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CONTINUITY OF THE SPECTRAL FACTORIZATION MAPPING

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Abstract

It is shown that the matrix spectral factorization mapping is sequentially continuous from L^p to H^{2p} (where $1 \leqslant p < \infty$), under the additional assumption of uniform integrability of the logarithms of the spectral densities to be factorized. It is shown, moreover, that this condition is necessary for continuity, as well as sufficient.

1. *Introduction*

Spectral factorization [**2**, **11**, Chapter 6] is the process by which a positive matrix-valued function w, on the unit circle in $\mathbb{C}$, is expressed in the form $w(e^{i\theta}) = \rho(e^{i\theta})^* \rho(e^{i\theta})$, for a certain complex matrix-valued function ρ, where $\theta \in [0, 2\pi)$. There are many contexts in which this factorization naturally arises, for example stochastic processes, linear quadratic control design, and H^∞ robust control.

We require the function ρ to have a certain analytic extension to the open unit disc in $\mathbb{C}$, and with a suitable constraint on $\rho(0)$ we can ensure that the spectral factor ρ is unique. In this way, we obtain a mapping from w to this canonical choice of ρ, which we term the *spectral factorization mapping*. (It is also possible to define this mapping for functions on the imaginary line in $\mathbb{C}$, for which the spectral factors are required to have analytic extensions to the right half-plane.)

There are many situations in which it is desirable to know the continuity properties of this mapping. One reason for this is that for applications it is common to use approximation methods, when working with spectral factors, and this makes it important to be able to provide error estimates. In this paper, we will address some of the continuity issues.

Our work will improve on some the results of earlier authors. Specifically, we show that L^p convergence of a sequence of positive matrix-valued functions is necessary and sufficient for H^{2p} convergence of their spectral factors, where $1 \leqslant p < \infty$, provided a certain uniform log-integrability condition is placed on this sequence. This condition is fairly weak, and it turns out to be satisfied automatically in many situations.

In the case $p = \infty$, it is already well known that continuity of the spectral factorization mapping fails. (The extra condition above is redundant here.) This is shown by Anderson [**1**], who provides a counterexample. However, it is also shown in [**1**] that L^∞ continuity can be recovered by placing a suitable constraint on the derivatives of the functions to be factorized. Additionally, $L^\infty \to H^2$ continuity is established in [**1**]. This latter result also follows as a corollary of our results, as we shall observe in the penultimate section.

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In [**7**], Jacob *et al.* work in the half-plane rather than the disc, and consider spectral factorizations of functions on the imaginary line, which extend analytically to some vertical strip about that line. Amongst other things, they establish an $\mathrm{L}^p \to \mathrm{H}^p$ type continuity result, under the additional assumption that the functions concerned and their inverses are uniformly bounded. However, they deal exclusively with scalar-valued functions.

Other continuity results have been obtained by Jacob and Partington [**6**], who look at spectral factorization on decomposing Banach algebras. This class includes the Wiener algebra of all absolutely convergent Fourier series. There they establish local Lipschitz continuity of the spectral factorization mapping for such algebras. This provides a rather different class of results to ours.

In this paper, we work exclusively in the disc. However, analoguous results will hold for the half-plane. This conversion can be achieved in the standard way, by means of a conformal mapping between the disc and the half-plane.

The outline of this paper is as follows. In the next section, we introduce the notation that will be used throughout. In Section 3, we define the spectral factorization mapping and state some of its properties, and it is here that we state the main theorem of this paper precisely. Section 4 will be used to derive an accompanying proposition. In Section 5, we will establish some lemmas which will be needed to prove the theorem. In Section 6, we will complete the proof and then use it to derive some related results. Finally, we make some concluding remarks in Section 7.

2. *Notation and conventions*

Let $\mathbb{D}$ denote the open unit disc in the complex plane and let $\mathbb{T}$ denote its boundary, the unit circle. Let μ denote normalized Lebesgue measure on $\mathbb{T}$. We will be working with Hardy spaces of matrix-valued analytic functions on $\mathbb{D}$, together with their measurable boundary functions.

Let $n \in \mathbb{N}$ and let $p \in [1, \infty]$. Throughout this text, we shall keep n fixed. We shall write $\mathrm{L}^p(\mathcal{L}(\mathbb{C}^n))$ to mean the Lebesgue space of measurable functions from $\mathbb{T}$ to the space of $n \times n$ complex matrices, $\mathcal{L}(\mathbb{C}^n)$, with finite $\|\cdot\|_p$ norm, given by

$$\|f\|_p = \begin{cases} \left(\dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} \|f(e^{i\theta})\|_p^p \, \mathrm{d}\theta \right)^{1/p} & \text{if } 1 \leqslant p < \infty \\ \operatorname*{ess\,sup}_{\theta \in [0, 2\pi)} \|f(e^{i\theta})\|_\infty & \text{if } p = \infty, \end{cases}$$

where $f : \mathbb{T} \longrightarrow \mathcal{L}(\mathbb{C}^n)$ is any measurable function. In this definition, we choose to use the $\|\cdot\|_p$ norm on $\mathcal{L}(\mathbb{C}^n)$, rather than an arbitrary norm. This will make the above norms easier for us to work with, but it makes no difference to the norm topology. We define the $\|\cdot\|_p$ norm on $\mathcal{L}(\mathbb{C}^n)$ as

$$\|X\|_p = \begin{cases} \left(\operatorname{Tr} (X^* X)^{p/2} \right)^{1/p} & \text{if } 1 \leqslant p \leqslant \infty \\ \displaystyle\sup_{\|v\| \leqslant 1} \|Xv\| & \text{if } p = \infty, \end{cases}$$

where X is any element of $\mathcal{L}(\mathbb{C}^n)$. These norms satisfy the relations

$$\begin{aligned} \|X^*\|_p &= \|X\|_p \\ |\operatorname{Tr} X| &\leqslant \|X\|_1 \\ \|XY\|_1 &\leqslant \|X\|_p \|Y\|_q \end{aligned}$$

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for any $X, Y \in \mathcal{L}(\mathbb{C}^n)$ and any $p, q \in [1, \infty]$ conjugate indices (that is, $1/p + 1/q = 1$). A proof of these well-known matrix relations can be found in [**3**, Section 11.9].

For $p, q \in [1, \infty]$ conjugate indices and functions f and g in $\mathrm{L}^p(\mathcal{L}(\mathbb{C}^n))$ and $\mathrm{L}^q(\mathcal{L}(\mathbb{C}^n))$ respectively, we define the inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Tr}\left(g(e^{i\theta})^* f(e^{i\theta})\right) \mathrm{d}\theta.$$

The above integrand is dominated by the mapping $\theta \longmapsto \|f(e^{i\theta})\|_p \|g(e^{i\theta})\|_q$, which is integrable, and so this inner product is well defined.

By combining the earlier matrix relations with the usual (scalar) formulation of Hölder's inequality, we obtain the following analogue of Hölder's inequality:

$$|\langle f, g \rangle| \leqslant \|g^* f\|_1 \leqslant \|f\|_p \|g\|_q,$$

where g^*, the hermitian conjugate of g, is taken pointwise, that is $g^*(z) = g(z)^*$ for all $z \in \mathbb{T}$. We will frequently use expressions such as $\det g$, $\operatorname{Tr} g$, fg, g^{-1}, etc. These are always intended to be taken pointwise.

