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MOMENTS FOR MULTIDIMENSIONAL MANDELBROT CASCADES

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Abstract

We consider the distributional equation $\boldsymbol{Z} \stackrel{\mathrm{D}}{=} \sum_{k=1}^{N} \boldsymbol{A}_k \boldsymbol{Z}(k)$, where N is a random variable taking value in $\mathbb{N}_0 = \{0, 1, \ldots\}$, $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots$ are $p \times p$ nonnegative random matrices, and $\boldsymbol{Z}, \boldsymbol{Z}(1), \boldsymbol{Z}(2), \ldots$, are independent and identically distributed random vectors in $\mathbb{R}_+^p$ with $\mathbb{R}_+ = [0, \infty)$, which are independent of $(N, \boldsymbol{A}_1, \boldsymbol{A}_2, \ldots)$. Let $\{\boldsymbol{Y}_n\}$ be the multidimensional Mandelbrot martingale defined as sums of products of random matrices indexed by nodes of a Galton–Watson tree plus an appropriate vector. Its limit $\boldsymbol{Y}$ is a solution of the equation above. For $\alpha > 1$, we show a sufficient condition for $\mathbb{E}\|\boldsymbol{Y}\|^\alpha \in (0, \infty)$. Then for a nondegenerate solution $\boldsymbol{Z}$ of the distributional equation above, we show the decay rates of $\mathbb{E}\mathrm{e}^{-\boldsymbol{t}\cdot\boldsymbol{Z}}$ as $\|\boldsymbol{t}\| \to \infty$ and those of the tail probability $\mathbb{P}(\boldsymbol{y}\cdot\boldsymbol{Z} \le x)$ as $x \to 0$ for given $\boldsymbol{y} = (y^1, \ldots, y^p) \in \mathbb{R}_+^p$, and the existence of the harmonic moments of $\boldsymbol{y}\cdot\boldsymbol{Z}$. As an application, these results concerning the moments (of positive and negative orders) of $\boldsymbol{Y}$ are applied to a special multitype branching random walk. Moreover, for the case where all the vectors and matrices of the equation above are complex, a sufficient condition for the L^α convergence and the αth-moment of the Mandelbrot martingale $\{\boldsymbol{Y}_n\}$ are also established.

Keywords: Moments; Mandelbrot martingales; multiplicative cascades; harmonic moments; multibranching random walks

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1. Introduction

We consider a multidimensional Mandelbrot martingale $\{\boldsymbol{Y}_n\}$ defined as sums of products of random matrices (weights) indexed by nodes of a Galton–Watson tree plus an appropriate vector. We are interested in the existence of the moments of positive and negative orders of its limit $\boldsymbol{Y}$. For the one-dimensional case, the classical model of Mandelbrot [34] corresponds to the case where the tree is a fixed r-ary tree ($r \ge 2$ being a constant), and all the weights are one-dimensional random variables. This classical model and its variations were studied by many authors in different contexts; see, for example, Bingham and Doney [14], [15] for branching processes and general age-dependent branching processes, Barral [3], Guivarc´h [20], and Kahane and Peyrière [27] for multiplicative cascades, Biggins [7]–[9] and Biggins and Kyprianou [11] for branching random walks, Durrett and Liggett [19] for some infinite particle systems, and Rösler [39] for the Quicksort algorithm. A general one-dimensional model (called Mandelbrot cascades), which unifies the study of cascades and branching random walks, was presented by Liu [32], where a number of applications were shown. The model considered here is a generalization of the model presented in [32] to the multidimensional case. Similar to the one-dimensional case, our model corresponds to a multitype branching random walk,

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which has attracted recent attention in the literature; see, for example, Biggins [10], Biggins and Rahimzadeh Sani [12], and Kyprianou and Rahimzadeh Sani [29].

The model is described as follows. We consider the distributional equation of $\boldsymbol{Z}$,

$$\boldsymbol{Z} \stackrel{\mathrm{D}}{=} \sum_{k=1}^{N} \boldsymbol{A}_k \boldsymbol{Z}(k), \tag{1.1}$$

where N is a random variable taking value in $\mathbb{N}_0 = \{0, 1, \ldots\}$, $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots,$ are $p \times p$ nonnegative random matrices, $\boldsymbol{Z}, \boldsymbol{Z}(1), \boldsymbol{Z}(2), \ldots$, which are independent of $(N, \boldsymbol{A}_1, \boldsymbol{A}_2, \ldots)$, are independent and identically distributed (i.i.d.) random vectors in $\mathbb{R}_+^p$ with $\mathbb{R}_+ = [0, \infty)$. We say that a matrix $\boldsymbol{A}$ is *finite* if all entries of $\boldsymbol{A}$ are finite, and say that $\boldsymbol{A}$ is *strictly positive* if for some positive integer n, all entries of $\boldsymbol{A}^n$ are positive. When a matrix $\boldsymbol{A}$ is finite and strictly positive, the Perron–Frobenius theorem shows that $\boldsymbol{A}$ has a positive maximal eigenvalue ρ and has associated positive right and left eigenvectors $\boldsymbol{v} = (v_1, \ldots, v_p)$ and $\boldsymbol{u} = (u_1, \ldots, u_p)$. Moreover, $\boldsymbol{u}$ and $\boldsymbol{v}$ can be normalized such that $\sum_{i=1}^p u_i = \sum_{i=1}^p u_i v_i = 1$. Throughout this paper the following assumption holds.

Assumption 1.1. *The matrix $\boldsymbol{M} := \mathbb{E} \sum_{k=1}^N \boldsymbol{A}_k$ is finite and strictly positive with the maximum-modulus eigenvalue 1 and the corresponding left and right eigenvectors $\boldsymbol{U} = (U_1, \ldots, U_p)$ and $\boldsymbol{V} = (V_1, \ldots, V_p)$ normalized such that $\sum_{i=1}^p U_i = \sum_{i=1}^p U_i V_i = 1$.*

If $\boldsymbol{A}_k$ are i.i.d. and independent of N, this is the case considered in Buraczewski *et al.* [16], where sufficient and necessary conditions were shown for nondegeneracy of the solution of (1.1) and its positive moments. The right-tail behavior was also studied when N is a constant. In this paper we consider the more general model in which we do not care about the relations among N and $\boldsymbol{A}_k$. We shall focus on the existence of the solution of the αth-moment ($\alpha > 1$) of (1.1), and, furthermore, the existence of its harmonic moments. As shown in the literature concerning the one-dimensional case, such moment results would be helpful in order to study related asymptotic properties such as tail probabilities and large deviations.

First, we note that there naturally exists a solution of (1.1). In fact, we can construct a solution (denoted by $\boldsymbol{Y}$) in the following way. Let $\mathbb{N} = \{1, 2, \ldots\}$ and write

$$I = \bigcup_{n=0}^{\infty} \mathbb{N}^n$$

for the set of all finite sequences $u = u_1 \cdots u_n$ with $u_i \in \mathbb{N}$, where by convention $\mathbb{N}^0 = \{\varnothing\}$ contains the null sequence $\varnothing$. If $u = u_1 \cdots u_n \in I$, we write $|u| = n$ for the length of u; if $u = u_1 \cdots u_n, v = v_1 \cdots v_m \in I$, we write $uv = u = u_1 \cdots u_n v_1 \cdots v_m$ for the sequences obtained by juxtaposition. In particular, $u\varnothing = \varnothing u = u$.

Let $\{(N_u, \boldsymbol{A}_{u1}, \boldsymbol{A}_{u2}, \ldots)\}$ be a family of independent copies of $(N, \boldsymbol{A}_1, \boldsymbol{A}_2, \ldots)$ indexed by the finite sequence $u \in I$. For simplicity, we write $(N, \boldsymbol{A}_1, \boldsymbol{A}_2, \ldots)$ for $(N_\varnothing, \boldsymbol{A}_{\varnothing 1}, \boldsymbol{A}_{\varnothing 2}, \ldots)$. Let $\mathbb{T}$ be the Galton–Watson tree with defining elements (N_u) $(u \in I)$:

(i) $\varnothing \in \mathbb{T}$;

(ii) if $u \in \mathbb{T}$ then $uk \in \mathbb{T}$ if and only if $1 \le k \le N_u$;

(iii) if $uk \in \mathbb{T}$ then $u \in \mathbb{T}$.

Here the null sequence $\varnothing$ is the root of the tree $\mathbb{T}$, which can be regarded as the initial particle, uk represents the kth child of u, and N_u represents the number of offspring of the particle u.

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Each node of the tree $\mathbb{T}$ is marked with the random vector $(N_u, \boldsymbol{A}_{u1}, \boldsymbol{A}_{u2}, \ldots)$. We can imagine that the random matrix $\boldsymbol{A}_{uk}$ is the 'weight' associated with the edge (u, uk) linking the nodes u and uk if $u \in \mathbb{T}$ and $1 \le k \le N_u$; the values $\boldsymbol{A}_{uk}$ for $k > N_u$ are of no influence for our purpose, and will be taken as 0 for convenience.

Let $\mathbb{T}_n = \{u \in \mathbb{T}\colon |u| = n\}$ be the set of a sequence u in $\mathbb{T}$ with length $|u| = n$. Put

$$\boldsymbol{X}_{\varnothing} = \boldsymbol{I} \quad \text{and} \quad \boldsymbol{X}_u = \boldsymbol{A}_{u_1}\boldsymbol{A}_{u_1u_2}\cdots\boldsymbol{A}_{u_1\cdots u_n} \quad \text{if } u = u_1\cdots u_n \in I \text{ for } n \ge 1,$$

and define

$$\boldsymbol{Y}_0 = \boldsymbol{V} \quad \text{and} \quad \boldsymbol{Y}_n = \sum_{u\in\mathbb{T}_n} \boldsymbol{X}_u \boldsymbol{V} \quad \text{for } n \ge 1. \tag{1.2}$$

It is not difficult to verify that $\boldsymbol{Y}_n = (Y_{n,1}, \ldots, Y_{n,p})$ is a nonnegative martingale with respect to the filtration

$$\mathcal{F}_n = \sigma((N_u, \boldsymbol{A}_{u1}, \boldsymbol{A}_{u2}, \ldots)\colon |u| < n),$$

the σ-field that contains all information up to generation n. We call $\{\boldsymbol{Y}_n\}$ a *multidimensional Mandelbrot martingale*. It reduces to the classical Mandelbrot martingale when the dimension $p = 1$. Clearly, there exists a nonnegative random vector $\boldsymbol{Y} = (Y_1, \ldots, Y_p) \in \mathbb{R}_+^p$ such that

$$\boldsymbol{Y} = \lim_{n\to\infty} \boldsymbol{Y}_n \quad \text{almost surely (a.s.)}$$

with $\mathbb{E}Y_i \le V_i$ for all $1 \le i \le p$ by Fatou's lemma. Note that

$$\boldsymbol{Y}_n = \sum_{u\in\mathbb{T}_1} \boldsymbol{A}_u \boldsymbol{Y}_{n-1}(u), \tag{1.3}$$

where $\{\boldsymbol{Y}_n(u)\}$ $(u \in \mathbb{T}_k)$ are independent copies of $\boldsymbol{Y}_n$ and independent of $\mathcal{F}_k$. Denote $\boldsymbol{Y}(u) = \lim_{n\to\infty} \boldsymbol{Y}_n(u)$. Letting $n \to \infty$ in (1.3), we have

$$\boldsymbol{Y} = \sum_{k=1}^{N} \boldsymbol{A}_k \boldsymbol{Y}(k), \tag{1.4}$$

which means that $\boldsymbol{Y}$ is a solution of (1.1).

