \mathsf{EQ}_X) \tag{8}$$

form a cocone of Eq_X. This establishes the unique

$$\widetilde{s} : RX \longrightarrow RY \quad \text{with } \widetilde{s} \cdot e^\sharp = \overline{e}^\sharp \cdot \mathsf{inl} \quad (e \text{ in } \mathsf{EQ}_X) \tag{9}$$

This morphism $\widetilde{s}$ fulfills

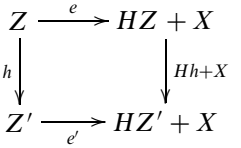
$$\varepsilon_Y \cdot \widetilde{s} \cdot e^\sharp = (\varepsilon_Y \cdot \overline{e}^\sharp) \cdot \mathsf{inl} = \overline{e}^\dagger \cdot \mathsf{inl}. \tag{10}$$

We finally establish that

$$\overline{e}^\dagger \cdot \mathsf{inl} = \widehat{\varepsilon_Y \cdot s} \cdot e^\dagger = \widehat{\varepsilon_Y \cdot s} \cdot \varepsilon_X \cdot e^\sharp, \tag{11}$$

which proves, together with (10), that (6) commutes, since both sides are equal when composed with the injections $e^\sharp$ of the colimit RX. This concludes the proof.

Proof that (8) is a cocone. Suppose that a morphism of EQ_X is given:



https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

We are going to prove the commutativity of the following square:

$$\begin{array}{ccc} Z+V & \xrightarrow{\bar{e}} & H(Z+V)+Y \\ {\scriptstyle h+V}\downarrow & & \downarrow{\scriptstyle H(h+V)+Y} \\ Z'+V & \xrightarrow[\bar{e'}]{} & H(Z'+V)+Y \end{array} \tag{12}$$

That means that $h+V$ is a morphism from $\bar{e}$ to $\bar{e'}$ in EQ_Y, proving $\bar{e}^\sharp = \bar{e'}^\sharp \cdot (h+V)$, which establishes

$$\bar{e}^\sharp \cdot \mathsf{inl} = \bar{e'}^\sharp \cdot \mathsf{inl} \cdot h,$$

as desired.

We consider, in (12), the components of $Z+V$ separately. The equality of the right-hand components of (12) is obvious form the definition of $\bar{e}$ and $\bar{e'}$. For the left-hand components, consider the following squares:

$$\begin{array}{ccccccc} Z & \xrightarrow{e} & HZ+X & \xrightarrow{HZ+fs'} & HZ+HV+Y & \xrightarrow{[H\mathsf{inl},H\mathsf{inr}]+Y} & H(Z+V)+Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle Hh+X} & & & & \downarrow{\scriptstyle H(h+V)+Y} \\ Z' & \xrightarrow[e']{} & HZ'+X & \xrightarrow[HZ'+fs']{} & HZ'+HV+Y & \xrightarrow[[H\mathsf{inl},H\mathsf{inr}]+Y]{} & H(Z'+V)+Y \end{array}$$

The left-hand square commutes by assumption and the right-hand one obviously does. Hence, the outer square commutes too.

Proof of (11). We shall now show that $\bar{e}^\dagger = [\widehat{\varepsilon_Y s} \cdot e^\dagger, f^\dagger]$, which establishes (11) above. By Proposition 3.5, we have to show that the square

$$\begin{array}{ccc} Z+V & \xrightarrow{\bar{e}} & H(Z+V)+Y \\ {\scriptstyle [\widehat{\varepsilon_Y s}\, e^\dagger, f^\dagger]}\downarrow & & \downarrow{\scriptstyle H[\widehat{\varepsilon_Y s}\, e^\dagger, f^\dagger]+Y} \\ TY & = & HTY+Y \end{array} \tag{13}$$

commutes.

We consider the components of the coproduct $Z+V$ separately. For the right-hand component of (13), apply Proposition 3.5 again to obtain

$$f^\dagger \equiv V \xrightarrow{f} HV+Y \xrightarrow{Hf^\dagger+Y} HTY+Y = TY$$

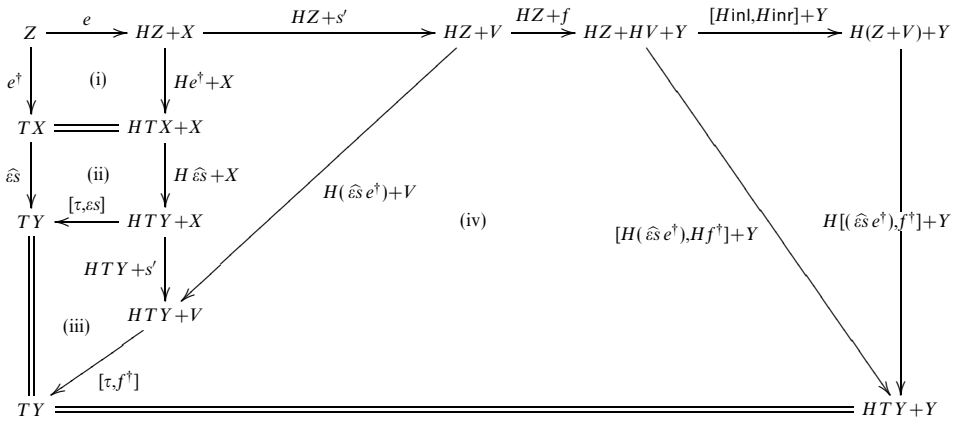
thus, the diagram

$$\begin{array}{ccccc} V & \xrightarrow{f} & HV+Y & \xrightarrow{H\mathsf{inr}+Y} & H(Z+V)+Y \\ {\scriptstyle f^\dagger}\downarrow & & & \searrow^{Hf^\dagger+Y} & \downarrow{\scriptstyle H[\widehat{s}\, e^\dagger, f^\dagger]+Y} \\ TY & = & = & = & HTY+Y \end{array}$$

commutes.

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

For the left-hand component of (13), consider the following commutative diagram:



Square (i) commutes once more by Proposition 3.5. For square (ii), use the definition of $\widehat{(_)}$, see Substitution Theorem 1.7: we have

$$\widehat{\varepsilon_Y s}\,[\tau_X, \eta_X] = [\tau_Y H(\widehat{\varepsilon_Y s}), \varepsilon_Y s] = [\tau_Y, \varepsilon_Y s](H(\widehat{\varepsilon_Y s}) + X).$$

Square (iii) commutes since

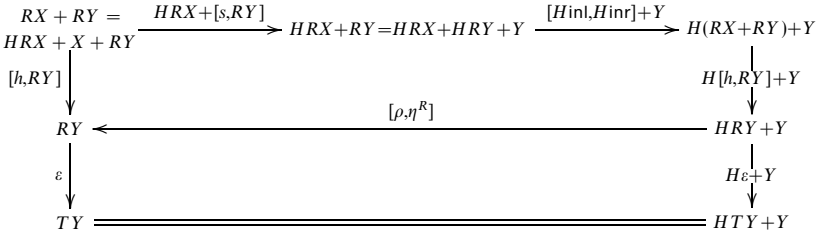
$$\varepsilon_Y \cdot s = \varepsilon_Y \cdot f^{\sharp} \cdot s' = f^{\dagger} \cdot s'. \tag{14}$$

The right-hand component of (iv) commutes by Proposition 3.5, and the left-hand one does trivially. The two remaining triangles obviously commute. Hence the outer square commutes as desired.

(II) **Uniqueness:** We are going to prove that for every homomorphism of H-algebras $h : RX \longrightarrow RY$ extending s, the following identity holds:

$$\varepsilon_Y [h, RY] = \varepsilon_Y [\widetilde{s}, RY]. \tag{15}$$

Since ε_Y is a monomorphism, it follows that $h = \widetilde{s}$. Observe that the diagram



(16)

commutes. In fact, the lower square commutes because $\tau_Y \cdot H\varepsilon_Y = \varepsilon_Y \cdot \rho_Y$ (ε_Y is a homomorphism with $\varepsilon_Y \cdot \eta^R_Y = \eta_Y$ by Remark 4.14(ii)). For the upper square consider the three components of $HRX + X + RY$: the left-hand component commutes because $\rho_Y \cdot Hh = h \cdot \rho_X$ (h is a homomorphism by assumption), the middle component does because h extends s, that is,

$$s = h \cdot \eta^R_X,$$

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

and the right-hand component commutes trivially. Thus, $\varepsilon_Y[h, RY]$ is a homomorphism into the final coalgebra TY. This holds for all H-algebra homomorphisms extending s, and, in particular, for $\widetilde{s}$. By coinduction, we conclude (15). □

Corollary 4.17. We obtain a finitary Kleisli triple $(R, \eta^R, \widetilde{(_)})$ on $\mathscr{A}$.

Proof. In fact, the axioms (i)–(iii) of Definition 2.4 follow immediately from the uniqueness of $\widetilde{(_)}$:

(i) $\widetilde{s} \cdot \eta_X = s$ by definition of $\widetilde{s}$.

(ii) $\widetilde{\eta_X} = id$ because id is a homomorphism extending η_X.

(iii) $\widetilde{r}\,\widetilde{s} = \widetilde{\widetilde{r}\,s}$ because $\widetilde{r}\,\widetilde{s}$ is a homomorphism extending $\widetilde{r}\,s$: from $\widetilde{s}\,\eta_X = s$ derive

$$(\widetilde{r}\,\widetilde{s})\eta_X = \widetilde{r}\,s$$

□

Definition 4.18. The finitary monad R defined by the above finitary Kleisli triple is called a *rational monad* generated by H.

Notation 4.19. The unit of the rational monad is denoted by

$$\eta^R : Id \longrightarrow R,$$

observe that for Y finitely presentable, η^R_Y are the above morphisms $\mathsf{inr}^\sharp$. We further use

$$\mu^R : RR \longrightarrow R$$

to denote the multiplication, given by

$$\mu^R_Y = \widetilde{id_{RY}} : RRY \longrightarrow RY$$

for finitely presentable Y.

The morphisms $\rho_Y : HRY \longrightarrow RY$ for Y finitely presentable in Notation 4.12 yield a natural transformation

$$\rho : HR \longrightarrow R.$$

In fact, $\mathscr{A}$ is a free cocompletion of $\mathscr{A}_{fp}$ under filtered colimits (Adámek and Rosický 1994), therefore the above natural transformation $\rho_Y : HRY \longrightarrow RY$ for $Y \in \mathscr{A}_{fp}$ extends uniquely to $\rho : HR \longrightarrow R$. We have

$$R = HR + Id$$

with the above injections ρ and η^R. Since coproduct injections are monomorphisms, the components of the last two natural transformations are monomorphisms. And since H preserves monomorphisms, the natural transformation

$$\rho^* \equiv H \xrightarrow{H\eta^R} HR \xrightarrow{\rho} R$$

also has monomorphic components. This is analogous to $\tau^* : H \longrightarrow T$ in Notation 3.2.

Remark 4.20. The fact that RY is a filtered union of all images of solutions of finitary flat equations extends from finitely presentable objects Y to all objects. In fact, both of the formulas

$$RY = \bigcup \mathrm{im}(e^\dagger)$$

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

and

$$RY = \operatorname{colim} \mathrm{Eq}_Y$$

extend from finitely presentable objects Y to all objects. Given any object Y in $\mathscr{A}$, we use, again,

$$\mathsf{EQ}_Y$$

to denote the category of all coalgebras of $H(_)+Y$ carried by finitely presentable objects X and by

$$\mathrm{Eq}_Y : \mathsf{EQ}_Y \longrightarrow \mathscr{A}$$

the forgetful functor. This is a filtered diagram and we show that it has colimit

$$RY = \operatorname{colim} \mathrm{Eq}_Y.$$

Express $Y = \operatorname{colim}_{t\in I} Y_t$ as a filtered colimit of finitely presentable objects Y_t with colimit cocone $y_t : Y_t \xrightarrow{t\in I} Y$, $t \in I$. Then each $e : X \longrightarrow HX + Y = \operatorname{colim}_{t\in I}(HX + Y_t)$ factors as

$$X \xrightarrow{e_0} HX + Y_t \xrightarrow{HX+y_t} HX + Y$$

for some $t \in I$. We have $e_0^\sharp : X \longrightarrow RY_t$ as in Notation 4.7(i). Since R is finitary, we have $RY = \operatorname{colim}_{t\in I} RY_t$ with colimit injections $Ry_t : RY_t \longrightarrow RY$. We put

$$e^\sharp \equiv X \xrightarrow{e_0^\sharp} RY_t \xrightarrow{Ry_t} RY$$

This is independent of the above factorisation of e. To verify this, we use the fact that the diagram (Y_t) is filtered: it is sufficient to prove that for every connecting morphism $y_{t,t'} : Y_t \longrightarrow Y_{t'}$, if e_1 is defined by the commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{e_0} & HX + Y_t \\ & \searrow^{e_1} & \downarrow{\scriptstyle HX+y_{t,t'}} \\ & & HX + Y_{t'} \end{array} \tag{17}$$

we have

$$e^\sharp \equiv X \xrightarrow{e_1^\sharp} RY_{t'} \xrightarrow{Ry_{t'}} RY.$$

In fact, since $Ry_{t'} = Ry_{t,t'} \cdot Ry_t$, it is sufficient to prove

$$e_1^\sharp = Ry_{t,t'} \cdot e_0^\sharp. \tag{18}$$

Recall that

$$Ry_{t,t'} = \widetilde{s}$$

for

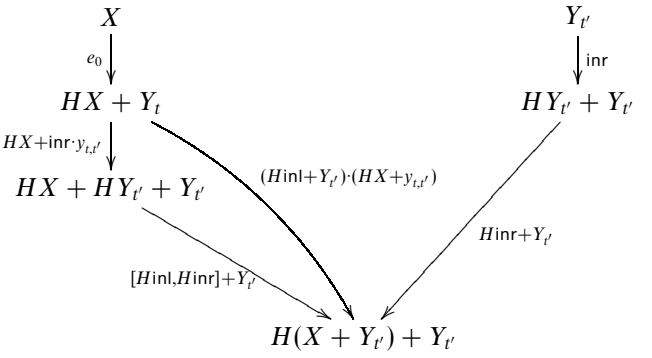
$$s \equiv Y_t \xrightarrow{y_{t,t'}} Y_{t'} \xrightarrow{\eta^R} RY_{t'}.$$

By (9),

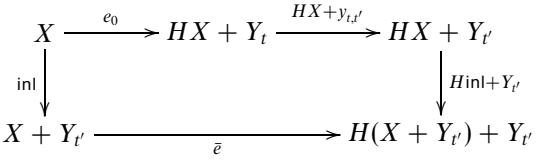
$$Ry_{t,t'} \cdot e_0^\sharp = \bar{e}^\sharp \cdot \mathsf{inl} \tag{19}$$

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

where $\bar{e}$ has, by (7), the following components:



Consequently, the diagram

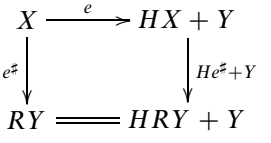


commutes. This proves that inl is a morphism of $\mathsf{EQ}_{Y_{t'}}$, thus, $\bar{e}^\sharp \cdot \mathsf{inl} = e_1^\sharp$, which together with (19) proves (18).