For $p \in [1, \infty]$, we write $\mathrm{H}^p(\mathcal{L}(\mathbb{C}^n))$ to mean the Hardy space of matrix-valued functions in $\mathrm{L}^p(\mathcal{L}(\mathbb{C}^n))$, with an analytic extension to $\mathbb{D}$. Thus $\mathrm{H}^p(\mathcal{L}(\mathbb{C}^n))$ is regarded as a closed subspace of $\mathrm{L}^p(\mathcal{L}(\mathbb{C}^n))$. Vector-valued and rectangular matrix-valued Hardy spaces $\mathrm{H}^p(\mathbb{C}^n)$ and $\mathrm{H}^p(\mathcal{L}(\mathbb{C}^m, \mathbb{C}^n))$ can be defined analogously and they will make a brief appearance in the next section. We will always be referring to the analytic extension implicitly, when we evaluate H^p functions at points in $\mathbb{D}$.

Let $\mathcal{P}(\mathbb{C}^n)$ denote the vector space of $\mathbb{C}^n$-valued trigonometric polynomials on $\mathbb{T}$. This is dense in $\mathrm{L}^p(\mathbb{C}^n)$ for all $p \in [1, \infty)$. Similarly, we define $\mathcal{P}^+(\mathbb{C}^n)$ to be the space of $\mathbb{C}^n$-valued polynomials over $\mathbb{C}$, considered as a subspace of $\mathcal{P}(\mathbb{C}^n)$. We shall only use these spaces briefly in the next section.

3. *Spectral factorization*

In this section, we review matrix spectral factorization and discuss some of its basic properties. We begin by introducing two related concepts.

DEFINITION 3.1. A matrix-valued function $\rho \in \mathrm{H}^2(\mathcal{L}(\mathbb{C}^n))$ is said to be *outer* if the set of products $\rho\mathcal{P}^+(\mathbb{C}^n)$ is dense in $\mathrm{H}^2(\mathbb{C}^n)$.

For any $0 \leqslant m \leqslant n$, we say that a matrix-valued function $\theta \in \mathrm{H}^\infty(\mathcal{L}(\mathbb{C}^m, \mathbb{C}^n))$ is *inner* if it is an isometry almost everywhere on $\mathbb{T}$.

A matrix-valued function $b \in \mathrm{H}^2(\mathcal{L}(\mathbb{C}^n))$ will be outer if and only if every inner factor of b is constant and unitary almost everywhere on $\mathbb{T}$. To show this, we consider the H^2 closure of the set $b\mathcal{P}^+(\mathbb{C}^n)$. This is a shift-invariant subspace of $\mathrm{H}^2(\mathbb{C}^n)$, so by the vector-valued Beurling–Lax theorem [**8**, p. 14], it is equal to $\theta\mathrm{H}^2(\mathbb{C}^m)$ for some $0 \leqslant m \leqslant n$ and an inner function $\theta \in \mathrm{H}^\infty(\mathcal{L}(\mathbb{C}^m, \mathbb{C}^n))$. Thus if b is not outer then θ cannot be constant and unitary, but θ is clearly an inner factor of b. Conversely, suppose that b has an inner factor θ which is non-constant or non-unitary. Then θ has no H^∞ inverse, which implies that $\theta\mathrm{H}^2(\mathbb{C}^n)$ is not dense. Since this space contains $b\mathcal{P}^+(\mathbb{C}^n)$, b cannot be outer.

Any outer function $\rho \in \mathrm{H}^2(\mathcal{L}(\mathbb{C}^n))$ is almost everywhere invertible. This assertion follows because the set $\rho\mathcal{P}(\mathbb{C}^n)$ is dense in $\mathrm{L}^2(\mathbb{C}^n)$, which implies that $\rho(z)$ has dense range for almost all $z \in \mathbb{T}$.

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We will also occasionally need to use the following fact about outer functions. If ρ and ρ' are outer functions in $\mathrm{H}^2(\mathcal{L}(\mathbb{C}^n))$ such that $\rho'\rho^{-1}$ lies in $\mathrm{L}^2(\mathcal{L}(\mathbb{C}^n))$, then $\rho'\rho^{-1}$ lies in $\mathrm{H}^2(\mathcal{L}(\mathbb{C}^n))$ and it is also outer. We prove this as follows.

Let $V = \rho\mathcal{P}^+(\mathbb{C}^n)$. The set $(\rho'\rho^{-1})V$ is clearly a dense subset of $\mathrm{H}^2(\mathbb{C}^n)$, but V is also dense in $\mathrm{H}^2(\mathbb{C}^n)$, so by taking H^2 limits of points in V, we find that $(\rho'\rho^{-1})\mathrm{H}^2(\mathbb{C}^n)$ is a dense subset of $\mathrm{H}^1(\mathbb{C}^n)$. This shows that $\rho'\rho^{-1} \in \mathrm{H}^2(\mathcal{L}(\mathbb{C}^n))$. We also know that $\mathrm{H}^2(\mathbb{C}^n) \cap (\rho'\rho^{-1})\mathrm{H}^2(\mathbb{C}^n)$ is dense in $\mathrm{H}^2(\mathbb{C}^n)$, since it contains $(\rho'\rho^{-1})V$. This implies that $\rho'\rho^{-1}$ has no non-trivial inner factor, and is therefore outer.

Definition 3.2. A positive matrix-valued function $w \in \mathrm{L}^1(\mathcal{L}(\mathbb{C}^n))$ is said to be a *spectral density* if it can be written in the form $\rho^*\rho$, for some matrix-valued outer function $\rho \in \mathrm{H}^2(\mathcal{L}(\mathbb{C}^n))$. We call this a *spectral factorization* of w and the functions ρ and ρ^* are called the *spectral factors* of w.

If w is a spectral density, then its spectral factors will be unique up to some unitary constant. Indeed, if $w = \rho^*\rho = \rho'^*\rho'$ for outer functions $\rho, \rho' \in \mathrm{H}^2(\mathcal{L}(\mathbb{C}^n))$, then the function $\rho'\rho^{-1}$ is seen to be an isometry almost everywhere. It therefore lies in $\mathrm{L}^2(\mathcal{L}(\mathbb{C}^n))$, proving that it is an outer function in $\mathrm{H}^2(\mathcal{L}(\mathbb{C}^n))$. Hence $\rho'\rho^{-1}$ is both inner and outer. This implies that it is an almost everywhere unitary constant.

There will always be a unique spectral factor $\rho \in \mathrm{H}^2(\mathcal{L}(\mathbb{C}^n))$ for which $\rho(0)$ is a positive matrix. We shall call this the *canonical spectral factor* of w and denote it $\Phi(w)$. This defines a mapping

$$w \longmapsto \Phi(w),$$

called the *spectral factorization mapping*.

Not every positive matrix-valued function in $\mathrm{L}^1(\mathcal{L}(\mathbb{C}^n))$ is a spectral density. The following theorem, taken from [**4**], will provide the necessary and sufficient conditions for a spectral factorization to exist.

Theorem 3.3 (Helson and Lowdenslager). *Let w be a positive matrix-valued function in $\mathrm{L}^1(\mathcal{L}(\mathbb{C}^n))$. A necessary and sufficient condition for w to have a factorization of the form*

$$w = b^*b,$$

where b is in $\mathrm{H}^2(\mathcal{L}(\mathbb{C}^n))$ with $\det b(0) \neq 0$, is that

$$\int_{\mathbb{T}} \mathrm{Tr}\, \log w \, \mathrm{d}\mu > -\infty. \tag{1}$$

If this condition is satisfied, we can choose b so that

$$\int_{\mathbb{T}} \log|\det b| \, \mathrm{d}\mu = \log|\det b(0)|. \tag{2}$$

Equation (2) is in fact an extremal condition that is satisfied precisely when b is outer. For arbitrary H^2 functions, (2) becomes an inequality with the left-hand side greatest. This is just an extension of Jensen's inequality, treated for scalar-valued H^1 functions in [**5**]. We refer the reader to [**4**, pp. 193–195] for more details relating to the above theorem.