Example 1.1. (*Multitype branching random walk.*) We define a multitype branching random walk (MBRW) with p-types as follows. A single particle $\varnothing$ of type $i \in \{1, 2, \ldots, p\}$ is located at the origin of real line $\mathbb{R}$. It gives birth to children of the first generation, which are scattered on $\mathbb{R}$, according to a vector point process $\boldsymbol{L}_i = (L_{i1}, L_{i2}, \ldots, L_{ip})$, where L_{ij} is the point process counting the number of particles of type $j \in \{1, 2, \ldots, p\}$ born to the particle of type i. These particles of the first generation reproduce particles to form the second generation. The displacements of the offspring of a particle of type j, relative to their parent's position, are given by the point process $\boldsymbol{L}_j$. These particles of the second generation reproduce children to form the next generation, and so on. All particles behave independently. We denote the position of a particle u by S_u and the type of u by $\tau(u)$, then the position of uk, the kth child of u, satisfies

$$S_{uk} = S_u + l_{uk},$$

where l_{uk} denotes the displacement of uk relative to u whose distribution is determined by $L_{\tau(u)\tau(uk)}$.

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Assume that $N_i := \sum_{j=1}^{p} L_{ij}(\mathbb{R})$ has the same distribution for all $1 \le i \le p$, which means that all particles produce offspring according to the same distribution if we do not care about the type. Under this assumption, all particles $u \in I$ associated with the number of their offspring N_u form a Galton–Watson tree $\mathbb{T}$. We remark that this assumption is not necessary in usual MBRWs, so the example presented here is just a special case of an MBRW. For more information and results about the usual MBRW; see [10], [12], and [29].

For $t \in \mathbb{R}$, define the matrix $\tilde{\boldsymbol{M}}(t) = (\tilde{M}_{ij}(t))$ as

$$\tilde{M}_{ij}(t) = \mathbb{E} \sum_{\substack{u \in \mathbb{T}_1 \\ \tau(u)=j}} \mathrm{e}^{-tS_u} \quad (\tau(\varnothing) = i).$$

Assume that $\tilde{\boldsymbol{M}}(t)$ defined above is finite and strictly positive. Denote the positive maximal eigenvalue of $\tilde{\boldsymbol{M}}(t)$ by $\tilde{\rho}(t)$ and the associated normalized positive left and right eigenvectors by $\tilde{\boldsymbol{U}}(t) = (\tilde{U}_1(t), \dots, \tilde{U}_p(t))$ and $\tilde{\boldsymbol{V}}(t) = (\tilde{V}_1(t), \dots, \tilde{V}_p(t))$, respectively. For each $i = 1, 2, \dots, p$, let

$$W_{n,i}(t) := \frac{\sum_{u \in \mathbb{T}_n} \tilde{V}_{\tau(u)}(t) \mathrm{e}^{-tS_u}}{\tilde{V}_i(t) \tilde{\rho}(t)^n} \qquad (\tau(\varnothing) = i).$$

It is known that for each $i = 1, 2, \dots, p$, $\{W_{n,i}(t)\}$ forms a nonnegative martingale with mean 1, hence it converges a.s. to a nonnegative random variable $W_i(t)$ with $\mathbb{E}W_i(t) \le 1$. Write

$$\boldsymbol{Y}_n = (W_{n,1}(t)\tilde{V}_1(t), W_{n,2}(t)\tilde{V}_2(t), \dots, W_{n,p}(t)\tilde{V}_p(t)).$$

We can verify that the martingale $\{\boldsymbol{Y}_n\}$ is just the Mandelbrot martingale defined in (1.2) if we put the random matrix $\boldsymbol{A}_k = ((\boldsymbol{A}_k)_{ij})$ with

$$(\boldsymbol{A}_k)_{ij} = \frac{\mathrm{e}^{-tS_k}}{\tilde{\rho}(t)} \mathbf{1}_{\{\tau(k)=j, \tau(\varnothing)=i\}},$$

and the limit of $\boldsymbol{Y}_n$ is $\boldsymbol{Y} = (W_1(t)\tilde{V}_1(t), W_2(t)\tilde{V}_2(t), \dots, W_p(t)\tilde{V}_p(t))$.

2. Main results

Let $\boldsymbol{Y}$ be the limit of the Mandelbrot martingale $\{\boldsymbol{Y}_n\}$. We first discuss the existence of the αth-moment ($\alpha > 1$) of $\boldsymbol{Y}$, which implies its nondegeneracy. Define the matrix $\boldsymbol{M}_n(t) = ((\boldsymbol{M}_n(t))_{ij})$ as

$$(\boldsymbol{M}_n(t))_{ij} := \mathbb{E} \sum_{u \in \mathbb{T}_n} [(\boldsymbol{X}_u)_{ij}]^t.$$

When $\boldsymbol{M}_n(t)$ is finite and strictly positive, we denote its maximum-modulus eigenvalue by $\rho_n(t)$ and the corresponding normalized positive left and right eigenvectors by $\boldsymbol{U}_n(t) = (U_{n,1}(t), \dots, U_{n,p}(t))$ and $\boldsymbol{V}_n(t) = (V_{n,1}(t), \dots, V_{n,p}(t))$ satisfying

$$\sum_{i=1}^{p} U_{n,i}(t) = \sum_{i=1}^{p} U_{n,i}(t) V_{n,i}(t) = 1.$$

We declare that throughout this paper the notation norm $\|\boldsymbol{A}\|$ represents any one of the matrix norms if $\boldsymbol{A}$ is a matrix, and $\|\boldsymbol{u}\| = \sum_{j=1}^{p} |u_j|$ is the L^1-norm of $\boldsymbol{u} = (u_1, \dots, u_p)$ if $\boldsymbol{u}$ is a vector.

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Theorem 2.1. (Moments.) *Let* $\alpha > 1$.

(i) *If* $\mathbb{E}\| \sum_{k=1}^{N} \boldsymbol{A}_k\|^\alpha < \infty$ *and* $p^{(\alpha-1)}\rho_n(\alpha) < 1$ *for some positive integer* n, *then*

$$0 < \mathbb{E}\|\boldsymbol{Y}\|^\alpha < \infty \quad \textit{and} \quad \mathbb{E}\boldsymbol{Y} = \boldsymbol{V}.$$

(ii) *Conversely, if* $0 < \mathbb{E}\|\boldsymbol{Y}\|^\alpha < \infty$ *then* $\mathbb{E}\| \sum_{k=1}^{N} \boldsymbol{A}_k\|^\alpha < \infty$ *and* $\rho_n(\alpha) \le 1$ *for all* $n \ge 1$. *If, additionally,*

$$\mathbb{P}(\textit{for all } k \in \{1, 2, \ldots, N\},\ \boldsymbol{A}_k \textit{ has a positive column vector}) > 0, \tag{2.1}$$

then $\rho_n(\alpha) < 1$ *for all* $n \ge 1$.

Remark 2.1. (i) For $\alpha > 1$, under Assumption 1.1, the condition $\mathbb{E}\| \sum_{k=1}^{N} \boldsymbol{A}_k\|^\alpha < \infty$ ensures that $\boldsymbol{M}_n(t)$ is finite and strictly positive for all $t \in [1, \alpha]$ and for all n, so that $\rho_n(t)$ exists. We can see that

$$\rho(t)^n \le \rho_n(t) \le p^{(t-1)(n-1)}\rho(t)^n.$$

Besides, we remark that the condition $\mathbb{E}\| \sum_{k=1}^{N} \boldsymbol{A}_k\|^\alpha < \infty$ is equivalent to $\mathbb{E}\|\boldsymbol{Y}_1\|^\alpha < \infty$.

(ii) Under Assumption 1.1, $\mathbb{E}\|\boldsymbol{Y}\|^\alpha > 0$ is equivalent to $\mathbb{E}(Y_i)^\alpha > 0$ for all $i \in \{1, \ldots, p\}$. Indeed, by (1.4), we see that $\mathbb{E}\boldsymbol{Y}$ is an eigenvector associated to the eigenvalue 1 if it is nonzero, i.e. $\mathbb{E}\boldsymbol{Y} \neq \boldsymbol{0}$, so that $\mathbb{E}\boldsymbol{Y} = c\boldsymbol{V}$ for some constant $c > 0$, which implies that $\mathbb{E}\boldsymbol{Y}$ is positive.

In Theorem 2.1(i) we show a sufficient condition for the existence of the αth-moment ($\alpha > 1$) of $\boldsymbol{Y}$, or, equivalently, the L^α convergence of the martingale $\{\boldsymbol{Y}_n\}$ to its limit $\boldsymbol{Y}$. Similar results were obtained by many authors in special cases and in different contexts; see, for example, [16], [27], [32] for multiplicative cascades, [14], [15] for branching process and the Crump–Mode–Jagers process, [2], [8], [9], [12] for branching random walks, [19] for smoothing processes, [36] for random fractals, and [26] for branching processes in random environments.

If $\mathbb{E}(Y_i)^\alpha < \infty$ for some $\alpha > 1$, it is obvious that $\mathbb{E}Y_i = V_i$ and $\mathbb{P}(Y_i > 0) > 0$. As $\boldsymbol{Y}$ is a solution of (1.1), Theorem 2.1(i) in fact also gives the existence of a nontrivial solution of (1.1). Moreover, if $p^{(\alpha-1)}\rho_n(\alpha) < 1$ for some positive integer n, Theorem 2.1 implies that $0 < \mathbb{E}\|\boldsymbol{Y}\|^\alpha < \infty$ if and only if $\mathbb{E}\|\boldsymbol{Y}_1\|^\alpha < \infty$, which reveals that $\boldsymbol{Y}_1$ and $\boldsymbol{Y}$ would have the same asymptotic properties. In particular, for $p = 1$, if $\mathbb{P}(\text{for all } k \in \{1, 2, \ldots, N\},\ A_k > 0) > 0$, from Theorem 2.1, we see that

$$0 < \mathbb{E}Y^\alpha < \infty \quad \text{if and only if } \mathbb{E}Y_1^\alpha < \infty \text{ and } \rho_1(\alpha) < 1.$$

This result was obtained by Liu [32, Theorem 2.1] with the help of a size-biased measure. Here, in our proof, we shall present a different idea based on inequalities for martingales. Our method, which is available for both $p = 1$ and $p > 1$, also avoids the trouble of finding an appropriate size-biased measure for the case where $p > 1$. We mention that this method can also be applied to the complex case where $\boldsymbol{A}_k$ are complex random matrices and $\boldsymbol{Z}$ and $\boldsymbol{Z}(k)$ are complex random vectors; see Section 6.

For the special case where $\boldsymbol{A}_k$ are i.i.d. and independent of N with a change of measure, Buraczewski *et al.* [16, Theorem 2.7] showed (under a certain hypothesis) that, for $\alpha > 1$,

$$\kappa(\alpha)\mathbb{E}N < 1 \quad \text{if and only if } 0 < \mathbb{E}\|\boldsymbol{Y}\|^\alpha < \infty,$$

where $\kappa(s) := \lim_{n\to\infty} \mathbb{E}\|\boldsymbol{A}_1 \cdots \boldsymbol{A}_n\|^s$. Comparing this result with Theorem 2.1, we discover that $\rho_n(\alpha) \le [\kappa(\alpha)\mathbb{E}N]^n \le p^{\alpha-1}\rho_n(\alpha)$ for all n; therefore, $p^{\alpha-1}\rho_n(\alpha) < 1$ implies that $\kappa(\alpha)\mathbb{E}N < 1$.

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Now we consider the existence of the harmonic moments of $\boldsymbol{Y}$, i.e. $\mathbb{E}(Y_i)^{-\lambda} < \infty$ for each $i \in \{1, 2, \dots, p\}$, where $\lambda > 0$. We shall deal with a more general case with a general nontrivial solution of (1.1), denoted still by $\boldsymbol{Z}$ instead of $\boldsymbol{Y}$. Note that if $\boldsymbol{Z}$ is a nontrivial solution of (1.1) with finite mean $\mathbb{E}\|\boldsymbol{Z}\| < \infty$ then, according to [16], $\boldsymbol{Z}$ is proportional to $\boldsymbol{Y}$.

