J. London Math. Soc. (2) 73 (2006) 657–668 © 2006 London Mathematical Society
doi:10.1112/S002461070602299X

HOCHSCHILD (CO)HOMOLOGY DIMENSION

YANG HAN

Dedicated to Professor Yingbo Zhang on the occasion of her 60th birthday

Abstract

In 1989 Happel asked the question whether, for a finite-dimensional algebra A over an algebraically closed field k, $\operatorname{gl.dim} A < \infty$ if and only if $\operatorname{hch.dim} A < \infty$. Here, the Hochschild cohomology dimension of A is given by $\operatorname{hch.dim} A := \inf\{n \in \mathbb{N}_0 \mid \dim HH^i(A) = 0 \text{ for } i > n\}$. Recently Buchweitz, Green, Madsen and Solberg gave a negative answer to Happel's question. They found a family of pathological algebras A_q for which $\operatorname{gl.dim} A_q = \infty$ but $\operatorname{hch.dim} A_q = 2$. These algebras are pathological in many aspects. However, their Hochschild homology behaviors are not pathological any more; indeed one has $\operatorname{hh.dim} A_q = \infty = \operatorname{gl.dim} A_q$. Here, the Hochschild homology dimension of A is given by $\operatorname{hh.dim} A := \inf\{n \in \mathbb{N}_0 \mid \dim HH_i(A) = 0 \text{ for } i > n\}$. This suggests posing a seemingly more reasonable conjecture by replacing the Hochschild cohomology dimension in Happel's question with the Hochschild homology dimension: $\operatorname{gl.dim} A < \infty$ if and only if $\operatorname{hh.dim} A < \infty$ if and only if $\operatorname{hh.dim} A = 0$. The conjecture holds for commutative algebras and monomial algebras. In the case where A is a truncated quiver algebra, these conditions are equivalent to the condition that the quiver of A has no oriented cycles. Moreover, an algorithm for computing the Hochschild homology of any monomial algebra is provided. Thus the cyclic homology of any monomial algebra can be read off when the underlying field is characteristic 0.

Let k be a fixed field. All algebras considered here are of the form $A = kQ/I$ where Q is a finite quiver and I is an admissible ideal of the path algebra kQ. We refer to [**1**] for the theory of quivers and their representations. It is well known that, if k is an algebraically closed field, up to Morita equivalence, any finite-dimensional k-algebra can be written in this form (cf. [**11**]).

1. *Hochschild (co)homology*

In this section we recall some definitions and properties on Hochschild (co)homology.

1.1. The complex

$$C_*(A) = (A^{\otimes *+1}, b_*) : \cdots \to A^{\otimes n+1} \xrightarrow{b_n} A^{\otimes n} \to \cdots \to A^{\otimes 2} \xrightarrow{b_1} A \to 0$$

with

$$b_n(a_0, a_1, \ldots, a_n) := \sum_{i=0}^{n-1} (-1)^i (a_0, \ldots, a_i a_{i+1}, \ldots, a_n) + (-1)^n (a_n a_0, a_1, \ldots, a_{n-1}),$$

where $(a_0, \ldots, a_n) := a_0 \otimes \ldots \otimes a_n$ and $\otimes := \otimes_k$, is called the *Hochschild complex* or *cyclic bar complex* of A. Note that $A^{\otimes n+1}$ is in degree n.

Received 30 August 2004; revised 23 August 2005.

2000 *Mathematics Subject Classification* 16E40, 16E10, 16G10.

The author is sponsored by Project 10201004 NSFC and Project SRF for ROCS SEM.

https://doi.org/10.1112/S002461070602299X Published online by Cambridge University Press

The homology group $HH_n(A) := H_n(C_*(A))$ is called the nth *Hochschild homology group* of A (cf. [**21**]). Clearly $HH_n(A)$ is a k-vector space. We denote $\dim_k HH_n(A)$ by $hh_n(A)$.

1.2. The cohomology group $HH^n(A) := H^n(C^*(A))$ of the complex

$$C^*(A) = (\mathrm{Hom}_k(A^{\otimes *}, A), d^*) : 0 \to A \xrightarrow{d^0} \mathrm{Hom}_k(A, A) \xrightarrow{d^1} \cdots$$
$$\cdots \to \mathrm{Hom}_k(A^{\otimes n}, A) \xrightarrow{d^n} \mathrm{Hom}_k(A^{\otimes n+1}, A) \to \cdots$$

with

$$(d^n(f))(a_0, a_1, \ldots, a_n) := a_0 f(a_1, \ldots, a_n) + \sum_{i=0}^{n-1} (-1)^{i+1} f(a_0, \ldots, a_i a_{i+1}, \ldots, a_n) + (-1)^{n+1} f(a_0, \ldots, a_{n-1}) a_n$$

is called the nth *Hochschild cohomology group* of A (cf. [**14**]). Clearly $HH^n(A)$ is a k-vector space. We denote $\dim_k HH^n(A)$ by $hh^n(A)$.

1.3. The chain complex

$$C'_*(A) = (A^{\otimes *+2}, b'_*) : \cdots \to A^{\otimes n+2} \xrightarrow{b'_n} A^{\otimes n+1} \to \cdots \to A^{\otimes 3} \xrightarrow{b'_1} A^{\otimes 2} \to 0$$

with

$$b'_n(a_0, a_1, \ldots, a_{n+1}) := \sum_{i=0}^{n} (-1)^i (a_0, \ldots, a_i a_{i+1}, \ldots, a_{n+1})$$

is called the *bar complex* or *standard complex* of A.