Thus, as a corollary of Theorem 3.3, we have the following proposition, characterizing spectral densities in $\mathrm{L}^1(\mathcal{L}(\mathbb{C}^n))$.

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PROPOSITION 3.4. *Let $w \in \mathrm{L}^1(\mathcal{L}(\mathbb{C}^n))$ be a positive matrix-valued function. The following are equivalent.*

(1) *w is a spectral density.*
(2) *$\log w$ is an integrable function defined almost everywhere on $\mathbb{T}$.*
(3) *$\log\det w$ is an integrable function.*
(4) *$\langle \log\det w, 1\rangle > -\infty$.*

Moreover, the above inner product satisfies

$$2\log|\det\rho(0)| = \langle \log\det w, 1\rangle = \langle \log w, 1\rangle \tag{3}$$

whenever ρ is a spectral factor of w.

Proof. (1) $\Leftrightarrow$ (4). As we remarked above, equation (2) of Theorem 3.3 is equivalent to the H^2 factor of w being outer. Thus w is a spectral density if and only if relation (1) is satisfied, but since $\mathrm{Tr}\log w$ and $\log\det w$ are equal, the integral in (1) is just $\langle \log\det w, 1\rangle$, so this is greater than $-\infty$ if and only if w is a spectral density.

(4) $\Rightarrow$ (2). Since $\log\det w > -\infty$ almost everywhere, $w(z)$ is invertible and therefore has a self-adjoint logarithm for almost all $z \in \mathbb{T}$. For all $x \in (0,\infty)$, we have $|\log x| \leqslant 2x - \log x$. By applying this inequality to the eigenvalues of $w(z)$ and then summing them, we find that

$$\|\log w(z)\|_1 \leqslant 2\mathrm{Tr}\, w(z) - \mathrm{Tr}\,\log w(z) = 2\,\mathrm{Tr}\, w(z) - \log\det w(z),$$

for almost all $z \in \mathbb{T}$. By integrating over $\mathbb{T}$, we obtain the inequality:

$$\|\log w\|_1 \leqslant 2\langle w, 1\rangle - \langle \log\det w, 1\rangle,$$

which provides a finite upper bound for $\|\log w\|_1$, since w is integrable.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4). These are trivial.
Now from equation (2) of Theorem 3.3, for any spectral factor ρ we have

$$\log|\det\rho(0)| = \langle \log|\det\rho|, 1\rangle.$$

However $|\det\rho|^2$ is equal to $\det\rho^*\rho$, so the right-hand side of the above equation is one half of $\langle \log\det w, 1\rangle = \langle \mathrm{Tr}\,\log w, 1\rangle = \langle \log w, 1\rangle$, establishing equation (3). □

Equation (3) of Proposition 3.4 will be fundamental to many of the results we shall later obtain. We will look at some of the continuity properties of the function

$$\Phi : \{\text{spectral densities in } \mathrm{L}^1(\mathcal{L}(\mathbb{C}^n))\} \longrightarrow \mathrm{H}^2(\mathcal{L}(\mathbb{C}^n)),$$

which maps a given spectral density to its canonical spectral factor.
The main result we shall obtain is the following.

THEOREM 3.5. *Let $w_r, w \in \mathrm{L}^1(\mathcal{L}(\mathbb{C}^n))$ be spectral densities, for all $r \in \mathbb{N}$. The following are equivalent.*

(1) *$w_r \to w$ in L^1 as $r \to \infty$ and $\{\log\det w_r : r \in \mathbb{N}\}$ is uniformly integrable.*
(2) *$\Phi(w_r) \to \Phi(w)$ in H^2 as $r \to \infty$.*

The above terminology may be unfamiliar. For a collection $\mathcal{G}$ of measurable functions on an arbitrary measure space $(X, \mathcal{E}, \nu)$, we say that $\mathcal{G}$ is *uniformly*

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integrable if it is bounded in L^1 (that is, the L^1 norms of the functions in $\mathcal{G}$ form a bounded subset of $\mathbb{R}$) and for all $\varepsilon > 0$, there is some $\delta > 0$ such that

$$\int_E |f| \, \mathrm{d}\nu < \varepsilon$$

for all $f \in \mathcal{G}$ and $E \in \mathcal{E}$ with $\nu(E) < \delta$. Even though this property is stronger than boundedness in L^1, it is weaker than boundedness in L^p, for any $p > 1$, provided the underlying measure space is finite. There are equivalent definitions of uniform integrability. Consult [**10**, pp. 270–279] for a standard treatment of this topic.

In the next section, we will show that the uniform log-integrability condition for $(\det w_r)_{r\in\mathbb{N}}$ in Theorem 3.5, is equivalent to several other conditions on the sequence $(w_r)_{r\in\mathbb{N}}$.

4. *A preliminary result*

First of all, we shall introduce some more notation.

For any matrix $A \in \mathcal{L}(\mathbb{C}^n)$, write $|A|$ to mean the positive matrix $(A^*A)^{1/2}$. Now, for any self-adjoint matrices $A, B \in \mathcal{L}(\mathbb{C}^n)$, we define the following operations:

$$A \vee B = \tfrac{1}{2}(A + B + |A - B|)$$
$$A \wedge B = \tfrac{1}{2}(A + B - |A - B|).$$

Note that $\vee$ and $\wedge$ are commutative but not associative (for $n > 1$).

For any self-adjoint matrices $A, B, C \in \mathcal{L}(\mathbb{C}^n)$, the following identities hold.

$$-(A \wedge B) = (-A) \vee (-B)$$
$$(A + C) \vee (B + C) = (A \vee B) + C.$$

In addition, the following statements hold.

$$A \wedge B \leqslant A, B \leqslant A \vee B$$
$$C = A \vee B \qquad \text{whenever } A, B \leqslant C \leqslant A \vee B.$$

In other words, $A \vee B$ is a minimal upper bound for A and B. Minimality follows from the fact that if C is chosen as above, we have

$$0 \leqslant (A \vee B) - C \leqslant (B - A) \vee 0, \quad (A - B) \vee 0.$$

It follows that

$$\ker((A \vee B) - C) \geqslant \ker((A - B) \vee 0), \quad \ker((B - A) \vee 0).$$

However the ranges of $(A - B) \vee 0$ and $(B - A) \vee 0$ are orthogonal, and so their kernels span $\mathbb{C}^n$. Hence $C = A \vee B$.

Likewise, $A \wedge B$ is a maximal lower bound for A and B. Note that A and B will not have universal upper and lower bounds in general. Indeed, their existence would contradict the non-associativity of $\wedge$ and $\vee$.

For self-adjoint matrix-valued functions $f, g : \mathbb{T} \longrightarrow \mathcal{L}(\mathbb{C}^n)$, let $f \wedge g$, $f \vee g$ be the functions given by

$$(f \wedge g)(z) = f(z) \wedge g(z)$$
$$(f \vee g)(z) = f(z) \vee g(z)$$

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for all $z \in \mathbb{T}$. We will use the binary operators $\wedge$ and $\vee$ in the proof of the main result of this section, and then later in the proof of the main theorem.

We will need the following lemma.

LEMMA 4.1. *Let $A, B \in \mathcal{L}(\mathbb{C}^n)$ be positive matrices and suppose that $A \leqslant B$. Then the following inequalities hold.*

(1) $\sqrt{A} \leqslant \sqrt{B}$.

(2) $\log A \leqslant \log B$, *provided that A and B are invertible.*

The first assertion is a special case of the Löwner–Heinz inequality [**9**], which states that $A^p \leqslant B^p$ for any positive operators $A \leqslant B$, whenever $p \in [0,1]$. Here we provide a separate proof for the matrix case, with $p = 1/2$.

