J. Plasma Physics (1998), vol. 60, part 3, pp. 449–467. Printed in the United Kingdom
© 1998 Cambridge University Press

Magnetically confined plasma flows with helical symmetry

L. J. PALUMBO† and A. M. PLATZECK‡

Facultad de Ciencias Astronómicas y Geofísicas, Universidad Nacional de La Plata, Paseo del Bosque s/n, 1900, La Plata, Argentina

(Received 24 April 1997 and in revised form 12 December 1997)

Stationary flows with an ignorable coordinate are analysed and the differential equation for the only independent current function of the problem is obtained for the polytropic, incompressible and adiabatic cases. The possibility of confinement for the stationary symmetric plasma fluxes is then investigated and the confinement conditions are given. Some examples are solved for the particular case of helical symmetry; for general flows it is possible to construct magnetically confined plasma columns only with cylindrical shape; whereas if the plasma velocity has the direction of the ignorable coordinate, we show that it is possible to construct magnetically confined plasma columns with helical shape and plasma flux along the helices.

1. Introduction

In line with a previous paper on static equilibria (Palumbo and Platzeck 1993), the main purpose of this work is to seek confined columns of magnetized plasma with stationary flux where some symmetry is supposed. We start by making an analysis of the stationary solutions of the MHD equations with a symmetry. This has already been discussed by Agim and Tataronis (1985) and Tsinganos (1982*a*, *b*), but in such a way that the results obtained are inappropriate for the purpose of the present paper. Other authors who have analysed stationary symmetric flows are Morozov and Solov'ev (1963), Solov'ev (1967) and Del Zanna and Chiuderi (1996); for static equilibria with a symmetry, see also Edenstrasser (1980*a*, *b*).

In the presence of a symmetry, a coordinate system can be chosen in such a way that one of the coordinates is ignorable, so that current functions can be defined for those quantities with zero divergence. It can be seen that in the stationary case there is only one independent current function, so that only its differential equation need be found. We choose for this purpose the current function of the magnetic field, i.e. the magnetic flux ψ. We analyse the polytropic, incompresible and adiabatic cases for general flows, and the more restricted case of plasma velocity in the direction of the ignorable coordinate. We then obtain the conditions for confinement of stationary plasma flows.

We show that, for the particular case of helical symmetry, confinement is possible. For this symmetry and supposing different conditions, some solutions for the magnetic flux, magnetic field, plasma velocity, density and pressure are

† Present address: Via Calcesana 171, 56010 Ghezzano (PI), Italy.
‡ Member of the Carrera del Investigador Científico, CONICET, Argentina.

https://doi.org/10.1017/S0022377898006965 Published online by Cambridge University Press

given. Using these solutions, we evaluate the possibility of constructing magnetically confined plasma columns. For a general flow, but assuming that the plasma has constant density, we find that confinement is possible for some axially symmetric solutions, corresponding to cylindrical columns; this appears as a particular case of cylindrical confinement, which is also possible when $\rho = \rho(r)$. Another situation analysed in detail is the case in which the plasma velocity is in the direction of the ignorable coordinate. Here the possibilities of achieving confinement are greater: we show that it is possible to construct magnetically confined plasma columns with helical shape and with the plasma flowing along the helices, for a variety of situations.

As in the static case, it is necessary to match the solutions obtained with those for the external medium: here a vacuum and a force-free plasma will be chosen. An analysis is made to determine which part of the field is of external origin and which part comes from the confined plasma inner currents.

There is observational evidence of helical structures in astrophysical plasmas as well as of twisting motions (House and Berger 1987; Ökten and Çakmak 1990; Schmieder *et al.* 1985; Vršnak *et al.* 1991).

2. Stationary flows with one ignorable coordinate

The equations satisfied by an ideal MHD plasma (Priest 1982), when stationarity and an ignorable coordinate are considered, can be transformed into an equivalent system of relations between Jacobians of the current functions of the physical quantities involved in the problem. Combining Faraday's and Ohm's laws and taking the curl of the equation of motion in the absence of gravitational forces, the following equations for the velocity and the magnetic field are obtained:

$$\boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}) = 0, \tag{1}$$

$$\boldsymbol{\nabla} \times (\mathbf{v} \times \boldsymbol{\omega}) - \frac{1}{\rho^2} \boldsymbol{\nabla} P \times \boldsymbol{\nabla} \rho + \frac{1}{c} \boldsymbol{\nabla} \times \left(\frac{\mathbf{J} \times \mathbf{B}}{\rho} \right) = 0, \tag{2}$$

where $\boldsymbol{\omega}$ is the vorticity. Now, considering that one coordinate γ is ignorable and that $\rho\mathbf{v}$, $\mathbf{B}$, $\boldsymbol{\omega}$ and $\mathbf{J}$ have zero divergence, the current functions for those quantities can be defined so that

$$\rho \mathbf{v} = \boldsymbol{\nabla} \chi \times \mathbf{e}^{\gamma} + \rho v^{\gamma} \mathbf{e}_{\gamma}, \tag{3}$$

$$\mathbf{B} = \boldsymbol{\nabla} \psi \times \mathbf{e}^{\gamma} + B^{\gamma} \mathbf{e}_{\gamma}, \tag{4}$$

$$\boldsymbol{\omega} = \boldsymbol{\nabla} v_{\gamma} \times \mathbf{e}^{\gamma} + \omega^{\gamma} \mathbf{e}_{\gamma}, \tag{5}$$

$$\mathbf{J} = \frac{c}{4\pi} \boldsymbol{\nabla} B_{\gamma} \times \mathbf{e}^{\gamma} + J^{\gamma} \mathbf{e}_{\gamma}. \tag{6}$$

Taking into account that $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{v}$ and $\mathbf{J} = (c/4\pi)\, \boldsymbol{\nabla} \times \mathbf{B}$, the expression in curvilinear coordinates for the curl of a vector, and the relations between co- and contravariant components of a vector (Santaló 1961), (3) and (4) give

$$\omega^{\gamma} = G(v_{\gamma}) - \frac{1}{\rho g_{\gamma\gamma}} D^2 \chi + \frac{1}{\rho^2 g_{\gamma\gamma}} \boldsymbol{\nabla} \rho \cdot \boldsymbol{\nabla} \chi, \tag{7}$$

$$J^{\gamma} = \frac{c}{4\pi} \left[G(B_{\gamma}) - \frac{1}{g_{\gamma\gamma}} D^2 \psi \right], \tag{8}$$

https://doi.org/10.1017/S0022377898006965 Published online by Cambridge University Press

where the elliptic operator D^2 and G are given by

$$D^2\psi = \frac{g_{\gamma\gamma}}{g^{1/2}}\left[\frac{\partial}{\partial\alpha}\left(\frac{g^{\alpha\alpha}g^{1/2}}{g_{\gamma\gamma}}\frac{\partial\psi}{\partial\alpha}\right)+\frac{\partial}{\partial\beta}\left(\frac{g^{\beta\beta}g^{1/2}}{g_{\gamma\gamma}}\frac{\partial\psi}{\partial\beta}\right)\right.$$
$$\left.+\frac{\partial}{\partial\alpha}\left(\frac{g^{\alpha\beta}g^{1/2}}{g_{\gamma\gamma}}\frac{\partial\psi}{\partial\beta}\right)+\frac{\partial}{\partial\beta}\left(\frac{g^{\alpha\beta}g^{1/2}}{g_{\gamma\gamma}}\frac{\partial\psi}{\partial\alpha}\right)\right], \quad (9)$$

$$G(f) = \frac{1}{g_{\gamma\gamma}}\Gamma f+\frac{1}{g_{\gamma\gamma}g^{1/2}}\left(g_{\beta\gamma}\frac{\partial f}{\partial\alpha}-g_{\alpha\gamma}\frac{\partial f}{\partial\beta}\right), \quad (10)$$

with

$$\Gamma = \frac{g_{\gamma\gamma}}{g^{1/2}}\left[\frac{\partial}{\partial\alpha}\left(\frac{g_{\beta\gamma}}{g_{\gamma\gamma}}\right)-\frac{\partial}{\partial\beta}\left(\frac{g_{\alpha\gamma}}{g_{\gamma\gamma}}\right)\right].$$