The bar complex $C'_*(A)$ is a projective resolution of A as an A^e-module where $A^e := A \otimes A^{op}$. Since $A \otimes_{A^e} C'_*(A) = C_*(A)$ and $\mathrm{Hom}_{A^e}(C'_*(A), A) = C^*(A)$, we have $HH_n(A) = \mathrm{Tor}_n^{A^e}(A, A)$ and $HH^n(A) = \mathrm{Ext}^n_{A^e}(A, A)$. These two formulas are very useful to calculate Hochschild (co)homology by constructing a convenient projective resolution of A over A^e.

LEMMA 1. *Let A and B be two algebras.*
(1) *We have*

$$HH_n\left(\begin{bmatrix} A & {}_AM_B \\ 0 & B \end{bmatrix}\right) = HH_n(A) \oplus HH_n(B)$$

for any A-B-bimodule M. In particular $HH_n(A \times B) = HH_n(A) \oplus HH_n(B)$.
(2) *If A and B are derived equivalent then $HH_n(A) = HH_n(B)$. In particular iterated tilting and Morita equivalence preserves Hochschild homology.*
(3) *We have $HH_n(A \otimes B) = \bigoplus_{i+j=n} HH_i(A) \otimes HH_j(B)$.*

Proof. (1) See [**21**, Theorem 1.2.15].
(2) See [**16**, Theorem 2.2].
(3) See [**22**, Chapter VIII, Theorem 7.4]. □

LEMMA 2. *Let A and B be two algebras.*
(1) *We have $HH^n(A \times B) = HH^n(A) \oplus HH^n(B)$.*
(2) *If A and B are derived equivalent then $HH^n(A) = HH^n(B)$. In particular iterated tilting and Morita equivalence preserves Hochschild cohomology.*
(3) *We have $HH^n(A \otimes B) = \bigoplus_{i+j=n} HH^i(A) \otimes HH^j(B)$.*

https://doi.org/10.1112/S002461070602299X Published online by Cambridge University Press

Proof. (1) See [**6**, Chapter IX, Theorem 5.3].
(2) See [**26**, Proposition 2.5].
(3) See [**22**, Chapter VIII, Theorem 7.4]. □

REMARK 1. For Hochschild cohomology in general we have no formula similar to Lemma 1, part (1). An obvious counterexample is the Kronecker algebra

$$\begin{bmatrix} k & k^2 \\ 0 & k \end{bmatrix}.$$

Its Hochschild cohomology groups can be obtained by [**13**, Proposition 1.6]. The relation between the triangular matrix algebra

$$\begin{bmatrix} A & {}_AM_B \\ 0 & B \end{bmatrix}$$

and A, B was observed in [**25**].

2. *Hochschild (co)homology dimension*

In this section we give the definitions and some properties of the Hochschild (co)homology dimension.

DEFINITION 1 [**4**, Definition 3.1]. The *Hochschild homology dimension* of A is $\operatorname{hh.dim} A := \inf\{n \in \mathbb{N}_0 \mid hh_i(A) = 0 \text{ for } i > n\}$.

PROPOSITION 1. *Let A and B be two algebras.*
(1) *We have*

$$\operatorname{hh.dim} \begin{bmatrix} A & {}_AM_B \\ 0 & B \end{bmatrix} = \max\{\operatorname{hh.dim} A, \operatorname{hh.dim} B\}$$

for any A-B-bimodule M. In particular

$$\operatorname{hh.dim}(A \times B) = \max\{\operatorname{hh.dim} A, \operatorname{hh.dim} B\}.$$

(2) *If A and B are derived equivalent then* $\operatorname{hh.dim} A = \operatorname{hh.dim} B$. *In particular iterated tilting and Morita equivalence preserves the Hochschild homology dimension.*
(3) *We have* $\operatorname{hh.dim}(A \otimes B) = \operatorname{hh.dim} A + \operatorname{hh.dim} B$.

Proof. The proofs follow from Lemma 1. □

DEFINITION 2. The *Hochschild cohomology dimension* of A is $\operatorname{hch.dim} A := \inf\{n \in \mathbb{N}_0 \mid hh^i(A) = 0 \text{ for } i > n\}$.

PROPOSITION 2. *Let A and B be two algebras.*
(1) *We have* $\operatorname{hch.dim}(A \times B) = \max\{\operatorname{hch.dim} A, \operatorname{hch.dim} B\}$.
(2) *If A and B are derived equivalent then* $\operatorname{hch.dim} A = \operatorname{hch.dim} B$. *In particular iterated tilting and Morita equivalence preserves the Hochschild cohomology dimension.*
(3) *We have* $\operatorname{hch.dim}(A \otimes B) = \operatorname{hch.dim} A + \operatorname{hch.dim} B$.

Proof. The proofs follow from Lemma 2. □

REMARK 2. For some special classes of algebras we have some other nice properties of the Hochschild cohomology dimension. If A is hereditary then $\operatorname{hch.dim} A \leqslant 1$;

https://doi.org/10.1112/S002461070602299X Published online by Cambridge University Press

in this case $\operatorname{hch.dim} A = 0$ if and only if Q is a tree (cf. [**13**, Proposition 1.6 and Corollary 1.6]). If A is radical square zero then $\operatorname{hch.dim} A = 0$ if and only if Q is a tree (cf. [**13**, Proposition 2.3]).

3. *Relation with global dimension*

In this section we consider the relation between the Hochschild (co)homology dimension and the global dimension of an algebra. This is the main part of the paper.