Proof of Lemma 4.1. *Part* (1). Let λ be the minimum non-negative real number such that $\sqrt{B} + \lambda I \geqslant \sqrt{A}$, or equivalently, such that $\lambda I \geqslant \sqrt{A} - \sqrt{B}$. Thus we may suppose that λ is (the maximum) eigenvalue of $\sqrt{A} - \sqrt{B}$, since otherwise $\lambda = 0$ and we are done.

Let $v \in \mathbb{C}^n$ be any eigenvector corresponding to λ. Then $(\sqrt{B} + \lambda I)v$ is equal to $\sqrt{A}v$. In particular their norms are equal, so that

$$\langle (B + 2\lambda\sqrt{B} + \lambda^2 I)v, v\rangle = \|(\sqrt{B} + \lambda I)v\|_2^2 = \|\sqrt{A}v\|_2^2 = \langle Av, v\rangle.$$

Rearranging, we have

$$\lambda^2 \|v\|_2^2 + 2\lambda\langle \sqrt{B}v, v\rangle + \langle (B - A)v, v\rangle = 0.$$

By the positivity of $B - A$ and $\sqrt{B}$, each of the terms of the above expression are non-negative. This implies that $\lambda = 0$, proving the first assertion.

Part (2). It is well known that the mapping

$$x \longmapsto r\big(x^{1/r} - 1\big)$$

converges to $x \longmapsto \log x$, locally uniformly in $x \in (0, \infty)$, as r tends to ∞. Provided that the matrices A and B are invertible, their spectra lie in $(0, \infty)$. Then by the spectral mapping theorem we have

$$\begin{aligned} \big\| r\big(A^{1/r} - 1\big) - \log A \big\| &= \max_{x \in \sigma(A)} \big| r\big(x^{1/r} - 1\big) - \log x \big| \\ &\to 0 \qquad \text{as } r \to \infty. \end{aligned}$$

The same happens for B in place of A. In particular, this shows that

$$\begin{aligned} \log A &= \lim_{k \to \infty} 2^k \big(A^{2^{-k}} - 1\big), \\ \log B &= \lim_{k \to \infty} 2^k \big(B^{2^{-k}} - 1\big). \end{aligned}$$

We have $A \leqslant B$ by hypothesis, so by induction on k, it follows from the first assertion that

$$A^{2^{-k}} \leqslant B^{2^{-k}} \qquad \text{for all } k \in \mathbb{N}.$$

Subtracting 1 from both sides, multiplying by 2^k and taking limits, we find that

$$\log A \leqslant \log B,$$

as required. □

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We remark that there are examples of positive matrices $A, B \in \mathcal{L}(\mathbb{C}^2)$, such that $A \leqslant B$, but A^p and B^p are incomparable for any $p > 1$.

We are now ready to prove the main result of this section, which is stated as follows.

PROPOSITION 4.2. *Let $w_r, w \in \mathrm{L}^1(\mathcal{L}(\mathbb{C}^n))$ be spectral densities, for all $r \in \mathbb{N}$. Suppose that $w_r \to w$ in L^1 as $r \to \infty$. Then the following are equivalent.*

(1) $\det \Phi(w_r)(0) \to \det \Phi(w)(0)$ *as* $r \to \infty$.
(2) $\log w_r \to \log w$ *in* L^1 *as* $r \to \infty$.
(3) $\log \det w_r \to \log \det w$ *in* L^1 *as* $r \to \infty$.
(4) $\{\log \det w_r : r \in \mathbb{N}\}$ *is uniformly integrable.*

Proof. (1) $\Rightarrow$ (2). For any $r \in \mathbb{N}$, the functions w_r, w and $w_r \vee w$ are almost everywhere positive invertible matrix-valued, and therefore have self-adjoint logarithms defined almost everywhere on $\mathbb{T}$. Also, since $w_r, w \leqslant w_r \vee w$, we have the relations:

$$\log w_r, \log w \leqslant \log(w_r \vee w) \qquad \text{for all } r \in \mathbb{N},$$

as a consequence of Lemma 4.1.

We first show that $\log(w_r \vee w) \to \log w$ in L^1 as $r \to \infty$, as follows. For all $x \in (0, \infty)$, the non-negative real number, $(\log x) \vee 0$, is less than or equal to x. This implies that the positive matrix-valued function

$$\log(w_r \vee w) \vee 0$$

is dominated by $w_r \vee w$, so we obtain the chain of inequalities

$$\log w \leqslant \log(w_r \vee w) \leqslant \log(w_r \vee w) \vee 0 \leqslant w_r \vee w,$$

but since w is a spectral density and $w_r, w \in \mathrm{L}^1(\mathcal{L}(\mathbb{C}^n))$, the terms $\log w$ and $w_r \vee w$ above, are both integrable. Moreover, it is clear from the definition, $2(w_r \vee w) = w_r + w + |w_r - w|$, that $w_r \vee w \to w$ in L^1 as $r \to \infty$, so the collection of functions $\{(w_r \vee w) - \log w : r \in \mathbb{N}\}$ is uniformly integrable. Subtracting $\log w$ from the terms in the above chain of inequalities, we obtain

$$0 \leqslant \log(w_r \vee w) - \log w \leqslant (w_r \vee w) - \log w,$$

from which we deduce that $\{\log(w_r \vee w) - \log w : r \in \mathbb{N}\}$ is uniformly integrable. However since $\log(w_r \vee w) \to \log w$ in measure as $r \to \infty$, we have

$$\log(w_r \vee w) \to \log w \quad \text{in } \mathrm{L}^1 \qquad \text{as } r \to \infty.$$

By Minkowski's inequality, we have

$$\begin{aligned} \|\log w_r - \log w\|_1 &\leqslant \|\log(w_r \vee w) - \log w_r\|_1 + \|\log(w_r \vee w) - \log w\|_1 \\ &= \langle \log(w_r \vee w) - \log w_r, 1\rangle + \langle \log(w_r \vee w) - \log w, 1\rangle \\ &= \langle \log w - \log w_r, 1\rangle + 2\langle \log(w_r \vee w) - \log w, 1\rangle \\ &= \langle \log w - \log w_r, 1\rangle + 2 \underbrace{\|\log(w_r \vee w) - \log w\|_1}_{\to 0 \text{ as } r \to \infty}. \end{aligned}$$

Thus to establish (2), it is sufficient to prove that $\langle \log w_r, 1\rangle$ tends to $\langle \log w, 1\rangle$ as $r \to \infty$. This is the point where we need to use (1).

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By equation (3) of Proposition 3.4, we have the identities

$$2\log\det\Phi(w)(0) = \langle \log w, 1\rangle$$
$$2\log\det\Phi(w_r)(0) = \langle \log w_r, 1\rangle$$

for all $r\in\mathbb{N}$, but since $\det\Phi(w_r)(0)\to\det\Phi(w)(0)$ as $r\to\infty$, the continuity of log implies that $\langle\log w_r,1\rangle$ tends to $\langle\log w,1\rangle$ as $r\to\infty$, proving (2).

(2) $\Rightarrow$ (3). The trace operation is L^1 norm-decreasing, so

$$\begin{aligned}\|\log\det w_r - \log\det w\|_1 &= \|\mathrm{Tr}(\log w_r - \log w)\|_1\\ &\leqslant \|\log w_r - \log w\|_1\\ &\to 0 \qquad \text{as } r\to\infty.\end{aligned}$$

(3) $\Rightarrow$ (4). This is trivial. (L^1 convergence always implies uniform integrability.)