It is not difficult to see that the morphisms $e^\sharp$ form a cocone of Eq_Y. It follows from Proposition 4.6 that this is a colimit cocone. In fact, given a cocone $(c_e : X \longrightarrow C)_{e:X\longrightarrow HX+Y}$ of Eq_Y, we obtain a cocone of Eq_{Y_t} (for each t) by assigning to every $e_0 : X \longrightarrow HX + Y_t$ the morphism c_e, where $e = (HX + y_t)e_0$. This yields a unique $f_t : RY_t \longrightarrow C$, and we obtain a cocone of $(RY_t)_t$, which uniquely factors by way of $f : RY \longrightarrow C$. Then $fe^\sharp = c_e$ and f is unique with this property. Consequently, $RY = \operatorname{colim} \mathrm{Eq}_Y$.

The fact that this implies $RY = \bigcup \mathrm{im}(e^\dagger)$ is proved exactly as in Proposition 4.6.

Lemma 4.21. For every finitary flat equation morphism $e : X \longrightarrow HX + Y$ the diagram



commutes.

Proof. See Corollary 4.13 for finitely presentable objects Y. The extension to arbitrary objects Y (by way of $e^\sharp = Ry_t \cdot e_0^\sharp$ above) is routine. □

Remark 4.22. R is a submonad of T (that is, a subobject in the category of all monads and homomorphisms).

Indeed, we use $c : T_0 \longrightarrow T$ to denote a finitary coreflection of T, see Corollary 2.8. From (5) and (6) above, the morphisms ε_Y form a homomorphisms of finitary

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

Kleisli-triples, or, equivalently, a homomorphism of finitary monads. Hence we obtain a monad homomorphism $\varepsilon = c \cdot \varepsilon_0 : R \longrightarrow T$. It is easy to check that for every finitary flat equation morphism $e : X \longrightarrow HX + Y$ (where Y is any object of $\mathscr{A}$), we have

$$\varepsilon_Y e^{\sharp} = e^{\dagger}.$$

Just use the fact that $HX + Y$ is a filtered colimit of $HX + Y'$ for all $Y' \longrightarrow Y$ in $\mathscr{A}_{fp}/Y$, and since X is finitely presentable, e factors through some $e' : X \longrightarrow HX + Y'$, $Y' \in \mathscr{A}_{fp}$. Thus ε_Y is a monomorphism since Remark 4.20 shows that RY is the union of images of solutions of finitary flat equation morphisms.

We are going to prove that the rational monad is iterative (see Definition 1.16). The main part of this is the following solution theorem, dealing with *rational equation morphisms*, that is, morphisms

$$e : X \longrightarrow R(X + Y) \quad (X \text{ finitely presentable}),$$

which are *guarded*, that is, factor through the coproduct injection of $R(X + Y) = HR(X + Y) + X + Y$:

$$\begin{array}{ccc} X & \xrightarrow{\ e\ } & R(X+Y) \\ & \searrow & \uparrow{\scriptstyle [\rho_{X+Y},\, \eta^R_{X+Y} \cdot \mathsf{inr}]} \\ & & HR(X+Y)+Y \end{array}$$

Theorem 4.23 (Rational Solution Theorem). Every rational equation morphism $e : X \longrightarrow R(X + Y)$ has a unique solution. That is, there exists a unique morphism

$$e^{\ddagger} : X \longrightarrow RY$$

such that the triangle

$$\begin{array}{ccc} X & \xrightarrow{\ e^{\ddagger}\ } & RY \\ {\scriptstyle e}\downarrow & \nearrow_{\widetilde{[e^{\ddagger}, \eta^R_Y]}} & \\ R(X+Y) & & \end{array} \tag{20}$$

commutes.

Proof. (I) **Existence:** Since e is guarded we have a factorisation

$$\begin{array}{ccc} X & \xrightarrow{\ e\ } & R(X+Y) \\ & \searrow^{e_0} & \uparrow{\scriptstyle [\rho, \eta^R \cdot \mathsf{inr}]} \\ & & HR(X+Y)+Y \end{array} \tag{21}$$

From the filtered colimit $RY = \operatorname{colim} \mathrm{Eq}_Y$ we obtain a filtered colimit

$$HR(X + Y) + Y = \operatorname{colim} H\mathrm{Eq}_{X+Y} + Y,$$

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

since $H(_) + Y$ preserves filtered colimits. It follows that e_0, whose domain is finitely presentable, factors through some colimit arrow $Hg^\sharp + Y$, that is, there exists an object

$$g : W \longrightarrow HW + X + Y \quad \text{of } \mathsf{EQ}_{X+Y}$$

and a factorisation

$$\begin{array}{ccc} X & \xrightarrow{e_0} & HR(X+Y)+Y \\ & \searrow^{w} & \uparrow{\scriptstyle Hg^\sharp+Y} \\ & & HW+Y \end{array} \tag{22}$$

This defines an object

$$h \equiv W + X \xrightarrow{[g,\mathsf{inm}]} HW + X + Y \xrightarrow{[\mathsf{inl},w,\mathsf{inr}]} HW + Y \xrightarrow{H\mathsf{inl}+Y} H(W+X)+Y$$

of EQ_Y, since W and X are finitely presentable. (Here, inm denotes the injection into the middle summand.) We define

$$e^\ddagger \equiv X \xrightarrow{\mathsf{inr}} W + X \xrightarrow{h^\sharp} RY$$

and prove that (20) commutes. For that, consider the solution $h^\dagger : W + X \longrightarrow TY$ of h.

Note that since e is guarded for R, the equation morphism

$$\begin{array}{ccccc} X & \xrightarrow{e} & R(X+Y) & \xrightarrow{\varepsilon} & T(X+Y) \\ & \searrow^{e_0} & \uparrow{\scriptstyle [\rho,\eta^R\cdot\mathsf{inr}]} & & \uparrow{\scriptstyle [\tau,\eta\cdot\mathsf{inr}]} \\ & & HR(X+Y)+Y & \xrightarrow[H\varepsilon+Y]{} & HT(X+Y)+Y \end{array} \tag{23}$$

is guarded for T. In fact, the above square commutes, because ε_{X+Y} is a homomorphism with $\varepsilon_{X+Y} \cdot \eta^R_{X+Y} = \eta_{X+Y}$, see Remark 4.14(ii). We show that

$$(\varepsilon_{X+Y} \cdot e)^\dagger = h^\dagger \cdot \mathsf{inr}, \tag{24}$$

that is

$$(\varepsilon_{X+Y} \cdot e)^\dagger = \varepsilon_Y \cdot h^\sharp \cdot \mathsf{inr}.$$

This proves that the outer square of the diagram

$$\begin{array}{ccccc} X & \xrightarrow{e^\ddagger} & RY & \xrightarrow{\varepsilon} & TY \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle \mu^R} & & \uparrow{\scriptstyle \mu} \\ R(X+Y) & \xrightarrow{R[e^\ddagger,\eta^R]} & RRY & \xrightarrow{\varepsilon R} & TRY \\ {\scriptstyle \varepsilon}\downarrow & \nearrow{\scriptstyle T[e^\ddagger,\eta^R]} & & & \searrow{\scriptstyle T\varepsilon} \\ T(X+Y) & & \xrightarrow[T[\varepsilon e^\ddagger,\eta]]{} & & TTY \end{array} \tag{25}$$

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

commutes. Consequently, the upper left-hand square also commutes (which is the desired equality (20)) because the other inner parts of that diagram clearly commute (recall that ε is a monad morphism, see Remark 4.22), and ε is a monomorphism.

Proof of (24). We shall show that

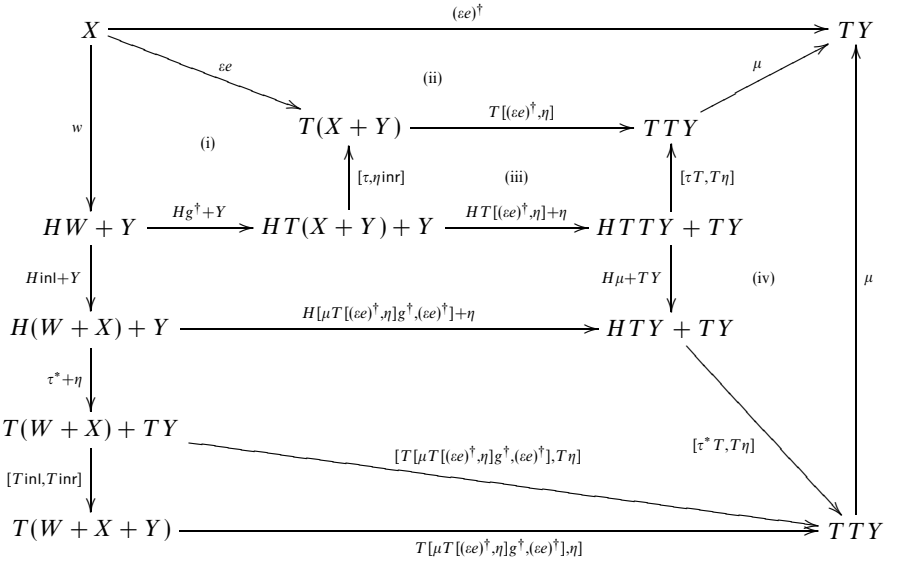
$$h^\dagger = [\mu_Y\, T[(\varepsilon_{X+Y}\, e)^\dagger, \eta_Y]\, g^\dagger, (\varepsilon_{X+Y}\, e)^\dagger],$$

from which the required result follows. Thus it is our task to prove that the diagram

$$\begin{array}{ccc} W+X & \xrightarrow{[\mu_Y\, T[(\varepsilon_{X+Y}\, e)^\dagger, \eta_Y]\, g^\dagger, (\varepsilon_{X+Y}\, e)^\dagger]} & TY \\ {\scriptstyle [T\mathsf{inl}, T\mathsf{inr}]\cdot(\tau^*_{W+X}+\eta_Y)\cdot h}\downarrow & & \uparrow{\scriptstyle \mu_Y} \\ T(W+X+Y) & \xrightarrow[T[\mu_Y\, T[(\varepsilon_{X+Y}\, e)^\dagger, \eta_Y]\, g^\dagger, (\varepsilon_{X+Y}\, e)^\dagger, \eta_Y]]{} & TTY \end{array} \tag{26}$$

commutes.