3.1. *Happel's question*

The question posed by Happel [**13**] was whether $\operatorname{hch.dim} A < \infty$ *if and only if* $\operatorname{gl.dim} A < \infty$.

Let $Q = (Q_0, Q_1)$ where Q_0 (respectively Q_1) is the set of vertices (respectively arrows) in Q. For any arrow $a \in Q_1$ denote by $s(a)$ and $e(a)$ the starting point and the ending point of a respectively. A path $a_1 \dots a_n$ with $n \geqslant 1$ and $\{a_1, \dots, a_n\} \subseteq Q_1$ is called an *oriented cycle* or a *cycle* of length n if $s(a_i) = e(a_{i+1})$ for $i = 1, \dots, n-1$ and $s(a_n) = e(a_1)$. Happel's question has a positive answer for commutative algebras and *truncated quiver algebras*, that is, the algebras kQ/k^nQ where $n \geqslant 2$ and k^nQ denotes the ideal of kQ generated by all paths of length at least n (cf. [**8**]).

PROPOSITION 3 [**2**, Corollary]. *Let A be a commutative algebra. Then the following conditions are equivalent:*

(1) $\operatorname{hch.dim} A < \infty$;
(2) $\operatorname{hch.dim} A = 0$;
(3) $\operatorname{gl.dim} A < \infty$;
(4) $\operatorname{gl.dim} A = 0$.

PROPOSITION 4 [**20**, Theorem 3]. *Let $\operatorname{char} k = 0$ and A be a truncated quiver algebra. Then the following are equivalent:*

(1) $\operatorname{hch.dim} A < \infty$;
(2) $\operatorname{gl.dim} A < \infty$;
(3) *the quiver of A has no oriented cycles.*

However, in general Happel's question does not always have a positive answer.

3.2. *Pathological algebras*

Let $A := A_q := k\langle x, y\rangle/(x^2, xy + qyx, y^2)$ with $q \in k$. The algebra A is very pathological for some special q. It is used to construct counterexamples for a few problems:

(1) By constructing a minimal projective resolution of A over A^e, Buchweitz, Green, Madsen and Solberg [**5**] show that $\operatorname{hch.dim} A = 2$ in the case q is neither zero nor a root of unity, but $\operatorname{gl.dim} A = \infty$. This provides a negative answer to Happel's question.

(2) In the case q is not a root of unity, the trivial extension of A_q is a local symmetric algebra whose AR-quiver contains a bounded infinite DTr-orbit. Thus Liu and Schulz [**18**] give a negative answer to Ringel's problem: whether the number

https://doi.org/10.1112/S002461070602299X Published online by Cambridge University Press

of modules having the same length in a connected component of the AR-quiver is always finite.

(3) Rickard (see [**29**]) showed that A_q and $A_{q'}$ are socle equivalent self-injective algebras which are not stably equivalent in the case $q' \neq q$ or q^{-1}.

We refer to [**27**, p. 498, Remark] for more aspects. Though these algebras are quite pathological in many aspects, their Hochschild homology does not behave pathologically any more.

PROPOSITION 5. *For any $q \in k$,* $\mathrm{hh.dim}\, A = \infty = \mathrm{gl.dim}\, A$.

Proof. Recall the minimal projective resolution constructed in [**5**].

For any $n \geqslant 0$ construct $\{f_i^n\}_{i=0}^n \subseteq A^{\otimes n}$: first $\{f_0^0\} \subseteq A^{\otimes 0} = k$ with $f_0^0 := 1$; second $f_0^1 := x$ and $f_1^1 := y$; third $f_{-1}^n := 0 =: f_{n+1}^n$; and for any $n \geqslant 2$, $f_i^n := f_{i-1}^{n-1} \otimes y + q^i f_i^{n-1} \otimes x$ with $i \in \{0, 1, \ldots, n\}$. Here A is identified with $k \otimes_k A$. Note that $f_i^n = x \otimes f_i^{n-1} + q^{n-i} y \otimes f_{i-1}^{n-1}$.

Let $P^n := \bigoplus_{i=0}^n A \otimes_k f_i^n \otimes_k A \subseteq A^{\otimes(n+2)}$ for all $n \geqslant 0$. Set $\tilde{f}_i^n := 1 \otimes f_i^n \otimes 1$ for $n \geqslant 1$ and $\tilde{f}_0^0 := 1 \otimes 1$. Define $\delta^n : P^n \to P^{n-1}$ by letting

$$\delta^n(\tilde{f}_i^n) := x\tilde{f}_i^{n-1} + q^{n-i} y \tilde{f}_{i-1}^{n-1} + (-1)^n \tilde{f}_{i-1}^{n-1} y + (-1)^n q^i \tilde{f}_i^{n-1} x.$$

Then (P^*, δ^*) is a minimal projective resolution of A over A^e.