Let $\boldsymbol{Z}$ be a nontrivial solution of (1.1). So we have $\mathbb{P}(\boldsymbol{Z} > \boldsymbol{0}) > 0$, where $\boldsymbol{Z} > \boldsymbol{0}$ means that $Z_i > 0$ for all $i = 1, 2, \dots, p$. Assume that (2.1) holds, and

$$\mathbb{P}(N = 0) = 0, \qquad \mathbb{P}(N = 1) < 1. \tag{2.2}$$

In fact, the assumption in (2.1) is used to ensure that the probability $\mathbb{P}(\boldsymbol{Z} = \boldsymbol{0})$ is a solution of $f(q) = q$, where $f(s) = \mathbb{E}s^N$ $(0 \le s \le 1)$ is the generating function of N. Since $\mathbb{P}(\boldsymbol{Z} = \boldsymbol{0}) < 1$, under the assumptions in (2.2) and by the uniqueness of solution, we have $\mathbb{P}(\boldsymbol{Z} = \boldsymbol{0}) = 0$, or, namely, $\mathbb{P}(\boldsymbol{Z} > \boldsymbol{0}) = 1$.

Let

$$\phi(\boldsymbol{t}) = \mathbb{E}\mathrm{e}^{-\boldsymbol{t}\cdot\boldsymbol{Z}}, \qquad \boldsymbol{t} = (t_1, \dots, t_p) \in \mathbb{R}_+^p,$$

be the Laplace transform of $\boldsymbol{Z}$, where we write $\boldsymbol{u} \cdot \boldsymbol{v} = \sum_{j=1}^p u_j v_j$ for the inner product of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. We are interested in the decay rate of $\phi(\boldsymbol{t})$ as $\|\boldsymbol{t}\| \to \infty$ and that of the left-tail probability $\mathbb{P}(\boldsymbol{y} \cdot \boldsymbol{Z} \le x)$ as $x \to 0$ for given $\boldsymbol{y} = (y_1, \dots, y_p) \in \mathbb{R}_+^p$, as well as the harmonic moment $\mathbb{E}(\boldsymbol{y} \cdot \boldsymbol{Z})^{-\lambda}$ for $\lambda > 0$. Set

$$\underline{m} := \operatorname{ess\,inf} N.$$

Then $\underline{m} \ge 1$ since $\mathbb{P}(N = 0) = 0$. We have the following result.

Theorem 2.2. (Harmonic moments.) *Assume that (2.1) and (2.2) hold. Write* $a_{ij} = (\boldsymbol{A}_1)_{ij}$.

(i) *If*

$$\mathbb{E}\biggl(\min_i \sum_{j=1}^p a_{ij}\biggr)^{-\lambda} < \infty, \qquad \mathbb{E}\biggl[\biggl(\min_i \sum_{j=1}^p a_{ij}\biggr)^{-\lambda} \mathbf{1}_{\{N=1\}}\biggr] < 1 \quad \textit{for some } \lambda > 0,$$

then $\phi(\boldsymbol{t}) = O(\|\boldsymbol{t}\|^{-\lambda})$ $(\|\boldsymbol{t}\| \to \infty)$, *and for every fixed nonzero* $\boldsymbol{y} = (y_1, \dots, y_p) \in \mathbb{R}_+^p$,

$$\mathbb{P}(\boldsymbol{y} \cdot \boldsymbol{Z} \le x) = O(x^{\lambda}) \quad (x \to 0), \qquad \mathbb{E}(\boldsymbol{y} \cdot \boldsymbol{Z})^{-\lambda_1} < \infty \quad (0 < \lambda_1 < \lambda).$$

(ii) *If, additionally,* $\underline{m} > 1$ *and* $\mathbb{E}[\prod_{i=1}^m (\min_i \sum_{j=1}^p (\boldsymbol{A}_k)_{ij})^{-\lambda}] < \infty$, *then we have* $\phi(\boldsymbol{t}) = O(\|\boldsymbol{t}\|^{-\underline{m}\lambda})$ $(\|\boldsymbol{t}\| \to \infty)$,

$$\mathbb{P}(\boldsymbol{y} \cdot \boldsymbol{Z} \le x) = O(x^{\underline{m}\lambda}) \quad (x \to 0), \qquad \mathbb{E}(\boldsymbol{y} \cdot \boldsymbol{Z})^{-\underline{m}\lambda_1} < \infty \quad (0 < \lambda_1 < \lambda).$$

The results for the harmonic moment would be useful for the multifractal analysis of Mandelbrot measures as in [3], [37]. Similar conclusions in different contexts were shown in [3], [18], [33], [37] for multiplicative cascades, [38] for Galton–Watson processes, and [21], [25] for branching processes in random environments.

For $p = 1$, Theorem 2.2 coincides with the results of Liu [33, Theorems 2.1 and 2.4]. Similar to [33, Theorem 2.5] we also have the following results concerning the exponential decay rate of $\phi(\boldsymbol{t})$. The same results were well known for Galton–Watson processes (see [13] and [22]) and for Mandelbrot cascades (see [23] and [33]).

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Theorem 2.3. (Exponential case.) *Assume that* $\underline{m} \geq 2$ *and* $\min_{i,j}(A_k)_{ij} \geq \underline{a}$ *a.s. for some constant* $\underline{a} > 0$ *and all* $1 \leq k \leq \underline{m}$, *and* $\mathbb{P}(\boldsymbol{Y}_1 = \boldsymbol{V}) < 1$. *Write* $\gamma = -\log \underline{m} / \log(\underline{a} p)$. *Then* $0 < \gamma < 1$ *and the following assertions hold.*

(i) *There exists a constant* $C_1 > 0$ *such that for all* $\|\boldsymbol{t}\| > 0$ *large enough,*

$$\phi(\boldsymbol{t}) \leq \exp\{-C_1 \|\boldsymbol{t}\|^{\gamma}\},$$

and for every fixed nonzero $\boldsymbol{y} = (y_1, \ldots, y_p) \in \mathbb{R}_+^p$, *there exists a constant* $C_{1,\boldsymbol{y}} > 0$ *such that for all* $x > 0$ *small enough,*

$$\mathbb{P}(\boldsymbol{y} \cdot \boldsymbol{Z} \leq x) \leq \exp\{-C_{1,\boldsymbol{y}} x^{-\gamma/(1-\gamma)}\}.$$

(ii) *For* $\varepsilon > 0$ *satisfying* $(\underline{a} + \varepsilon) p \underline{m} < 1$, *if*

$$\mathbb{P}(N = \underline{m}, \max_{ij}(A_k)_{ij} \leq \underline{a} + \varepsilon \text{ for all } 1 \leq k \leq \underline{m}) > 0,$$

then there exists a constant $C_2 > 0$ *such that for all* $\|\boldsymbol{t}\| > 0$ *large enough,*

$$\phi(\boldsymbol{t}) \geq \exp\{-C_2 \|\boldsymbol{t}\|^{\gamma(\varepsilon)}\},$$

and for every fixed $\boldsymbol{y} = (y_1, \ldots, y_p) \in \mathbb{R}_+^p$, *there exists a constant* $C_{2,\boldsymbol{y}} > 0$ *such that for all* $x > 0$ *small enough,*

$$\mathbb{P}(\boldsymbol{y} \cdot \boldsymbol{Z} \leq x) \geq \exp\{-C_{2,\boldsymbol{y}} x^{-\gamma(\varepsilon)/(1-\gamma(\varepsilon))}\},$$

where $\gamma(\varepsilon) = -\log \underline{m} / \log[(\underline{a} + \varepsilon) p] \in (0, 1)$.

Finally, as applications of the moment results for the limit $\boldsymbol{Y}$ of the Mandelbrot martingale $\{\boldsymbol{Y}_n\}$, we consider the MBRW described in Example 1.1 and show the sufficient conditions for the existence of moments (of positive and negative orders) of $W_i(t)$ for each $i = 1, 2, \ldots, p$ and for $t \in \mathbb{R}$ fixed. For MBRW, it is obvious that (2.1) is satisfied. Applying Theorem 2.1 yields the result for the moments of positive orders, and Theorem 2.2 yields the result for the moments of negative orders.

Corollary 2.1. (Application to MBRW.) *We consider the MBRW described in Example 1.1.*

(i) *Let* $\alpha > 1$. *If* $\max_i \mathbb{E}(W_{1,i}(t))^{\alpha} < \infty$ *and* $p^{\alpha-1} \tilde{\rho}(\alpha t)/\tilde{\rho}(t)^{\alpha} < 1$, *then we have* $\max_i \mathbb{E}[W_i(t)]^{\alpha} < \infty$.

(ii) *Assume that (2.2) holds. Denote by* S_1^i *the displacement of the first child 1 of the initial particle* $\varnothing$ *of type* $i \in \{1, \ldots, p\}$. *Let* $\lambda > 0$. *If* $\max_i \mathbb{E} e^{-(\lambda+\varepsilon)t S_1^i} < \infty$ *and* $\mathbb{E} \max_i e^{-(\lambda+\varepsilon) S_1^i} \mathbf{1}_{\{N=1\}} < 1$ *for some* $\varepsilon > 0$, *then* $\max_i \mathbb{E}[W_i(t)]^{-\lambda} < \infty$.

In Corollary 2.1(i) we give a sufficient condition for the existence of the αth-moment of $W_i(t)$. In fact, if we deal with the martingale $\{W_{n,i}(t)\}$ directly following the ideas in the proof of Theorem 2.1, the condition $p^{\alpha-1} \tilde{\rho}(\alpha t)/\tilde{\rho}(t)^{\alpha} < 1$ can be weakened to $\tilde{\rho}(\alpha t)/\tilde{\rho}(t)^{\alpha} < 1$ (see Huang [24], where the authors show that $\max_i \mathbb{E}(W_{1,i}(t))^{\alpha} < \infty$ and $\tilde{\rho}(\alpha t)/\tilde{\rho}(t)^{\alpha} < 1$ is a necessary and sufficient condition for $\max_i \mathbb{E}[W_i(t)]^{\alpha} < \infty$).

The rest of this paper is arranged as follows. In the next section we shall establish two auxiliary inequalities for the martingale $\{\boldsymbol{Y}_n^{(t)}\}$, which will be used in Section 4 for the proof of Theorem 2.1. In Section 5 we shall prove Theorems 2.2 and 2.3. Finally, we shall consider the complex case in Section 6, where we shall show sufficient conditions for the L^{α} convergence and the αth-moment of the Mandelbrot martingale $\{\boldsymbol{Y}_n\}$.

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3. The martingale $\{Y_n^{(t)}\}$

In the proof of Theorem 2.1 we shall follow the lines of Huang and Liu [26] or Alsmeyer *et al.* [2]. In contrast to [26], the exposition here will be simpler since there is no affection of environments, but we should pay attention to eigenvalues of the related matrices to confront the rise of the dimension.

