Math. Proc. Camb. Phil. Soc. (1979), **86**, 279
Printed in Great Britain

Strong and absolute summabilities of functions based on (D, α) summability methods

By BABBAN PRASAD MISHRA
and ANAND PRAKASH SRIVASTAVA
The University, Gorakhpur, India

(*Received* 15 *August* 1978, *revised* 13 *February* 1979)

1. *Introduction.* In 1966, Kuttner(4) constructed (D, α) summability methods for functions. In this paper, strong and absolute summabilities based on (D, α) summability methods are defined and some of their properties investigated.

Let $f(x)$ be any function which is Lebesgue integrable in $(0, x)$ for any (finite) x and which is bounded in some right-hand neighbourhood of the origin. Let $\alpha > 0$. If, for $y > 0$, the integral

$$g(y) = g^{(\alpha)}(y) = \alpha y \int_0^\infty \frac{x^{\alpha-1} f(x)}{(x+y)^{\alpha+1}}\, dx \tag{1·1}$$

exists and, if $g(y) \to s$ as $y \to \infty$, we say that $f(x) \to s(D, \alpha)$ as $x \to \infty$.

Integrals of the form $\int_a^\infty$ are throughout to be taken as $\lim_{x\to\infty} \int_a^x$, $\int_a^x$ being a Lebesgue integral.

We note that, for any fixed $y > 0$, $\alpha > 0$, it is necessary and sufficient for the convergence of (1·1) that

$$\int_1^\infty \frac{f(x)}{x^2}\, dx \tag{1·2}$$

should converge. If (1·2) converges, write, for $x > 0$,

$$F(x) = \int_x^\infty \frac{f(t)}{t^2}\, dt; \tag{1·3}$$

we note that $F(x) = o(1)$ as $x \to \infty$. Further, (since $f(t)$ is bounded in a right-hand neighbourhood of the origin) we have

$$F(x) = O\left(\frac{1}{x}\right) \tag{1·4}$$

as $x \to 0+$. It follows that, in the integration by parts which follows, the integrated terms tend to 0 as $x \to 0$ and as $x \to \infty$. We have

$$\begin{aligned} g^{(\alpha)}(y) &= -\alpha y \int_0^\infty \frac{x^{\alpha+1}}{(x+y)^{\alpha+1}}\, dF(x) \\ &= \alpha(\alpha+1) y^2 \int_0^\infty \frac{x^\alpha}{(x+y)^{\alpha+2}} F(x)\, dx. \end{aligned} \tag{1·5}$$

Now, it is easy to justify differentiation under the integral sign in (1·5), and a straightforward calculation then shows that

$$y \frac{d}{dy} g^{(\alpha)}(y) = \alpha\{g^{(\alpha+1)}(y) - g^{(\alpha)}(y)\} \quad (\alpha > 0) \tag{1·6}$$

0305-0041/79/0000-6590 $03.50 © 1979 Cambridge Philosophical Society

https://doi.org/10.1017/S0305004100056103 Published online by Cambridge University Press

holds for all $y > 0$. If

$$f(x) = \int_0^x \phi(t)\,dt, \tag{1·7}$$

where $\phi(t)$ is Lebesgue integrable in each finite interval $x \geqslant 0$, then we define

$$\tau^{(\alpha)}(y) = \alpha y \int_0^\infty \frac{x^\alpha \phi(x)}{(x+y)^{\alpha+1}}\,dx \tag{1·8}$$

if this converges. For any $\alpha > 0, y > 0$, it is necessary and sufficient for the convergence of (1·8) that

$$\int_1^\infty \frac{\phi(x)}{x}\,dx \tag{1·9}$$

should converge. The convergence of (1·9) implies $f(x) = o(x)$ and that $\int^\infty \frac{f(x)}{x^2}\,dx$ converges. If (1·7) holds and (1·9) converges (as noted this implies the convergence of (1·2)), then

$$\alpha\{g^{(\alpha+1)}(y) - g^{(\alpha)}(y)\} = \tau^{(\alpha)}(y) \quad (\alpha > 0). \tag{1·10}$$

In Hölder's inequality q denotes a number conjugate to $p > 1$ i.e. $(1/p) + (1/q) = 1$. The case $p = 1, q = \infty$ is interpreted in accordance with the usual conventions.