(4) $\Rightarrow$ (1). We have already established the identities

$$2\log\det\Phi(w)(0) = \langle\log\det w,1\rangle = \int_{\mathbb{T}}\log\det w\,\mathrm{d}\mu$$
$$2\log\det\Phi(w_r)(0) = \langle\log\det w_r,1\rangle = \int_{\mathbb{T}}\log\det w_r\,\mathrm{d}\mu.$$

However $\log\det w_r$ converges to $\log\det w$ in measure as r tends to ∞, so by the uniform integrability of $\{\log\det w_r : r\in\mathbb{N}\}$ and the integrability of $\log\det w$, we find that

$$\int_{\mathbb{T}}\log\det w_r\,\mathrm{d}\mu \to \int_{\mathbb{T}}\log\det w\,\mathrm{d}\mu \qquad \text{as } r\to\infty.$$

In other words, $\log\det\Phi(w_r)(0)\to\log\det\Phi(w)(0)$ as $r\to\infty$. This proves (1). □

5. *Some inequalities*

In this section we will establish in succession, two inequalities which will lead to a proof of Theorem 3.5. The first of these will show that in some appropriate sense, spectral densities which are close to 1 have particular spectral factors which are also close to 1. This will lead to the second result, which deals with two spectral densities which are close to each other, and deduces an inequality concerning their canonical spectral factors.

It will be useful to introduce some more notation here. Recall that μ denotes normalized Lebesgue measure on $\mathbb{T}$. This makes the triple $(\mathbb{T},\mathcal{B}(\mathbb{T}),\mu)$ into a probability space, where $\mathcal{B}(\mathbb{T})$ denotes the σ-algebra of Borel subsets of $\mathbb{T}$, so any measurable function

$$X:\mathbb{T}\longrightarrow\mathcal{L}(\mathbb{C}^n)$$

may be regarded as a matrix-valued random variable. In the case when X is integrable, we have a notion of expectation and variance, given as

$$\begin{aligned}\mathbb{E}X &= \int_{\mathbb{T}} X\,\mathrm{d}\mu\\ \mathrm{var}X &= \frac{1}{n}\|X-\mathbb{E}X\|_2^2\\ &= \frac{1}{n}\mathrm{Tr}\,\mathbb{E}((X-\mathbb{E}X)^*(X-\mathbb{E}X)).\end{aligned} \tag{4}$$

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We have the identities

$$\mathbb{E}(X^*(\mathbb{E}X)) = \mathbb{E}((\mathbb{E}X^*)X) = \mathbb{E}((\mathbb{E}X^*)(\mathbb{E}X)) = (\mathbb{E}X)^*(\mathbb{E}X),$$

from which we obtain the alternative formula

$$\begin{aligned}\operatorname{var} X &= \frac{1}{n}\operatorname{Tr}(\mathbb{E}(X^*X) - (\mathbb{E}X)^*(\mathbb{E}X)) \\ &= \frac{1}{n}\big(\|X\|_2^2 - \|\mathbb{E}X\|_2^2\big). \end{aligned} \tag{5}$$

When X is square integrable, this is just the statement that $\mathbb{E}X$ is orthogonal to $X - \mathbb{E}X$. In fact, $\mathbb{E}X$ is precisely the orthogonal projection of X onto the matrix-valued constant functions.

In the next lemma, we shall be considering the variance of a spectral factor ρ, for a given spectral density $w \in \mathrm{L}^1(\mathcal{L}(\mathbb{C}^n))$. The two key facts we shall use are

$$\begin{aligned}\mathbb{E}\rho &= \rho(0), \\ \|\rho\|_2^2 &= \|\rho^*\rho\|_1 = \|w\|_1.\end{aligned}$$

This shows that the variance of ρ is small provided w is close to 1 and $\rho(0)$ is close to the identity. From this we can deduce that ρ is close to 1, as well. It turns out that the determinant of $\rho(0)$ will provide the constraint we need to show that $\rho(0)$ is close to the identity, so it is involved in the inequality which follows.

LEMMA 5.1. *Let $w \in \mathrm{L}^1(\mathcal{L}(\mathbb{C}^n))$ be a spectral density and let $\rho \in \mathrm{H}^2(\mathcal{L}(\mathbb{C}^n))$ be a spectral factor of w. Suppose that $\rho(0)$ has positive spectrum. Then*

$$\frac{1}{n}\|\rho - 1\|_2^2 \leqslant \frac{1}{n}\|w\|_1 - 2(\det \rho(0))^{1/n} + 1 \tag{6}$$

$$\leqslant \frac{1}{n}\|w - 1\|_1 + 2\big(1 - (\det \rho(0))^{1/n}\big). \tag{7}$$

This upper bound tends to 0 whenever $w \to 1$ in L^1 and $\langle \log \det w, 1\rangle \to 0$.

Proof. Every eigenvalue of $\rho(0)$ is positive. By applying the AM–GM (arithmetic mean – geometric mean) inequality to these eigenvalues, we find that

$$(\det \rho(0))^{1/n} \leqslant \frac{1}{n}\operatorname{Tr}\rho(0) = \frac{1}{n}\langle \rho(0), 1\rangle.$$

Therefore

$$\begin{aligned}\frac{1}{n}\|\rho(0) - 1\|_2^2 &= \frac{1}{n}\langle \rho(0) - 1, \rho(0) - 1\rangle \\ &= \frac{1}{n}\|\rho(0)\|_2^2 - \frac{2}{n}\langle \rho(0), 1\rangle + 1 \\ &\leqslant \frac{1}{n}\|\rho(0)\|_2^2 - 2(\det \rho(0))^{1/n} + 1. \end{aligned} \tag{8}$$

From the variance formula (5), we have

$$\begin{aligned}\operatorname{var}\rho &= \frac{1}{n}\big(\|\rho\|_2^2 - \|\rho(0)\|_2^2\big), \\ \operatorname{var}(\rho - 1) &= \frac{1}{n}\big(\|\rho - 1\|_2^2 - \|\rho(0) - 1\|_2^2\big).\end{aligned}$$

However ρ and $\rho - 1$ have the same variance, by the definition given in (4), so by

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adding this variance to both sides of inequality (8), we find that

$$\frac{1}{n}\|\rho - 1\|_2^2 \leqslant \frac{1}{n}\|\rho\|_2^2 - 2(\det \rho(0))^{1/n} + 1.$$

Substituting $\|w\|_1$ for $\|\rho\|_2^2$, we obtain (6). By the triangle inequality,

$$\frac{1}{n}\|w\|_1 \leqslant \frac{1}{n}(\|w-1\|_1 + \|1\|_1) = \frac{1}{n}\|w-1\| + 1,$$

so by substituting the right-hand side into (6), we obtain inequality (7).

Recalling identity (3) from Proposition 3.4,

$$2\log|\det\rho(0)| = \langle \log\det w, 1\rangle,$$

we find that $\det\rho(0) \to 1$ whenever $\langle \log\det w, 1\rangle \to 0$. Therefore the upper bound given in (7) tends to 0, whenever $w \to 1$ in L^1 as well. □

We make two remarks about the above lemma. Firstly, a spectral factor ρ, for which $\rho(0)$ has positive spectrum, will not be the canonical spectral factor in general, that is, $\rho(0)$ need not be positive. In fact, a matrix is diagonalizable and has positive spectrum if and only if it is the product of two positive, invertible matrices. We shall use (and prove) this fact in the next lemma.

Secondly, for functions w and ρ as in the statement of the lemma, we have the following chain of inequalities:

$$(\det\rho(0))^{1/n} \leqslant \frac{1}{n}\langle\rho(0), 1\rangle \leqslant \frac{1}{n}\|\rho(0)\|_1 \leqslant \frac{1}{\sqrt{n}}\|\rho(0)\|_2 \leqslant \left(\frac{1}{n}\|w\|_1\right)^{1/2}.$$

When $w \to 1$ in L^1 and $\langle \log\det w, 1\rangle \to 0$, the leftmost and rightmost terms of the above chain tend to unity. Hence all the terms tend to unity, and this forces ρ to tend to 1 in H^2. This summarizes the main idea of the proof.