Substituting (3)–(6) into (1) and (2), employing Jacobian notation and using (A 8) and (A 5) from the Appendix one gets

$$\boldsymbol{\nabla}\left(\frac{1}{\rho g^{1/2}}\left[\chi,\psi\right]\right)\times\mathbf{e}^{\gamma}-\frac{1}{g^{1/2}}\left(\left[\psi,v^{\gamma}\right]+\left[\frac{B^{\gamma}}{\rho},\chi\right]\right)\mathbf{e}_{\gamma} = 0, \quad (11)$$

$$\boldsymbol{\nabla}\left\{\frac{1}{\rho g^{1/2}}\left(\left[\chi,v_{\gamma}\right]+\frac{1}{4\pi}\left[B_{\gamma},\psi\right]\right)\right\}\times\mathbf{e}^{\gamma}-\frac{1}{g^{1/2}}\left(\left[v_{\gamma},v^{\gamma}\right]-\left[\chi,\frac{\omega^{\gamma}}{\rho}\right]\right.$$
$$\left.-\left[P,\frac{1}{\rho}\right]-\frac{1}{c}\left[\frac{J^{\gamma}}{\rho},\psi\right]+\left[\frac{1}{4\pi\rho}B^{\gamma},B_{\gamma}\right]\right)\mathbf{e}_{\gamma} = 0. \quad (12)$$

Taking into account that a vector product with $\mathbf{e}^{\gamma}$ is independent of $\mathbf{e}_{\gamma}$, the following relations among the Jacobians can be obtained from (11) and (12):

$$[\chi,\psi] = 0, \quad (13)$$

$$[\chi,v_{\gamma}]+\frac{1}{4\pi}[B_{\gamma},\psi] = 0, \quad (14)$$

$$[\psi,v^{\gamma}]+\left[\frac{B^{\gamma}}{\rho},\chi\right] = 0, \quad (15)$$

$$[v_{\gamma},v^{\gamma}]-\left[\chi,\frac{\omega^{\gamma}}{\rho}\right]-\frac{1}{c}\left[\frac{J^{\gamma}}{\rho},\psi\right]+\left[\frac{B^{\gamma}}{4\pi\rho},B_{\gamma}\right]-\left[P,\frac{1}{\rho}\right] = 0. \quad (16)$$

What one actually obtains from (11) and (12) is that the gradients of (13) and (14) are zero; but (13) can be obtained directly from Ohm's law, supposing the E_{γ} component of the electric field to be zero, and (14) comes from the γ component of the equation of motion.

From (13) and considering the case of variable ψ only, it is clear that (see (A 2))

$$\chi = \chi(\psi). \quad (17)$$

This implies that the magnetic surfaces $\psi = \text{const}$ coincide with those of constant χ so that the magnetic field and the velocity are tangent to the same surfaces.

It is convenient to analyse separately the cases of variable and constant χ.

https://doi.org/10.1017/S0022377898006965 Published online by Cambridge University Press

2.1. χ variable with the magnetic flux

When $\chi = \chi(\alpha, \beta)$, in principle all the velocity components are different from zero (see (3)). From the system of relations (14)–(16) between Jacobians, the differential equation for the magnetic flux and relations for the covariant components B_γ and v_γ can be deduced. From (14), and taking into account (17) and (A 3),

$$\left[-v_\gamma \chi' + \frac{B_\gamma}{4\pi}, \psi\right] = 0, \tag{18}$$

where $\chi' = d\chi/d\psi$. A first integration yields

$$-v_\gamma \chi' + \frac{1}{4\pi} B_\gamma = F_1(\psi), \tag{19}$$

where F_1 is an arbitrary function of ψ. Operating on (15) in the same way gives

$$-v^\gamma + \frac{B^\gamma \chi'}{\rho} = F_2(\psi). \tag{20}$$

Then, from (19) and (20), provided that $4\pi\chi'^2$ is different from ρ (the Alfvén limit), one gets

$$B_\gamma = \frac{4\pi(F_1 - g_{\gamma\gamma}\chi' F_2)}{1 - \dfrac{4\pi\chi'^2}{\rho}}, \tag{21}$$

$$v_\gamma = \frac{\dfrac{4\pi\chi'}{\rho} F_1 - g_{\gamma\gamma} F_2}{1 - \dfrac{4\pi\chi'^2}{\rho}}. \tag{22}$$

Finally, (16) will supply the differential equation for the magnetic flux, since it includes the second-order differential expressions J^γ and ω^γ (see (7) and (8)). To perform a first integration of (16), it is convenient to put this equation into the form of a single Jacobian with ψ equated to zero. From (19), (20), (A 3) and (A 4), it can be shown that

$$[v_\gamma, v^\gamma] + \left[\frac{B^\gamma}{4\pi\rho}, B_\gamma\right] = \left[-v_\gamma F_2' + \frac{v_\gamma B^\gamma \chi''}{\rho} + \frac{B^\gamma}{\rho} F_1', \psi\right]. \tag{23}$$

Introducing this expression into (16), one has

$$\left[-v_\gamma F_2' + \frac{v_\gamma B^\gamma \chi''}{\rho} + \frac{B^\gamma F_1'}{\rho} + \frac{\omega^\gamma \chi'}{\rho} - \frac{J^\gamma}{c\rho}, \psi\right] - \left[P, \frac{1}{\rho}\right] = 0. \tag{24}$$

To transform the Jacobian of the pressure with density it is convenient to make additional assumptions. The polytropic, incompressible and adiabatic cases will be considered.

2.1.1. Polytropic case. Since $P = P(\rho)$ is assumed, automatically one has

$$\left[P, \frac{1}{\rho}\right] = 0, \tag{25}$$

https://doi.org/10.1017/S0022377898006965 Published online by Cambridge University Press

so that a first integration of (24) yields

$$v_\gamma F_2' - \frac{v_\gamma B^\gamma \chi''}{\rho} - \frac{B^\gamma F_1'}{\rho} - \frac{\omega^\gamma \chi'}{\rho} + \frac{J^\gamma}{c\rho} = F_3(\psi). \tag{26}$$

Given that the operator ω^γ contains the differential expression $D^2\chi$ and that $\chi = \chi(\psi)$, it is convenient to put

$$D^2\chi = \chi' D^2\psi + \chi''|\nabla\psi|^2. \tag{27}$$

Introducing (7)–(10), (19) and (27) into (26), and after some algebra, the following differential equation is obtained:

$$\frac{1}{4\pi}\left(1 - \frac{4\pi\chi'^2}{\rho}\right) D^2\psi - \frac{\chi'\chi''|\nabla\psi|^2}{\rho} + \frac{\chi'^2}{\rho^2}\nabla\rho\cdot\nabla\psi + B_\gamma(F_1' + v^\gamma\chi'') - \rho g_{\gamma\gamma} v_\gamma F_2' - \Gamma F_1 = -\rho g_{\gamma\gamma} F_3, \tag{28}$$

where B_γ and v_γ are given by (21) and (22), and χ, F_1, F_2 and F_3 are arbitrary functions of ψ. Once these functions and the dependence of P on ρ have been given, the density is determined through the equation of motion, which for the stationary case is

$$\tfrac{1}{2}\rho\,\nabla v^2 - \rho\mathbf{v}\times\boldsymbol{\omega} = -\nabla P + \frac{1}{c}\mathbf{J}\times\mathbf{B}. \tag{29}$$

Employing (A 7) and (14), the following equation of motion results:

$$\tfrac{1}{2}\rho\,\nabla v^2 + \omega^\gamma\nabla\chi - \rho v^\gamma\nabla v_\gamma = -\nabla P - \frac{1}{4\pi}B^\gamma\nabla B_\gamma + \frac{1}{c}J^\gamma\nabla\psi. \tag{30}$$

Introducing (26), (19) and (20), one finally has

$$\tfrac{1}{2}\nabla v^2 + \frac{\nabla P}{\rho} = F_3\,\nabla\psi - \nabla(F_2\,v_\gamma). \tag{31}$$