Applying the functor $A \otimes_{A^e} -$ to the minimal projective resolution we have $A \otimes_{A^e} (P^*, \delta^*) = (\bigoplus_{i=0}^* A \otimes_k f_i^*, \tau^*)$ where τ^n is defined by

$$\tau^n(\lambda \otimes f_i^n) = \lambda x \otimes f_i^{n-1} + (-1)^n q^i x\lambda \otimes f_i^{n-1} + q^{n-i}\lambda y \otimes f_{i-1}^{n-1} + (-1)^n y\lambda \otimes f_{i-1}^{n-1}.$$

Note that A has a basis $\{1, y, x, yx\}$. Thus $A \otimes_k f_i^n$ has a basis $\{1 \otimes f_i^n, y \otimes f_i^n, x \otimes f_i^n, yx \otimes f_i^n\}$. Furthermore $\bigoplus_{i=0}^n A \otimes_k f_i^n$ has a basis $\{1 \otimes f_i^n, y \otimes f_i^n, x \otimes f_i^n, yx \otimes f_i^n\}_{i=0}^n$. If $n \geqslant 1$ then $\bigoplus_{i=0}^{n-1} A \otimes_k f_i^{n-1}$ has a basis $\{1 \otimes f_i^{n-1}, y \otimes f_i^{n-1}, x \otimes f_i^{n-1}, yx \otimes f_i^{n-1}\}_{i=0}^{n-1}$. Moreover

$$\begin{aligned}
\tau^n(1 \otimes f_i^n) &= (1 + (-1)^n q^i)(x \otimes f_i^{n-1}) + (q^{n-i} + (-1)^n)(y \otimes f_{i-1}^{n-1}),\\
\tau^n(y \otimes f_i^n) &= (1 + (-1)^{n+1} q^{i+1})(yx \otimes f_i^{n-1}),\\
\tau^n(x \otimes f_i^n) &= (-q^{n-i+1} + (-1)^n)(yx \otimes f_{i-1}^{n-1})
\end{aligned}$$

and $\tau^n(yx \otimes f_i^n) = 0$. Write τ^n as a matrix. It is not difficult to see that

$$\mathrm{rank}(\tau^n) \leqslant \begin{cases} 2n-1, & \text{if } n \text{ odd;} \\ 2n+1, & \text{if } n \text{ even.} \end{cases}$$

Thus

$$\mathrm{rank}(\tau^{n+1}) \leqslant \begin{cases} 2n+3, & \text{if } n \text{ odd;} \\ 2n+1, & n \text{ even.} \end{cases}$$

Therefore

$$\begin{aligned}
hh_n(A) &= \dim_k \mathrm{Ker}(\tau^n) - \dim_k \mathrm{Im}(\tau^{n+1})\\
&= \dim_k P^n - \dim_k \mathrm{Im}(\tau^n) - \dim_k \mathrm{Im}(\tau^{n+1})\\
&= 4(n+1) - \mathrm{rank}(\tau^n) - \mathrm{rank}(\tau^{n+1})\\
&\geqslant 4(n+1) - (4n+2)\\
&= 2.
\end{aligned}$$

https://doi.org/10.1112/S002461070602299X Published online by Cambridge University Press

Indeed $hh_n(A) \geqslant 2$ for all n since

$$hh_0(A) = \dim_k A/[A,A] = \begin{cases} 4, & \text{if } q = -1; \\ 3, & \text{if } q \neq -1. \end{cases}$$ □

3.3. Proposition 5 suggests posing a seemingly more reasonable conjecture by replacing hch.dim A in Happel's question with hh.dim A. Indeed the following result, which is essentially due to Keller, makes us expect more. Denote by E the maximal semisimple subalgebra of A, that is, $E := kQ_0$, and by r the Jacobson radical of A. Then one has $A = E \oplus r$.

PROPOSITION 6. *If* $\operatorname{gl.dim} A < \infty$ *then*

$$hh_i(A) = \begin{cases} |Q_0|, & \text{if } i = 0; \\ 0, & \text{if } i \geqslant 1. \end{cases}$$

In particular, $\operatorname{hh.dim} A = 0$.

Proof. If $\operatorname{gl.dim} A < \infty$ then by [**17**, Proposition 2.5] we have $HC_n(A,r) = 0$ for all $n \geqslant 0$. Using the long exact sequence

$$\begin{gathered} \cdots \to HH_n(A,r) \to HC_n(A,r) \to HC_{n-2}(A,r) \to \cdots \\ \cdots \to HH_2(A,r) \to HC_2(A,r) \to HC_0(A,r) \to HH_1(A,r) \\ \to HC_1(A,r) \to 0 \to HH_0(A,r) \to HC_0(A,r) \to 0 \end{gathered}$$

(cf. [**21**, E2.2.4]), we have $HH_n(A,r) = 0$ for all $n \geqslant 0$. Using the long exact sequence

$$\begin{gathered} \cdots \to HH_n(A,r) \to HH_n(A) \to HH_n(E) \to \cdots \\ \cdots \to HH_0(A,r) \to HH_0(A) \to HH_0(E) \to 0 \end{gathered}$$

(cf. [**21**, Section 1.1.16]), we have $HH_n(A) = HH_n(E)$ for all $n \geqslant 0$. Thus the proposition follows from

$$hh_i(E) = \begin{cases} |Q_0| & \text{if } i = 0; \\ 0 & \text{if } i \geqslant 1 \end{cases}$$

(cf. [**21**, E1.1.1]). □

3.4. The conjecture is stated as follows.

CONJECTURE 1. The following conditions are equivalent:
(1) $\operatorname{hh.dim} A < \infty$;
(2) $\operatorname{hh.dim} A = 0$;
(3) $\operatorname{gl.dim} A < \infty$.

3.5. Now we show that Conjecture 1 holds for commutative algebras. We refer to [**24**] for commutative algebra theory. The following result is essentially due to Avramov and Vigueé-Poirrier.

THEOREM 1. *Let A be a commutative algebra. Then the following conditions are equivalent:*

https://doi.org/10.1112/S002461070602299X Published online by Cambridge University Press

(1) hh.dim $A < \infty$;
(2) hh.dim $A = 0$;
(3) gl.dim $A < \infty$;
(4) gl.dim $A = 0$.