For $t \in \mathbb{R}$ fixed, define the random matrix $\boldsymbol{A}_k^{(t)} = ((\boldsymbol{A}_k^{(t)})_{ij})$ with $(\boldsymbol{A}_k^{(t)})_{ij} := [(\boldsymbol{A}_k)_{ij}]^t$. Write

$$\boldsymbol{M}(t) = \boldsymbol{M}_1(t) = \mathbb{E}\sum_{k=1}^{N} \boldsymbol{A}_k^{(t)},$$

its positive maximal eigenvalue $\rho(t) = \rho_1(t)$, and the corresponding normalized positive left and right eigenvectors $\boldsymbol{U}(t) = \boldsymbol{U}_1(t)$ and $\boldsymbol{V}(t) = \boldsymbol{V}_1(t)$. Define

$$\boldsymbol{Y}_0^{(t)} = \boldsymbol{V}(t) \quad \text{and} \quad \boldsymbol{Y}_n^{(t)} = \frac{\sum_{u\in\mathbb{T}_n} \boldsymbol{X}_u^{(t)} \boldsymbol{V}(t)}{\rho(t)^n} \quad \text{for } n \geq 1,$$

where $\boldsymbol{X}_{\varnothing}^{(t)} = \boldsymbol{I}$ and $\boldsymbol{X}_u^{(t)} = \boldsymbol{A}_{u_1}^{(t)} \boldsymbol{A}_{u_1u_2}^{(t)} \cdots \boldsymbol{A}_{u_1\cdots u_n}^{(t)}$ if $u = u_1 \cdots u_n \in I$ for $n \geq 1$. Clearly, $\boldsymbol{Y}_n^{(t)} = (Y_{n,1}^{(t)}, \ldots, Y_{n,p}^{(t)})$ is a nonnegative martingale with mean $\boldsymbol{V}(t)$; hence, it converges a.s. to a random vector $\boldsymbol{Y}^{(t)} = (Y_1^{(t)}, \ldots, Y_p^{(t)})$. In particular, when $t = 1$, we have $\boldsymbol{X}_u^{(1)} = \boldsymbol{X}_u$, $\rho(1) = 1$, and $\boldsymbol{V}(1) = \boldsymbol{V}$; hence, $\boldsymbol{Y}_n^{(1)} = \boldsymbol{Y}_n$ and $\boldsymbol{Y}^{(1)} = \boldsymbol{Y}$. The critical idea of the proof of Theorem 2.1 is to note the double martingale structure (cf. [1] for more information) of the martingale $\{\boldsymbol{Y}_n^{(t)}\}$ and apply the following Burkholder inequality.

Lemma 3.1. (Burkholder's inequality; see, for example, [17].) *Let $\{S_n\}$ be an L^1 one-dimensional martingale with $S_0 = 0$. Let*

$$Q_n = \left(\sum_{k=1}^{n} (S_k - S_{k-1})^2\right)^{1/2}, \qquad Q = \left(\sum_{n=1}^{\infty} (S_n - S_{n-1})^2\right)^{1/2}.$$

Then for all $\alpha > 1$,

$$c_\alpha \parallel Q_n \parallel_\alpha \leq \parallel S_n \parallel_\alpha \leq C_\alpha \parallel Q_n \parallel_\alpha, \qquad c_\alpha \parallel Q \parallel_\alpha \leq \sup_n \parallel S_n \parallel_\alpha \leq C_\alpha \parallel Q \parallel_\alpha,$$

where $\| \cdot \|_\alpha := (\mathbb{E}|\cdot|^\alpha)^{1/\alpha}$, $c_\alpha = (\alpha - 1)/18\alpha^{3/2}$, and $C_\alpha = 18\alpha^{3/2}/(\alpha - 1)^{1/2}$.

With the help of Burkholder's inequality, we obtain two lemmas (inequalities) below to the martingale $\{\boldsymbol{Y}_n^{(t)}\}$ which will be used in the proof of Theorem 2.1.

Lemma 3.2. *Let $\alpha > 1$. Fix $t \in \mathbb{R}$. If $\max_i \mathbb{E}[Y_{1,i}^{(t)}]^\alpha < \infty$ then for each $i = 1, \ldots, p$ and $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$,*

(i) *for $\alpha \in (1, 2]$,*

$$\mathbb{E}|Y_{n+1,i}^{(t)} - Y_{n,i}^{(t)}|^\alpha \leq C p^{(\alpha-1)n} \left[\frac{\rho(\alpha t)}{\rho(t)^\alpha}\right]^n; \tag{3.1}$$

(ii) *for $\alpha > 2$,*

$$\mathbb{E}|Y_{n+1,i}^{(t)} - Y_{n,i}^{(t)}|^\alpha \leq C p^{\alpha n/2} \left[\frac{\rho(2t)^{\alpha/2}}{\rho(t)^\alpha}\right]^n \mathbb{E}[Y_{n,i}^{(2t)}]^{\alpha/2}, \tag{3.2}$$

where C is a constant depending on α, p, and t (and it may be different from line to line).

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Proof. We need to consider only $n \geq 1$ since the results hold naturally for $n = 0$. Decompose $Y_{n,i}^{(t)}$ as

$$Y_{n,i}^{(t)} = \frac{1}{\rho(t)^n} \sum_{j=1}^{p} \sum_{u \in \mathbb{T}_n} (\boldsymbol{X}_u^{(t)})_{ij} V_j(t) = \frac{1}{\rho(t)^n} \sum_{j=1}^{p} \sum_{u \in \mathbb{T}_{n-1}} (\boldsymbol{X}_u^{(t)})_{ij} Y_{1,j}^{(t)}(u),$$

where $\boldsymbol{Y}_1^{(t)}(u)$ is a version of $\boldsymbol{Y}_1^{(t)}$ at root u. Hence,

$$Y_{n+1,i}^{(t)} - Y_{n,i}^{(t)} = \frac{1}{\rho(t)^n} \sum_{j=1}^{p} \sum_{u \in \mathbb{T}_n} (\boldsymbol{X}_u^{(t)})_{ij} [Y_{1,j}^{(t)}(u) - V_j(t)].$$

By Burkholder's inequality,

$$\begin{aligned} \mathbb{E}|Y_{n+1,i}^{(t)} - Y_{n,i}^{(t)}|^\alpha &\leq \frac{p^{\alpha-1}}{\rho(t)^{\alpha n}} \sum_{j=1}^{p} \mathbb{E}\bigg| \sum_{u \in \mathbb{T}_n} (\boldsymbol{X}_u^{(t)})_{ij} [Y_{1,j}^{(t)}(u) - V_j(t)] \bigg|^\alpha \\ &\leq \frac{C}{\rho(t)^{\alpha n}} \sum_{j=1}^{p} \mathbb{E}\bigg(\sum_{u \in \mathbb{T}_n} [(\boldsymbol{X}_u^{(t)})_{ij}]^2 [Y_{1,j}^{(t)}(u) - V_j(t)]^2 \bigg)^{\alpha/2}. \end{aligned}$$

Noting the fact that

$$\begin{aligned} &\mathbb{E}\bigg(\sum_{u \in \mathbb{T}_n} [(\boldsymbol{X}_u^{(t)})_{ij}]^2 [Y_{1,j}^{(t)}(u) - V_j(t)]^2 \bigg)^{\alpha/2} \\ &\quad \leq \begin{cases} \mathbb{E}\bigg(\sum_{u \in \mathbb{T}_n} [(\boldsymbol{X}_u^{(t)})_{ij}]^\alpha \bigg) \mathbb{E}|Y_{1,j}^{(t)} - V_j(t)|^\alpha & \text{for } \alpha \in (1, 2], \\ \mathbb{E}\bigg(\sum_{u \in \mathbb{T}_n} [(\boldsymbol{X}_u^{(t)})_{ij}]^2 \bigg)^{\alpha/2} \mathbb{E}|Y_{1,j}^{(t)} - V_j(t)|^\alpha & \text{for } \alpha > 2, \end{cases} \end{aligned}$$

we have

$$\begin{aligned} &\mathbb{E}|Y_{n+1,i}^{(t)} - Y_{n,i}^{(t)}|^\alpha \\ &\quad \leq \begin{cases} \dfrac{C}{\rho(t)^{\alpha n}} \displaystyle\sum_{j=1}^{p} \mathbb{E}\bigg(\sum_{u \in \mathbb{T}_n} [(\boldsymbol{X}_u^{(t)})_{ij}]^\alpha \bigg) \mathbb{E}|Y_{1,j}^{(t)} - V_j(t)|^\alpha & \text{for } \alpha \in (1, 2], \\ \dfrac{C}{\rho(t)^{\alpha n}} \displaystyle\sum_{j=1}^{p} \mathbb{E}\bigg(\sum_{u \in \mathbb{T}_n} [(\boldsymbol{X}_u^{(t)})_{ij}]^2 \bigg)^{\alpha/2} \mathbb{E}|Y_{1,j}^{(t)} - V_j(t)|^\alpha & \text{for } \alpha > 2. \end{cases} \end{aligned} \tag{3.3}$$

Note that, for $u \in \mathbb{T}_n$,

$$[(\boldsymbol{X}_u^{(t)})_{ij}]^\alpha \leq p^{(\alpha-1)(n-1)} (\boldsymbol{X}_u^{(\alpha t)})_{ij} \quad \text{for all } \alpha > 1,$$

and

$$\sum_{u \in \mathbb{T}_n} (\boldsymbol{X}_u^{(t)})_{ij} \leq \frac{\rho(t)^n}{V_j(t)} Y_{n,i}^{(t)}.$$

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Thus,

$$\mathbb{E}\sum_{u\in\mathbb{T}_n}[(\boldsymbol{X}_u^{(t)})_{ij}]^\alpha \le p^{(\alpha-1)(n-1)}\mathbb{E}\sum_{u\in\mathbb{T}_n}(\boldsymbol{X}_u^{(\alpha t)})_{ij} \le \frac{V_i(\alpha t)}{V_j(\alpha t)}p^{(\alpha-1)(n-1)}\rho(\alpha t)^n. \tag{3.4}$$

Applying (3.4) to the first inequality of (3.3), and noting that $\max_i \mathbb{E}[Y_{1,i}^{(t)}]^\alpha < \infty$, we obtain (3.1). To obtain (3.2), we need only to see that

$$\begin{aligned}\mathbb{E}\bigg(\sum_{u\in\mathbb{T}_n}[(\boldsymbol{X}_u^{(t)})_{ij}]^2\bigg)^{\alpha/2} &\le p^{\alpha(n-1)/2}\mathbb{E}\bigg(\sum_{u\in\mathbb{T}_n}(\boldsymbol{X}_u^{(2t)})_{ij}\bigg)^{\alpha/2}\\ &\le V_j(2t)^{-\alpha/2}p^{\alpha(n-1)/2}\rho(2t)^{\alpha n/2}\mathbb{E}[Y_{n,i}^{(2t)}]^{\alpha/2},\end{aligned}$$

and combine this inequality with the second inequality of (3.3). □

Lemma 3.3. *Let $\alpha > 1$. Fix $t \in \mathbb{R}$. If $\max_i[\mathbb{E}Y_{1,i}^{(t)}]^\alpha < \infty$ then for each $i = 1, \dots, p$ and $n \in \mathbb{N} = \{1, 2, \dots\}$,*

$$\mathbb{E}[Y_{n,i}^{(t)}]^\alpha \le Cn^{1+((2^m-1)/2^m)\alpha}\bigg[\max_{1\le l\le m}\bigg\{1,\, p^{\alpha-1}\frac{\rho(\alpha t)}{\rho(t)^\alpha},\, p^{((2^l-1)/2^l)\alpha}\frac{\rho(2^l t)^{\alpha/2^l}}{\rho(t)^\alpha}\bigg\}\bigg]^n \tag{3.5}$$

for $\alpha \in (2^m, 2^{m+1}]$, where $m \in \mathbb{N}_0$.