We consider the components of the coproduct $W + X$ separately. For the right-hand one, we obtain the diagram (27):



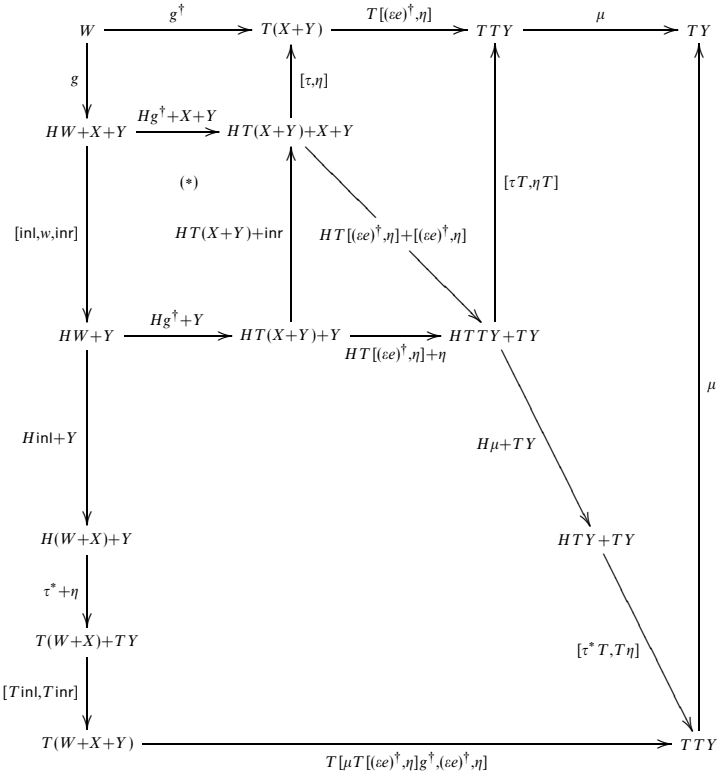
(27)

It commutes: in fact, square (i) commutes by (22) and (23), since $\varepsilon_{X+Y}\, g^\sharp = g^\dagger$. Square (ii) is just the definition of the solution $(\varepsilon_{X+Y}\, e)^\dagger$. Square (iii) is easily seen to commute when the components of $HT(X + Y) + Y$ are considered: for the left-hand components commutativity is obvious; for the right-hand ones we have

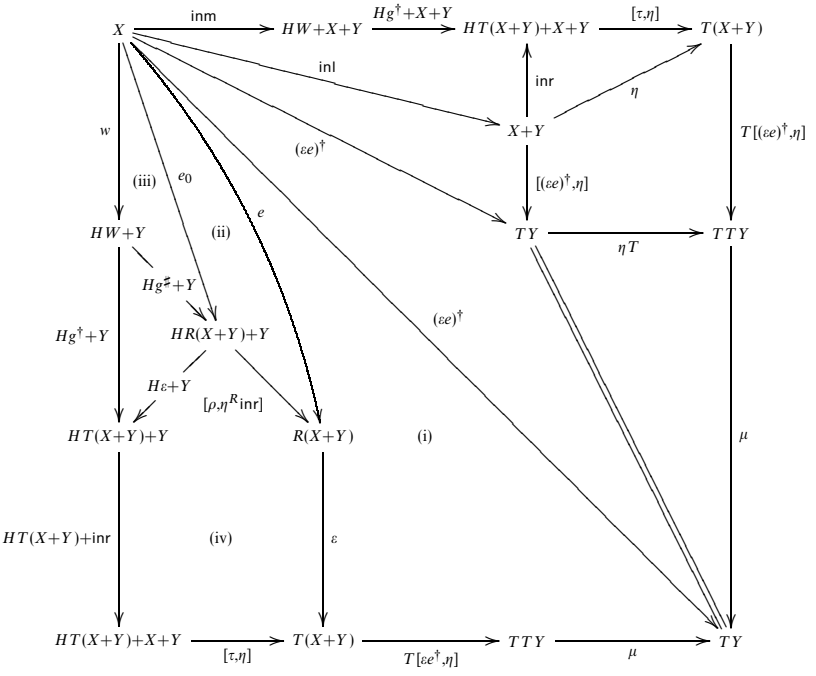
$$T[(\varepsilon_{X+Y}\, e)^\dagger, \eta_Y] \cdot \eta_{X+Y} \cdot \mathsf{inr} = T[(\varepsilon_{X+Y}\, e)^\dagger, \eta_Y] \cdot T\mathsf{inr} \cdot \eta_Y = T\eta_Y \cdot \eta_Y.$$

For (iv), recall that $\mu_Y \cdot \tau^*_{TY} = \tau_Y$, and then use the fact that μ_Y is an H-algebra homomorphism (see Corollary 1.8) in order to see that the left-hand components commute; for the right-hand ones commutativity is obvious. All the other parts of the above diagram clearly commute, and thus the whole diagram does.

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press



(28)



(29)

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

Finally, for the left-hand component of (26), we must show that the outward square of the diagram (28) commutes. Note that the lower part and the right-hand part of this diagram is the same as in the diagram (27), and therefore they commute. The upper left-hand square commutes by Proposition 3.5. Of the remaining three parts, we only need to consider square (∗); the other parts obviously commute. Consider the components of $HW + X + Y$ separately. The first and the last components obviously commute. The middle component need not commute *per se*, but it does when extended by the passage to TY in the upper right-hand corner of (28), which is sufficient for our purposes. In other words, we are to prove that the outward square of the diagram (29) commutes (here, inm denotes the injection into the middle summand).

In fact: (i) commutes by definition of $(_)^\dagger$; (ii) commutes due to (21); (iii) does by (22); (iv) follows from ε_{X+Y} being a homomorphism of H-algebras (see Remark 4.14) and (5). The remaining inner parts commute trivially.

(II) **Uniqueness:** From the Solution Theorem 1.11, we know that $(\varepsilon_{X+Y} e)^\dagger$ is unique, and it is thus sufficient to verify that whenever $e^\ddagger$ is a solution in the sense of (20), we have

$$(\varepsilon_{X+Y} e)^\dagger = \varepsilon_Y e^\ddagger : X \longrightarrow TY.$$

Since ε_Y is a monomorphism by definition, this determines $e^\ddagger$. Thus, we just have to observe the commutativity of diagram (25). □

Remark 4.24. In the above proof we have seen that every rational equation morphism

$$e : X \longrightarrow R(X + Y)$$

reduces to a finitary flat equation morphism in the following sense: there is a finitely presentable object W (of 'additional' variables) and a finitary flat equation morphism

$$h : W + X \longrightarrow H(W + X) + Y$$

whose solution $h^\dagger : W + X \longrightarrow TY$ defines that of e via

$$e^\dagger = h^\dagger \mathsf{inr} : X \longrightarrow TY.$$

Or, equivalently, the rational solution $h^\ddagger : W + X \longrightarrow RY$ of h defines that of e via

$$e^\ddagger = h^\ddagger \mathsf{inr} : X \longrightarrow RY.$$

Corollary 4.25. The rational monad is iterative.

Proof. In fact, R is ideal due to Notation 4.19: we have

$$R = HR + Id$$

with injections ρ and η^R, and for the natural transformation

$$\mu^R : RR \longrightarrow R$$

with components $\mu^R_X = \widetilde{id_{RX}}$, we consider

$$H\mu^R : HRR \longrightarrow HR,$$

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

and find that the appropriate diagram

$$\begin{array}{ccc} HRR & \xrightarrow{H\mu^R} & HR \\ {\scriptstyle \rho R}\downarrow & & \downarrow{\scriptstyle \rho} \\ RR & \xrightarrow[\mu^R]{} & R \end{array}$$

commutes, since μ^R_X (being of the form $\widetilde{(_)}$) is a homomorphism of H-algebras for every finitely presentable X.

Now the Rational Solution Theorem tells us that R is iterative, see Definition 1.16. □

Remark 4.26. Solutions for the monads R and T are closely related:

(i) For every guarded equation morphism $e : X \longrightarrow R(X + Y)$ for R, we have seen in the proof of Theorem 4.23 that $\varepsilon_{X+Y} \cdot e : X \longrightarrow T(X + Y)$ is also guarded, and the rational solution $e^\ddagger$ of e is determined by

$$\varepsilon_Y \cdot e^\ddagger = (\varepsilon_{X+Y} \cdot e)^\dagger.$$

(ii) Conversely, guarded equation morphisms $e : X \longrightarrow T(X + Y)$, which factor through $R(X + Y)$,

$$\begin{array}{ccc} X & \xrightarrow{e} & T(X+Y) \\ & {\scriptstyle e_0}\searrow & \uparrow{\scriptstyle \varepsilon} \\ & & R(X+Y) \end{array}$$

have the property that e_0 is guarded for R, and, therefore, the solution of e is determined by the rational solution of e_0:

$$e^\dagger = \varepsilon_Y \cdot e_0{}^\ddagger.$$

In fact, since e is guarded,

$$\begin{array}{ccc} X & \xrightarrow{e} & T(X+Y) \\ & {\scriptstyle e'}\searrow & \uparrow{\scriptstyle [\tau,\eta\cdot \mathsf{inr}]} \\ & & HT(X+Y)+Y \end{array}$$

we just have to observe that the square

$$\begin{array}{ccc} HR(X+Y)+Y & \xrightarrow{H\varepsilon+Y} & HT(X+Y)+Y \\ {\scriptstyle [\rho,\eta^R\cdot \mathsf{inr}]}\downarrow & & \downarrow{\scriptstyle [\tau,\eta\cdot \mathsf{inr}]} \\ R(X+Y) & \xrightarrow[\varepsilon]{} & T(X+Y) \end{array}$$

is a pullback. To verify this, observe that both the components of $HR(X+Y)+Y$ yield pullbacks: the left-hand one is the pullback in Corollary 4.15, for the right-hand one, use the fact that $\mathscr{A}$ is extensive, see Section 2.3. The universal property of pullbacks

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

yields, because of

$$[\tau_{X+Y}, \eta_{X+Y} \cdot \mathsf{inr}] \cdot e' = e = \varepsilon_{X+Y} \cdot e_0,$$

the existence of $f : X \longrightarrow HR(X+Y)+Y$ with $e' = (H\varepsilon_{X+Y} + Y) \cdot f$ and

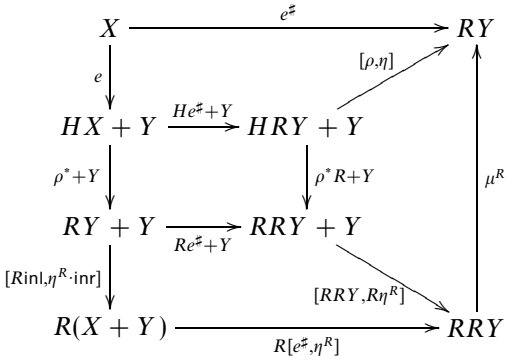
$$e_0 = [\rho, \eta^R \cdot \mathsf{inr}] \cdot f$$

which proves that e_0 is guarded with respect to R.

Remark 4.27. Every finitary flat equation morphism $e : X \longrightarrow HX + Y$ can be identified with the corresponding guarded rational equation morphism

$$X \xrightarrow{e} HX + Y \xrightarrow{\rho^* + Y} RX + Y \xrightarrow{[R\mathsf{inl}, \eta^R \cdot \mathsf{inr}]} R(X+Y).$$

The solution of the latter (in the sense of Theorem 4.23) is simply $e^\sharp$. In fact, the diagram



commutes; see Lemma 4.21 for the upper square.

Example 4.28. Let Σ be a signature with Σ_n finite, for every n, and $\Sigma_n = \emptyset$ for all $n \geqslant n_0$. Let $H = H_\Sigma$ be a (strongly finitary) polynomial functor in Set corresponding to Σ, see Section 1.2. Then TY is the Σ-algebra of finite and infinite Σ-labelled trees. We have constructed RY above by considering all images of solutions of finitary flat equations $e : X \longrightarrow H_\Sigma X + Y$ – that is, RY is the subalgebra of all rational trees in TY.

Thus, in the case of strongly finitary polynomial functors, the rational monad is the monad of rational trees. This was proved in Elgot *et al.* (1978) to be a free iterative monad on H.

Recall the notions of ideal and iterative monad from Definition 1.13 and Definition 1.16, respectively. We show that the rational monad R can be characterised as a free iterative monad, when ideal monad morphisms are taken as the 'right' morphisms.

Definition 4.29. Let S be an ideal monad.

(a) A natural transformation from a functor H to S is called *ideal* if it factors through $\sigma^S : S' \longrightarrow S$. For example, for the rational monad, the above natural transformation

$$\rho^* \equiv H \xrightarrow{H\eta^R} HR \xrightarrow{\rho} R$$

is ideal.

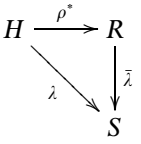
https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

(b) Given another ideal monad $\bar{S}$, a monad morphism $\varphi : S \longrightarrow \bar{S}$ is called *ideal* if it has the form

$$\varphi = \varphi' + id$$

for some natural transformation $\varphi' : S' \longrightarrow \bar{S}'$.

Theorem 4.30. The rational monad R is a free iterative monad on H. That is, given an iterative monad S and an ideal transformation $\lambda : H \longrightarrow S$, there exists a unique ideal monad morphism $\bar{\lambda} : R \longrightarrow S$ for which the triangle



commutes.

Proof. (I) **Existence of** $\bar{\lambda}$: By Remark 2.9, it is sufficient to give the components

$$\bar{\lambda}'_Y : HRY \longrightarrow S'Y$$

of $\bar{\lambda}' : HR \longrightarrow S'$ (such that $\bar{\lambda} = \bar{\lambda}' + id$) for finitely presentable objects Y and to prove that for every arrow $s : Y \longrightarrow RZ$ (Y, Z finitely presentable) the square

$$\begin{array}{ccc} RY & \xrightarrow{\bar{s}} & RZ \\ {\scriptstyle \bar{\lambda}=\bar{\lambda}'+Y}\downarrow & & \downarrow{\scriptstyle \bar{\lambda}=\bar{\lambda}'+Z} \\ SY & \xrightarrow[\widehat{\bar{\lambda}s}]{} & SZ \end{array} \tag{30}$$

commutes (where $\widehat{\bar{\lambda}_Z s} = \mu^S_Z \cdot S(\bar{\lambda}_Z s)$, as usual).