Proof. By Proposition 6 it is enough to show that (1) $\Rightarrow$ (4). Since A is commutative, it is the direct product of local algebras. Thus without loss of generality we may assume that A is local. If hh.dim $A < \infty$ then by [**3**, Theorem] A is smooth. By [**24**, p. 216, Lemma 1], A is regular. Furthermore gl.dim $A = 0$ by Serre's theorem (cf. [**24**, Theorem 19.2]). □

3.6. Now we prove that Conjecture 1 holds for *monomial algebras* (sometimes called *zero relation algebras*), that is, the algebras $A = kQ/I$ with I generated by some paths of length at least 2.

A cycle $a_1 \dots a_n$ with $\{a_1, \dots, a_n\} \subseteq Q_1$ is said to be *basic* if the vertices $s(a_1), \dots, s(a_n)$ are distinct from each other (cf. [**19**]). A cycle is said to be *proper* if it is not a power ($\geqslant 2$) of any cycle (cf. [**9**]). Note that proper cycles are called basic cycles in [**28**]. Denote by Q_n^c (respectively Q_n^p) the set of cycles (respectively proper cycles) of length n and $Q^c := \bigcup_{n\geqslant 1} Q_n^c$ (respectively $Q^p := \bigcup_{n\geqslant 1} Q_n^p$). The truncated quiver algebra whose quiver is a basic cycle is called a *truncated basic cycle algebra*. The Hochschild homology of a truncated basic cycle is known (cf. [**28**] and [**30**]). If A is a monomial algebra then A can be naturally defined to be an $\mathbb{N}_0$-graded algebra by using the length of the paths. Thus the ith Hochschild homology group $HH_n(A)$ becomes an $\mathbb{N}_0$-graded k-vector space by $\deg_{\mathbb{N}_0} a_0 \otimes \dots \otimes a_n := \sum_{i=0}^{n} \deg_{\mathbb{N}_0} a_i$ for $a_i \in A, i = 0, 1, \dots, n$. The degree j part of $HH_i(A)$ is denoted by $HH_{i,j}(A)$. Let C_q be the cyclic group of order q generated by g. Then C_q acts on Q_q^c by $g \cdot (a_1 \dots a_{q-1} a_q) := (a_q a_1 \dots a_{q-1})$. Under this action the orbit of the cycle Y is denoted by O_Y.

PROPOSITION 7 [**28**, Theorem 2]. *Let $A = kQ/k^nQ$ be a truncated quiver algebra and $q = cn + e$ for $0 \leqslant e \leqslant n-1$. Then*

$$HH_{p,q}(A) = \begin{cases} k^{a_q}, & \\ & \text{if } 1 \leqslant e \leqslant n-1 \text{ and } 2c \leqslant p \leqslant 2c+1; \\ \bigoplus_{r|q} \left(k^{(n,r)-1} \oplus \operatorname{Ker}\left(\cdot \frac{n}{(n,r)} : k \to k \right) \right)^{b_r}, & \\ & \text{if } e = 0 \text{ and } 0 < 2c-1 = p; \\ \bigoplus_{r|q} \left(k^{(n,r)-1} \oplus \operatorname{Coker}\left(\cdot \frac{n}{(n,r)} : k \to k \right) \right)^{b_r}, & \\ & \text{if } e = 0 \text{ and } 0 < 2c = p; \\ k^{|Q_0|}, & \\ & \text{if } p = q = 0; \\ 0, & \\ & \text{otherwise.} \end{cases}$$

Here $a_q := |Q_q^c/C_q|$, $(n, r) := \gcd(n, r)$ and $b_r := |Q_r^p/C_r|$.

https://doi.org/10.1112/S002461070602299X Published online by Cambridge University Press

COROLLARY 1. *Let $A = kQ/k^nQ$ where Q is a basic cycle of length l. Then*

$$hh_p(A) = \begin{cases} l + \left[\dfrac{n-1}{l}\right], \\ \qquad \text{if } p = 0; \\ \left[\dfrac{[p/2]n+n-1}{l}\right] - \left[\dfrac{[p/2]n}{l}\right], \\ \qquad \text{if } (p \geqslant 1) \wedge \left(l \not\left| \left[\dfrac{p+1}{2}\right] n\right.\right); \\ \left[\dfrac{[p/2]n+n-1}{l}\right] - \left[\dfrac{[p/2]n}{l}\right] + (n,l) - 1, \\ \qquad \text{if } (p \geqslant 1) \wedge \left(l \left| \left[\dfrac{p+1}{2}\right] n\right.\right) \wedge \left(\operatorname{char} k \not\left| \dfrac{n}{(n,l)}\right.\right); \\ \left[\dfrac{[p/2]n+n-1}{l}\right] - \left[\dfrac{[p/2]n}{l}\right] + (n,l), \\ \qquad \text{if } (p \geqslant 1) \wedge \left(l \left| \left[\dfrac{p+1}{2}\right] n\right.\right) \wedge \left(\operatorname{char} k \left| \dfrac{n}{(n,l)}\right.\right). \end{cases}$$

Here $[x]$ denotes the largest integer i satisfying $i \leqslant x$.

COROLLARY 2. *If $A = kQ/k^nQ$ where Q is a basic cycle of length l, then* $\operatorname{hh.dim} A = \infty$.