Proof. For $m = 0$, $\alpha \in (1, 2]$. Applying Burkholder's inequality to the martingale $\{Y_{n,i}^{(t)}\}$ and by Lemma 3.2, for each $i = 1, \dots, p$ and $n \in \mathbb{N}$,

$$\begin{aligned}\mathbb{E}|Y_{n,i}^{(t)} - 1|^\alpha &\le C\sum_{k=0}^{n-1}\mathbb{E}|Y_{k+1,i}^{(t)} - Y_{k,i}^{(t)}|^\alpha\\ &\le C\sum_{k=0}^{n-1}p^{(\alpha-1)k}\bigg(\frac{\rho(\alpha t)}{\rho(t)^\alpha}\bigg)^k\\ &\le Cn\bigg[\max\bigg\{1,\, p^{\alpha-1}\frac{\rho(\alpha t)}{\rho(t)^\alpha}\bigg\}\bigg]^n.\end{aligned}$$

Thus,

$$\mathbb{E}[Y_{n,i}^{(t)}]^\alpha \le Cn\bigg[\max\bigg\{1,\, p^{\alpha-1}\frac{\rho(\alpha t)}{\rho(t)^\alpha}\bigg\}\bigg]^n.$$

So (3.5) holds for $m = 0$. Now suppose that (3.5) holds for some $m \ge 0$, we shall prove that it still holds for $m + 1$. For $\alpha \in (2^{m+1}, 2^{m+2}]$, we have $\alpha/2 \in (2^m, 2^{m+1}]$. Since $\max_i \mathbb{E}[Y_{1,i}^{(t)}]^\alpha < \infty$ ensures that $\max_i \mathbb{E}[Y_{1,i}^{(2t)}]^{\alpha/2} < \infty$, by induction, we have for each $i = 1, \dots, p$ and $k \in \mathbb{N}$,

$$\begin{aligned}&\mathbb{E}[Y_{k,i}^{(2t)}]^{\alpha/2}\\ &\quad\le Ck^{1+((2^m-1)/(2^{m+1}))\alpha}\bigg[\max_{1\le l\le m}\bigg\{1,\, p^{\alpha/2-1}\frac{\rho(\alpha t)}{\rho(2t)^{\alpha/2}},\, p^{((2^l-1)/2^{l+1})\alpha}\frac{\rho(2^{l+1}t)^{\alpha/2^{l+1}}}{\rho(2t)^{\alpha/2}}\bigg\}\bigg]^k.\end{aligned} \tag{3.6}$$

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Hence, combining (3.2) and (3.6), we obtain for each $i = 1, \dots, p$ and $k \in \mathbb{N}$,

$$\mathbb{E}|Y_{k+1,i}^{(t)} - Y_{k,i}^{(t)}|^\alpha \le Ck^{1+((2^m-1)/(2^{m+1}))\alpha}\left[\max_{1\le l\le m+1}\left\{p^{\alpha-1}\frac{\rho(\alpha t)}{\rho(t)^\alpha}, p^{((2^l-1)/2^l)\alpha}\frac{\rho(2^l t)^{\alpha/2^l}}{\rho(t)^\alpha}\right\}\right]^k. \tag{3.7}$$

By Burkholder's inequality and Minkowski's inequality, and applying (3.7), we have

$$\begin{aligned}
&\mathbb{E}|Y_{n,i}^{(t)} - 1|^\alpha \\
&\quad\le C\left(\sum_{k=0}^{n-1}(\mathbb{E}|Y_{k+1,i}^{(t)} - Y_{k,i}^{(t)}|^\alpha)^{2/\alpha}\right)^{\alpha/2} \\
&\quad\le C\Bigg(1 + \sum_{k=1}^{n-1} k^{(1+((2^m-1)/(2^{m+1}))\alpha)(2/\alpha)} \\
&\qquad\times\left[\max_{1\le l\le m+1}\left\{p^{\alpha-1}\frac{\rho(\alpha t)}{\rho(t)^\alpha}, p^{((2^l-1)/2^l)\alpha}\frac{\rho(2^l t)^{\alpha/2^l}}{\rho(t)^\alpha}\right\}\right]^{2k/\alpha}\Bigg)^{\alpha/2} \\
&\quad\le Cn^{1+((2^m-1)/(2^{m+1}))\alpha+\alpha/2}\left[\max_{1\le l\le m+1}\left\{1, p^{\alpha-1}\frac{\rho(\alpha t)}{\rho(t)^\alpha}, p^{((2^l-1)/2^l)\alpha}\frac{\rho(2^l t)^{\alpha/2^l}}{\rho(t)^\alpha}\right\}\right]^n,
\end{aligned}$$

which implies that (3.5) holds for $m + 1$. This completes the proof. □

Remark 3.1. Lemmas 3.2(ii) and 3.3 also hold with β in place of 2 for any $\beta \in (1, 2]$. To see this fact, observing that

$$\mathbb{E}\left(\sum_{u\in\mathbb{T}_n}[(\boldsymbol{X}_u^{(t)})_{ij}]^2\right)^{\alpha/2} \le \mathbb{E}\left(\sum_{u\in\mathbb{T}_n}[(\boldsymbol{X}_u^{(t)})_{ij}]^\beta\right)^{\alpha/\beta}$$

in the proof of Lemma 3.2, we just need to repeat the proofs of Lemmas 3.2(ii) and 3.3 with β in place of 2 for the case where $\alpha > 2$.

4. Proof of Theorem 2.1

Now we provide the proof of Theorem 2.1 by using the inequalities for the martingale $\{\boldsymbol{Y}_n^{(t)}\}$ (Lemmas 3.2 and 3.3) which are obtained in Section 3.

Proof of Theorem 2.1. The proof of Theorem 2.1(i) is composed of two steps.

Step 1. We shall show that if $\mathbb{E}\|\sum_{k=1}^N \boldsymbol{A}_k\|^\alpha < \infty$ and $p^{(\alpha-1)}\rho(\alpha) < 1$, then for each i, $\mathbb{E}Y_i = V_i$ and $\mathbb{E}(Y_i)^\alpha < \infty$, which implies that $\mathbb{E}\boldsymbol{Y} = \boldsymbol{V}$ and $0 < \mathbb{E}\|\boldsymbol{Y}\|^\alpha < \infty$. In fact, it suffices to prove that $\sup_n \mathbb{E}(Y_{n,i})^\alpha < \infty$ for each i, which is equivalent to $Y_{n,i} \to Y_i$ in L^α, so that $\mathbb{E}Y_i = V_i$ and $0 < \mathbb{E}(Y_i)^\alpha < \infty$. The condition $\mathbb{E}\|\sum_{k=1}^N \boldsymbol{A}_k\|^\alpha < \infty$, or, equivalently, $\max_i \mathbb{E}(Y_{1,i})^\alpha < \infty$, ensures the finiteness of $\boldsymbol{M}(t)$ for all $t \in [1, \alpha]$. Moreover, since $\boldsymbol{M}(1) = \boldsymbol{M}$ is strictly positive, by the log-convexity of $(\boldsymbol{M}(t))_{ij}$, we have $\boldsymbol{M}(t)$ is strictly positive for all $t \in [1, \alpha]$, so $\rho(t)$ exists for all $t \in [1, \alpha]$. For $\alpha \in (1, 2]$, by Burkholder's inequality and Lemma 3.2,

$$\sup_n \mathbb{E}|Y_{n,i} - 1|^\alpha \le C\sum_{n=0}^\infty \mathbb{E}|Y_{n+1,i} - Y_{n,i}|^2 \le C\sum_{n=0}^\infty p^{(\alpha-1)n}\rho(\alpha)^n < \infty.$$

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For $\alpha > 2$, by Burkholder's inequality and Minkowski's inequality,

$$\sup_n \mathbb{E}|Y_{n,i} - 1|^\alpha \le C\left(\sum_{n=0}^{\infty}(\mathbb{E}|Y_{n+1,i} - Y_{n,i}|^\alpha)^{2/\alpha}\right)^{\alpha/2}.$$

We shall show that the series $\sum_{n=0}^{\infty}(\mathbb{E}|Y_{n+1,i} - Y_{n,i}|^\alpha)^{2/\alpha} < \infty$. Observe that we have $\max_i \mathbb{E}[Y_{1,i}^{(2)}]^{\alpha/2} < \infty$ since $\max_i \mathbb{E}(Y_{1,i})^\alpha < \infty$. By Lemma 3.3, for $\alpha \in (2^m, 2^{m+1}]$ $(m \in \mathbb{N}_0)$ and $n \ge 1$,

$$\mathbb{E}[Y_{n,i}^{(2)}]^{\alpha/2} \le Cn^\gamma\left[\max_{1\le l\le m-1}\left\{1,\ p^{\alpha/2-1}\frac{\rho(\alpha)}{\rho(2)^{\alpha/2}},\ p^{((2^l-1)/2^{l+1})\alpha}\frac{\rho(2^{l+1})^{\alpha/2^{l+1}}}{\rho(2)^{\alpha/2}}\right\}\right]^n, \tag{4.1}$$

where $\gamma = 1 + ((2^{m-1} - 1)/2^m)\alpha \le \alpha/2$. By Lemma 3.2 and (4.1),

$$\begin{aligned}\mathbb{E}|Y_{n+1,i} - Y_{n,i}|^\alpha &\le Cp^{\alpha n/2}\rho(2)^{\alpha n/2}\mathbb{E}[Y_{n,i}^{(2)}]^{\alpha/2}\\ &\le Cn^{\alpha/2}\Big[\max_{1\le l\le m}\{p^{\alpha-1}\rho(\alpha),\ p^{((2^l-1)/2^l)\alpha}\rho(2^l)^{\alpha/2^l}\}\Big]^n.\end{aligned}$$

Therefore,

$$\sum_{n=0}^{\infty}(\mathbb{E}|Y_{n+1,i} - Y_{n,i}|^\alpha)^{2/\alpha} \le C\sum_n n\Big[\max_{1\le l\le m}\{p^{\alpha-1}\rho(\alpha),\ p^{((2^l-1)/2^l)\alpha}\rho(2^l)^{\alpha/2^l}\}\Big]^{2n/\alpha}.$$

The series in the right-hand side of the inequality above converges if and only if

$$\max_{1\le l\le m}\{p^{\alpha-1}\rho(\alpha),\ p^{((2^l-1)/2^l)\alpha}\rho(2^l)^{\alpha/2^l}\} < 1. \tag{4.2}$$

Note that $\rho(t)$ is log-convex since $(\boldsymbol{M}(t))_{ij}$ is log-convex (see [28]). We have for all $\beta \in (1, \alpha)$, $\rho(\beta) \le \rho(\alpha)^{(\beta-1)/(\alpha-1)}$. Thus,

$$p^{((\beta-1)/\beta)\alpha}\rho(\beta)^{\alpha/\beta} \le [p^{\alpha-1}\rho(\alpha)]^{\alpha(\beta-1)/\beta(\alpha-1)} < 1,$$

and so (4.2) holds from this fact.

Step 2. We shall prove that if $\mathbb{E}\|\sum_{k=1}^N \boldsymbol{A}_k\|^\alpha < \infty$ and $p^{(\alpha-1)}\rho_r(\alpha) < 1$ for some r, then for each i, $\mathbb{E}Y_i = V_i$ and $\mathbb{E}(Y_i)^\alpha < \infty$. Let $\bar{N} := N_r$ be the population of the rth-generation and $\bar{\boldsymbol{A}}_i := \boldsymbol{X}_{u^i}$, where u^i denotes the ith particle of the rth-generation. We consider $(\bar{N}, \bar{\boldsymbol{A}}_1, \bar{\boldsymbol{A}}_2, \ldots)$ in place of $(N, \boldsymbol{A}_1, \boldsymbol{A}_2, \ldots)$. Clearly, $\bar{\boldsymbol{M}} := \mathbb{E}\sum_{i=1}^{\bar{N}} \bar{\boldsymbol{A}}_i$ is finite and strictly positive with the maximum-modulus eigenvalue 1 and the corresponding eigenvectors $\bar{\boldsymbol{U}} = \boldsymbol{U}$ and $\bar{\boldsymbol{V}} = \boldsymbol{V}$. Let $\bar{\mathbb{T}}$ be the corresponding Galton–Watson tree and $\bar{\mathbb{T}}_n = \{u \in \bar{\mathbb{T}}\colon |u| = n\}$. Define

$$\bar{\boldsymbol{Y}}_n := \sum_{u\in\bar{\mathbb{T}}_n} \bar{\boldsymbol{X}}_u \boldsymbol{V} \quad \text{with } \bar{\boldsymbol{X}}_u := \bar{\boldsymbol{A}}_{u_1}\cdots\bar{\boldsymbol{A}}_{u_1\cdots u_n} \text{ for } u \in \bar{\mathbb{T}}_n.$$

Similarly, we define $\bar{\boldsymbol{M}}(t)$ and $\bar{\rho}(t)$ as in the beginning of Section 3. It is easy to see that $\bar{\boldsymbol{Y}}_n$ has the same distribution as $\boldsymbol{Y}_{nr}$; therefore, $\bar{\boldsymbol{Y}} := \lim_{n\to\infty}\bar{\boldsymbol{Y}}_n$ a.s. has the same distribution as $\boldsymbol{Y}$. To obtain $\mathbb{E}Y_i = V_i$ and $\mathbb{E}(Y_i)^\alpha < \infty$ by step 1, we need only to verify that $\mathbb{E}\|\sum_{i=1}^{\bar{N}} \bar{\boldsymbol{A}}_i\|^\alpha < \infty$ and $p^{(\alpha-1)}\bar{\rho}(\alpha) < 1$. The latter is obvious since $\bar{\boldsymbol{M}}(t) = \boldsymbol{M}_r(t)$ and so $\bar{\rho}(\alpha) = \rho_r(\alpha)$. To verify the former, we note that $\mathbb{E}\|\sum_{i=1}^{\bar{N}} \bar{\boldsymbol{A}}_i\|^\alpha < \infty$ is equivalent to $\max_i \mathbb{E}(\bar{Y}_{1,i})^\alpha < \infty$, which holds by Lemma 3.3 since $\mathbb{E}(\bar{Y}_{1,i})^\alpha = \mathbb{E}(Y_{r,i})^\alpha$.