To define $\bar{\lambda}'_Y$, whose domain is

$$HRY = \operatorname{colim} H\mathrm{Eq}_Y,$$

consider an arbitrary object $e : X \longrightarrow HX + Y$ of EQ_Y. Then the following, obviously guarded, equation morphism

$$\lambda \cdot He \equiv HX \xrightarrow{He} H(HX + Y) \xrightarrow{\lambda'} S'(HX + Y) \xrightarrow{\sigma} S(HX + Y) \tag{31}$$

has the unique solution $(\lambda_Y \cdot He)^\dagger : HX \longrightarrow SY$. Since we have, by Remark 3.6, a factorisation through σ_Y, say $(\lambda \cdot He)^\dagger = \sigma_Y \cdot \check{e}$, we can define $\bar{\lambda}'_Y$ by

$$\bar{\lambda}'_Y \cdot He^\sharp = \check{e} \quad (\text{for all } e : X \longrightarrow HX + Y \text{ in } \mathsf{EQ}_Y). \tag{32}$$

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

This is well-defined because all $\check{e}$ form a cocone of $H\mathrm{Eq}_Y$. In fact, for every morphism $h : e \longrightarrow e'$ in EQ_Y, we have a commutative diagram

$$\begin{array}{ccccc} HX & \xrightarrow{He} & H(HX+Y) & \xrightarrow{\lambda} & S(HX+Y) \\ {\scriptstyle Hh}\downarrow & & {\scriptstyle H(Hh+Y)}\downarrow & & \downarrow{\scriptstyle S(Hh+Y)} \\ HX' & \xrightarrow[He']{} & H(HX'+Y) & \xrightarrow[\lambda]{} & S(HX'+Y) \end{array}$$

This implies $(\lambda \cdot e)^\dagger = (\lambda \cdot e')^\dagger \cdot Hh$, see Lemma 3.7. Since σ_Y is a monomorphism, we conclude

$$\check{e} = \check{e}' \cdot Hh,$$

as required.

In order to prove that (30) commutes, we are to show that for every object $e : X \longrightarrow HX + Y$ of EQ_Y, we have

$$\widehat{\bar{\lambda}_Z s}\ \bar{\lambda}_Y e^\sharp = \bar{\lambda}_Z\ \widetilde{s}\ e^\sharp.$$

As in the proof of Theorem 4.16, factor

$$s = f^\sharp s' \tag{33}$$

and define $\bar{e} : Z + V \longrightarrow H(Z+V) + Y$ by (7). That is, $\bar{e}$ has the following components:

$$\bar{e} \cdot \mathsf{inl} \equiv X \xrightarrow{e} HX + Y \xrightarrow{HX + fs'} HX + HV + Z \xrightarrow{[H\mathsf{inl}, H\mathsf{inr}]+Z} H(X+V) + Z$$

and

$$\bar{e} \cdot \mathsf{inr} \equiv V \xrightarrow{f} HV + Z \xrightarrow{H\mathsf{inr}+Z} H(X+V) + Z.$$

Note that for all e in EQ_Y, we have

$$\bar{\lambda}_Y e^\sharp = \left[(\lambda He)^\dagger, \eta_Y^S\right] \cdot e. \tag{34}$$

Indeed, consider the following diagram:

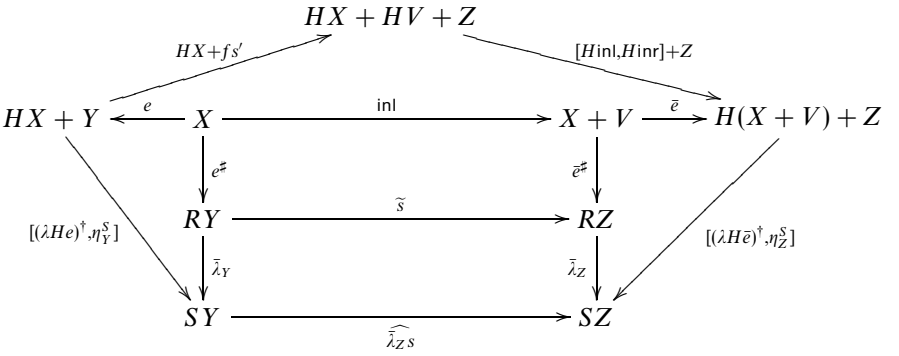
$$\begin{array}{ccccc} X & \xrightarrow{e} & HX+Y & = & HX+Y \\ {\scriptstyle e^\sharp}\downarrow & & {\scriptstyle He^\sharp+Y}\downarrow & & \\ RY & \xrightarrow[i]{\cong} & HRY+Y & & \downarrow{\scriptstyle [(\lambda He)^\dagger, \eta_Y^S]} \\ & \searrow^{\bar{\lambda}_Y} & \downarrow{\scriptstyle [\sigma_Y \bar{\lambda}'_Y, \eta_Y^S]} & & \\ & & SY & = & SY \end{array}$$

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

The upper square commutes by the definition of the isomorphism $i : RY \longrightarrow HRY + Y$ (see Remark 4.11), the lower triangle is the definition of $\bar{\lambda}_Y$ on the components of the coproduct $RY = HRY + Y$, and the right-hand square commutes, since

$$\sigma_Y \cdot \bar{\lambda}'_Y \cdot He^\sharp = (\lambda He)^\dagger.$$

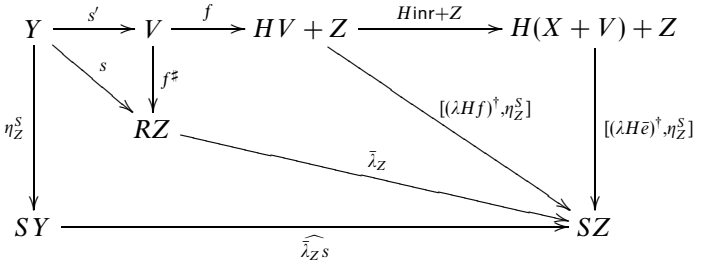
Now consider the following diagram:



(35)

Observe that the upper part commutes by the definition of $\bar{e}$, and the right-hand and left-hand parts commute by (34) for e and $\bar{e}$, respectively. Of the remaining two inner squares, the upper one commutes by definition of $\tilde{s}$, see (9). We shall show below that the outer shape commutes. Thus, the lower square commutes when precomposed with $e^\sharp$, for any e in EQ_Y, which establishes the commutativity of (30).

In order to show that the outer shape of (35) commutes, we consider the components of $HX + Y$ separately. For the right-hand one we obtain the following diagram:



(36)

Observe that all parts of this diagram, except perhaps the right-hand triangle, clearly commute: we have $\bar{\lambda}_Z f^\sharp = [(\lambda Hf)^\dagger, \eta^S_Z] \cdot f$ by (34) and the other two parts are obvious. Now notice that $\mathsf{inr} : f \longrightarrow \bar{e}$ is a morphism in EQ_Z. Therefore the square

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

$$\begin{array}{ccccc} HV & \xrightarrow{Hf} & H(HV+Z) & \xrightarrow{\lambda} & S(HV+Z) \\ {\scriptstyle H\mathsf{inr}}\downarrow & & & & \downarrow{\scriptstyle S(H\mathsf{inr}+Z)} \\ H(X+V) & \xrightarrow[H\bar{e}]{} & H(H(X+V)+Z) & \xrightarrow[\lambda]{} & S(H(X+V)+Z) \end{array}$$

commutes. Hence

$$(\lambda H\bar{e})^\dagger = (\lambda Hf)^\dagger \cdot H\mathsf{inr}$$

by Lemma 3.7, which establishes the commutativity of (36).

For the left-hand component of the outer shape of (35) we are to establish the commutativity of the following diagram:

$$\begin{array}{ccc} HX & \xrightarrow{H\mathsf{inl}} & H(X+V) \\ {\scriptstyle (\lambda He)^\dagger}\downarrow & & \downarrow{\scriptstyle (\lambda H\bar{e})^\dagger} \\ SY & \xrightarrow[\widehat{\bar{\lambda}_Z s}]{} & SZ \end{array} \quad \text{for all } e \text{ in } \mathsf{EQ}_Y \tag{37}$$

In order to prove this, consider the following arrow $h : HX + HV \longrightarrow H(HX + HV + Z)$ defined by

$$h \cdot \mathsf{inl} \equiv HX \xrightarrow{He} H(HX+Y) \xrightarrow{H(HX+fs')} H(HX+HV+Z)$$

and

$$h \cdot \mathsf{inr} \equiv HV \xrightarrow{Hf} H(HV+Z) \xrightarrow{H\mathsf{inr}} H(HX+HV+Z).$$

Note that we have

$$H([H\mathsf{inl}, H\mathsf{inr}] + Z) \cdot h = H\bar{e} \cdot [H\mathsf{inl}, H\mathsf{inr}]. \tag{38}$$

To show the commutativity of (37), we shall prove below that for the solution of the guarded equation morphism $\lambda_{HX+HV+Z} \cdot h$, the following two claims hold:

$$(\lambda h)^\dagger = (\lambda H\bar{e})^\dagger \cdot [H\mathsf{inl}, H\mathsf{inr}] \tag{39}$$

and

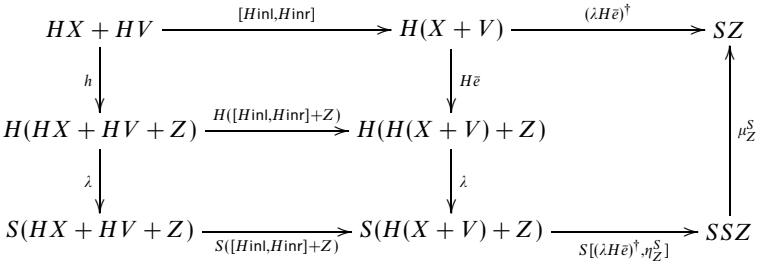
$$(\lambda h)^\dagger = [\widehat{\bar{\lambda}_Z s} \cdot (\lambda He)^\dagger, (\lambda Hf)^\dagger]. \tag{40}$$

Observe that the left-hand components yield (37), which concludes the proof of the existence of the ideal monad morphism $\bar{\lambda} : R \longrightarrow S$.

Proof of (39). Consider the commutative diagram (41). The upper left-hand square is just (38), the lower square commutes by naturality of λ. The right-hand square is the definition of $(\lambda H\bar{e})^\dagger$. Thus $(\lambda H\bar{e})^\dagger \cdot [H\mathsf{inl}, H\mathsf{inr}]$ solves λh, and thus (39) holds.

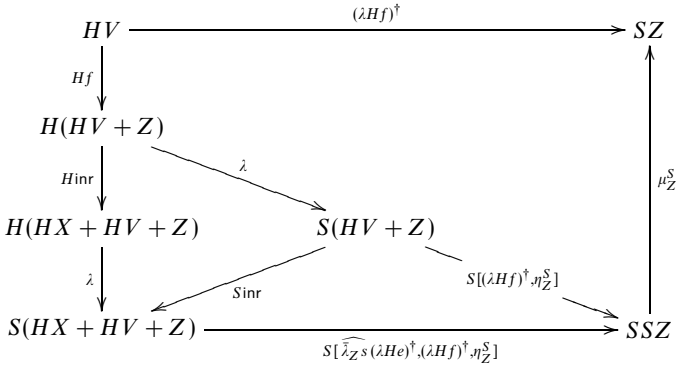
https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

Proof of (40). As for (39), we use the definition of the solution $(\lambda h)^\dagger$, but we consider the



(41)

components of the coproduct $HX+HV$ separately. For the right-hand component of (40), consider the commutative diagram (42):



(42)

For the left-hand component of (40), we analyse the diagram (43). All inner parts of this diagram except, perhaps, the lower one obviously commute. For the lower part, remove S and consider the components of $HX + Y$. For the left-hand one, nothing needs to be shown, the right-hand one commutes since by (33) and (34) we have

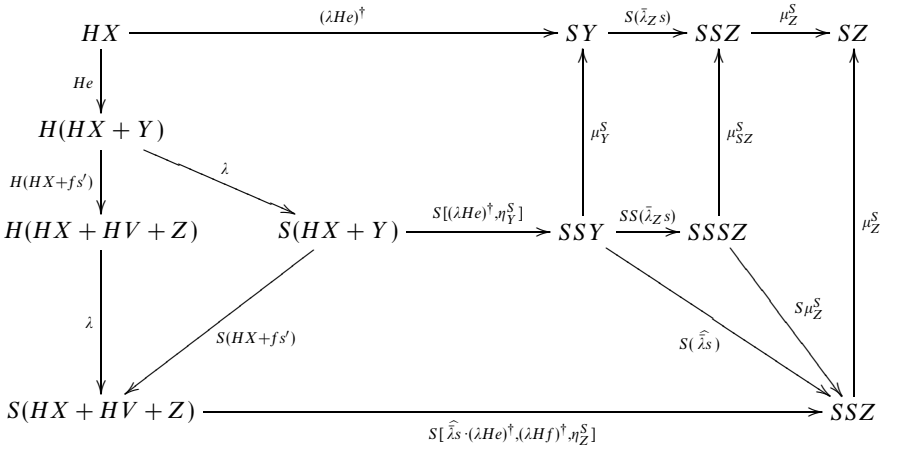
$$\widehat{\bar{\lambda}_Z s} \cdot \eta^S_Y = \bar{\lambda}_Z \cdot s = \bar{\lambda}_Z \cdot f^\sharp \cdot s' = \left[(\lambda Hf)^\dagger, \eta^S_Z\right] \cdot f \cdot s'.$$

Therefore (43) commutes, concluding the proof of (40).