Taking into account that $P = P(\rho)$ and using $p(\rho)$ and $f_3(\psi)$ for the primitives of P'/ρ and F_3 respectively gives

$$p(\rho) = -\tfrac{1}{2}v^2 + f_3 - F_2\,v_\gamma, \tag{32}$$

where

$$v^2 = \frac{\chi'^2|\nabla\psi|^2}{\rho^2 g_{\gamma\gamma}} + \frac{v_\gamma^2}{g_{\gamma\gamma}}. \tag{33}$$

2.1.2. Incompressible case. The incompressibility condition for the stationary case is

$$\mathbf{v}\cdot\nabla\rho = 0, \tag{34}$$

which implies (see (A 6))

$$[\rho, \chi] = 0. \tag{35}$$

Given that χ is assumed to be non-constant, and since $\chi = \chi(\psi)$, it follows that

$$\rho = \rho(\psi). \tag{36}$$

https://doi.org/10.1017/S0022377898006965 Published online by Cambridge University Press

In this case, the Jacobian of P with ρ is nonzero, but it can be written in the form

$$\left[P, \frac{1}{\rho}\right] = \left[-\frac{P\rho'}{\rho^2}, \psi\right]. \tag{37}$$

Then a first integration of (24) can be performed, and, following the same steps as in Sec. 2.1.1, one arrives at

$$\frac{1}{4\pi}\left(1-\frac{4\pi\chi'^2}{\rho}\right)D^2\psi - \frac{\chi'\chi''|\nabla\psi|^2}{\rho} + \frac{\chi'^2}{\rho^2}\nabla\rho\cdot\nabla\psi + B_\gamma(F_1' + v_\gamma\chi'')$$
$$-\rho g_{\gamma\gamma}v_\gamma F_2' - \Gamma F_1 + g_{\gamma\gamma}\frac{P\rho'}{\rho} = -\rho g_{\gamma\gamma}F_3. \tag{38}$$

The expression for the pressure is obtained from the equation of motion. Proceeding with (30) in a similar way as in Sec. 2.1.1 gives

$$\tfrac{1}{2}\nabla v^2 + \frac{\nabla P}{\rho} = F_3\nabla\psi - \nabla(F_2 v_\gamma) + \frac{P\nabla\rho}{\rho^2}, \tag{39}$$

so that

$$P = \rho f_3 - \rho F_2 v_\gamma - \tfrac{1}{2}\rho v^2, \tag{40}$$

where v^2 is given by (33).

A particular case of incompressibility is that of constant density ρ_0. The differential equation for ψ is then simplified:

$$\frac{1}{4\pi}\left(1-\frac{4\pi\chi'^2}{\rho_0}\right)D^2\psi - \frac{\chi'\chi''|\nabla\psi|^2}{\rho_0} - \Gamma F_1$$
$$+\frac{1}{1-\dfrac{4\pi\chi'^2}{\rho_0}}(\rho_0 g_{\gamma\gamma}^2 F_2 F_2' + 4\pi F_1 F_1')$$
$$+\frac{4\pi}{\rho_0}\frac{\chi'\chi''}{\left(1-\dfrac{4\pi\chi'^2}{\rho_0}\right)^2}(4\pi F_1^2 + \rho_0 g_{\gamma\gamma}^2 F_2^2)$$
$$= -\rho_0 g_{\gamma\gamma}\left[F_3 - \frac{d}{d\psi}\left(\frac{\dfrac{4\pi}{\rho_0}\chi' F_1 F_2}{1-\dfrac{4\pi}{\rho_0}\chi'^2}\right)\right] = -\rho_0 g_{\gamma\gamma}F_4(\psi). \tag{41}$$

It can be seen that in this equation the pressure does not appear explicitly: when ρ is constant, the equation for the pressure is uncoupled from the differential equation for the magnetic flux. Once this latter equation has been solved, the pressure can be obtained from (see (40), (41) and (22))

$$P = \rho_0 f_4 + \frac{\rho_0 g_{\gamma\gamma}F_2^2}{1-\dfrac{4\pi\chi'^2}{\rho_0}} - \frac{\rho_0 v^2}{2}, \tag{42}$$

where f_4 is the primitive of F_4.

https://doi.org/10.1017/S0022377898006965 Published online by Cambridge University Press

The differential equation for ψ will be solved assuming helical symmetry. The reason for this particular choice is that this is the most general form of spatial symmetry (Solov'ev 1967). It will be convenient to work in helical coordinates $r = r$, $\xi = a\theta - z$ and $\gamma = z$, where (r, θ, z) are the cylindrical coordinates. Let us define the nondimensional coordinates

$$x = \frac{r}{|a|}, \qquad y = \frac{\xi}{|a|}. \tag{43}$$

Then the components of the metric tensor (for $a > 0$) are

$$g_{11} = 1, \qquad g_{22} = g_{23} = x^2, \qquad g_{33} = 1 + x^2, \qquad g_{12} = g_{13} = 0, \qquad g^{1/2} = x.$$

The following model will be assumed for the arbitrary functions of the magnetic flux:

$$\left.\begin{aligned} \chi(\psi) &= \chi^{(1)}\psi, \\ F_1(\psi) &= F_1^{(1)}\psi, \\ F_2(\psi) &= F_2^{(0)}, \\ F_4(\psi) &= F_4^{(0)}, \end{aligned}\right\} \tag{44}$$

where $\chi^{(1)}$, $F_1^{(1)}$, $F_2^{(0)}$ and $F_4^{(0)}$ are constants. Substituting this model into (41) yields

$$x\frac{\partial}{\partial x}\left(\frac{x}{1+x^2}\frac{\partial\psi}{\partial x}\right) + \frac{\partial^2\psi}{\partial y^2} - \frac{2ax^2}{(1+x^2)^2}\frac{F_1^{(1)}\psi}{\dfrac{1}{4\pi} - \dfrac{(\chi^{(1)})^2}{\rho_0}} + \frac{a^2x^2}{1+x^2}\frac{(F_1^{(1)})^2\psi}{\left(\dfrac{1}{4\pi} - \dfrac{(\chi^{(1)})^2}{\rho_0}\right)^2} = -\frac{F_4^{(0)}\rho_0 a^2x^2}{\dfrac{1}{4\pi} - \dfrac{(\chi^{(1)})^2}{\rho_0}}, \tag{45}$$

whose solution is (Tsinganos 1982*b*)

$$\psi(x,y) = \sum_{m=-\infty}^{\infty} \{A_m[akJ_m(Cx) - CxJ'_m(Cx)] + B_m[akN_m(Cx) - CxN'_m(Cx)]\}\exp(imy) - \frac{F_4^{(0)}\rho_0}{kF_1^{(1)}}\left(1 + \frac{2}{ak} + x^2\right), \tag{46}$$

where

$$k = \frac{F_1^{(1)}}{\dfrac{1}{4\pi} - \dfrac{(\chi^{(1)})^2}{\rho_0}}, \tag{47}$$

$$C = (a^2k^2 - m^2)^{1/2}. \tag{48}$$

Now, introducing the chosen model in (22) and (33) and substituting into (42), the pressure becomes

$$P = P_0 + \rho_0 F_4^{(0)}\psi + \frac{\rho_0 g_{\gamma\gamma} F_2^{(0)2}}{1 - \dfrac{4\pi(\chi^{(1)})^2}{\rho_0}} - \frac{(\chi^{(1)})^2|\nabla\psi|^2}{2\rho_0 g_{\gamma\gamma}} - \frac{\rho_0}{2}\frac{\left(\dfrac{4\pi\chi^{(1)}}{\rho_0}F_1^{(1)}\psi - g_{\gamma\gamma}F_2^{(0)}\right)^2}{g_{\gamma\gamma}\left(1 - \dfrac{4\pi(\chi^{(1)})^2}{\rho_0}\right)^2}. \tag{49}$$

https://doi.org/10.1017/S0022377898006965 Published online by Cambridge University Press

The components of the magnetic field and the velocity can be obtained from (3), (4), (21) and (22).