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Now we prove Theorem 2.1(ii). Suppose that $0 < \mathbb{E}\|\boldsymbol{Y}\|^\alpha < \infty$, which implies that $\max_i \mathbb{E}(Y_i)^\alpha < \infty$ and $\boldsymbol{Y}$ is nondegenerate. As $\boldsymbol{Y}$ is a nontrivial solution of (1.1), we have $\mathbb{E}\boldsymbol{Y} = \boldsymbol{M}\mathbb{E}\boldsymbol{Y}$ with $\mathbb{E}\boldsymbol{Y} \neq \boldsymbol{0}$, which means that $\mathbb{E}\boldsymbol{Y}$ is an eigenvector corresponding to the eigenvalue 1, and so $\mathbb{E}\boldsymbol{Y} = c\boldsymbol{V}$ for some constant $c > 0$. By (1.1), for each i,

$$Y_i = \sum_{k=1}^{N} (\boldsymbol{A}_k \boldsymbol{Y}(k))_i = \sum_{k=1}^{N} \sum_{j=1}^{p} (\boldsymbol{A}_k)_{ij} Y_j(k).$$

By Jensen's inequality, for each i,

$$\mathbb{E}(Y_i)^\alpha \geq \mathbb{E}\bigg[\mathbb{E}\bigg(\sum_{k=1}^{N}\sum_{j=1}^{p}(\boldsymbol{A}_k)_{ij}Y_j(k) \;\bigg|\; \mathcal{F}_1\bigg)\bigg]^\alpha = c^\alpha \mathbb{E}\bigg[\mathbb{E}\sum_{k=1}^{N}(\boldsymbol{A}_k\boldsymbol{V})_i\bigg]^\alpha = c^\alpha \mathbb{E}(Y_{1,i})^\alpha.$$

Thus, $\max_i \mathbb{E}(Y_i)^\alpha < \infty$ implies that $\max_i \mathbb{E}(Y_{1,i})^\alpha < \infty$, or, equivalently, $\mathbb{E}\|\sum_{i=1}^{N} \boldsymbol{A}_i\|^\alpha < \infty$.

Next, we consider $\rho_n(t)$. Since

$$(Y_i)^\alpha = \bigg[\sum_{j=1}^{p}\sum_{u\in\mathbb{T}_n}(\boldsymbol{X}_u)_{ij}Y_j(u)\bigg]^\alpha \geq \sum_{j=1}^{p}\sum_{u\in\mathbb{T}_n}[(\boldsymbol{X}_u)_{ij}]^\alpha [Y_j(u)]^\alpha, \tag{4.3}$$

we obtain

$$\mathbb{E}(Y_i)^\alpha \geq \sum_{j=1}^{p}\mathbb{E}\sum_{u\in\mathbb{T}_n}[(\boldsymbol{X}_u)_{ij}]^\alpha \mathbb{E}(Y_j)^\alpha = \sum_{j=1}^{p}(\boldsymbol{M}_n(\alpha))_{ij}\mathbb{E}(Y_j)^\alpha. \tag{4.4}$$

Thus,

$$\sum_{i=1}^{p} U_{n,i}(\alpha)\mathbb{E}(Y_i)^\alpha \geq \sum_{j=1}^{p}\mathbb{E}(Y_j)^\alpha \sum_{i=1}^{p} U_{n,i}(\alpha)(\boldsymbol{M}_n(\alpha))_{ij} = \rho_n(\alpha)\sum_{j=1}^{p} U_{n,j}(\alpha)\mathbb{E}(Y_j)^\alpha, \tag{4.5}$$

which leads to $\rho_n(\alpha) \leq 1$. If, additionally, (2.1) holds then for each i,

$$\mathbb{P}\bigg(\sum_{j=1}^{p}\sum_{u\in\mathbb{T}_n}\mathbf{1}_{\{(\boldsymbol{X}_u)_{ij}>0\}} = 0 \text{ or } 1\bigg) < 1.$$

Hence, the strict inequality in (4.3) holds with positive probability, and so both (4.4) and (4.5) are strict inequalities, which leads to $\rho_n(\alpha) < 1$. □

5. Proof of Theorems 2.2 and 2.3

We shall prove Theorems 2.2 and 2.3 based on (1.1) with ideas from Liu [33]. Recall that $\phi(\boldsymbol{t}) = \mathbb{E}\mathrm{e}^{-\boldsymbol{t}\cdot\boldsymbol{Z}}$ is the Laplace transform of the nontrivial solution $\boldsymbol{Z}$ to (1.1). By (1.1), $\phi(\boldsymbol{t})$ satisfies the functional equation

$$\phi(\boldsymbol{t}) = \mathbb{E}\prod_{k=1}^{N}\phi(\boldsymbol{t}\boldsymbol{A}_k). \tag{5.1}$$

Our proofs are based on this equation.

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To prove Theorem 2.2, the following lemma is necessary.

Lemma 5.1. *Let $\phi\colon \mathbb{R}_+^p \mapsto \mathbb{R}_+$ be a bounded function, and $\boldsymbol{A} = (a_{ij})$ be a nonzero matrix such that for some $0 < q < 1$, $t_\varepsilon > 0$, and all $\boldsymbol{t}$ satisfying $\|\boldsymbol{t}\| > t_\varepsilon$,we have*

$$\phi(\boldsymbol{t}) \le q\mathbb{E}\phi(\boldsymbol{t}\boldsymbol{A}). \tag{5.2}$$

If $q\mathbb{E}(\min_i \sum_j a_{ij})^{-\lambda} < 1$ then $\phi(\boldsymbol{t}) = O(\|\boldsymbol{t}\|^{-\lambda})$ ($\|\boldsymbol{t}\| \to \infty$).

Proof. Assume that ϕ is bounded by a constant K. For $\boldsymbol{t} \neq \boldsymbol{0}$, if $\|\boldsymbol{t}\| \le t_\varepsilon$ then $\phi(\boldsymbol{t}) \le K \le Kt_\varepsilon^\lambda \|\boldsymbol{t}\|^{-\lambda}$, which yields by (5.2). Then

$$\phi(\boldsymbol{t}) \le q\mathbb{E}\phi(\boldsymbol{t}\boldsymbol{A}) + C\|\boldsymbol{t}\|^{-\lambda} \quad \text{for all } \boldsymbol{t} \neq \boldsymbol{0}, \tag{5.3}$$

where C is a general positive constant. Let $\{\boldsymbol{A}_k\}$ be a family of i.i.d. copies of $\boldsymbol{A}$. By induction on (5.3), we have

$$\phi(\boldsymbol{t}) \le q^n\mathbb{E}\phi(\boldsymbol{t}\boldsymbol{A}_1\cdots\boldsymbol{A}_n) + C\left[\sum_{k=1}^{n-1}(q^{k-1}\mathbb{E}\|\boldsymbol{t}\boldsymbol{A}_1\cdots\boldsymbol{A}_{k-1}\|)^{-\lambda} + \|\boldsymbol{t}\|^{-\lambda}\right] \quad \text{for all } \boldsymbol{t} \neq \boldsymbol{0}. \tag{5.4}$$

Note that for any matrix $\boldsymbol{A} = (a_{ij})$ and vector $\boldsymbol{t}$, we have

$$\|\boldsymbol{t}\boldsymbol{A}\| = \sum_j (\boldsymbol{t}\boldsymbol{A})_j = \sum_j \sum_i t_i a_{ij} \ge \sum_i t_i \min_i \sum_j a_{ij} = \|\boldsymbol{t}\|\left(\min_i \sum_j a_{ij}\right).$$

Thus, by the independency of $\{\boldsymbol{A}_k\}$,

$$\mathbb{E}\|\boldsymbol{t}\boldsymbol{A}_1\cdots\boldsymbol{A}_k\|^{-\lambda} \le \|\boldsymbol{t}\|^{-\lambda}\mathbb{E}\left(\prod_{l=1}^{k}\left(\min_i \sum_j (\boldsymbol{A}_l)_{ij}\right)\right)^{-\lambda} = \|\boldsymbol{t}\|^{-\lambda}\left[\mathbb{E}\left(\min_i \sum_j a_{ij}\right)^{-\lambda}\right]^k. \tag{5.5}$$

Combining (5.5) with (5.4) and letting $n \to \infty$ leads to $\phi(\boldsymbol{t}) = O(\|\boldsymbol{t}\|^{-\lambda})$ ($\|\boldsymbol{t}\| \to \infty$). □

Proof of Theorem 2.2. We first prove Theorem 2.2(i). We prove only the assertions for $\phi(\boldsymbol{t})$. The other assertions follow by applying [31, Lemma 4.4] to the random variable $\boldsymbol{y} \cdot \boldsymbol{Z}$.

Let $N_\delta = \sum_{k=1}^{N} \mathbf{1}_{\{\min_i \sum_j (\boldsymbol{A}_k)_{ij} > \delta\}}$ for $\delta > 0$. Then $N_\delta \uparrow N$ as $\delta \downarrow 0$. Since $\phi(\boldsymbol{t}) = \mathbb{E}e^{-\boldsymbol{t}\cdot\boldsymbol{Z}} \to 0$ as $\|\boldsymbol{t}\| \to \infty$, there exists $t_\varepsilon > 0$ such that for $\|\boldsymbol{t}\| > t_\varepsilon$, $\phi(\boldsymbol{t}) < \varepsilon$. For $\|\boldsymbol{t}\| > t_\varepsilon/\delta$, if $\min_i \sum_j (\boldsymbol{A}_k)_{ij} > \delta$, we have

$$\|\boldsymbol{t}\boldsymbol{A}_k\| \ge \|\boldsymbol{t}\| \min_i \sum_j (\boldsymbol{A}_k)_{ij} > t_\varepsilon.$$

By (5.1), we have

$$\phi(\boldsymbol{t}) \le \mathbb{E}\phi(\boldsymbol{t}\boldsymbol{A}_1)(\varepsilon^{N_\delta - 1}\mathbf{1}_{\{N_\delta \ge 1\}} + \mathbf{1}_{\{N_\delta = 0\}}) = q_{\varepsilon,\delta}\mathbb{E}\phi(\boldsymbol{t}\tilde{\boldsymbol{A}}),$$

where $q_{\varepsilon,\delta} = \mathbb{E}(\varepsilon^{N_\delta - 1}\mathbf{1}_{\{N_\delta \ge 1\}} + \mathbf{1}_{\{N_\delta = 0\}})$ and $\tilde{\boldsymbol{A}} = (\tilde{a}_{ij})$ is a random matrix whose distribution is determined by $\mathbb{E}g(\tilde{\boldsymbol{A}}) = (q_{\varepsilon,\delta})^{-1}\mathbb{E}g(\boldsymbol{A}_1)(\varepsilon^{N_\delta - 1}\mathbf{1}_{\{N_\delta \ge 1\}} + \mathbf{1}_{\{N_\delta = 0\}})$ for all bounded and measurable functions g on $\mathbb{R}_+^{p^2}$. We can see that by the dominated convergence theorem,

$$\lim_{\varepsilon \downarrow 0}\lim_{\delta \downarrow 0} q_{\varepsilon,\delta} = \lim_{\varepsilon \downarrow 0}\mathbb{E}\varepsilon^{N-1}\mathbf{1}_{\{N \ge 1\}} = \mathbb{P}(N = 1) < 1,$$

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and since $\mathbb{E}(\min_i \sum_j a_{ij})^{-\lambda} < \infty$, we have

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0} q_{\varepsilon,\delta}\mathbb{E}\left(\min_i \sum_j \tilde{a}_{ij}\right)^{-\lambda} &= \lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0}\mathbb{E}\left(\min_i \sum_j a_{ij}\right)^{-\lambda}(\varepsilon^{N_\delta-1}\mathbf{1}_{\{N_\delta\geq 1\}} + \mathbf{1}_{\{N_\delta=0\}})\\ &= \lim_{\varepsilon\downarrow 0}\mathbb{E}\left(\min_i \sum_j a_{ij}\right)^{-\lambda}\mathbf{1}_{\{N\geq 1\}}\\ &= \mathbb{E}\left(\min_i \sum_j a_{ij}\right)^{-\lambda}\mathbf{1}_{\{N=1\}}\\ &< 1.\end{aligned}$$

By Lemma 5.1, $\phi(\boldsymbol{t}) = O(\|\boldsymbol{t}\|^{-\lambda})$ $(\|\boldsymbol{t}\| \to \infty)$.