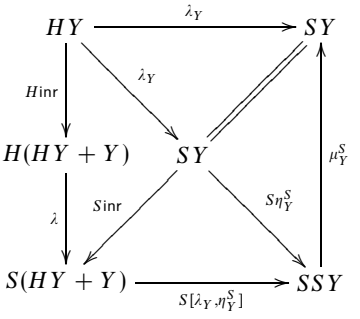
(II) **Equality** $\bar{\lambda} \cdot \rho^* = \lambda$**.** It suffices to check that $\bar{\lambda}_Y \cdot \rho^*_Y = \lambda_Y$ for objects Y in $\mathscr{A}_{fp}$, see Remark 2.2(iii). Notice that for $e = \mathsf{inr} : Y \longrightarrow HY + Y$ we have $(\lambda_{HY+Y} H\mathsf{inr})^\dagger = \lambda_Y$. Indeed, the diagram (44) commutes. That means we can choose $\lambda'_Y : HY \longrightarrow S'Y$ as our $\check{e}$ (see (32)). But now we have

$$\bar{\lambda}'_Y H\eta^R_Y = \bar{\lambda}'_Y H\mathsf{inr}^\sharp = \lambda'_Y.$$

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

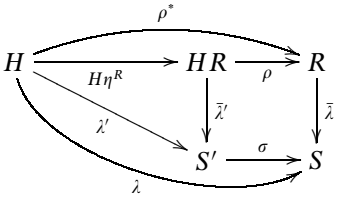


(43)



(44)

Extending by σ_Y, we obtain the desired equation. In fact, the diagram



commutes.

(III) **Uniqueness of $\bar{\lambda}$.** Suppose that $v = v' + id$ is another ideal monad morphism with $v \cdot \rho^* = \lambda$. We have to show that v'_Y satisfies the defining equation of $\bar{\lambda}'_Y$, that is,

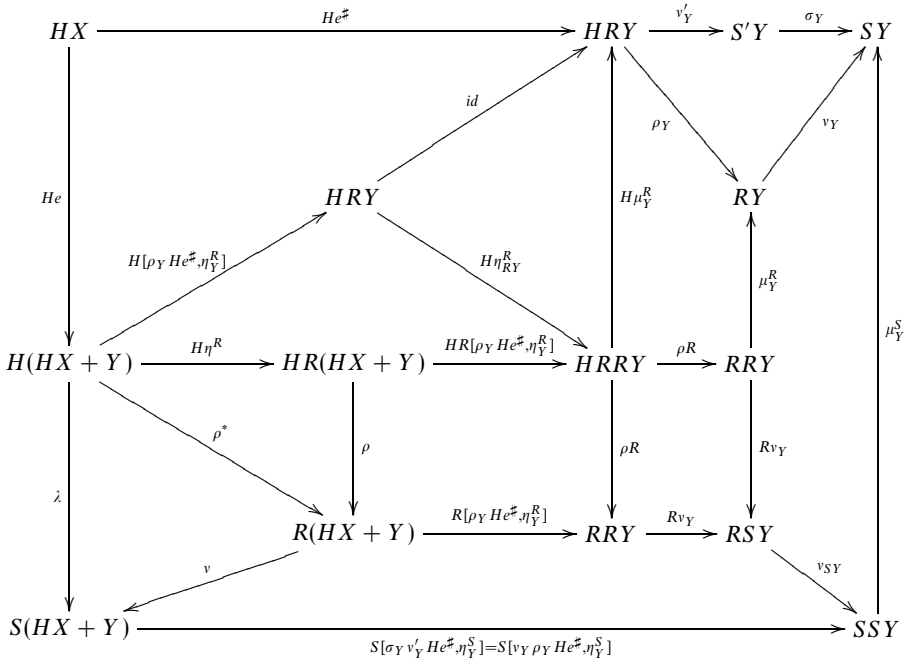
$$v'_Y \cdot He^\sharp = \check{e} : HX \longrightarrow S'Y$$

for all objects Y of $\mathscr{A}_{fp}$ and e of EQ_Y. Since the coproduct injection $\sigma_Y : S'Y \longrightarrow SY$ is monomorphic, it suffices to show that

$$\sigma_Y \cdot v'_Y \cdot He^\sharp = \sigma_Y \cdot \check{e} = (\lambda He)^\dagger. \tag{45}$$

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

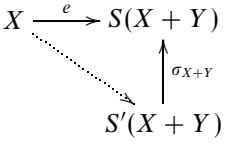
We use the definition of the solution $(\lambda He)^\dagger$ to show that this is indeed the case. All parts of the diagram (46) commute:



(46)

For the upper left-hand triangle, recall from Corollary 4.13 that $[\rho_Y, \eta^R_Y]\cdot(He^\sharp + Y)\cdot e = e^\sharp$, for the lower part use naturality of ν and $\nu_Y \eta^R_Y = \eta^S_Y$, for the rest commutativity is obvious. This completes the proof of (45). □

Remark 4.31. Recall that Elgot defined *iterative theories* by the (seemingly weaker) condition that all *ideal equation morphisms*, that is, all $e : X \longrightarrow S(X + Y)$ that factor through σ_{X+Y},

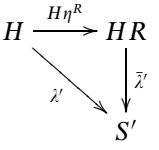


have unique solutions. This, however, does not influence the above result. That is, the rational monad is also a free iterative monad in the sense of Elgot. This follows from the above proof: the only guarded equation we used for the iterative monad S was the equation morphism $\lambda \cdot He$, see (31), and that is, indeed, ideal.

Remark 4.32. Our definition of ideal monad (as a monad S of the form $S = S' + Id$) rests on our assumption that coproduct injections are monomorphic in $\mathscr{A}$. Without this assumption, we would have to define an ideal monad by means of a functor $S' : \mathscr{A} \longrightarrow \mathscr{A}$ and natural transformations $\mu' : S'S \longrightarrow S'$ such that the functor $S = S' + Id$ together with the natural transformations $\mathsf{inr} : Id \longrightarrow S$ and $\mu \equiv SS = S'S + S \xrightarrow{[\mu', S]} S' + Id = S$ is

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

a monad. In that sense it is important to note that in the above theorem we actually proved the following:

> For every natural transformation $\lambda' : H \longrightarrow S'$, there exists a unique natural transformation $\bar{\lambda}' : HR \longrightarrow S'$ such that $\bar{\lambda}' + id$ is an ideal monad homomorphism and the triangle
>
> 
>
>
> commutes.

In fact, both the uniqueness, and the naturality of $\bar{\lambda}'_X : HRX \longrightarrow S'X$ follow from the uniqueness and naturality of $\bar{\lambda}_X : RX \longrightarrow SX$ (again through our assumptions that the coproduct injections ρ_X and σ_X are monomorphisms).

5. Rational monad – finitary case

Assumptions 5.1. Throughout this section $\mathscr{A}$ denotes a strongly LFP category and H a finitary endofunctor preserving monomorphisms.

We are going to prove that, once again, the rule

$$RY = \bigcup \mathrm{im}(e^\dagger)$$

defines a free iterative monad on H, called the rational monad of H.

Remark 5.2. We know from Corollary 2.16 that H is a directed colimit of strongly finitary functors preserving monomorphisms:

$$H = \operatorname*{colim}_{i \in I} H_i \quad (I \text{ a directed poset}).$$

We use

$$\alpha_{i,j} : H_i \longrightarrow H_j \quad (i \leqslant j)$$

to denote the connecting morphisms and

$$\beta_i : H_i \longrightarrow H \quad (i \in I)$$

to denote the colimit cocone.

Let

$$\rho_i^* : H_i \longrightarrow R_i \quad (i \in I)$$

be the rational monad of H_i as constructed in Section 4. For each $i \leqslant j$ we have an ideal natural transformation

$$H_i \xrightarrow{\alpha_{i,j}} H_j \xrightarrow{\rho_j^*} R_j$$

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

for which there exists, by Theorem 4.30, a unique extension to an ideal monad homomorphism $\bar{\alpha}_{i,j} : R_i \longrightarrow R_j$:

$$\begin{array}{ccc} H_i & \xrightarrow{\alpha_{i,j}} & H_j \\ {\scriptstyle \rho_i^*}\downarrow & & \downarrow{\scriptstyle \rho_j^*} \\ R_i & \xrightarrow[\bar{\alpha}_{i,j}]{} & R_j \end{array}$$

The unicity makes it clear that we obtain a directed diagram of iterative monads $(R_i \mid i \in I)$ (with connecting morphisms $\bar{\alpha}_{i,j}$), and we put

$$R = \operatorname*{colim}_{i \in I} R_i.$$

That is, we define an endofunctor R of $\mathscr{A}$ as a colimit (object-wise) of the endofunctors R_i, that is, $RY = \operatorname*{colim}_{i \in I} R_i Y$ for all Y in $\mathscr{A}$. Next we prove that a colimit of a filtered diagram of iterative monads is an iterative monad (Proposition 5.5). An easy consequence is that the above monad R is actually a free iterative monad on H. Finally, we prove that $RY = \operatorname{colim} \mathrm{Eq}_Y$.

Proposition 5.3 (Aczel *et al.* 2002). Ideal monad homomorphisms $\lambda : S \longrightarrow \bar{S}$ between iterative monads preserve solutions. That is, if $e : X \longrightarrow S(X + Y)$ is guarded for S and has the solution $e^\dagger : X \longrightarrow SY$, then $\lambda_{X+Y} \cdot e$ is guarded for $\bar{S}$ and has the solution $\lambda_Y \cdot e^\dagger$ with respect to $\bar{S}$.

Corollary 5.4. For all $i \leqslant j$ in I and any finitary flat equation $f : X \longrightarrow H_i X + Y$, put

$$g \equiv X \xrightarrow{f} H_i X + Y \xrightarrow{\alpha_{i,j}+Y} H_j X + Y.$$

Then the solution of f with respect to R_i denoted by $f_i^\sharp : X \longrightarrow R_i Y$ (see Remark 4.27), is related to the solution g_j of g with respect to R_j by

$$\begin{array}{ccc} & X & \\ {\scriptstyle f_i^\sharp}\swarrow & & \searrow{\scriptstyle g_j^\sharp} \\ R_i Y & \xrightarrow[\bar{\alpha}_{i,j}]{} & R_j Y \end{array}$$

Proof. Apply Proposition 5.3 to $\lambda = \bar{\alpha}_{i,j}$. □

Proposition 5.5. For every directed collection of iterative monads S_i ($i \in I$, I a directed poset) and ideal monad morphisms $\lambda_{i,j} : S_i \longrightarrow S_j$ ($i \leqslant j$ in I), there is a unique structure of an iterative monad on the functor

$$S = \operatorname*{colim}_{i \in I} S_i$$

turning the colimit morphisms into ideal monad homomorphisms.

Remark. The above colimit is, of course, formed objectwise: we define SX by $SX = \operatorname*{colim}_{i \in I} S_i X$ for all objects X in $\mathscr{A}$, and analogously for morphisms.

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

Proof. (I) We are given ideal monads (S_i, η_i, μ_i) with appropriate $\sigma_i : S_i' \longrightarrow S_i$ and $\mu_i' : S_i' S_i \longrightarrow S_i'$ for $i \in I$ and we form a colimit

$$(\lambda_i : S_i \longrightarrow S \mid i \in I)$$

in the functor category $[\mathscr{A}, \mathscr{A}]$. It is proved in Section 4 of Kelly and Power (1993) that there is a unique structure (S, η, μ) of a monad on the functor S such that all the transformations λ_i are monad homomorphisms, forming a colimit cocone in the category of all finitary monads on $\mathscr{A}$.

(II) (S, η, μ) is an ideal monad. In fact, we have a directed diagram of all S_i', $i \in I$ and all $\lambda_{i,j}' : S_i' \longrightarrow S_j'$, $i \leqslant j$, such that

$$S_i = S_i' + Id$$

and

$$\lambda_{i,j} = \lambda_{i,j}' + id$$

for all $i \leqslant j$ in I. (To show that $(\lambda_{i,j}')$ form a directed diagram, use the fact that $(\lambda_{i,j})$ do and that all σ_i are monomorphisms, being coproduct injections of $S_i = S_i' + Id$.) Put

$$S' = \operatorname*{colim}_{i \in I} S_i'$$

with the colimit cocone denoted by $\lambda_i' : S_i' \longrightarrow S'$, $i \in I$. Define $\sigma : S' \longrightarrow S$ as the unique natural transformation such that diagrams

$$\begin{array}{ccc} S' & \xrightarrow{\sigma} & S \\ {\scriptstyle \lambda_i'}\uparrow & & \uparrow{\scriptstyle \lambda_i} \\ S_i' & \xrightarrow[\sigma_i]{} & S_i \end{array}$$

commute for every $i \in I$. (This is, once more, forced on us by the requirement that λ_i is an ideal monad homomorphism.)