2.1.3. Adiabatic case. In this case, the entropy is assumed to be constant along the lines of flux; that is, the sources and sinks of energy are negligible or cancel among themselves. In the stationary case, this condition can be written as

$$\mathbf{v}\cdot\boldsymbol{\nabla}S = 0, \tag{50}$$

where S is the entropy per unit mass. Proceeding similarly to the incompressible case, there results

$$S = S(\psi). \tag{51}$$

From the thermodynamic relation

$$\boldsymbol{\nabla}H = T\,\boldsymbol{\nabla}S + \frac{\boldsymbol{\nabla}P}{\rho}, \tag{52}$$

where H is the enthalpy per unit mass, the Jacobian of P with ρ can be transformed to give

$$\begin{aligned}\left[P,\frac{1}{\rho}\right] &= g^{1/2}\left(\boldsymbol{\nabla}P\times\boldsymbol{\nabla}\left(\frac{1}{\rho}\right)\right)\cdot\mathbf{e}^{\gamma} = g^{1/2}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}H - T\boldsymbol{\nabla}S)\cdot\mathbf{e}^{\gamma}\\ &= [T,S]\\ &= [TS',\psi],\end{aligned} \tag{53}$$

where (A 5) from the Appendix has been employed. Introducing (53) into (24) and performing a first integration gives

$$v_{\gamma}F_2' - \frac{v_{\gamma}B^{\gamma}\chi'}{\rho} - \frac{B^{\gamma}F_1'}{\rho} - \frac{\omega^{\gamma}\chi'}{\rho} + \frac{J^{\gamma}}{c\rho} + TS' = F_3(\psi), \tag{54}$$

so that the differential equation for the magnetic flux becomes

$$\begin{aligned}\frac{1}{4\pi}\left(1-\frac{4\pi\chi'^2}{\rho}\right)D^2\psi - \frac{\chi'\chi''|\boldsymbol{\nabla}\psi|^2}{\rho} + \frac{\chi'^2}{\rho^2}\boldsymbol{\nabla}\rho\cdot\boldsymbol{\nabla}\psi + B_{\gamma}(F_1' + v_{\gamma}\chi'')\\ -\rho g_{\gamma\gamma}v_{\gamma}F_2' - \Gamma F_1 - \rho g_{\gamma\gamma}TS' = -\rho g_{\gamma\gamma}F_3.\end{aligned} \tag{55}$$

The expressions for ρ and T are obtained from the equation of motion and the equation of state. Substituting (54) into (30) and using (19) and (20), there results

$$\tfrac{1}{2}\boldsymbol{\nabla}v^2 + \frac{\boldsymbol{\nabla}P}{\rho} = F_3\,\boldsymbol{\nabla}\psi - \boldsymbol{\nabla}(F_2\,v_{\gamma}) - TS'\boldsymbol{\nabla}\psi. \tag{56}$$

Employing (52), this expression transforms

$$\tfrac{1}{2}\boldsymbol{\nabla}v^2 + \boldsymbol{\nabla}H = F_3\,\boldsymbol{\nabla}\psi - \boldsymbol{\nabla}(F_2\,v_{\gamma}), \tag{57}$$

and integrating gives

$$H = -\tfrac{1}{2}v^2 + f_3 - F_2\,v_{\gamma}. \tag{58}$$

https://doi.org/10.1017/S0022377898006965 Published online by Cambridge University Press

Finally, making use of the properties of an ideal gas, ρ and T can be expressed as functions of H and S.

2.2. Constant χ

When χ is constant, the only nonzero contravariant component of the velocity is v^γ, that is, the component along the ignorable coordinate.

It is convenient to treat this case separately, because although the adiabaticity and incompressibility conditions are satisfied automatically and there is no need to impose them, one cannot conclude that $\rho = \rho(\chi)$ or $S = S(\chi)$.

From (14) and (15), one gets

$$B_\gamma = B_\gamma(\psi), \tag{59}$$

$$v^\gamma = v^\gamma(\psi). \tag{60}$$

Given the simplicity of this case, it is more direct to derive the differential equation for ψ from the equation of motion, (30), which for constant χ is

$$\boldsymbol{\nabla} P = \tfrac{1}{2}\rho (v^\gamma)^2\, \boldsymbol{\nabla} g_{\gamma\gamma} + \left(\frac{1}{c} J^\gamma - \frac{1}{4\pi} B^\gamma B'_\gamma\right) \boldsymbol{\nabla}\psi. \tag{61}$$

It can easily be seen from this equation that $P = P(\psi, g_{\gamma\gamma})$ and that

$$\tfrac{1}{2}\rho\,(v^\gamma)^2 = \frac{\partial P}{\partial g_{\gamma\gamma}}, \tag{62}$$

$$\frac{1}{c} J^\gamma - \frac{1}{4\pi} B^\gamma B'_\gamma = \frac{\partial P}{\partial \psi}. \tag{63}$$

Equation (63) is the differential equation for ψ:

$$D^2\psi - \Gamma B_\gamma + B'_\gamma B_\gamma = -4\pi g_{\gamma\gamma} \frac{\partial P}{\partial \psi}. \tag{64}$$

Equation (62) gives the density from the models for $P(\psi, g_{\gamma\gamma})$ and $v^\gamma(\psi)$; note that, in general, ρ is not function of ψ only.

Now, (64) will be solved for a given model of B_γ and P, assuming helical symmetry. Then, taking

$$P = P_0\left(1 - \frac{\psi}{\psi_0}\right) + \tfrac{1}{2}\rho_0\, v_0^2\, g_{\gamma\gamma} \frac{\psi_b - \psi}{\psi_0}, \tag{65}$$

$$B_\gamma = k\psi, \tag{66}$$

the differential equation for ψ in helical coordinates is

$$x\frac{\partial}{\partial x}\left(\frac{x}{1+x^2}\frac{\partial \psi}{\partial x}\right) + \frac{\partial^2 \psi}{\partial y^2} - 2\frac{a x^2}{(1+x^2)^2} k\psi + \frac{a^2 k^2 x^2}{1+x^2}\psi = 4\pi a^2 x^2 \left[\frac{P_0}{\psi_0} + \frac{\rho_0 v_0^2}{2\psi_0}(1+x^2)\right], \tag{67}$$

whose solution is (Agim and Tataronis 1985)

$$\psi(x,y) = \sum_{m=-\infty}^{\infty} \{A_m[akJ_m(Cx) - CxJ'_m(Cx)] + B_m[akN_m(Cx) - CxN'_m(Cx)]\}\exp(imy) + C_0 + C_2 x^2 + C_4 x^4, \tag{68}$$

https://doi.org/10.1017/S0022377898006965 Published online by Cambridge University Press

where

$$C = (a^2k^2 - m^2)^{1/2}, \tag{69}$$

$$C_0 = \frac{4\pi P_0}{k^2\psi_0}\left[1 + \frac{2}{ak} - \frac{v_0^2\rho_0}{2P_0a^3k^3}(16 + 4ak - 2a^2k^2 - a^3k^3)\right], \tag{70}$$

$$C_2 = \frac{4\pi P_0}{k^2\psi_0}\left[1 - \frac{v_0^2\rho_0}{a^2k^2P_0}(4 - ak - a^2k^2)\right], \tag{71}$$

$$C_4 = \frac{4\pi P_0}{k^2\psi_0}\frac{v_0^2\rho_0}{2P_0}. \tag{72}$$

To have a complete model, it is necessary to give an expression for v^γ that is compatible with (62). For instance, if an expression

$$v^\gamma = v_0\left(\frac{\psi_b - \psi}{\psi_0 - \psi}\right)^{1/2} \tag{73}$$

is chosen, the density will be

$$\rho = \rho_0\left(1 - \frac{\psi}{\psi_0}\right). \tag{74}$$

3. Magnetically confined plasma columns

To construct confined plasma columns from a given solution of the equations analysed in the previous sections, it is necessary to find a closed surface S where the gas pressure is constant or zero and the velocity is tangent; besides, P must be positive inside the volume bounded by S.

The problem posed will make physical sense only if the medium external to the column is defined so that negative-pressure regions will not appear. A vacuum and a force-free plasma are considered as possible external media. In the first case, to the confinement conditions given above, the condition that the current density **J** be tangent to S should be added. Instead, if the external medium is a force-free plasma, i.e. a static fluid, it is convenient that the plasma column velocity be zero on S. According to a corollary of the virial theorem (Shafranov 1966), a plasma cannot be confined magnetically without the presence of either a magnetic field originating outside the plasma column or a nonzero total current inside it. Shafranov's argument for the static case is also valid for the stationary one, as can be seen from the general form of the virial theorem. The results shown in Sec. 3.3 are in agreement with this.