As a consequence of Lemma 5.1, we have following result.

LEMMA 5.2. *Let $w, w' \in \mathrm{L}^1(\mathcal{L}(\mathbb{C}^n))$ be spectral densities and let $0 < \eta \leqslant R < \infty$ be given constants. Suppose that $w \geqslant \eta$, and let $P \in \mathrm{L}^\infty(\mathcal{L}(\mathbb{C}^n))$ be the projection-valued function given pointwise by the Borel functional calculus as*

$$P = \chi_{[0,R]}(w),$$

where $\chi_{[0,R]}$ is the characteristic function of the interval $[0, R]$. Let ρ, ρ' denote the canonical spectral factors $\Phi(w), \Phi(w')$ respectively. Then

$$\|(\rho' - \rho)P\|_2^2 \leqslant R\left(\frac{\|w' - w\|_1}{\eta} + 2n\left[1 - \left(\frac{\det\rho'(0)}{\det\rho(0)}\right)^{1/n}\right]\right).$$

Proof. Let $\rho'' = \rho'\rho^{-1}$. This function lies in $\mathrm{L}^2(\mathcal{L}(\mathbb{C}^n))$ since ρ^{-1} is bounded, but ρ and ρ' are outer and, as we showed in Section 3, this implies that ρ'' is an outer function in $\mathrm{H}^2(\mathcal{L}(\mathbb{C}^n))$. Therefore ρ'' is a spectral factor of the spectral density $w'' \in \mathrm{L}^1(\mathcal{L}(\mathbb{C}^n))$ given by

$$w'' = \rho''^*\rho'' = \rho^{-1^*} w' \rho^{-1}.$$

Now since $\rho''(0) = \rho'(0)\rho^{-1}(0)$ is a product of two positive invertible matrices, it has positive spectrum. Indeed, if $P, Q \in \mathcal{L}(\mathbb{C}^n)$ are positive and invertible, then PQ is conjugate to $Q^{1/2}PQ^{1/2}$, which is manifestly positive and invertible and therefore has positive spectrum. (Conversely, if $X \in \mathcal{L}(\mathbb{C}^n)$ is diagonalizable with positive spectrum, then $X = L^{-1}DL$ for some $D, L \in \mathcal{L}(\mathbb{C}^n)$ invertible with D positive and diagonal. Therefore X is the product of $L^{-1}DL^{-1^*}$ and L^*L, both of which are positive and invertible.)

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Thus by Lemma 5.1, we have the inequality

$$\begin{aligned}\|\rho'' - 1\|_2^2 &\leqslant \|w'' - 1\|_1 + 2n\big(1 - (\det \rho''(0))^{1/n}\big)\\ &= \|w'' - 1\|_1 + 2n\left[1 - \left(\frac{\det \rho'(0)}{\det \rho(0)}\right)^{1/n}\right], \end{aligned} \tag{9}$$

but $w'' - 1$ is equal to ${\rho^{-1}}^*(w' - w)\rho^{-1}$, and so

$$\|w'' - 1\|_1 \leqslant \|{\rho^{-1}}^*\|_\infty \|w' - w\|_1 \|\rho^{-1}\|_\infty \leqslant \eta^{-1}\|w' - w\|_1. \tag{10}$$

Now, $\rho' - \rho$ is equal to $(\rho'' - 1)\rho$, so we have

$$\begin{aligned}\|(\rho' - \rho)P\|_2^2 &\leqslant \|\rho'' - 1\|_2^2 \|\rho P\|_\infty^2\\ &= \|PwP\|_\infty \|\rho'' - 1\|_2^2\\ &\leqslant R\|\rho'' - 1\|_2^2. \end{aligned} \tag{11}$$

By combining (10) and (11) with inequality (9), we obtain the desired inequality, proving the lemma. □

6. *The main theorem*

We now have all the tools needed to prove Theorem 3.5. Recalling the statement of the theorem, we are given a spectral density w, together with a sequence of spectral densities $(w_r)_{r\in\mathbb{N}}$, and we aim to show that the following two statements are equivalent.

(1) $w_r \to w$ in L^1 as $r \to \infty$ and $\{\log\det w_r : r \in \mathbb{N}\}$ is uniformly integrable.
(2) $\Phi(w_r) \to \Phi(w)$ in H^2 as $r \to \infty$.

We will use Proposition 4.2 to reformulate the first statement. This will enable us to prove that (2) implies (1). The other direction is harder, and for this we will make use of Lemma 5.2.

Proof of Theorem 3.5. (2) ⇒ (1). Let ρ, ρ_r denote the canonical spectral factors $\Phi(w), \Phi(w_r)$ respectively, for each $r \in \mathbb{N}$. The function $w_r - w$ can be written in the form:

$$\frac{(\rho_r + \rho)^*(\rho_r - \rho)}{2} + \frac{(\rho_r - \rho)^*(\rho_r + \rho)}{2},$$

so by Hölder's inequality, together with the triangle inequality, we have

$$\begin{aligned}\|w_r - w\|_1 &\leqslant \tfrac{1}{2}\|(\rho_r + \rho)^*(\rho_r - \rho)\|_1 + \tfrac{1}{2}\|(\rho_r - \rho)^*(\rho_r + \rho)\|_1\\ &\leqslant \tfrac{1}{2}\|\rho_r + \rho\|_2\|\rho_r - \rho\|_2 + \tfrac{1}{2}\|\rho_r - \rho\|_2\|\rho_r + \rho\|_2\\ &\leqslant (\|\rho_r\|_2 + \|\rho\|_2)\|\rho_r - \rho\|_2 \end{aligned}$$

for all $r \in \mathbb{N}$. However the sequence of reals

$$(\|\rho_r\|_2 + \|\rho\|_2)_{r\in\mathbb{N}}$$

is bounded since $(\rho_r)_{r\in\mathbb{N}}$ is convergent in H^2. Therefore $\|w_r - w\|_1$ tends to 0 as $r \to \infty$. This establishes the first part of (1). Now as r tends to ∞,

$$\det \Phi(w_r)(0) \to \det \Phi(w)(0),$$

since $\Phi(w_r) \to \Phi(w)$ in H^2, so by Proposition 4.2, the set $\{\log\det w_r : r \in \mathbb{N}\}$ is uniformly integrable. This establishes the second part of (1).

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(1) $\Rightarrow$ (2). Fix constants $0<\eta\leqslant R<\infty$. Let $\tilde{w}$ be the spectral density $w\vee\eta$ and let $\tilde{\rho}$ denote its canonical spectral factor $\Phi(\tilde{w})$. Let $P\in \mathrm{L}^\infty(\mathcal{L}(\mathbb{C}^n))$ be the projection-valued function given by

$$P=\chi_{[0,R]}(w)=\chi_{[0,R]}(\tilde{w}).$$

By Lemma 5.2, we have the inequalities:

$$\|(\rho-\tilde{\rho})P\|_2^2\leqslant\frac{R}{\eta}\|w-\tilde{w}\|_1+2nR\left|1-\left(\frac{\det\rho(0)}{\det\tilde{\rho}(0)}\right)^{1/n}\right|, \tag{12}$$

$$\|(\rho_r-\tilde{\rho})P\|_2^2\leqslant\frac{R}{\eta}\|w_r-\tilde{w}\|_1+2nR\left|1-\left(\frac{\det\rho_r(0)}{\det\tilde{\rho}(0)}\right)^{1/n}\right|, \tag{13}$$

for all $r\in\mathbb{N}$. We shall now obtain an upper bound on $\|\rho_r-\rho\|_2$, in terms of the left-hand sides of (12) and (13).