Since the diagonal functor $I \longrightarrow I \times I$ is cofinal, the composite $S'S$ is a colimit of the diagram of $S_i' S_i$, $i \in I$, with connecting morphisms

$$S_i' S_i \xrightarrow{S_i' \lambda_{i,j}} S_i' S_j \xrightarrow{\lambda_{i,j}' S_j} S_j' S_j \quad \text{for } i \leqslant j$$

and colimit cocone

$$S_i' S_i \xrightarrow{S_i' \lambda_i} S_i' S \xrightarrow{\lambda_i' S} S'S \quad \text{for } i \in I.$$

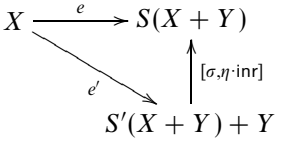
Then $\mu' : S'S \longrightarrow S'$ is defined by commutativity of the squares

$$\begin{array}{ccc} S'S & \xrightarrow{\mu'} & S' \\ {\scriptstyle \lambda_i' S}\uparrow & & \uparrow{\scriptstyle \lambda_i'} \\ S_i' S & & \\ {\scriptstyle S_i' \lambda_i}\uparrow & & \\ S_i' S_i & \xrightarrow[\mu_i']{} & S_i' \end{array}$$

for all $i \in I$. The verification that (S, η, μ) is an ideal monad is quite mechanical.

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

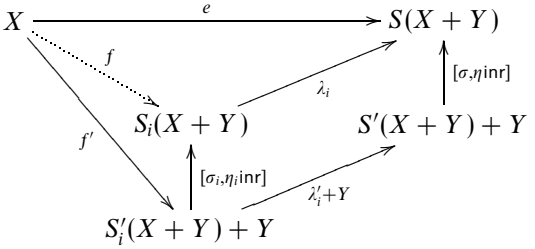
(III) (S, η, μ) is iterative. In fact, let



be a guarded equation morphism. Since X is a finitely presentable object, e' factors through one of the colimit morphisms of the filtered colimit

$$S'(X+Y)+Y = \operatorname*{colim}_{i\in I} S'_i(X+Y)+Y$$

with colimit cocone formed by $(\lambda'_i)_{X+Y} + Y$, $i \in I$. That is, we have $i \in I$ and f' such that the diagram



(47)

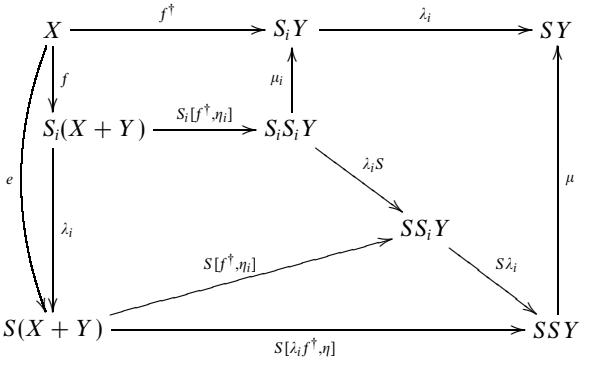
commutes. This defines a guarded equation morphism $f : X \longrightarrow S_i(X+Y)$ for S_i. We use $f^\dagger : X \longrightarrow S_i Y$ to denote the unique solution with respect to S_i. Then the composite

$$X \xrightarrow{f^\dagger} S_i Y \xrightarrow{\lambda_i} SY$$

is a solution of e with respect to S. In fact, the diagram (48) obviously commutes. It remains to show that solutions are unique for S. In fact, let

$$h : X \longrightarrow SY$$

be a solution of e. We prove $h = \lambda_i \cdot f^\dagger$. Observe first that the solution $\lambda_i \cdot f^\dagger$ is independent of the above factorisation $e' = ((\lambda'_i)_{X+Y} + Y) \cdot f'$. In fact, choose any other factorisation,



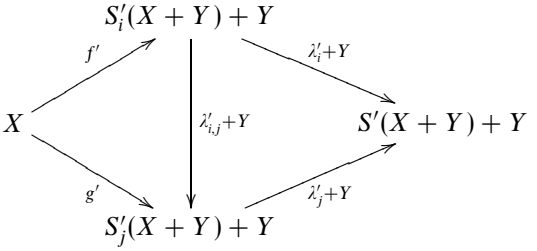
(48)

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

through $j \in I$:

$$e' = ((\lambda'_j)_{X+Y} + Y) \cdot g' \quad \text{for some } g' : X \longrightarrow S'_j(X+Y) + Y.$$

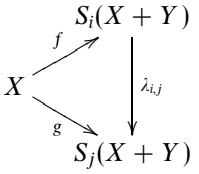
Since I is directed, we can assume the existence of some $(\lambda'_{i,j})_{X+Y} : S'_i(X+Y) \longrightarrow S'_j(X+Y)$ such that the diagram



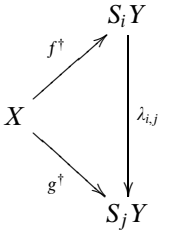
(49)

commutes.

Put $g = [(\sigma_j)_{X+Y}, (\eta_j)_{X+Y} \cdot \mathsf{inr}] \cdot g'$, analogously to (47). Notice that the triangle



commutes. By Proposition 5.3 applied to $\lambda = \lambda_{i,j}$, the triangle



commutes, where, $g^\dagger : X \longrightarrow S_jY$ denotes the unique solution with respect to S_j. This proves

$$(\lambda_j)_Y \cdot g^\dagger = (\lambda_j)_Y \cdot (\lambda_{i,j})_Y f^\dagger = (\lambda_i)_Y \cdot f^\dagger, \tag{50}$$

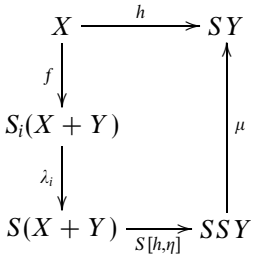
as desired.

We are ready to prove that solutions for S are unique. Let

$$h : X \longrightarrow SY$$

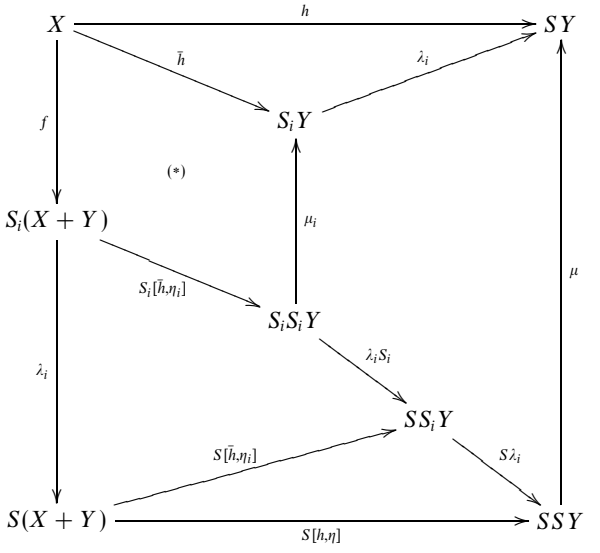
https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

be a solution, that is, let the diagram



(51)

commute. We can factor h through one of the colimit maps of the filtered colimit defining SY; without loss of generality we can assume that it is $(\lambda_i)_Y$ for the given $i \in I$ (recall the independence of $(\lambda_i)_Y \cdot f^\dagger$ of the given factorisation of e). We obtain the diagram (52):



(52)

The proof would be finished if we knew that $(*)$ in that diagram commutes: then $\bar{h} = f^\dagger$ implies $h = \lambda_i \cdot f^\dagger$, as required. However, we do not claim that commutativity. All we claim is that all the other inner parts of (52) commute (trivially) and the outer shape does by (51), thus,

$(\lambda_i)_Y$ merges the two sides of $(*)$.

This implies, since the domain X of $(*)$ is finitely presentable and $(\lambda_i)_Y$ is a colimit morphism of the filtered colimit defining SY, that some connecting morphism $(\lambda_{i,j})_Y$ for $i \leqslant j$ in I also merges the two sides of $(*)$. Put

$$g' \equiv X \xrightarrow{f'} S'_i(X+Y) + Y \xrightarrow{\lambda'_{i,j}+Y} S'_j(X+Y) + Y$$

and

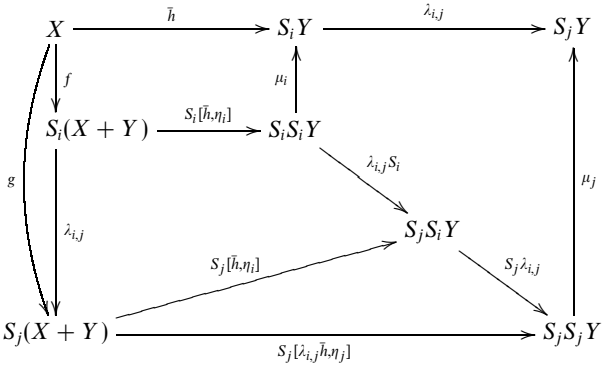
$$g \equiv X \xrightarrow{g'} S'_j(X+Y) + Y \xrightarrow{[\sigma_j, \eta_j \mathsf{inr}]} S_j(X+Y)$$

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

We claim that g, which obviously is a guarded equation morphism for S_j, has the solution

$$g^\dagger = (\lambda_{i,j})_Y \cdot \bar{h} : X \longrightarrow S_j Y. \tag{53}$$

To prove this, we observe that the diagram (54) commutes:



(54)

all the inner parts commute trivially except the upper left-hand square – and, by our choice of j, that square commutes when composed with $(\lambda_{i,j})_Y$. This proves (53). Therefore, we have, by (50),

$$(\lambda_i)_Y \cdot f^\dagger = (\lambda_j)_Y \cdot g^\dagger = (\lambda_j)_Y \cdot (\lambda_{i,j})_Y \bar{h} = (\lambda_i)_Y \bar{h} = h,$$

which concludes the proof. □

Definition 5.6. For every finitary functor H, expressed as a filtered colimit of strongly finitary functors H_i we define the *rational monad* R of H as the colimit of the corresponding diagram of the rational monads R_i of H_i,

$$R = \operatorname*{colim}_i R_i, \quad \text{with injections } \bar{\beta}_i : R_i \longrightarrow R$$

(see Remark 5.2).

We have yet to show that R is well-defined, that is, independent of the given representation of H as $H = \operatorname{colim} H_i$. This follows from the following proposition.

Proposition 5.7. The rational monad R of H is a free iterative monad on H. That is:
(a) R is an iterative monad;
(b) the natural transformation $\rho^* = \operatorname*{colim}_{i \in I} \rho_i^* : H \longrightarrow R$ is ideal;
and
(c) given an iterative monad S and an ideal natural transformation $\lambda : H \longrightarrow S$, there exists a unique ideal monad homomorphism $\bar{\lambda} : R \longrightarrow S$ with $\lambda = \bar{\lambda} \cdot \rho^*$.

Proof. For (a) see Proposition 5.5. Since each $\rho_i^* : H_i \longrightarrow R_i$ is an ideal transformation, that is, $\rho_i^* = \rho_i \cdot H\eta^{R_i}$, it follows that

$$\operatorname*{colim}_{i \in I} \rho_i^* = \left(\operatorname*{colim}_{i \in I} \rho_i \right) \cdot H \left(\operatorname*{colim}_{i \in I} \eta^{R_i} \right)$$

is also ideal. It remains to prove (c).

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

For each $i \in I$ we have an ideal transformation

$$H_i \xrightarrow{\beta_i} H \xrightarrow{\lambda} S,$$

which, by Theorem 4.30, yields a unique ideal monad homomorphism $\bar{\lambda}_i : R_i \longrightarrow S$ such that the square

$$\begin{array}{ccc} H_i & \xrightarrow{\rho_i^*} & R_i \\ {\scriptstyle \beta_i}\downarrow & & \downarrow{\scriptstyle \bar{\lambda}_i} \\ H & \xrightarrow[\lambda]{} & S \end{array} \tag{55}$$

commutes. The unicity makes it clear that $\bar{\lambda}_i$, $i \in I$, is a cocone of the diagram $(R_i \mid i \in I)$. Thus, we have a unique ideal monad homomorphism $\bar{\lambda} : R \longrightarrow S$ with

$$\bar{\lambda} \cdot \bar{\beta}_i = \bar{\lambda}_i \quad \text{for } i \in I. \tag{56}$$

Then, from $\rho^* \cdot \beta_i = \bar{\beta}_i \cdot \rho_i^*$ (recall $\rho^* = \operatorname{colim} \rho_i^*$), we obtain

$$\begin{aligned} (\bar{\lambda} \cdot \rho^*) \cdot \beta_i &= \lambda \cdot \bar{\beta}_i \cdot \rho_i^* \\ &= \bar{\lambda}_i \cdot \rho_i^* \qquad \text{by (56)} \\ &= \lambda \cdot \beta_i \qquad \text{by (55)} \end{aligned}$$

for all $i \in I$, which proves

$$\bar{\lambda} \cdot \rho^* = \lambda.$$

The unicity of $\bar{\lambda}$ is obvious: given an ideal homomorphism $\bar{\lambda}$ with $\bar{\lambda} \cdot \rho^* = \lambda$, we have $\bar{\lambda} \cdot \bar{\beta}_i : R_i \longrightarrow S$ is an ideal homomorphism with

$$\bar{\lambda} \cdot \bar{\beta}_i \cdot \rho_i^* = \bar{\lambda} \cdot \rho^* \cdot \beta_i = \lambda \cdot \beta_i,$$

which (since R_i is free on H_i) determines $\bar{\lambda} \cdot \bar{\beta}_i$ for $i \in I$, and this determines $\bar{\lambda}$. □

Recall that guarded equation morphisms $e : X \longrightarrow R(X+Y)$ with X finitely presentable are called rational.