Proof. If $(n,l) \geqslant 2$ then for any $p = 2ml - 1$ with $m \geqslant 1$ one has $hh_p(A) \geqslant 1$. If $(n,l) = 1$ and $l \leqslant n-1$ then

$$\left[\frac{[p/2]n+n-1}{l}\right] > \left[\frac{[p/2]n}{l}\right]$$

for any $p \geqslant 1$. Thus $hh_p(A) \geqslant 1$ for any $p \geqslant 1$. If $(n,l) = 1$ and $l > n-1$ then there are integers u and v such that $un + vl = 1$. Thus for any $p = 2(ml + u - 1)$ with $m \gg 0$ one has

$$\left[\frac{[p/2]n+n-1}{l}\right] = \left[\frac{[p/2]n+n-un-vl}{l}\right] = mn - v$$

and

$$\left[\frac{[p/2]n}{l}\right] < mn - v.$$

Therefore $hh_p(A) \geqslant 1$. □

Let $A = kQ/I$ be a monomial algebra. Let $C' = a_1 \dots a_n$ be a cycle in Q. Then there is a basic cycle quiver C with $|C_0| = n$ and a natural quiver morphism $C \to C' \to Q$. We define the relations on C by pulling back the relations in Q, that is, a path in C is a relation if and only if its image in Q is a relation. The algebra Z given by the quiver C and the relations is called a *cycle algebra overlying* A. A cycle algebra Z overlying A is said to be *minimal* if Z is not a covering of a smaller cycle algebra overlying A (cf. [**15**]). For covering theory we refer to [**12**] and [**23**].

https://doi.org/10.1112/S002461070602299X Published online by Cambridge University Press

Clearly minimal cycle algebras overlying A are just those cycle algebras overlying A constructed from proper cycles in Q.

Denote by $HC_n(A)$ the nth cyclic homology group of A (cf. [**21**]). Let $hc_n(A) := \dim_k HC_n(A)$.

THEOREM 2. *Let A be a monomial algebra. Then*

$$hh_n(A) = \begin{cases} \sum_Z hh_n(Z), & n \geqslant 1 \\ \sum_Z hh_n(Z) - \sum_Z |(Q_Z)_0| + |Q_0|, & n = 0 \end{cases}$$

and

$$hc_n(A) = \begin{cases} \sum_Z hc_n(Z), & n \text{ odd} \\ \sum_Z hc_n(Z) - \sum_Z |(Q_Z)_0| + |Q_0|, & n \text{ even} \end{cases}$$

where the index Z runs through all minimal cycle algebra over A and Q_Z denotes the quiver of Z.

Proof. By [**9**, Lemma 2.3] the Hochschild homology and the cyclic homology of A are just the Hochschild homology and the cyclic homology of the *E-normalized mixed complex* $\bar{C}_E(A) = (A \otimes_{E^e} (r^{\otimes_E n}), b, B)$ with b the Hochschild boundary and $B : A \otimes_{E^e} (r^{\otimes_E n}) \to A \otimes_{E^e} (r^{\otimes_E n+1})$ given by the formula

$$B(a_0, a_1, \ldots, a_n) = \sum_{i=0}^{n} (-1)^{in} (1, a_i, \ldots, a_n, (a_0)_r, \ldots, a_{i-1}),$$

where $a = a_E + a_r$ is the decomposition of an element a of A in the direct sum $A = E \oplus r$. Define the *E-reduced normalized mixed complex* ${}_{\text{red}}\bar{C}_E(A)$ by the split exact sequence of mixed complexes $0 \to \bar{C}_E(E) \to \bar{C}_E(A) \to {}_{\text{red}}\bar{C}_E(A) \to 0$ (cf. [**9**, p. 135]). Its homology and cyclic homology are called the nth *E-reduced Hochschild and cyclic homology groups* of A and denoted by ${}_{\text{red}}HH_n(A)$ and ${}_{\text{red}}HC_n(A)$ respectively.

In the case $n \geqslant 1$, since the set $\{(a_0, r_1, \ldots, r_n) \mid a_0$ a path in A; $r_1, \ldots, r_n$ paths in r; $a_0 r_1 \ldots r_n$ a cycle in $Q\}$ is a k-basis of $A \otimes_{E^e} (r^{\otimes_E n})$,

$$A \otimes_{E^e} (r^{\otimes_E n}) = \bigoplus_{(a_0, r_1, \ldots, r_n)} k(a_0 \otimes r_1 \otimes \ldots \otimes r_n),$$

where $(a_0, r_1, \ldots, r_n)$ runs through all path sequences with $a_0 r_1 \ldots r_n \in Q^c$. Next, organize $(a_0, r_1, \ldots, r_n)$ according to the cycle $a_0 r_1 \ldots r_n$, we have

$$A \otimes_{E^e} (r^{\otimes_E n}) = \bigoplus_{O_Y \text{ with } Y \in Q^p} \bigoplus_{(a_0, r_1, \ldots, r_n)} k(a_0 \otimes r_1 \otimes \ldots \otimes r_n),$$

where $(a_0, r_1, \ldots, r_n)$ runs through all path sequences with $a_0 r_1 \ldots r_n = X^m$ for some $X \in O_Y$ and some $m \geqslant 1$. According to the correspondence between minimal cycle algebras Z overlying A and the orbits of proper cycles in Q, we further have

$$A \otimes_{E^e} (r^{\otimes_E n}) \cong \bigoplus_{\text{minimal cycle algebra } Z \text{ overlying } A} Z \otimes_{F^e} (s^{\otimes_F n}),$$

https://doi.org/10.1112/S002461070602299X Published online by Cambridge University Press

where F is the maximal semisimple subalgebra of Z and s is the Jacobson radical of Z.