For the proof of Theorem2.2(ii), note that we have obtained $\phi(\boldsymbol{t}) \leq C\|\boldsymbol{t}\|^{\lambda}$ for all $\|\boldsymbol{t}\| > 0$ in the proof of Theorem 2.2(i), where C is a positive constant. By (5.1), we have

$$\phi(\boldsymbol{t}) \leq \mathbb{E}\prod_{k=1}^{\underline{m}}\phi(\boldsymbol{t}\boldsymbol{A}_k) \leq C^{\underline{m}}\mathbb{E}\prod_{k=1}^{\underline{m}}\|\boldsymbol{t}\boldsymbol{A}_k\|^{-\lambda} \leq C^{\underline{m}}\|\boldsymbol{t}\|^{-\underline{m}\lambda}\mathbb{E}\left[\prod_{i=1}^{\underline{m}}\left(\min_i \sum_{j=1}^{p}(\boldsymbol{A}_k)_{ij}\right)^{-\lambda}\right],$$

which implies that $\phi(\boldsymbol{t}) = O(\|\boldsymbol{t}\|^{-\underline{m}\lambda})(\|\boldsymbol{t}\| \to \infty)$. □

Proof of Theorem 2.3. By (1.1), we have

$$1 = \sum_{j=1}^{p} V_j = \mathbb{E}\sum_{k=1}^{N}\sum_{j,l=1}^{p}(\boldsymbol{A}_k)_{jl}V_l > \sum_{k=1}^{\underline{m}}\sum_{j,l=1}^{p}\underline{a}V_l = \underline{a}\underline{m}p.$$

The strict inequality holds because of $\mathbb{P}(\boldsymbol{Y}_1 = \boldsymbol{V}) < 1$. Therefore, we have

$$\gamma = -\frac{\log \underline{m}}{\log(\underline{a}p)} \in (0, 1).$$

For Theorems 2.3(i) and 2.3(ii), we prove only the results for $\phi(\boldsymbol{t})$. The assertions for $\mathbb{P}(\boldsymbol{y}\cdot\boldsymbol{Z} \leq x)$ follow from the assertions for $\mathbb{E}^{-t\boldsymbol{y}\cdot\boldsymbol{Z}}$ and the Tauberian theorem of exponential type (cf. [30]).

We first prove Theorem 2.3(i). Since for all k,

$$\phi(\boldsymbol{t}\boldsymbol{A}_k) = \mathbb{E}\exp\left\{-\sum_{i,j=1}^{p} t_i(\boldsymbol{A}_k)_{ij}Z_j\right\} \leq \mathbb{E}\exp\{-\underline{a}\|\boldsymbol{t}\|\boldsymbol{e}\cdot\boldsymbol{Z}\} = \phi(\underline{a}\|\boldsymbol{t}\|\boldsymbol{e}),$$

where $\boldsymbol{e} = (1, 1, \ldots, 1)$, by (5.1),

$$\phi(\boldsymbol{t}) \leq \mathbb{E}\prod_{k=1}^{\underline{m}}\phi(\boldsymbol{t}\boldsymbol{A}_k) \leq [\phi(\underline{a}\|\boldsymbol{t}\|\boldsymbol{e})]^{\underline{m}}. \tag{5.6}$$

By (5.6), we have $\phi(\underline{a}\|\boldsymbol{t}\|\boldsymbol{e}) \leq [\phi(\underline{a}^2 p\|\boldsymbol{t}\|\boldsymbol{e})]^{\underline{m}}$, so that $\phi(\boldsymbol{t}) \leq [\phi(\underline{a}^2 p\|\boldsymbol{t}\|\boldsymbol{e})]^{\underline{m}}$. By iteration, we obtain

$$\phi(\boldsymbol{t}) \leq [\phi(\underline{a}^k p^{k-1}\|\boldsymbol{t}\|\boldsymbol{e})]^{\underline{m}^k}. \tag{5.7}$$

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As $\underline{a}p < 1$ for $\|\boldsymbol{t}\| \geq p$, there exists an integer $k \geq 0$ such that $p/(\underline{a}p)^k \leq \|\boldsymbol{t}\| < p/(\underline{a}p)^{k+1}$. So this k satisfies

$$\frac{\log p - \log \|\boldsymbol{t}\|}{\log(\underline{a}p)} - 1 < k \leq \frac{\log p - \log \|\boldsymbol{t}\|}{\log(\underline{a}p)}. \tag{5.8}$$

For any $x \geq 1$, we see that

$$\phi(x\boldsymbol{e}) = \mathbb{E}\exp\left\{-x\sum_{j=1}^{p} Z_j\right\} \leq \mathbb{E}\exp\left\{-\sum_{j=1}^{p} Z_j\right\} = \phi(\boldsymbol{e}) < 1. \tag{5.9}$$

Since $\underline{a}^k p^{k-1}\|\boldsymbol{t}\| \geq 1$, by (5.7)–(5.9), we have

$$\log\phi(\boldsymbol{t}) \leq \underline{m}^k \log\phi(\boldsymbol{e}) \leq \exp\left\{\frac{\log \underline{m}}{\log(\underline{a}p)}(\log p - \log\|\boldsymbol{t}\|)\right\} = -C_1\|\boldsymbol{t}\|^{\gamma},$$

where $C_1 = -p^{-\gamma}\log\phi(\boldsymbol{e}) > 0$.

Now we turn to the proof of Theorem 2.3(ii). The proof is similar to that of Theorem 2.3(i). If $\max_{ij}(\boldsymbol{A}_k)_{ij} \leq \underline{a} + \varepsilon$ then

$$\phi(\boldsymbol{t}\boldsymbol{A}_k) \geq \phi((\underline{a}+\varepsilon)\|\boldsymbol{t}\|\boldsymbol{e}).$$

By (5.1),

$$\phi(\boldsymbol{t}) \geq \mathbb{E}\prod_{k=1}^{\underline{m}} \phi(\boldsymbol{t}\boldsymbol{A}_k)\mathbf{1}_{\{N=\underline{m},\max_{ij}(\boldsymbol{A}_k)_{ij}\leq \underline{a}+\varepsilon,\ \text{for all } k\}} \geq \rho[\phi((\underline{a}+\varepsilon)\|\boldsymbol{t}\|\boldsymbol{e})]^{\underline{m}},$$

where $\rho = \mathbb{P}(N = \underline{m}, \max_{ij}(\boldsymbol{A}_k)_{ij} \leq \underline{a} + \varepsilon, \text{ for all } k) > 0$. By iteration, we obtain

$$\phi(\boldsymbol{t}) \geq \rho^{\sum_{j=0}^{k-1}\underline{m}^j}[\phi((\underline{a}+\varepsilon)^k p^{k-1}\|\boldsymbol{t}\|\boldsymbol{e})]^{\underline{m}^k}. \tag{5.10}$$

As $(\underline{a}+\varepsilon)p < 1$ for $\|\boldsymbol{t}\| \geq p$, there exists an integer $k \geq 1$ such that $p/((\underline{a}+\varepsilon)p)^{k-1} \leq \|\boldsymbol{t}\| < p/((\underline{a}+\varepsilon)p)^k$. Since $\phi(x\boldsymbol{e}) \geq \phi(\boldsymbol{e})$ for any $x < 1$ and $(\underline{a}+\varepsilon)^k p^{k-1}\|\boldsymbol{t}\| < 1$, we have

$$\phi((\underline{a}+\varepsilon)^k p^{k-1}\|\boldsymbol{t}\|\boldsymbol{e}) \geq \phi(\boldsymbol{e}).$$

Therefore, (5.10) yields

$$\log\phi(\boldsymbol{t}) \geq \underline{m}^k\left(\log\phi(\boldsymbol{e}) + \underline{m}^{-k}\sum_{j=0}^{k-1}\underline{m}^j\log\rho\right) \geq \underline{m}^k\left(\log\phi(\boldsymbol{e}) + \frac{\log\rho}{\underline{m}-1}\right) \geq -C_2\|\boldsymbol{t}\|^{\gamma(\varepsilon)},$$

where $C_2 = -p^{-\gamma(\varepsilon)}\underline{m}(\log\phi(\boldsymbol{e}) + \log\rho/(\underline{m}-1)) > 0$. □

6. Moments for the complex case

In this section we consider the complex case, where in (1.1) all the matrices $\boldsymbol{A}_k$ and the vectors $\boldsymbol{Z}$, $\boldsymbol{Z}(k)$ are complex (with $\mathbb{C}$ in place of $\mathbb{R}_+$). Related work can be found in [4] and [5]. Here we are still interested in the existence of the αth-moment ($\alpha > 1$) of the solution. Besides Assumption 1.1, we assume, moreover, that

$$\hat{\boldsymbol{M}} := \mathbb{E}\sum_{k=1}^{N}\hat{\boldsymbol{A}}_k \quad \text{with } (\hat{\boldsymbol{A}}_k)_{ij} := |(\boldsymbol{A}_k)_{ij}|$$

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is finite and strictly positive. For $t \in \mathbb{R}$ fixed, let

$$\hat{\boldsymbol{M}}(t) := \mathbb{E}\sum_{k=1}^{N} \hat{\boldsymbol{A}}_k^{(t)} \quad \text{with } (\hat{\boldsymbol{A}}_k^{(t)})_{ij} := (\hat{\boldsymbol{A}}_k)_{ij}^t,$$

whose maximum-modulus eigenvalue is denoted by $\hat{\rho}(t)$ and the corresponding normalized left and right positive eigenvectors by $\hat{\boldsymbol{U}}(t)$, $\hat{\boldsymbol{V}}(t)$. Define

$$\hat{\boldsymbol{Y}}_n^{(t)} := \frac{\sum_{u\in\mathbb{T}_n} \hat{\boldsymbol{X}}_u^{(t)} \hat{\boldsymbol{V}}(t)}{\hat{\rho}(t)^n} \quad \text{with } \hat{\boldsymbol{X}}_u^{(t)} := \hat{\boldsymbol{A}}_{u_1}^{(t)} \cdots \hat{\boldsymbol{A}}_{u_1\cdots u_n}^{(t)} \text{ for } u \in \mathbb{T}_n.$$

Obviously, $\{\hat{\boldsymbol{Y}}_n^{(t)}\}$ has the same structure as the martingale $\{\boldsymbol{Y}_n^{(t)}\}$ of the real case for which we have established inequalities in Section 3; therefore, we can apply these results (Lemmas 3.2 and 3.3) to the martingale $\{\hat{\boldsymbol{Y}}_n^{(t)}\}$.