It is evident that **v** (or **J**) will be tangent to S only if χ (or B_γ) is constant over S. The cases $\chi =$ const everywhere and $\chi = \chi(\psi)$ will be studied separately, and the examples given in the above sections will be analysed to determine whether it is possible to construct confined plasma columns from the obtained solutions.

3.1. χ constant everywhere

As we said above, in this case the only nonzero component of the velocity is v^γ, and then the velocity is tangent to any surface $f(\alpha, \beta) =$ const. Given that the velocity does not impose any condition over S, the surface of confinement can

https://doi.org/10.1017/S0022377898006965 Published online by Cambridge University Press

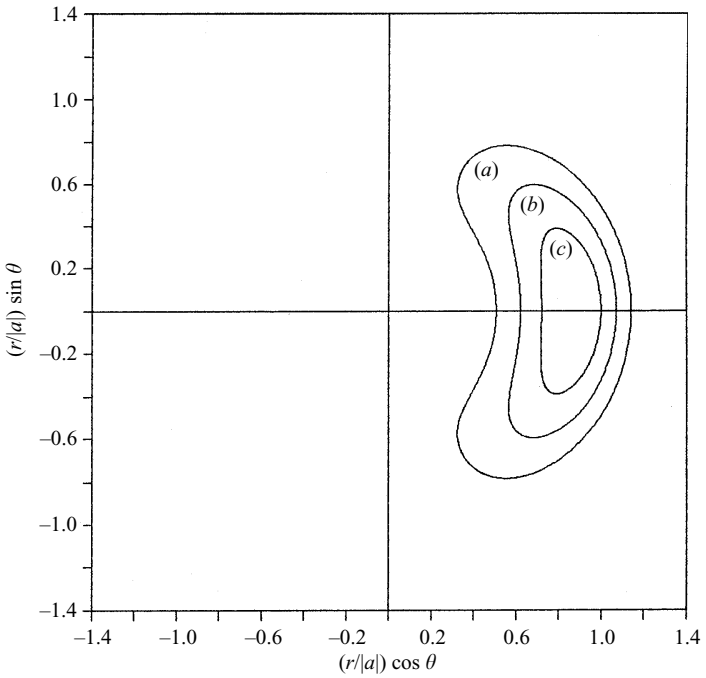


Figure 1. Cross-sections at $z = 0$ of the surfaces $P = \text{const}$ for a vacuum-surrounded plasma column, with $m = 1$, $ak = 4$, $4\pi P_0/k^2\psi_0^2 = 0.7$, $\rho_0 v_0^2/2P_0 = 0.5$, $\psi_b/\psi_0 = 1$, $A_1/\psi_0 = -A_{-1}/\psi_0 = -0.5$ and $B_1/\psi_0 = -B_{-1}/\psi_0 = -0.5$. The curves correspond to (*a*) $P/P_0 = 0$, (*b*) $P/P_0 = 1.5$ and (*c*) $P/P_0 = 2.5$.

be defined so that the pressure will be constant over it. As was seen in Sec. 2.2, the pressure is given by an arbitrary function of ψ and $g_{\gamma\gamma}$. If the column is immersed in a vacuum or in a force-free plasma, this model should be such that a surface of constant pressure coincides with a surface of constant ψ in order to fulfil the conditions of constant B_γ or zero velocity on it (cf. (59) and (60)); the only case in which B_γ can be constant on S, keeping ψ variable, is when the arbitrary function $B_\gamma(\psi)$ is chosen to be constant.

In Sec. 2.2, a model with helical symmetry and constant χ was given. Taking as the external surface S of the plasma column that where $\psi \equiv \psi_b$, $\mathbf{J}$ turns out to be tangent to S. In the case in which the column is matched with the solution in vacuum, the pressure on the boundary will have to be zero; if the matching is with a force-free plasma, it will have to be a positive constant. For this, ψ_b will be taken equal to ψ_0 or less than ψ_0 respectively. For a vacuum-surrounded plasma column, Fig. 1 represents cross-sections at $z = 0$ of the $P = \text{const}$ surfaces for the $m = 1$ mode. Figures 2 and 3 show P/P_0, v^γ/v_0 and ρ/ρ_0 as functions of x at $\theta = 0$ for two different models of v^γ:

$$v^\gamma = v_0, \tag{75}$$

which corresponds to a density (see (62))

$$\rho = \rho_0\left(1 - \frac{\psi}{\psi_0}\right), \tag{76}$$

and

https://doi.org/10.1017/S0022377898006965 Published online by Cambridge University Press

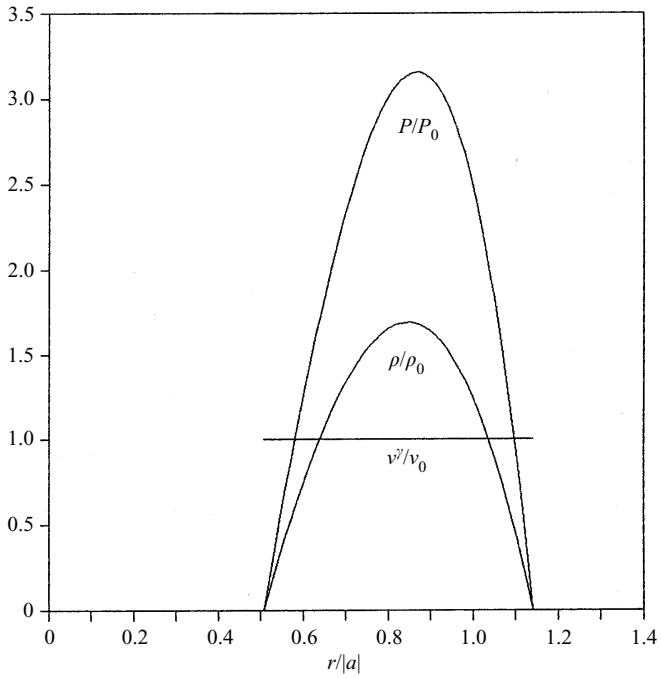


Figure 2. P/P_0, v^γ/v_0 and ρ/ρ_0 versus $x = r/|a|$ at $z = 0$, $\theta = 0$ for a vacuum-surrounded plasma column with $v^\gamma = v_0$ and the parameters considered in Fig. 1.

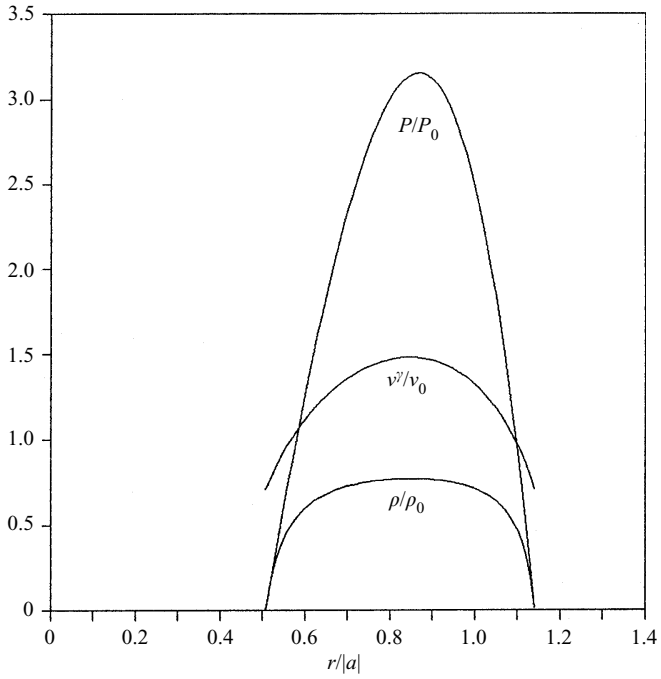


Figure 3. P/P_0, v^γ/v_0 and ρ/ρ_0 versus $x = r/|a|$ at $z = 0$, $\theta = 0$ for a vacuum-surrounded plasma column with $v^\gamma = v_0\ (1.5-\psi/\psi_0)^{1/2}$ and the parameters considered in Fig. 1.