We use C to denote a constant, not necessarily the same at each occurrence.

2. In this section, we consider strong summability.

Suppose that $\beta > 0$ and $p \geqslant 1$. Then we say that the function $f(x)$ is

(i) summable $[D, \beta-1]_p$ or strongly summable $(D, \beta-1)$ with index p, to the sum s, and we write $f(x) \to s[D, \beta-1]_p$, if

$$\int_0^y |g^{(\beta)}(t) - s|^p\,dt = o(y) \tag{2·1}$$

as $y \to \infty$,

(ii) bounded $[D, \beta-1]_p$ or strongly bounded $(D, \beta-1)$ with index p, if

$$\int_0^y |g^{(\beta)}(t)|^p\,dt = O(y) \tag{2·2}$$

as $y \to \infty$.

We omit the suffix p in the case $p = 1$. We may regard summability (D, β) and boundedness (D, β) as the cases $p = \infty$ of summability $[D, \beta-1]_p$ and boundedness $[D, \beta-1]_p$, respectively. It follows from Hölder's inequality that, for fixed $\beta > 0$, $[D, \beta-1]_p$ increases in strength with decreasing p.

Comparing lemma 4 of Kuttner (4) with lemma 2 (i) of Borwein (1), we see that, for $\alpha > \beta > 0$, the formula giving $g^{(\beta)}(x)$ in terms of $g^{(\alpha)}(t)$ is the same as that giving $\sigma_{\beta-1}(x)$ in terms of $\sigma_{\alpha-1}(t)$ (where $\sigma_\alpha(x)$ denotes the A_α transform of a sequence $\{s_n\}$). Hence the same argument as in (5)† gives the exact analogues of Theorems 1–3 of (5). In particular, we have the following result.

† In the statement of theorem 3 of (5), the inequalities to be satisfied by the parameters are wrongly stated; the correct inequalities are those implied by the division into cases at the top of p. 123. The statement of theorem 3 would allow the sign $>$ in $\mu > \mu' + (1/p) - (1/p')$ to be replaced by $\geqslant$ in case (3) a (when it ought not to), but would not allow it in case (3) b (when it ought to).

https://doi.org/10.1017/S0305004100056103 Published online by Cambridge University Press

THEOREM 1. *Let* $\beta > -1$. *Let*

$$1 \leqslant p \leqslant p' \leqslant \infty (p < \infty); \alpha > \beta + \frac{1}{p} - \frac{1}{p'},$$

where $>$ *may be replaced by* $\geqslant$ *in the last inequality in the cases*

$$1 < p, \quad p' < \infty;$$

or

$$1 = p, \quad p' = \infty.$$

Then

$$[D, \alpha]_p \Rightarrow [D, \beta]_{p'}.$$

Since, for $\beta > 0$,

$$[D, \beta]_p \Rightarrow (D, \beta) \Rightarrow [D, \beta - 1]_p$$

a now familiar argument, originally introduced by Hyslop (3), enables us to deduce the following result from (1·6).

THEOREM 2. *Necessary and sufficient conditions for the function* $f(x)$ *to be summable* $[D, \beta]_p$, $\beta > 0$ *to the sum* s *are that it be summable* (D, β) *to* s *and*

$$\int_0^x \left| y \frac{d}{dy} g^{(\beta)}(y) \right|^p dy = o(x), \quad \text{as} \quad x \to \infty.$$

The analogues of Mishra (5), theorems 6–9 follow by the same argument as in (5). In particular we have, as an analogue of (5), theorem 6.