Corollary 5.8. (Rational Solution Theorem) Every rational equation morphism $e : X \longrightarrow R(X + Y)$ has a unique solution. That is, there exists a unique morphism

$$e^\ddagger : X \longrightarrow RY$$

such that the triangle

$$\begin{array}{ccc} X & \xrightarrow{e^\ddagger} & RY \\ {\scriptstyle e}\downarrow & \nearrow_{\widehat{[e^\ddagger, \eta_Y^R]}} & \\ R(X+Y) & & \end{array}$$

commutes (where for $s = [e^\ddagger, \eta_Y^R]$ we denote $\widehat{s} = \mu_Y^R \cdot Rs$, as always).

In fact, if Y is finitely presentable, this follows immediately from Proposition 5.7(a). If Y is arbitrary, we express it as a filtered colimit of finitely presentable objects $Y = \operatorname*{colim}_{i \in I} Y_i$,

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

with a colimit cocone $y_i : Y_i \longrightarrow Y$. Since R is finitary, we obtain $R(X+Y) = \operatorname{colim}_{i\in I} R(X+Y_i)$. By the finite presentability of X, e factors through one of the colimit morphisms $R(X+y_i)$ through a guarded equation morphism $e_i : X \longrightarrow R(X+Y_i)$. It is a routine verification that the existence and uniqueness of the solution of e_i implies the existence and uniqueness of the solution of e.

Remark 5.9. The rational monad R fulfills

$$R = HR + Id$$

with coproduct injections

$$\rho : HR \longrightarrow R$$

(turning each object RY into an H-algebra) given by

$$\rho = \operatorname*{colim}_i \rho_i : \operatorname*{colim}_i H_i R_i \longrightarrow \operatorname*{colim}_i R_i$$

and

$$\eta : Id \longrightarrow R$$

(the unit of the monad).

In fact, filtered colimits commute with finite coproducts, so all this follows from $R_i = H_i R_i + Id$ with coproduct injections ρ_i and η_i.

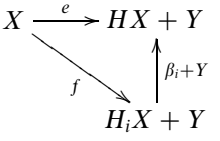
Remark 5.10. We are going to establish the formulas $RY = \bigcup \mathrm{im}(e^\dagger)$ and $RY = \operatorname{colim} \mathrm{Eq}_Y$ for all objects Y of $\mathscr{A}$ (cf. 4.2. and 4.6.). It is sufficient to prove the second one for all finitely presentable objects Y, the extension to all objects is exactly as in Remark 4.20.

For Y in $\mathscr{A}_{fp}$, the diagram $\mathrm{Eq}_Y^{(i)}$ of all finitary flat equations $e : X \longrightarrow H_i X + Y$ (X finitely presentable) has a colimit

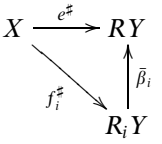
$$R_i Y = \operatorname{colim} \mathrm{Eq}_Y^{(i)}$$

with colimit cocone $e_i^\sharp : X \longrightarrow R_i Y$, see Section 4.

Consider an arbitrary finitary flat equation morphism $e : X \longrightarrow HX + Y$ for H. Since X is finitely presentable and $HX + Y$ is a directed colimit of $H_i X + Y$, $i \in I$, we have a factorisation as follows:



for some $i \in I$. Here f is a finitary flat equation morphism for H_i, and we use $f_i^\sharp : X \longrightarrow R_i Y$ to denote the corresponding colimit map of $R_i Y = \operatorname{colim} \mathrm{Eq}_Y^{(i)}$. Define $e^\sharp$ by



https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

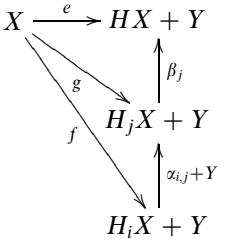
This is well-defined (that is, independent of the factorisation) and forms a colimit of the diagram Eq_Y, as we prove now.

Proposition 5.11. For every finitely presentable object Y we have

$$RY = \operatorname{colim} \mathrm{Eq}_Y$$

with the above colimit cocone $e^\sharp : X \longrightarrow RY$ (e in EQ_Y).

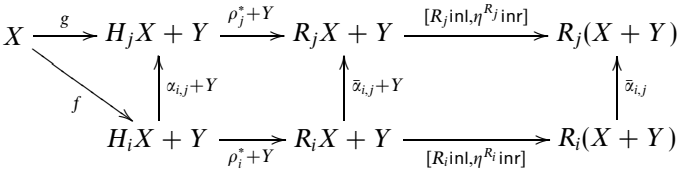
Proof. (I) **Independence of the factorisation:** Since I is a directed poset, all we have to prove is that given $i \leqslant j$ in I, and, using $g : X \longrightarrow H_jX + Y$ to denote the composite of f and $(\alpha_{i,j})_X + Y$,



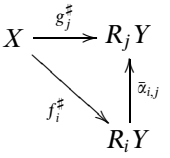
we have

$$\bar{\beta}_i \cdot f_i^\sharp = \bar{\beta}_j \cdot g_j^\sharp.$$

Since the diagram

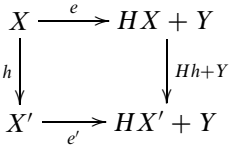


commutes, it follows from Corollary 5.4 that the solutions $f_i^\sharp : X \longrightarrow R_iY$ (with respect to R_i) and $g_j^\sharp : X \longrightarrow R_jY$ (with respect to R_j) also form a commutative triangle:



Combined with the fact that $\bar{\beta}_i = \bar{\beta}_j \cdot \bar{\alpha}_{i,j}$ (because $\beta_i = \beta_j \cdot \alpha_{i,j}$), this yields the desired equality $\bar{\beta}_i \cdot f_i^\sharp = \bar{\beta}_j \cdot g_j^\sharp$.

(II) **The morphisms $e^\sharp$ form a cocone of Eq_Y:** That is, given a morphism

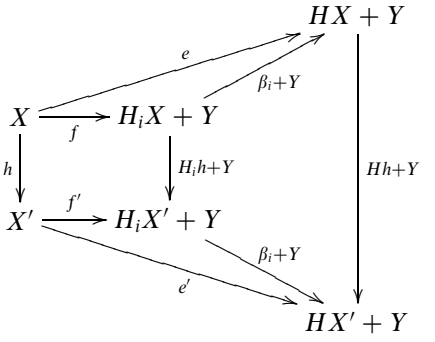


https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

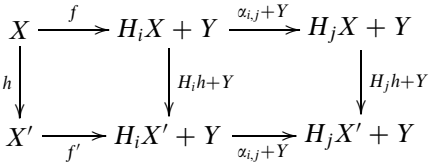
in EQ_Y, we have $e^\sharp = (e')^\sharp \cdot h$ holds. Since X and X' are finitely presentable, there exists $i \in I$ such that e factors as $e = ((\beta_i)_X + Y) \cdot f$ and e' as $e' = ((\beta_i)_{X'} + Y) \cdot f'$. Next, observe that

$$HX + Y = \operatorname*{colim}_{i \in I} H_i X + Y$$

is a filtered colimit with colimit cocone $(\beta_i)_X + Y$. The parallel pair $f' \cdot h, (H_i h + Y) \cdot f : X \longrightarrow H_i X' + Y$ gets merged by the colimit morphism $(\beta_i)_{X'} + Y$:



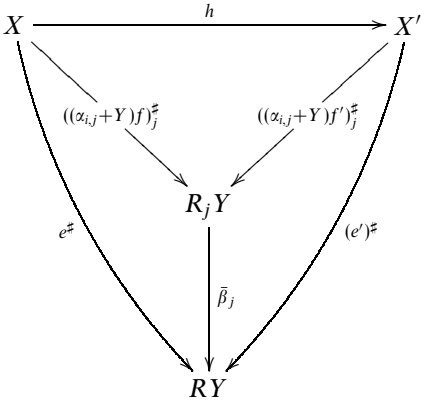
Since the domain X of that parallel pair is finitely presentable, it follows that some connecting morphism $(\alpha_{i,j})_{X'} + Y$ of the above filtered diagram also merges that parallel pair. That is, in the diagram



the outward square commutes. Consequently, we have a morphism h in the filtered diagram defining $R_j Y$, therefore,

$$(((\alpha_{i,j})_X + Y) \cdot f)^\sharp_j = (((\alpha_{i,j})_{X'} + Y) \cdot f')^\sharp_j \cdot h.$$

This proves that the desired triangle



commutes, since $e^\sharp$ and $(e')^\sharp$ are by (I) independent of the above factorisation.

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

(III) **Universal property.** Let a cocone $e^{@} : X \longrightarrow C$ (for e in EQ_Y) of the diagram Eq_Y be given. For each $i \in I$, we obtain a cocone of $D_Y^{(i)}$ (the diagram of all finitary flat equation morphisms of H_i with respect to Y) as follows: to every $f : X \longrightarrow H_iX + Y$, we assign

$$(((\beta_i)_X + Y) \cdot f)^{@} : X \longrightarrow C.$$

This is indeed a cocone, since for every morphism

$$\begin{array}{ccc} X & \xrightarrow{f} & H_iX + Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle H_ih+Y} \\ X' & \xrightarrow[f']{} & H_iX' + Y \end{array}$$

in $\mathsf{EQ}_Y^{(i)}$, we have a morphism

$$\begin{array}{ccccc} X & \xrightarrow{f} & H_iX + Y & \xrightarrow{\beta_i+Y} & HX + Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle H_ih+Y} & & \downarrow{\scriptstyle Hh+Y} \\ X' & \xrightarrow[f']{} & H_iX' + Y & \xrightarrow[\beta_i+Y]{} & HX' + Y \end{array}$$

in EQ_Y, thus, $(((\beta_i)_X + Y) \cdot f)^{@} = (((\beta_i)_{X'} + Y) \cdot f')^{@} \cdot h$. Therefore, we get a unique

$$c_i : R_iY \longrightarrow C$$

with

$$c_i \cdot f_i^{\sharp} = (((\beta_i)_X + Y) \cdot f)^{@} \quad \text{for all } f \text{ in } \mathsf{EQ}_Y^{(i)}.$$

The uniqueness of c_i, for each $i \in I$, makes it obvious that these morphisms form a cocone of the diagram $(R_iY \mid i \in I)$. Thus, there is a unique

$$c : RY \longrightarrow C$$

with

$$c \cdot (\bar{\beta}_i)_Y = c_i \quad \text{for all } i \in I.$$

This morphism fulfills

$$c \cdot e^{\sharp} = e^{@} \quad \text{for each } e \text{ in } \mathsf{EQ}_Y.$$

In fact, we can factor e via $f : X \longrightarrow H_iX + Y$ as above, and then

$$\begin{aligned} c \cdot e^{\sharp} &= c \cdot (\bar{\beta}_i)_Y \cdot f_i^{\sharp} && \text{by the definition of } e^{\sharp} \\ &= c_i \cdot f_i^{\sharp} && \text{by the definition of } c \\ &= (((\beta_i)_X + Y) \cdot f)^{@} && \text{by the definition of } c_i \\ &= e^{@}. \end{aligned}$$

To prove that c is unique, we only have to observe that given a morphism $c : RY \longrightarrow C$ with $c \cdot e^{\sharp} = e^{@}$ for each e in EQ_Y, it follows that $c \cdot (\bar{\beta}_i)_Y = c_i$ for each $i \in I$ (which determines c). In fact, R_iY is a colimit of $\mathrm{Eq}_Y^{(i)}$ with colimit morphisms $f_i^{\sharp}$, and from

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

$c \cdot e^\sharp = e^{@}$, we conclude, for each $f : X \longrightarrow H_i X + Y$ in $\mathsf{EQ}_Y^{(i)}$, that

$$\begin{aligned} c \cdot (\bar{\beta}_i)_Y \cdot f_i^\sharp &= c \cdot (((\beta_i)_X + Y) \cdot f)^\sharp \\ &= (((\beta_i)_X + Y) \cdot f)^{@} \\ &= c_i \cdot f_i^\sharp \end{aligned}$$

□

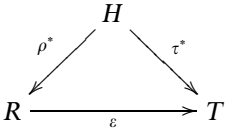
Remark 5.12. Recall that since H is iteratable, it generates the monad T of Corollary 1.8. Together with

$$\tau^* \equiv H \xrightarrow{H\eta} HT \xrightarrow{\tau} T,$$

this is a free completely iterative monad on H. We use

$$\varepsilon : R \longrightarrow T$$

to denote the unique ideal monad homomorphism with



commutative, see Proposition 5.7. By Proposition 5.3, ε preserves solutions, thus, it satisfies (due to Remark 4.27)

$$\varepsilon_Y \cdot e^\sharp = e^\dagger \quad \text{for all } e : X \longrightarrow HX + Y \text{ in } \mathsf{EQ}_Y,$$

for all objects Y of $\mathscr{A}$. Since RY is the union of all images of finitary flat equation morphisms (see Remark 5.10), it follows that ε_Y is a monomorphism (*viz.*, the monomorphism representing that union). This proves the following corollary.