For the complex case, we have the following result compared to Theorem 2.1(i).

Theorem 6.1. (Complex case.) *Assume that all the vectors $\boldsymbol{Z}$, $\boldsymbol{Z}(k)$ and the matrix $\boldsymbol{A}_k$ are complex. Let $\alpha > 1$. If $\mathbb{E}\|\sum_{k=1}^N \hat{\boldsymbol{A}}_k\|^\alpha < \infty$ and either of the following assertions holds:*

(i) *$\alpha \in (1, 2]$ and $p^{(\alpha-1)}\hat{\rho}(\alpha) < 1$;*

(ii) *$\alpha > 2$ and $\max\{p^{\alpha-1}\hat{\rho}(\alpha), p^{\alpha/\beta}\hat{\rho}(\beta)\} < 1$ for some $\beta \in (1, 2]$,*

then $\sup_n \mathbb{E}\|\boldsymbol{Y}_n\|^\alpha < \infty$, and $\{\boldsymbol{Y}_n\}$ converges a.s. and in L^α to a random vector $\boldsymbol{Y}$, so that $\mathbb{E}\boldsymbol{Y} = \boldsymbol{V}$ and $0 < \mathbb{E}\|\boldsymbol{Y}\|^\alpha < \infty$.

In particular, for the $p = 1$ case, it is easy to see that $V = 1$ and $\hat{\rho}(t) = \hat{m}(t) = \mathbb{E}\sum_{k=1}^N |A_k|^t$.

Corollary 6.1. (The $p = 1$ case.) *Let $p = 1$ and $\alpha > 1$. If $\mathbb{E}(\sum_{k=1}^N |A_k|)^\alpha < \infty$ and either of the following assertions holds:*

(i) *$\alpha \in (1, 2]$ and $\hat{\rho}(\alpha) < 1$;*

(ii) *$\alpha > 2$ and $\max\{\hat{\rho}(\alpha), \hat{\rho}(\beta)\} < 1$ for some $\beta \in (1, 2]$,*

then $\sup_n \mathbb{E}|Y_n|^\alpha < \infty$ and $\{Y_n\}$ converges a.s. and in L^α to a random variable Y, so that $\mathbb{E}Y = 1$ and $0 < \mathbb{E}Y^\alpha < \infty$.

The proof of Theorem 6.1 is similar to that of Theorem 2.1. We first show several lemmas for the martingale $\{\boldsymbol{Y}_n^{(t)}\}$.

Lemma 6.1. *Let $\alpha > 1$. Fix $t \in \mathbb{R}$. Assume that $\max_i \mathbb{E}|Y_{1,i}^{(t)}|^\alpha < \infty$. Then for each $i = 1, 2, \ldots, p$ and $n \in \mathbb{N}_0$,*

(i) *if $\alpha \in (1, 2]$,*

$$\mathbb{E}|Y_{n+1,i}^{(t)} - Y_{n,i}^{(t)}|^\alpha \le Cp^{(\alpha-1)n}\left[\frac{\hat{\rho}(\alpha t)}{|\rho(t)|^\alpha}\right]^n; \tag{6.1}$$

(ii) *if $\alpha > 2$, for any $\beta \in (1, 2]$,*

$$\mathbb{E}|Y_{n+1,i}^{(t)} - Y_{n,i}^{(t)}|^\alpha \le Cp^{\alpha n/2}\left[\frac{\hat{\rho}(\beta t)^{\alpha/\beta}}{|\rho(t)|^\alpha}\right]^n \mathbb{E}[\hat{Y}_{n,i}^{(\beta t)}]^{\alpha/\beta}, \tag{6.2}$$

where C is a constant depending on α, p, and t.

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Proof. Note that $|(\boldsymbol{X}_u^{(t)})_{ij}| \le (\hat{\boldsymbol{X}}_u^{(t)})_{ij}$. Applying Burkholder's inequality, we obtain

$$\begin{aligned}\mathbb{E}|Y_{n+1,i}^{(t)} - Y_{n,i}^{(t)}|^\alpha &\le \frac{C}{|\rho(t)|^{\alpha n}} \sum_{j=1}^{p} \mathbb{E}\bigg(\sum_{u\in\mathbb{T}_n} |(\boldsymbol{X}_u^{(t)})_{ij}|^2 |Y_{1,j}^{(t)}(u) - V_j(t)|^2 \bigg)^{\alpha/2} \\ &\le \frac{C}{|\rho(t)|^{\alpha n}} \sum_{j=1}^{p} \mathbb{E}\bigg(\sum_{u\in\mathbb{T}_n} [(\hat{\boldsymbol{X}}_u^{(t)})_{ij}]^2 |Y_{1,j}^{(t)}(u) - V_j(t)|^2 \bigg)^{\alpha/2}.\end{aligned}$$

Then repeat the proof of Lemma 3.2 (with β in place of 2 for the case where $\alpha > 2$). □

Applying Lemma 3.3 (with β in place of 2) to (6.2), we immediately obtain the following lemma.

Lemma 6.2. *Let $\alpha > 1$. Fix $t \in \mathbb{R}$. If $\max_i \mathbb{E}|Y_{1,i}^{(t)}|^\alpha < \infty$ and $\max_i \mathbb{E}|\hat{Y}_{1,i}^{(\beta t)}|^{\alpha/\beta} < \infty$ for some $\beta \in (1, 2]$, then for each $i = 1, \dots, p$ and $n \in \mathbb{N}$,*

$$\mathbb{E}|Y_{n+1,i}^{(t)} - Y_{n,i}^{(t)}|^\alpha \le C n^{\alpha/\beta} \bigg[\max_{1\le l\le m} \bigg\{ p^{\alpha-1} \frac{\hat{\rho}(\alpha t)}{|\rho(t)|^\alpha}, p^{((\beta^l-1)/\beta^l)\alpha} \frac{\hat{\rho}(\beta^l t)^{\alpha/\beta^l}}{|\rho(t)|^\alpha} \bigg\} \bigg]^n$$

if $\alpha \in (\beta^m, \beta^{m+1}]$ for some $m \in \mathbb{N}_0$.

Combining Lemmas 6.1 and 6.2 leads to Lemma 6.3 below.

Lemma 6.3. *Let $\alpha > 1$. Assume that $\mathbb{E}\| \sum_{k=1}^N \hat{\boldsymbol{A}}_k\|^\alpha < \infty$. Then for each $i = 1, 2, \dots, p$ and $n \in \mathbb{N}$,*

(i) *if $\alpha \in (1, 2]$,*

$$\mathbb{E}|Y_{n+1,i} - Y_{n,i}|^\alpha \le C[p^{(\alpha-1)} \hat{\rho}(\alpha)]^n; \tag{6.3}$$

(ii) *if $\alpha > 2$, for any $\beta \in (1, 2]$,*

$$\mathbb{E}|Y_{n+1,i} - Y_{n,i}|^\alpha \le C n^{\alpha/\beta} [\max\{p^{\alpha-1} \hat{\rho}(\alpha), p^{\alpha/\beta} \hat{\rho}(\beta)^{\alpha/\beta}\}]^n.$$

Proof. First, we remark that $\hat{\rho}(t)$ exists for all $t \in [1, \alpha]$ since $\mathbb{E}\| \sum_{k=1}^N \hat{\boldsymbol{A}}_k\|^\alpha < \infty$ and $\hat{\boldsymbol{M}}(1) = \hat{\boldsymbol{M}}$ is finite and strictly positive. Furthermore,

$$\mathbb{E}\| \sum_{k=1}^{N} \hat{\boldsymbol{A}_k}\|^\alpha < \infty$$

implies that $\max_i \mathbb{E}|Y_{1,i}|^\alpha < \infty$ and $\max_i \mathbb{E}|\hat{Y}_{1,i}^{(\beta)}|^{\alpha/\beta} < \infty$. So (6.3) follows directly from (6.1). For $\alpha > 2$, by Lemma 6.2, we have

$$\begin{aligned}\mathbb{E}|Y_{n+1,i} - Y_{n,i}|^\alpha &\le C n^{\alpha/\beta} \bigg[\max_{1\le l\le m} \{p^{\alpha-1} \hat{\rho}(\alpha), p^{((\beta^l-1)/\beta^l)\alpha} \hat{\rho}(\beta^l)^{\alpha/\beta^l}\} \bigg]^n \\ &\le C n^{\alpha/\beta} \bigg[\sup_{\beta\le x\le\alpha} \{p^{1-1/x} \hat{\rho}(x)^{1/x}\} \bigg]^{\alpha n}\end{aligned}$$

if $\alpha \in (\beta^m, \beta^{m+1}]$ ($m \ge 1$ is an integer). Let

$$g(x) := \log(p^{1-1/x} \hat{\rho}(x)^{1/x}) = \bigg(1 - \frac{1}{x}\bigg) \log p + \frac{1}{x} \log \hat{\rho}(x).$$

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Clearly, $g(x)$ is derivable on $(1, \alpha)$ with derivative

$$g'(x) = \frac{h(x)}{x^2},$$

where $h(x) := \log p + x(\hat{\rho}'(x)/\hat{\rho}(x)) - \log \hat{\rho}(x)$.The log-convexity of $\hat{\rho}(x)$ implies that $h(x)$ is increasing; hence, $g(x)$ reaches its maximum on a closed interval at the extremity points. We have

$$\sup_{\beta \le x \le \alpha} \{p^{1-1/x} \hat{\rho}(x)^{1/x}\} = \max\{p^{1-1/\alpha} \hat{\rho}(\alpha)^{1/\alpha}, p^{1/\beta} \hat{\rho}(\beta)^{1/\beta}\}.$$

The proof is complete. □

Now we prove Theorem 6.1.

Proof of Theorem 6.1. By Lemma 6.3, we obtain the series

$$\sum_n (\mathbb{E}|Y_{n+1,i} - Y_{n,i}|^\alpha)^{1/\alpha} < \infty.$$

Observing that

$$(\mathbb{E}|Y_{n,i}|^\alpha)^{1/\alpha} \le \sum_{k=0}^{n-1} (\mathbb{E}|Y_{k+1,i} - Y_{k,i}|^\alpha)^{1/\alpha} + 1,$$

we immediately obtain $\sup_n \mathbb{E}|Y_{n,i}|^\alpha < \infty$. Note that

$$\mathbb{E} \sum_n |Y_{n+1,i} - Y_{n,i}| \le \sum_n (\mathbb{E}|Y_{n+1,i} - Y_{n,i}|^\alpha)^{1/\alpha} < \infty.$$

This fact leads to the a.s. convergence of the series $\sum_n |Y_{n+1,i} - Y_{n,i}|$, which shows that $\{Y_{n,i}\}$ is a Cauchy sequence in the sense a.s., so there exists a random variable Y_i such that $Y_{n,i} \to Y_i$ a.s. By Fatou's Lemma, we have

$$\begin{aligned}
\mathbb{E}|Y_{n,i} - Y_i|^\alpha &= \mathbb{E} \lim_{l \to \infty} |Y_{n+l,i} - Y_{n,i}|^\alpha \\
&\le \liminf_{l \to \infty} \mathbb{E}|Y_{n+l,i} - Y_{n,i}|^\alpha \\
&\le \left(\sum_{k=n}^{\infty} (\mathbb{E}|Y_{k+1,i} - Y_{k,i}|^\alpha)^{1/\alpha} \right)^\alpha \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Thus, $Y_{n,i} \to Y_i$ in L^α, so that $\mathbb{E}Y_i = V_i$ and $0 < \mathbb{E}(Y_i)^\alpha < \infty$. □

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