THEOREM 3. *Suppose that* $1 \leqslant p \leqslant \infty$, $\beta > -1$, $\delta > 0$. *If the function* $f(x)$ *is bounded* $[D, \beta + \delta]_p$ *and summable* $[D, \beta + \delta - 1]_p$, $\beta + \delta > 0$ *to the sum* s, *it is summable* $[D, \beta]_p$ *to* s.

3. We now consider absolute summability. Let $\alpha > 0, p \geqslant 1$. If

$$\int_1^\infty y^{p-1} \left| \frac{d}{dy} g^{(\alpha)}(y) \right|^p dy < \infty, \tag{3·1}$$

we say that the function $f(x)$ is summable $|D, \alpha|_p$ or absolutely summable (D, α) with index p. In the case $p = \infty$, we interpret (3·1) as meaning

$$y \frac{d}{dy} g^{(\alpha)}(y) = o(1).$$

In the theorems which we will prove involving this concept, there is no loss of generality in supposing that $f(x) = 0$ identically in some right-hand neighbourhood of the origin. Now once this assumption is made and assuming that (1·2) converges, $F(x)$ is bounded for all $x > 0$. Since

$$\frac{d}{dy} g^{(\alpha)}(y) = \alpha(\alpha + 1) y \int_0^\infty \frac{x^\alpha (2x - \alpha y)}{(x + y)^{\alpha + 3}} F(x) \, dx,$$

we now have, uniformly in $y > 0$,

$$\frac{d}{dy} g^{(\alpha)}(y) = O\left\{ y \int_0^\infty \frac{x^\alpha |2x - \alpha y|}{(x + y)^{\alpha + 3}} dx \right\} = O(1).$$

https://doi.org/10.1017/S0305004100056103 Published online by Cambridge University Press

Thus (3·1) is equivalent to

$$\int_0^\infty y^{p-1}\left|\frac{d}{dy}g^{(\alpha)}(y)\right|^p dy < \infty, \tag{3·2}$$

since the integral converges at the origin. Also, it is evident that summability $|D,\alpha|_1$ implies summability (D,α) but the converse cannot be assumed.

We have first the following theorem in the direction of increasing p and decreasing α.

THEOREM 4. *Let* $\beta > 0$. *Let*

$$1 \leqslant p \leqslant p' \leqslant \infty (p < \infty); \quad \alpha > \beta + \frac{1}{p} - \frac{1}{p'},$$

where $>$ *may be replaced by* $\geqslant$ *in the last inequality in the cases*

$$1 < p, \quad p' < \infty;$$

or

$$1 = p, \quad p' = \infty.$$

Then

$$|D,\alpha|_p \Rightarrow |D,\beta|_{p'}.$$

A lemma is necessary.

LEMMA. *If* $\alpha > \beta > 0$, *then for all* $y > 0$,

$$y\frac{d}{dy}g^{(\beta)}(y) = \frac{\Gamma(\alpha)}{\Gamma(\beta)\,\Gamma(\alpha-\beta)}y^{1-\alpha}\int_0^y t^\beta (y-t)^{\alpha-\beta-1}\left(\frac{d}{dt}g^{(\alpha)}(t)\right)dt.$$

Proof. Using (1·6), we see that the result to be proved is equivalent to

$$\beta(g^{(\beta+1)}(y) - g^{(\beta)}(y)) = \frac{\Gamma(\alpha+1)}{\Gamma(\beta)\,\Gamma(\alpha-\beta)}y^{1-\alpha}\int_0^y t^{\beta-1}(y-t)^{\alpha-\beta-1}\,(g^{(\alpha+1)}(t) - g^{(\alpha)}(t))\,dt. \tag{3·3}$$

By (4), lemma 4

$$g^{(\beta)}(y) = \frac{\Gamma(\alpha)}{\Gamma(\beta)\,\Gamma(\alpha-\beta)}y^{1-\alpha}\int_0^y t^{\beta-1}(y-t)^{\alpha-\beta-1}g^{(\alpha)}(t)\,dt. \tag{3·4}$$