The functions $(\rho_r-\rho)P$ and $(\rho_r-\rho)(1-P)$ are orthogonal, since

$$\begin{aligned}\langle(\rho_r-\rho)P,(\rho_r-\rho)1\rangle&=\langle(\rho_r-\rho)P^2,(\rho_r-\rho)1\rangle\\&=\langle(\rho_r-\rho)P,(\rho_r-\rho)P\rangle.\end{aligned}$$

Therefore by Pythagoras' theorem, we have

$$\begin{aligned}\|\rho_r-\rho\|_2^2-\|(\rho_r-\rho)P\|_2^2&=\|(\rho_r-\rho)(1-P)\|_2^2\\&\leqslant(\|\rho_r(1-P)\|_2+\|\rho(1-P)\|_2)^2\\&\leqslant\left[\|(1-P)w_r(1-P)\|_1^{1/2}+\|w(1-P)\|_1^{1/2}\right]^2\\&\leqslant\left[(\|w_r-w\|_1+\|w(1-P)\|_1)^{1/2}+\|w(1-P)\|_1^{1/2}\right]^2,\end{aligned}$$

which gives us the inequality

$$\begin{aligned}\|\rho_r-\rho\|_2^2\leqslant{}&(\|(\rho-\tilde{\rho})P\|_2+\|(\rho_r-\tilde{\rho})P\|_2)^2\\&+\left[(\|w_r-w\|_1+\|w(1-P)\|_1)^{1/2}+\|w(1-P)\|_1^{1/2}\right]^2.\end{aligned} \tag{14}$$

Let $\varepsilon>0$. Recall that the functions P and $\tilde{w}$ depend on R and η respectively. We wish to choose R,η and find $N\in\mathbb{N}$, such that for all $r\geqslant N$, the following five inequalities are satisfied:

$$\|w(1-P)\|_1\leqslant\frac{\varepsilon^2}{18} \tag{15}$$

$$\frac{\|w-\tilde{w}\|_1}{\eta}\leqslant\frac{\varepsilon^2}{18R} \tag{16}$$

$$\left|1-\left(\frac{\det\rho(0)}{\det\tilde{\rho}(0)}\right)^{1/n}\right|\leqslant\frac{\varepsilon^2}{36nR} \tag{17}$$

$$\left|\frac{[\det\rho_r(0)]^{1/n}-[\det\rho(0)]^{1/n}}{[\det\tilde{\rho}(0)]^{1/n}}\right|\leqslant\frac{\varepsilon^2}{36nR} \tag{18}$$

$$\frac{\|w_r-w\|_1}{\eta}\leqslant\frac{\varepsilon^2}{18R}. \tag{19}$$

We shall establish (15)–(19) in numerical order. We first choose R to satisfy (15), as follows shortly. Next, we look for an η which satisfies (16) and (17). Finally, we choose N such that (18) and (19) are satisfied.

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The positive scalar-valued function $\operatorname{Tr} w$ is integrable and therefore

$$\int_{E_k} \operatorname{Tr} w \, \mathrm{d}\mu \qquad (\text{where } E_k = \{z \in \mathbb{T} : \operatorname{Tr} w(z) > k\})$$

tends to zero whenever $k \to \infty$. Choose R to be any positive k for which the above integral is at most $\varepsilon^2/18$. Since $\|w(z)\|_\infty$ is at most R for any $z \in \mathbb{T} \setminus E_R$, the function $1 - P$ is supported on the set E_R. This implies that

$$\|w(1-P)\|_1 = \int_{\mathbb{T}} \operatorname{Tr}(w(1-P)) \, \mathrm{d}\mu \leqslant \int_{E_R} \operatorname{Tr} w \, \mathrm{d}\mu \leqslant \frac{\varepsilon^2}{18},$$

establishing (15).

For any $x \in (0, \infty)$, we have

$$\frac{(\eta \vee x) - x}{\eta} = \left(1 - \frac{x}{\eta}\right) \vee 0 \leqslant \left(-\log \frac{x}{\eta}\right) \vee 0 = \log(\eta \vee x) - \log x.$$

This implies that

$$\frac{(\eta \vee w) - w}{\eta} \leqslant \log(\eta \vee w) - \log w,$$

so by integrating the trace of both sides, we find that

$$\frac{\|\tilde{w} - w\|_1}{\eta} = \frac{\langle \tilde{w} - w, 1 \rangle}{\eta} \leqslant \langle \log \tilde{w} - \log w, 1 \rangle. \tag{20}$$

Now by equation (3) from Proposition 3.4, the right-hand side is just

$$2 \log \det \tilde{\rho}(0) - 2 \log \det \rho(0) = -2 \log \left(\frac{\det \rho(0)}{\det \tilde{\rho}(0)} \right),$$

and this is also an upper bound for the expression

$$2n \left[1 - \left(\frac{\det \rho(0)}{\det \tilde{\rho}(0)} \right)^{1/n} \right] \geqslant 0. \tag{21}$$

Now observe that $\tilde{w} \to w$ almost everywhere on $\mathbb{T}$ as $\eta \to 0$. This is because $w(z)$ is invertible for almost all $z \in \mathbb{T}$. Therefore $\operatorname{Tr}(\log \tilde{w} - \log w) \to 0$ almost everywhere as $\eta \to 0$, but this sequence of functions is non-negative and decreasing and therefore dominated in L^1, since $\log w$ and $\log \tilde{w}$ are integrable for all $\eta > 0$. Thus by the dominated convergence theorem, the integral

$$\langle \log \tilde{w} - \log w, 1 \rangle \;=\; \int_{\mathbb{T}} \operatorname{Tr}(\log \tilde{w} - \log w) \, \mathrm{d}\mu$$

tends to zero as $\eta \to 0$. Choose $\eta \leqslant R$ such that this integral is as most $\varepsilon^2/18R$. This is an upper bound for the left-hand sides of (20) and (21), and therefore establishes inequalities (16) and (17).

Now that η and R are fixed, we shall choose N. By hypothesis, $w_r \to w$ in L^1 as $r \to \infty$ and by Proposition 4.2, $\det \rho_r(0) \to \det \rho(0)$ as $r \to \infty$, so the left-hand sides of (18) and (19) tend to zero as $r \to \infty$. This allows us to choose $N \in \mathbb{N}$ such that inequalities (18) and (19) are satisfied for all $r \geqslant N$.

By the triangle inequality, we have

$$\frac{\|w_r - \tilde{w}\|_1}{\eta} \leqslant \frac{\|w_r - w\|_1}{\eta} + \frac{\|w - \tilde{w}\|_1}{\eta},$$

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and also

$$\left|1-\left(\frac{\det\rho_r(0)}{\det\tilde{\rho}(0)}\right)^{1/n}\right| \leqslant \left|1-\left(\frac{\det\rho(0)}{\det\tilde{\rho}(0)}\right)^{1/n}\right| + \left|\frac{[\det\rho_r(0)]^{1/n}-[\det\rho(0)]^{1/n}}{[\det\tilde{\rho}(0)]^{1/n}}\right|,$$

so from the upper bounds provided by (16)–(19), we obtain the following inequalities upon substitution into (12) and (13):

$$\|(\rho-\tilde{\rho})P\|_2^2 \leqslant \frac{\varepsilon^2}{18}(1+1)$$

$$\|(\rho_r-\tilde{\rho})P\|_2^2 \leqslant \frac{\varepsilon^2}{18}(1+1+1+1) \qquad \text{for all } r \geqslant N.$$