In the case $n = 0$, the k-vector space

$$r \otimes_{E^e} E = \bigoplus_{r_0 \in Q^c} k(r_0) = \bigoplus_{O_Y \text{ with } Y \in Q^p} \bigoplus_{r_0} k(r_0),$$

where r_0 runs through all cycles with $r_0 = X^m$ for some $X \in O_Y$ and some $m \geqslant 1$. Further

$$r \otimes_{E^e} E \cong \bigoplus_{\text{minimal cycle algebra } Z \text{ overlying } A} s \otimes_{F^e} F.$$

Clearly these isomorphisms induce an isomorphism of mixed complex from ${}_{\mathrm{red}}\bar{C}_E(A)$ to $\bigoplus_{Z\ \mathrm{red}} \bar{C}_F(Z)$ where Z runs through all minimal cycle algebras over A. By [**21**, Proposition 2.5.15] we have ${}_{\mathrm{red}}HH_n(A) \cong \bigoplus_{Z\ \mathrm{red}} HH_n(Z)$ and ${}_{\mathrm{red}}HC_n(A) \cong \bigoplus_{Z\ \mathrm{red}} HC_n(Z)$ where the index Z runs through all minimal cycle algebras over A. By [**9**, Lemma 2.4], we are done. □

LEMMA 3. *If the algebra A' is obtained from the algebra A by dropping an indecomposable projective A-module whose radical is also projective, then*

$$hh_n(A) = \begin{cases} hh_n(A'), & n \geqslant 1; \\ hh_0(A') + 1, & n = 0. \end{cases}$$

Proof. The inclusion map $A' \to A$ induces a natural chain map from the long exact sequence $\cdots \to HH_n(A, r) \to HH_n(A) \to HH_n(E) \to \cdots$ to the long exact sequence $\cdots \to HH_n(A', r') \to HH_n(A') \to HH_n(E') \to \cdots$, where r' denotes the Jacobson radical of A'. Thus the lemma follows from [**15**, Theorem 2.3]. □

ALGORITHM. The Hochschild homology of any monomial algebra A can be computed as follows:

(1) Determine the set $\mathcal{O}$ of the orbits of proper cycles in the quiver of A.

(2) Determine the set $\mathcal{M}$ of the minimal cycle algebras overlying A corresponding to the elements in $\mathcal{O}$.

(3) Apply the reduction of dropping projective modules in [**15**, p. 512], Lemma 3 and Corollary 1 to obtain $HH_n(Z)$ for each $Z \in \mathcal{M}$ and for all $n \geqslant 0$.

(4) Apply Theorem 2 to obtain $HH_n(A)$ for all $n \geqslant 0$.

In the case $\operatorname{char} k = 0$, the cyclic homology of A can be read off:

(5) Apply [**21**, Theorem 4.1.13] to obtain $HC_n(A)$.

THEOREM 3. *Let A be a monomial algebra. Then the following conditions are equivalent:*

(1) $\operatorname{hh.dim} A < \infty$;

(2) $\operatorname{hh.dim} A = 0$;

(3) $\operatorname{gl.dim} A < \infty$.

Proof. By Proposition 6 it is enough to show that (1) $\Rightarrow$ (3). If $\operatorname{gl.dim} A = \infty$ then by [**15**, Proposition 3.2] there is a minimal cycle algebra Z with $\operatorname{gl.dim} Z = \infty$. Via a series of dropping projective modules, Z can be reduced to a non-semisimple truncated basic cycle algebra Z' (cf. [**15**, p. 512]). By Corollary 2 and

https://doi.org/10.1112/S002461070602299X Published online by Cambridge University Press

[**15**, Theorem 2.3], one has $\mathrm{hh.dim}\, Z = \mathrm{hh.dim}\, Z' = \infty$. Thus by Theorem 2 one has $\mathrm{hh.dim}\, A \geqslant \mathrm{hh.dim}\, Z = \infty$. This is a contradiction. □

REMARK 3. Essentially, the equivalence between parts (2) and (3) of Theorem 3 is the main result of [**15**].

It is well known that the algebras whose Gabriel quiver have no oriented cycles are of finite global dimension (cf. [**10**, Corollary 6]) and Hochschild homology dimension 0 (cf. [**7**]). We show in the following that the converse statements hold for truncated quiver algebras.

THEOREM 4. *Let A be a truncated quiver algebra* (*in particular a radical square zero algebra*). *Then the following conditions are equivalent:*

(1) $\mathrm{hh.dim}\, A < \infty$;
(2) $\mathrm{hh.dim}\, A = 0$;
(3) $\mathrm{gl.dim}\, A < \infty$;
(4) *Q has no oriented cycles.*

Proof. (4) ⇒ (3). This is well known (cf. [**10**, Corollary 6]). By Proposition 6 it is enough to show that (1) ⇒ (4). If Q has an oriented cycle then we take the shortest cycle Q' in Q which must be a basic cycle. Let $I' = I \cap kQ'$. Then $Z := kQ'/I'$ is a truncated basic cycle algebra. By [**19**, Theorem 2], $HH_n(Z)$ is a direct summand of $HH_n(A)$ for all $n \geqslant 0$. By Corollary 2 we have $\mathrm{hh.dim}\, Z = \infty$ which implies $\mathrm{hh.dim}\, A = \infty$. This is a contradiction. □

REMARK 4. In general the finiteness of the global dimension does not imply that the Gabriel quiver of the algebra has no oriented cycle. The simplest counterexample is the algebra $A = kQ/I$ where $Q_0 = \{1, 2\}$, $Q_1 = \{\alpha : 1 \to 2,\ \beta : 2 \to 1\}$ and $I = \langle \beta\alpha \rangle$. Clearly $\mathrm{gl.dim}\, A = 2$.