https://doi.org/10.1017/S0022377898006965 Published online by Cambridge University Press

$$v^\gamma = v_0\left(\frac{\psi_v-\psi}{\psi_0}\right)^{1/2}, \qquad \psi_v > \psi_0, \tag{77}$$

from which

$$\rho = \rho_0\frac{\psi_0-\psi}{\psi_v-\psi}. \tag{78}$$

In the case of a match with a force-free solution, P/P_0, v^γ/v_0 and ρ/ρ_0 as functions of x are plotted in Fig. 4 for $\theta = 0$, with a velocity given by

$$v^\gamma = v_0\left(\frac{\psi_b-\psi}{\psi_0-\psi}\right)^{1/2}. \tag{79}$$

which yields a density

$$\rho = \rho_0\left(1-\frac{\psi}{\psi_0}\right). \tag{80}$$

It is also possible to obtain plasma columns with internal pressure lower than the external pressure of the force-free plasma; the only change that is needed is to put a minus instead of a plus between the two terms in the model for $P(\psi)$, (65). Figure 5 represents cross-sections at $z = 0$ of the $P =$ const surfaces for the $m = 1$ mode, and Fig. 6 shows P/P_0, v^γ/v_0 and ρ/ρ_0 as functions of x, taking as a model for v^γ the following:

$$v^\gamma = v_0\left(\frac{\psi-\psi_b}{\psi_0-\psi}\right)^{1/2}. \tag{81}$$

In the limit as v_0 tends to zero in these models, the results tend to those obtained for the static case (Palumbo and Platzeck 1993). The plasma columns constructed with nonaxisymmetric modes are helical in shape.

3.2. $\chi = \chi(\psi)$

In this case, constant χ means constant ψ. To see what happens with the pressure, the equation of motion is employed. Then substituting (21), (22) and (33) into (30) gives

$$\begin{aligned}\boldsymbol{\nabla}P = {} & \left(-\omega^\gamma\chi' + \frac{1}{c}J^\gamma\right)\boldsymbol{\nabla}\psi - \tfrac{1}{2}\rho\,\boldsymbol{\nabla}\left(\frac{\chi'^2|\boldsymbol{\nabla}\psi|^2}{\rho g_{\gamma\gamma}} + \frac{v_\gamma^2}{g_{\gamma\gamma}}\right) \\ & + \rho v^\gamma\boldsymbol{\nabla}\left(\frac{\dfrac{4\pi\chi'}{\rho}F_1 - g_{\gamma\gamma}F_2}{1-\dfrac{4\pi\chi'^2}{\rho}}\right) - \frac{1}{4\pi}B^\gamma\boldsymbol{\nabla}\left(\frac{4\pi(F_1 - g_{\gamma\gamma}\chi' F_2)}{1-\dfrac{4\pi\chi'^2}{\rho}}\right).\end{aligned} \tag{82}$$

The two possibilities $\chi' = 0$ or $\chi' \neq 0$ on S are treated separately.

3.2.1. $\chi' = 0$ *on* S. From (21), $B_\gamma =$ const on S, and so $\mathbf{J}$ turns out to be tangent to S. The pressure gradient along the plasma surface becomes

$$\boldsymbol{\nabla}P = \left(\frac{1}{c}J^\gamma - B^\gamma F_1' + B^\gamma g_{\gamma\gamma}F_2\chi''\right)\boldsymbol{\nabla}\psi + \tfrac{1}{2}\rho(v^\gamma)^2\,\boldsymbol{\nabla}g_{\gamma\gamma}. \tag{83}$$

https://doi.org/10.1017/S0022377898006965 Published online by Cambridge University Press

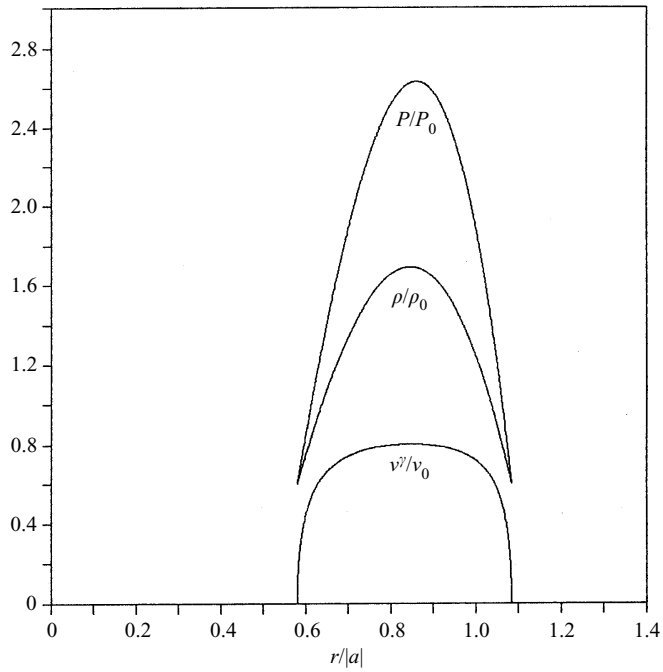


Figure 4. P/P_0, v^γ/v_0 and ρ/ρ_0 versus $x = r/|a|$ at $z = 0$, $\theta = 0$ for a column embedded in a force-free plasma with $v^\gamma = v_0[(\psi_b - \psi)/(\psi_0 - \psi)]^{1/2}$, $m = 1$, $ak = 4$, $4\pi P_0/k^2\psi_0^2 = 0.7$, $\rho_0 v_0^2/2P_0 = 0.5$, $\psi_b/\psi_0 = 0.4$, $A_1/\psi_0 = -A_{-1}/\psi_0 = -0.5$ and $B_1/\psi_0 = -B_{-1}/\psi_0 = -0.5$.

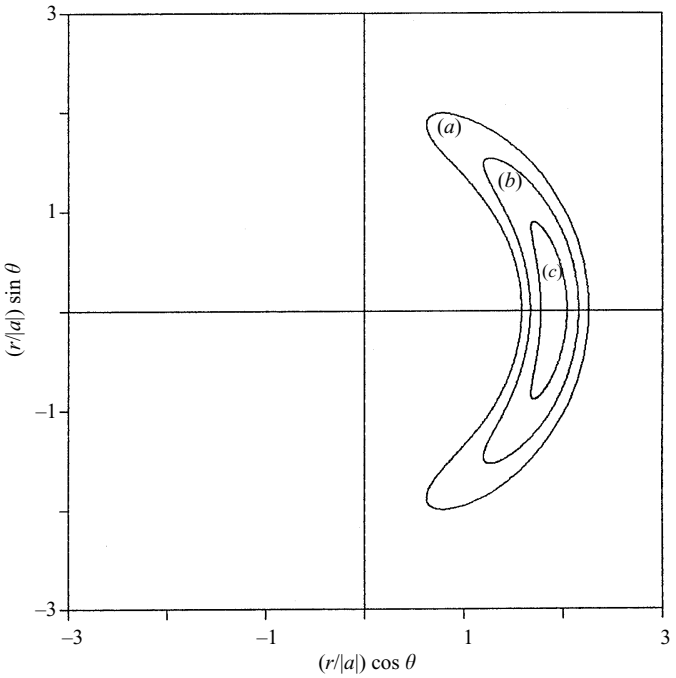


Figure 5. Cross-sections at $z = 0$ of the surfaces $P = \text{const}$ for a column embedded in a force-free plasma with $m = 1$, $ak = 4$, $4\pi P_0/k^2\psi_0^2 = 0.1$, $\rho_0 v_0^2/2P_0 = 0.05$, $\psi_b/\psi_0 = 0.5$, $A_1/\psi_0 = -A_{-1}/\psi_0 = -0.05$ and $B_1/\psi_0 = -B_{-1}/\psi_0 = -0.05$. The curves correspond to (*a*) $P/P_0 = 0.48$, (*b*) $P/P_0 = 0.4$ and (*c*) $P/P_0 = 0.33$.

https://doi.org/10.1017/S0022377898006965 Published online by Cambridge University Press