Inequality (19) implies that $\|w_r - w\|_1 \leqslant \varepsilon^2/18$ for all $r \geqslant N$, since η is chosen to be at most R. Thus from the above inequalities, together with (15), we find upon sustitution into (14) that

$$\begin{aligned}\|\rho_r-\rho\|_2^2 &\leqslant \frac{\varepsilon^2}{18}\times((\sqrt{1+1}+\sqrt{1+1+1+1})^2+(\sqrt{1+1}+\sqrt{1})^2)\\ &= \frac{\varepsilon^2}{18}\times 3(1+\sqrt{2})^2\end{aligned}$$

for all $r \geqslant N$. Therefore

$$\|\rho_r-\rho\|_2 \leqslant \left(\frac{1+\sqrt{2}}{\sqrt{6}}\right)\varepsilon < \varepsilon$$

for all $r \geqslant N$. This shows that $\rho_r \to \rho$ in H^2 as $r \to \infty$, proving (2). □

We make two remarks about this theorem. Firstly, unlike Lemmas 5.1 and 5.2, this result does not give quantitive information about the proximity of two spectral factors in terms of their spectral densities. This is because the values of R and η needed to establish inequalities (15)–(17) depend upon the distribution of the large values of $\operatorname{Tr} w$ and $-\log\det w$. Even if we constrain the norms of w and $\log\det w$, we can still find spectral densities w, such that R and $1/\eta$ need to be arbitrarily large.

Secondly, for an L^1 convergent sequence of spectral densities, we need only show that their spectral factors converge in measure, in order to establish their H^2 convergence. This is because of their uniform square-integrability. We make use of this idea in the next result, where we extend Theorem 3.5.

THEOREM 6.1. *Let $p \in [1,\infty)$ and let $w_r, w \in \mathrm{L}^p(\mathcal{L}(\mathbb{C}^n))$ be spectral densities, for all $r \in \mathbb{N}$. The following are equivalent.*

(1) *$w_r \to w$ in L^p as $r \to \infty$ and $\{\log\det w_r : r \in \mathbb{N}\}$ is uniformly integrable.*
(2) *$\Phi(w_r) \to \Phi(w)$ in H^{2p} as $r \to \infty$.*

Proof. (2) $\Rightarrow$ (1). Let ρ, ρ_r denote the canonical spectral factors $\Phi(w), \Phi(w_r)$ respectively, for each $r \in \mathbb{N}$. As in the proof of Theorem 3.5, we have the inequality

$$\|w_r - w\|_p \leqslant \tfrac{1}{2}\|(\rho_r+\rho)^*(\rho_r-\rho)\|_p + \tfrac{1}{2}\|(\rho_r-\rho)^*(\rho_r+\rho)\|_p.$$

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Let $q \in (1,\infty]$ be the index conjugate to p. For any functions $f, g \in \mathrm{L}^{2p}(\mathcal{L}(\mathbb{C}^n))$, we have, by Hölder's inequality,

$$\|fg\|_p = \sup_{\|h\|_q \leqslant 1} \|fgh\|_1 \leqslant \sup_{\|h\|_q \leqslant 1} \|f\|_{2p}\|g\|_{2p}\|h\|_q = \|f\|_{2p}\|g\|_{2p},$$

and so we find that

$$\|w_r - w\|_p \leqslant \|\rho_r + \rho\|_{2p}\|\rho_r - \rho\|_{2p} \to 0 \qquad \text{as } r \to \infty.$$

Now we apply Proposition 4.2 to find that $\{\log\det w_r : r \in \mathbb{N}\}$ is uniformly integrable, establishing (1).

(1) $\Rightarrow$ (2). Since $w_r \to w$ in L^1, we find by Theorem 3.5 that $\rho_r \to \rho$ in H^2, and in particular, $\rho_r - \rho \to 0$ in measure as $r \to \infty$.

Let $f_r : \mathbb{T} \longrightarrow [0,\infty)$ be given by

$$f_r(z) = \|\rho_r(z) - \rho(z)\|_{2p}^{2p} \qquad \text{for all } z \in \mathbb{T},\ r \in \mathbb{N}.$$

By the triangle inequality,

$$\begin{aligned} f_r(z) &\leqslant \left(\|\rho_r(z)\|_{2p} + \|\rho(z)\|_{2p}\right)^{2p} \\ &\leqslant 2^p\left(\|\rho_r(z)\|_{2p}^2 + \|\rho(z)\|_{2p}^2\right)^p \\ &\leqslant 2^p\left(\|w_r(z) - w(z)\|_p + 2\|w(z)\|_p\right)^p \\ &\leqslant 2^p \cdot 3^{p-1}\left(\|w_r(z) - w(z)\|_p^p + 2\|w(z)\|_p^p\right) \qquad \text{for all } z \in \mathbb{T}, \end{aligned}$$

and this upper bound is uniformly integrable over $r \in \mathbb{N}$, since $w_r - w \to 0$ in L^p as $r \to \infty$. Now since $(f_r)_{r\in\mathbb{N}}$ is uniformly integrable and convergent in measure to 0, it also tends to 0 in L^1. This implies that $\rho_r \to \rho$ in H^{2p} as $r \to \infty$, establishing (2). □

Note that the set $\{\log\det w_r : r \in \mathbb{N}\}$ in the first condition of the above theorem, will be uniformly integrable automatically if it is bounded in L^s, for any $s > 1$. Uniform integrability also follows when $\{\log w_r : r \in \mathbb{N}\}$ is bounded in L^s, or when the (convergent) spectral densities have inverses uniformly bounded in L^∞. Thus any of these conditions, together with L^p convergence of the w_r to some spectral density w, will be sufficient for $\Phi(w_r) \to \Phi(w)$ in H^{2p} as $r \to \infty$, for any $p \in [1,\infty)$.

In light of these remarks, the following is immediate from Theorem 6.1.

COROLLARY 6.2. *Let $w_r \in \mathrm{L}^1(\mathcal{L}(\mathbb{C}^n))$ be a spectral density for all $r \in \mathbb{N}$. Suppose that $w_r \to w$ in L^∞, where $w, w^{-1} \in \mathrm{L}^\infty(\mathcal{L}(\mathbb{C}^n))$. Then $\Phi(w_r) \to \Phi(w)$ in H^p for all $p \in [1,\infty)$.*

We also note that L^1 boundedness of the set $\{\log w_r : r \in \mathbb{N}\}$ will not guarantee uniform integrability of the functions $(\log\det w_r)_{r\in\mathbb{N}}$, even if the sequence of spectral densities $(w_r)_{r\in\mathbb{N}}$ is uniformly bounded (in L^∞) and convergent to 1 almost everywhere. Thus by Theorem 3.5, their canonical spectral factors will not converge to 1 in H^2. Since they are uniformly bounded, they do not even converge to 1 in measure.

7. *Concluding remarks*

Lemma 5.1 is the key ingredient used in the proof of Theorem 3.5. This is what establishes the $\mathrm{L}^1 \to \mathrm{H}^2$ continuity at the constant density 1. Thus the simple

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variance argument used to prove the lemma is perhaps the most important idea involved in the proof of the theorem. Much of the work needed to complete the proof is simply overcoming the technical difficulties involved in the translation from continuity at 1, to continuity at an arbitrary spectral density.

Theorem 6.1 and Proposition 4.2 complement each other. Together they provide many conditions equivalent to H^{2p} convergence of a sequence of canonical spectral factors. They also suggest that it is the L^1 convergence of the *logarithms* of the spectral densities, and not the densities themselves, which is important for the convergence of their respective spectral factors. Perhaps L^1 log-convergence alone, of any given sequence of spectral densities, is a sufficient condition for their canonical spectral factors to converge in measure.

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