Corollary 5.13. R is a submonad of the monad T via ε.

Theorem 5.14 (Rational Substitution Theorem). For every morphism

$$s : X \longrightarrow RY,$$

there exists a unique extension to a homomorphism

$$\widetilde{s} : RX \longrightarrow RY$$

of H-algebras (that is, a unique homomorphism with $s = \widetilde{s} \cdot \eta_Y^R$).

Proof. Existence is clear: $\widetilde{s} = \mu_Y \cdot Rs$. Uniqueness is proved precisely as in Theorem 4.16. □

6. The rational monad of a set functor

We are going to define a rational monad of every finitary functor H of Set. In the last section, this has been done whenever H preserves monomorphisms.

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

Now all monomorphisms $m : X \longrightarrow Y$ in Set with $X \neq \emptyset$ are split – choose $x_0 \in X$ and define $e : Y \longrightarrow X$ by

$$e(y) = \begin{cases} x, & \text{if } m(x) = y \\ x_0, & \text{otherwise.} \end{cases}$$

Then $em = id$. Thus, every functor preserves these monomorphisms. The only trouble-makers are, thus, the empty functions

$$\psi_X : \emptyset \longrightarrow X \quad (X \neq \emptyset).$$

Not all finitary functors preserve monomorphisms. For example, put $HX = 1$ (terminal object) for $X \neq \emptyset$ and $H\emptyset = 1 + 1$. This defines H uniquely, and $H\psi_1 : 1 + 1 \longrightarrow 1$ is, of course, no monomorphism. Luckily, iterative monads in Set do preserve monomorphisms.

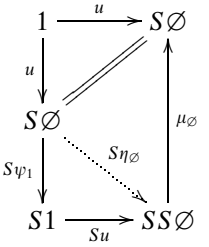
Proposition 6.1. Every iterative monad in Set preserves monomorphisms.

Proof. Let S be an iterative monad. Observe that $S\emptyset = S'\emptyset$ (since $S = S' + Id$).

If $S'\emptyset = \emptyset$, then $S\psi_X : \emptyset \longrightarrow SX$ is a monomorphism and the proof is concluded. Suppose $S'\emptyset \neq \emptyset$, choose $u' : 1 \longrightarrow S'\emptyset$ and put $u = \sigma_\emptyset \cdot u' : 1 \longrightarrow S\emptyset$. Then the composite

$$e \equiv 1 \xrightarrow{u} S\emptyset \xrightarrow{S\psi_1} S1$$

is a guarded equation morphism for the empty set $Y = \emptyset$ of parameters. And u is a solution of e, that is, the diagram



(where $u = [u, \eta_\emptyset] : 1 + \emptyset \longrightarrow S\emptyset$) commutes. In fact, we have $u \cdot \psi_1 = \eta_\emptyset : \emptyset \longrightarrow S\emptyset$ because $\mathsf{Set}(\emptyset, S\emptyset)$ is a singleton set; the rest is clear. This proves that $S\psi_1$ is a monomorphism: given $u, v : 1 \longrightarrow S\emptyset$ with $S\psi_1 \cdot u = S\psi_1 \cdot v = e$, then both u and v are solutions of e, and thus, $u = v$. Consequently, $S\psi_X$ is a monomorphism for every X. □

Remark 6.2. For every endofunctor H of Set, the equivalence relation on $H\emptyset$ given by

$$u \sim v \quad \text{iff} \quad H\psi_X(u) = H\psi_X(v)$$

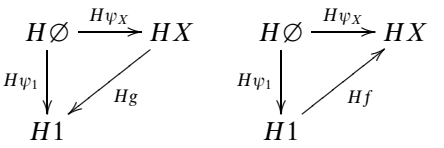
where X is a fixed non-empty set, is independent of the choice of X.

In brief,

$$H\psi_X(u) = H\psi_X(v) \quad \text{iff} \quad H\psi_1(u) = H\psi_1(v).$$

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

In fact, since $X \neq \emptyset$, we have functions $f : 1 \longrightarrow X$ and $g : X \longrightarrow 1$, and the triangles



commute (because $\emptyset$ is the initial object).

Definition 6.3. For every endofunctor H of Set, we define a quotient

$$\gamma : H \longrightarrow H^+$$

of H as follows.

For all non-empty sets X, we put $H^+X = HX$ and $\gamma_X = id$; for the empty set, let

$$\gamma_\emptyset : H\emptyset \longrightarrow H\emptyset/\sim = H^+\emptyset$$

be the canonical function of the equivalence $\sim$ of Remark 6.2.

For all functions $f : X \longrightarrow Y$ with $X \neq \emptyset$, put $H^+f = Hf$, and for the empty function put $H^+id_\emptyset = id_{H^+\emptyset}$ and

$$H^+\psi_X : [u] \mapsto H\psi_X(u) \quad \text{for all } u \in H\emptyset$$

if X is non-empty.

The above definition is a small modification of a procedure used by V. Trnková (Trnková 1971).

We now prove that H^+ is a reflection of H in the category of all endofunctors of Set preserving monomorphisms.

Lemma 6.4. For every endofunctor H of Set, the functor H^+ preserves monomorphisms, and $\gamma : H \longrightarrow H^+$ has the following universal property:

> For every natural transformation $\delta : H \longrightarrow K$, where K preserves monomorphisms, there exists a unique natural transformation $\delta^+ : H^+ \longrightarrow K$ with $\delta = \delta^+\gamma$.

Proof. (I) H^+ preserves monomorphisms (equivalently, it maps ψ_1 to a monomorphism), since $[u] \neq [v]$ holds iff $H\psi_1(u) \neq H\psi_1(v)$, for all $u, v \in H\emptyset$.

(II) Let $\delta : H \longrightarrow K$ be given. The equation $\delta = \delta^+\gamma$ forces us to define

$$\delta^+_X = \delta_X : HX \longrightarrow KX \quad \text{for all } X \neq \emptyset.$$

It remains to discuss $\delta^+_\emptyset : H\emptyset/\sim \longrightarrow K\emptyset$. All we have to prove is that for $u, v \in H\emptyset$, we have

$$u \sim v \quad \text{implies} \quad \delta_\emptyset(u) = \delta_\emptyset(v). \tag{57}$$

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

Then $\delta^+_\emptyset$ is uniquely determined by $\delta_\emptyset = \delta^+_\emptyset \cdot \gamma_\emptyset$ (*viz.*, $\delta^+_\emptyset([u]) = \delta_\emptyset(u)$), and the naturality of δ^+ is obvious. To show (57), use naturality of δ on ψ_1:

$$\begin{array}{ccc} H\emptyset & \xrightarrow{\delta_\emptyset} & K\emptyset \\ {\scriptstyle H\psi_1}\downarrow & & \downarrow{\scriptstyle K\psi_1} \\ H1 & \xrightarrow[\delta_1]{} & K1 \end{array}$$

From $u \sim v$ we conclude, since $H\psi_1(u) = H\psi_1(v)$, that $K\psi_1(\delta_\emptyset(u)) = K\psi_1(\delta_\emptyset(v))$. Since $K\psi_1$ is a monomorphism, it follows that $\delta_\emptyset(u) = \delta_\emptyset(v)$. □

Definition 6.5. For every finitary endofunctor H the *rational monad of* H is defined to be the rational monad $\rho^* : H^+ \longrightarrow R$ of the above reflection $\gamma : H \longrightarrow H^+$.

Corollary 6.6. The rational monad of H, together with the natural transformation

$$H \xrightarrow{\gamma} H^+ \xrightarrow{\rho^*} R$$

is a free iterative monad on H.

Proof. In fact, since ρ^* is an ideal natural transformation, so is $\rho^*\gamma$. Let S be an iterative monad and $\lambda : H \longrightarrow S$ be an ideal transformation, $\lambda = \sigma\lambda'$. Since the subfunction S' of S preserves monomorphisms (see Proposition 6.1), the natural transformation $\lambda' : H \longrightarrow S'$ extends uniquely to $(\lambda')^+ : H^+ \longrightarrow S'$, and we obtain an ideal natural transformation

$$\lambda^+ \equiv H^+ \xrightarrow{(\lambda')^+} S' \xrightarrow{\sigma} S.$$

By Proposition 5.7, this yields a unique ideal monad homomorphism $\overline{\lambda^+} : R \longrightarrow S$ extending λ^+, that is, such that

$$\begin{array}{ccccc} H & \xrightarrow{\gamma} & H^+ & \xrightarrow{\rho^*} & R \\ & \searrow^{\lambda} & & \searrow^{\lambda^+} & \downarrow{\scriptstyle \overline{\lambda^+}} \\ & & & & S \end{array}$$

commutes. □

7. Conclusion and future directions

We have constructed a free iterative monad on every monos-preserving, finitary endofunctor H of a 'set-like' category. In particular, every finitary endofunctor of Set generates a free iterative monad (or free iterative Lawvere theory). The method of our construction was coalgebraic, making heavy use of the fact that final coalgebras TX of the functors $H(_) + X$ form a free completely iterative monad on H.

The proof is surprisingly technical, and one question it naturally raises is whether a simpler proof can be found. When H is a polynomial functor, the existence of a free iterative theory was established by C. C. Elgot (Elgot 1975) and the fact that this

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

is the theory of rational trees was proved later in Elgot *et al.* (1978). Their method is fundamentally different from ours, and their proof was also quite involved.

The rational monad as presented here is based on solutions of finitary flat equations. This can be extended to solutions of finitary equations in much the same style as mentioned, for polynomial functors, in the Introduction (see Section 1.1), we have proved this is Adámek *et al.* (2002).

In the work of Bloom and Ésik (Bloom and Ésik 1993), equational properties of solutions are described and revealed to have a general pattern in various fields. See also a recent restatement of some of these results in Simpson and Plotkin (2000). It would be interesting to investigate these properties categorically. L. Moss (Moss 2001) has already taken first steps in that direction.

Acknowledgements

The present paper has evolved out of a very pleasant cooperation between the authors and Peter Aczel. In the joint paper Aczel *et al.* (2002), whose extended abstract was published as Aczel *et al.* (2001), we constructed a free completely iterative monad on every iteratable functor. The current paper has also been discussed with Peter, and the inspiration obtained from him is gratefully acknowledged. We are also grateful to the referees for their suggestions for improvements of presentation.

References

Aczel, P., Adámek, J. and Velebil, J. (2001) A coalgebraic view of infinite trees and iteration. *Electronic Notes in Theoretical Computer Science* **44** (1).

Aczel, P., Adámek, J., Milius, S. and Velebil, J. (2002) Infinite trees and completely iterative theories: A Coalgebraic View. (To appear in *Theoretical Computer Science*.)

Adámek, J. (1997) A categorical generalisation of Scott domains. *Mathematical Structures in Computer Science* **7** 419–443.

Adámek, J. and Koubek, V. (1995) On the greatest fixed point of a set functor. *Theoretical Computer Science* **150** 57–75.

Adámek, J., Milius, S. and Velebil, J. (2002) Some remarks on finitary and iterative monads (submitted).

Adámek, J. and Rosický, J. (1994) *Locally Presentable and Accessible Categories*, Cambridge University Press.

Adámek, J. and Trnková, V. (1990) *Automata and Algebras in Categories*, Kluwer Academic Publishers.

Bloom, S. L. and Ésik, Z. (1993) *Iteration Theories: The Equational Logic of Iterative Processes*, EATCS Monograph Series on Theoretical Computer Science, Springer-Verlag.

Carboni, A., Lack, S. and Walters, R. F. C. (1993) Introduction to extensive and distributive Categories. *J. Pure Appl. Algebra* **84** 145–158.

Elgot, C. C. (1975) Monadic computation and iterative algebraic theories. In: Rose, H. E. and Shepherdson, J. C. (eds.) *Logic Colloquium '73*, North-Holland 175–230.

Elgot, C. C., Bloom, S. L. and Tindell, R. (1978) On the algebraic structure of rooted trees. *J. Comp. Syst. Sciences* **16** 361–399.

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press

Gabriel, P. and Ulmer, F. (1971) Lokal präsentierbare Kategorien. *Springer-Verlag Lecture Notes in Mathematics* **221**.

Kelly, M. and Power, J. (1993) Adjunctions whose counits are coequalisers, and presentations of finitary enriched monads. *J. Pure Appl. Algebra* **89** 163–179.

Lambek, J. (1968) A fixpoint theorem for complete categories. *Math. Z.* **103** 151–161.

Manes, E. G. (1976) *Algebraic Theories*, Springer-Verlag.

Moss, L. (2001) Parametric corecursion. *Theoretical Computer Science* **260** (1-2) 139–163.

Moss, L. (2000) Recursion and corecursion have the same equational logic. (To appear in *Theorectical Computer Science*.)

Simpson, A. and Plotkin, G. (2000) Complete axioms for categorical fixed-point operators. *Fifteenth Annual IEEE Symposium on Logic in Computer Science* 30–41.

Trnková, V. On descriptive classification of set functors. *Comment. Math. Univ. Carolinae* **12** 143–175.

https://doi.org/10.1017/S0960129502003924 Published online by Cambridge University Press