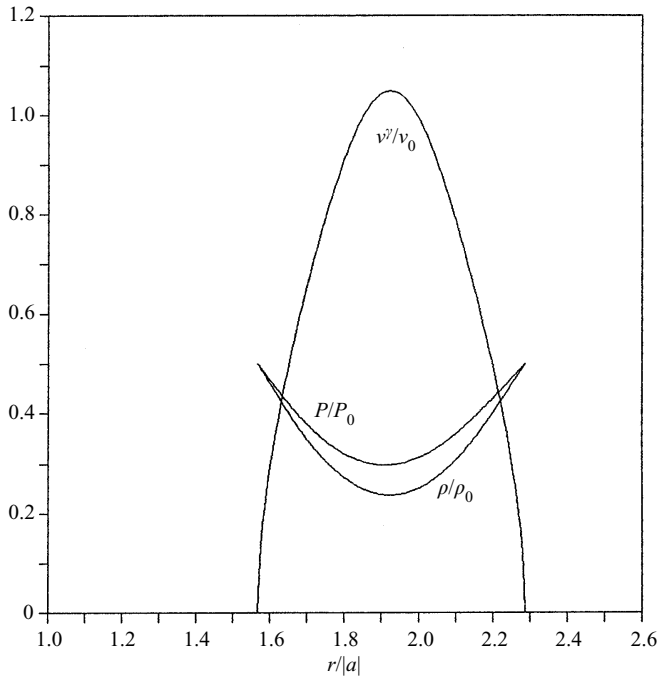


Figure 6. P/P_0, v^γ/v_0 and ρ/ρ_0 versus $x = r/|a|$ at $z = 0$, $\theta = 0$ for a column embedded in a force-free plasma with $v^\gamma = v_0[(\psi-\psi_b)/(\psi_0-\psi)]^{1/2}$ and the parameters considered in Fig. 5.

To have constant pressure over S, where ψ is known to be constant, ∇P must be parallel to $\nabla\psi$, and so for the existence of confinement one of the following three conditions should be fulfilled:

(i) $\rho = 0$;

(ii) $v^\gamma = 0$ (from which, $v = 0$);

(iii) $\nabla\psi$ is parallel to $\nabla g_{\gamma\gamma}$

along S.

3.2.2. $\chi' \not\equiv 0$ *on* S. It can be seen from (82) that if $\chi' \not\equiv 0$ then necessary conditions to have ∇P parallel to $\nabla\psi$ are that the latter should be parallel to $\nabla g_{\gamma\gamma}$ and that $\rho = \text{const}$ on S. As a result, B_γ and v_γ are constant on the external surface.

In Sec. 2.1.2, a model with helical symmetry and constant density was solved. This example belongs to the case $\chi' \not\equiv 0$, and then the only possibility to have confinement is that $\nabla\psi$ be parallel to $\nabla g_{\gamma\gamma}$. When helical coordinates are employed, $\nabla g_{\gamma\gamma}$ points in the r direction. This implies that only axisymmetric confined plasma columns can be constructed, because the only mode of the solution in which $\nabla\psi$ is parallel to $\nabla g_{\gamma\gamma}$ is the $m = 0$ mode (see (46)). So, the columns are z-axis cylinders, and neither the magnetic field nor the velocity has a radial component. For different sets of parameters of the chosen model, there exists the possibility of finding a confined column surrounded by a vacuum or embedded in a force-free plasma. Figure 7 shows P/P_0 and $|v|/v_0$ as functions of the radius for a confined plasma column in which the pressure is zero on the boundary.

https://doi.org/10.1017/S0022377898006965 Published online by Cambridge University Press

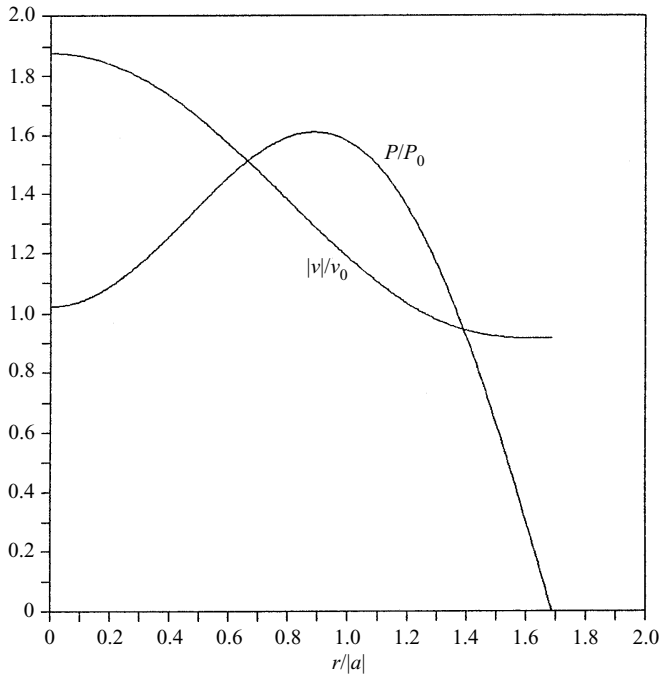


Figure 7. P/P_0 and $|v|/v_0$ versus $x = r/|a|$ for a cylindrical plasma column with constant density and zero pressure on the boundary, with $ak = 2$, $4\pi P_0/k^2\psi_0^2 = 0.1$, $4\pi(\chi^{(1)})^2/\rho_0 = 0.1$, $F_2^{(0)}/v_0 = 0.09$, $A_0/\psi_0 = -1$ and $B_0/\psi_0 = 0$, where $v_0 = (P_0/\rho_0)^{1/2}$ and $\psi_0 = -P_0/\rho_0 F_4^{(0)}$.

3.3. Matching of the confined columns with external media

As it was said at the start of this section, the confined plasma should be included in an external medium. To match the internal and the external solutions, it is necessary that at the column surface the magnetic field satisfy boundary conditions. Given the values of the magnetic flux and of B_γ in the column, one must be able to construct the solution in the external medium.

3.3.1. Vacuum-surrounded plasma columns. For a vacuum (Tsinganos 1982*b*; Palumbo and Platzeck 1993),

$$B_\gamma = B_\gamma^v = \text{const}, \tag{84}$$

and, in the case of helical symmetry, the magnetic flux is

$$\psi^v(x, y) = \psi_H^v(x, y) + \tfrac{1}{2}aB_\gamma^v x^2 \tag{85}$$

where

$$\psi_H^v(x, y) = A_0^v + B_0^v(\tfrac{1}{2}x^2 + \ln x) + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} [A_m^v\, xI'_m(|m|\, x) + B_m^v\, xK'_m(|m|\, x)]\exp(imy). \tag{86}$$

Thus the continuity of B_γ is ensured by choosing B_γ^v to be equal to the value that B_γ takes on the plasma surface, which, as seen earlier, must be constant. The continuity of the other components of the magnetic field is ensured by the continuity of ψ and $\nabla\psi$. Using the second Green identity (Sommerfeld 1949)

https://doi.org/10.1017/S0022377898006965 Published online by Cambridge University Press

and this condition, in the case of helical symmetry one obtains expressions for the coefficients A^v_m and B^v_m of the solution in vacuum, once the plasma solution ψ^p and its derivatives on the column surface have been given. This is done in a similar way as in the static case (Palumbo and Platzeck 1993), resulting in

$$A^v_m = \frac{1}{2\pi}\int_{C_0}\left\{\left[\frac{\partial\psi^p}{\partial y'} + im\left(\psi^p - \tfrac{1}{2}aB^v_\gamma x'^2\right)\right]K'_m(|m|\,x')\exp(-imy')\,dx'\right.$$
$$+\left[-\left(\frac{\partial\psi^p}{\partial x'} - aB^v_\gamma x'\right)\frac{x'^2}{1+x'^2}K'_m(|m|\,x')\right.$$
$$\left.\left.+\left(\psi^p - \tfrac{1}{2}aB^v_\gamma x'^2\right)|m|\,K_m(|m|\,x')\right]\exp(-imy')\,dy'\right\}, \tag{87}$$