References

1. M. AUSLANDER, I. REITEN and S. O. SMALØ, *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics 36 (Cambridge University Press, Cambridge, 1995).
2. L. L. AVRAMOV and S. IYENGAR, 'Gaps in Hochschild cohomology imply smoothness for commutative algebras', *Math. Res. Lett.* 12 (2005) 789–804.
3. L. L. AVRAMOV and M. VIGUEÉ-POIRRIER, 'Hochschild homology criteria for smoothness', *Int. Math. Res. Not.* 1 (1992) 17–25.
4. J. L. BLOCK, 'Cyclic homology of filtered algebras', *K-Theory* 1 (1987) 515–518.
5. R. O. BUCHWEITZ, E. L. GREEN, D. MADSEN and Ø. SOLBERG, 'Finite Hochschild cohomology without finite global dimension', *Math. Res. Lett.* 12 (2005) 805–816.
6. H. CARTAN and S. EILENBERG, *Homological algebra* (Princeton University Press, Princeton, NJ, 1956).
7. C. CIBILS, 'Hochschild homology of an algebra whose quiver has no oriented cycles', *Representation Theory, I*, Lecture Notes in Mathematics 1177 (Springer, Berlin, 1986) 55–59.
8. C. CIBILS, 'Rigidity of truncated quiver algebras', *Adv. Math.* 79 (1990) 18–42.
9. C. CIBILS, 'Cyclic and Hochschild homology of 2-nilpotent algebras', *K-Theory* 4 (1990) 131–141.
10. S. EILENBERG, H. NAGAO and T. NAKAYAMA, 'On the dimension of modules and algebras. IV. Dimension of residue rings of hereditary rings', *Nagoya Math. J.* 10 (1956) 87–95.
11. P. GABRIEL, 'Auslander-Reiten sequences and representation-finite algebras', *Representation Theory I*, Lecture Notes in Mathematics 831 (Springer, Berlin, 1980) 1–71.

https://doi.org/10.1112/S002461070602299X Published online by Cambridge University Press

12. P. Gabriel, 'The universal cover of a representation-finite algebra', *Representations of algebras*, Lecture Notes in Mathematics 903 (Springer, Berlin, 1980) 68–105.
13. D. Happel, 'Hochschild cohomology of finite-dimensional algebras', *Séminaire d'algèbre Paul Dubreil et Marie-Paul Malliavin, 39ème année*, Paris, 1987/1988, Lecture Notes in Mathematics 1404 (Springer, Berlin, 1989) 108–126.
14. G. Hochschild, 'On the cohomology groups of an associate algebra', *Ann. of Math.* (2) 46 (1945) 58–67.
15. K. Igusa and D. Zacharia, 'On the cyclic homology of monomial relation algebras', *J. Algebra* 151 (1992) 502–521.
16. B. Keller, 'Invariance of cyclic homology under derived equivalence', CMS Conf. Proc. 18 (American Mathematical Society, Providence, RI, 1996) 353–361.
17. B. Keller, 'Invariance and localization for cyclic homology of DG algebras', *J. Pure Appl. Algebra* 123 (1998) 223–273.
18. S. P. Liu and R. Schulz, 'The existence of bounded infinite DTr-orbits', *Proc. Amer. Math. Soc.* 122 (1994) 1003–1005.
19. S. X. Liu and P. Zhang, Hochschild homology of truncated algebras, *Bull. London Math. Soc.* 26 (1994) 427–430.
20. A. C. Locateli, 'Hochschild cohomology of truncated quiver algebras', *Comm. Algebra* 27 (1999) 645–664.
21. J. L. Loday, *Cyclic homology*, Grundlehren der Mathematischen Wissenschaften 301 (Springer, Berlin, 1992).
22. S. MacLane, *Homology*, Grundlehren der Mathematischen Wissenschaften 114, third corrected printing (Springer, Berlin, 1975).
23. R. Martinez-Villa and J. A. de la Peña, 'The universal cover of a quiver with relations', *J. Pure Appl. Algebra* 30 (1983) 277–292.
24. H. Matsumura, *Commutative ring theory*, translated by M. Reid (Cambridge University Press, Cambridge, 1986).
25. S. Michelena and M. Platzeck, 'Hochschild cohomology of triangular matrix algebras', *J. Algebra* 233 (2000) 502–525.
26. J. Rickard, 'Derived equivalences as derived functors', *J. London Math. Soc.* (2) 43 (1991) 37–48.
27. R. Schulz, 'A non-projective module without self-extensions', *Arch. Math.* (*Basel*) 62 (1994) 497–500.
28. E. Sköldberg, 'The Hochschild homology of truncated and quadratic monomial algebras', *J. London Math. Soc.* (2) 59 (1999) 76–86.
29. A. Skowroński and K. Yamagata, 'Socle deformations of self-injective algebras', *J. London Math. Soc.* (2) 72 (1996) 545–566.
30. P. Zhang, 'The Hochschild homology of a truncated basic cycle', *Algebra Colloq.* 5 (1998) 77–84.

Yang Han
Institute of Systems Science
Academy of Mathematics and Systems Science
Chinese Academy of Sciences
Beijing 100080
PR China

hany@iss.ac.cn

https://doi.org/10.1112/S002461070602299X Published online by Cambridge University Press