$$A^v_0 = \frac{1}{2\pi}\int_{C_0}\left\{\frac{\partial\psi^p}{\partial y'}\frac{1}{x'}(\tfrac{1}{2}x'^2 + \ln x')\,dx'\right.$$
$$+\left[-\left(\frac{\partial\psi^p}{\partial x'} - aB^v_\gamma x'\right)\frac{x'}{1+x'^2}(\tfrac{1}{2}x'^2 + \ln x')\right.$$
$$\left.\left.+(\psi^p - \tfrac{1}{2}aB^v_\gamma x'^2)\right]dy'\right\}. \tag{88}$$

$$B^v_0 = \frac{1}{2\pi}\int_{C_0}\left\{-\frac{\partial\psi^p}{\partial y'}\frac{1}{x'}dx' + \left(\frac{\partial\psi^p}{\partial x'} - aB^v_\gamma x'\right)\frac{x'}{1+x'^2}dy'\right\}, \tag{89}$$

where the curve C_0 is the intersection of S and the plane $z = 0$. The coefficients B^v_m are obtained from the expression for A^v_m by replacing K_m by I_m and K'_m by I'_m and changing sign. These expressions are valid when the helical plasma column includes the z axis, and give the solution in vacuum in a region between the maximum column radius and possible external sources. The existence of these sources is determined by the values of A^v_m, B^v_0 and B^v_γ: while the terms in K'_m come from the currents inside the plasma, the terms in I'_m originated from external sources; the term proportional to $\frac{1}{2}x^2 + \ln x$ has two sources, namely a constant inner current along the z axis and an external current that gives rise to a uniform field in the z direction; B^v_γ contributes only in this direction.

3.3.2. Columns embedded in a force-free plasma. For a force-free plasma (Agim and Tataronis 1985; Palumbo and Platzeck 1993),

$$B^{ff}_\gamma = k\psi^{ff} + B_0, \tag{90}$$

and, for constant k and helical symmetry,

$$\psi^{ff}(x,y) = \sum_{m=-\infty}^{\infty}[A^{ff}_m f_m(x) + B^{ff}_m g_m(x)]\exp(imy) - \frac{B_0}{k}, \tag{91}$$

with

$$f_m(x) = (ak+m)\,J_m(Cx) - CxJ_{m-1}(Cx), \tag{92}$$

$$g_m(x) = (ak+m)\,N_m(Cx) - CxN_{m-1}(Cx), \tag{93}$$

$$C = (a^2k^2 - m^2)^{1/2}. \tag{94}$$

From the analysis of the confinement conditions given earlier, in the case of columns immersed in a force-free plasma, both B_γ and ψ are constant on S; an adequate choice of k or B_0 will be enough to ensure the continuity of B_γ.

The particular combination of different modes of the solution for the force-free plasma that matches with the chosen solution for the plasma column can

https://doi.org/10.1017/S0022377898006965 Published online by Cambridge University Press

be obtained by employing the same method as in the case of vacuum-surrounded plasma columns, resulting in

$$A_m^{ff} = \frac{1}{4(a^2k^2-m^2)}\int_{C_0}\left\{-\frac{1}{x'}\left[g_m(x')\frac{\partial\psi^p}{\partial y'}+im\left(\psi^p+\frac{B_0}{k}\right)g_m(x')\right]dx'\right.$$
$$\left.+\frac{x'}{1+x'^2}\left[g_m(x')\frac{\partial\psi^p}{\partial x'}-\left(\psi^p+\frac{B_0}{k}\right)g'_m(x')\right]dy'\right\}\exp(-imy'). \quad (95)$$

B_m^{ff} is obtained from the expression for A_m^{ff} by replacing g_m and g'_m by f_m and f'_m respectively and changing sign. In this case, the terms in f_m and in g_m come both from sources situated inside and outside the column.

4. Conclusions

We have analysed the conditions under which solutions of the stationary MHD equations, with an ignorable coordinate, correspond to magnetically confined plasma flows; we have also considered a variety of helically symmetric solutions. Assuming that the plasma velocity to be along the ignorable coordinate, we have shown that helically symmetric confined plasma columns can exist. These columns are helicoidal in shape, and the plasma flux inside them is helicoidal too. Unphysical regions are avoided by matching the solution for the plasma columns with those for an external medium such as a vacuum or a force-free plasma. Different pressure and velocity profiles have been illustrated; when the external medium is a force-free plasma, the pressure inside the column may be greater or lower than the external pressure.

Appendix

Given the functions $F(\alpha,\beta)$, $G(\alpha,\beta)$ and $H(\alpha,\beta)$, using for the Jacobian of F and G the notation

$$[F,G] = \frac{\partial F}{\partial\alpha}\frac{\partial G}{\partial\beta}-\frac{\partial F}{\partial\beta}\frac{\partial G}{\partial\alpha}, \quad (A\ 1)$$

the following properties hold:

$$[F,G]=0 \text{ with } G\neq\text{const}\Rightarrow F=F(G), \quad (A\ 2)$$
$$[F,G(H(\alpha,\beta))] = G'[F,H] = [FG',H], \quad (A\ 3)$$
$$[F,GH]+[G,FH]+[H,FG]=0, \quad (A\ 4)$$
$$\nabla F\times\nabla G = \frac{1}{g^{1/2}}[F,G]\,\mathbf{e}_\gamma. \quad (A\ 5)$$

If F and G are the respective current functions of the vectors $\mathbf{V}$ and $\mathbf{W}$ then

$$\mathbf{V}\cdot\nabla H = \frac{1}{g^{1/2}}[F,H], \quad (A\ 6)$$

$$\mathbf{V}\times\mathbf{W} = \frac{1}{g^{1/2}}[F,G]\,\mathbf{e}^\gamma - W^\gamma\nabla F + V^\gamma\nabla G, \quad (A\ 7)$$

$$\nabla\times(H\mathbf{V}\times\mathbf{W}) = \nabla\left(\frac{H}{g^{1/2}}[F,G]\right)\times\mathbf{e}^\gamma-\frac{1}{g^{1/2}}([G,HV^\gamma]+[HW^\gamma,F])\,\mathbf{e}_\gamma. \quad (A\ 8)$$

https://doi.org/10.1017/S0022377898006965 Published online by Cambridge University Press

References

Agim, Y. Z. and Tataronis, J. A. 1985 *J. Plasma Phys.* **34**, 337.

Del Zanna, L. and Chiuderi, C. 1996 *Astron. Astrophys.* **310**, 341.

Edenstrasser, J. W. 1980*a* *J. Plasma Phys.* **24**, 299.

Edenstrasser, J. W. 1980*b* *J. Plasma Phys.* **24**, 515.

House, L. L. and Berger, M. A. 1987 *Astrophys. J.* **323**, 406.

Morozov, A. I. and Solov'ev, L. S. 1963 *Soviet Phys. JETP* **8**, 243.

Ökten, A. and Çakmak, H. 1990 *Solar Phys.* **128**, 365.

Palumbo, L. J. and Platzeck, A. M. 1993 *Astrophys. J.* **416**, 656.

Priest, E. R. 1982 *Solar Magnetohydrodynamics*. Reidel, Dordrecht.

Santaló, L. A. 1961 *Vectores y tensores con sus aplicaciones*. EUDEBA, Buenos Aires.

Schmieder, B., Raadu, M. A. and Malherbe, J. M. 1985 *Astron. Astrophys.* **142**, 249.

Shafranov, V. D. 1966 *Reviews of Plasma Physics*, Vol. 2 (ed. M. A. Leontovich), p. 103. Consultants Bureau, New York.

Solov'ev, L. S. 1967 *Reviews of Plasma Physics*, Vol. 3 (ed. M. A. Leontovich), p. 277. Consultants Bureau, New York.

Sommerfeld, A. 1949 *Partial Differential Equations in Physics*, p. 45. Academic Press, New York.

Tsinganos, K. C. 1982*a* *Astrophys. J.* **252**, 775.

Tsinganos, K. C. 1982*b* *Astrophys. J.* **259**, 820.

Vršnak, B., Ruždjak, V. and Rompolt, B. 1991 *Solar Phys.* **136**, 151.

https://doi.org/10.1017/S0022377898006965 Published online by Cambridge University Press