Replacing α by $\alpha+1$,

$$g^{(\beta)}(y) = \frac{\Gamma(\alpha+1)}{\Gamma(\beta)\,\Gamma(\alpha+1-\beta)}y^{-\alpha}\int_0^y t^{\beta-1}(y-t)^{\alpha-\beta}g^{(\alpha+1)}(t)\,dt. \tag{3·5}$$

Replacing α by $\alpha+1$ and β by $\beta+1$ in (3·4),

$$g^{(\beta+1)}(y) = \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)\,\Gamma(\alpha-\beta)}y^{-\alpha}\int_0^y t^{\beta}(y-t)^{\alpha-\beta-1}g^{(\alpha+1)}(t)\,dt. \tag{3·6}$$

Multiplying (3·4) by $-\alpha$, (3·5) by $\alpha-\beta$ and (3·6) by β and adding, we obtain (3·3), and the lemma is proved.

It is worth mentioning that, in the special case in which (1·7) holds, the result could be proved more simply by applying (4), lemma 4 with $f(x)$ replaced by $x\phi(x)$, and using (1·10).

It follows from the lemma that, for $\alpha > \beta > 0$, the formula giving $(d/dy)\,g^{(\beta)}(y)$ in terms of $(d/dt)\,g^{(\alpha)}(t)$ is the same as that giving $(d/dy)\,\sigma_{\beta-1}(y)$ in terms of $(d/dt)\,\sigma_{\alpha-1}(t)$ (with the notation of (6)). Hence the same argument as in (6), theorem 1, gives theorem

https://doi.org/10.1017/S0305004100056103 Published online by Cambridge University Press

4 for $p' < \infty$. However, in (6), theorem 1, we restricted ourselves to the case $p' < \infty$. The result does continue to hold in the case $p' = \infty$, provided that

$$p = 1; \quad \lambda \geqslant \mu + 1$$

or

$$p > 1; \quad \lambda > \mu + (1/p)$$

and we now give the proof for that case. Suppose first $p = 1$ so that $\lambda \geqslant \mu + 1$. We have

$$\tau_\mu(y) = Cy^{-\lambda} \int_0^y (y-t)^{\lambda-\mu-1} t^\mu \tau_\lambda(t)\, dt, \tag{3·7}$$

we are given that

$$\int_0^\infty |\tau_\lambda(t)| \frac{dt}{t} < \infty,$$

so that, given any $\epsilon > 0$, there is a T such that

$$\int_T^\infty |\tau_\lambda(t)| \frac{dt}{t} < \epsilon.$$

Once T has been fixed, the contribution to (3·7) of the range $(0, T)$ is $O(y^{-\mu-1}) = o(1)$ since $\mu > -1$. But in the range (T, y) (provided that y is large enough) the maximum value of $(y-t)^{\lambda-\mu-1} t^{\mu+1}$ occurs for $t = (\mu + 1/\lambda)\, y$ and equals Cy^λ. Hence the contribution to (3·7) of the range (T, y) is in modulus

$$\leqslant C \int_T^y \frac{1}{t} |\tau_\lambda(t)|\, dt < C\epsilon.$$

Hence the result. Now suppose that $p > 1$, so that we now have $\lambda > \mu + \frac{1}{p}$. We now choose T so that

$$\int_T^\infty |\tau_\lambda(t)|^p \frac{dt}{t} < \epsilon.$$

The contribution to (3·7) of the range $(0, T)$ is as before. That of the range (T, y) is (by Hölder's inequality) in modulus

$$\leqslant Cy^{-\lambda} \left(\int_T^y t^{-1} |\tau_\lambda(t)|^p\, dt \right)^{1/p} \left(\int_T^y (y-t)^{p(\lambda-\mu-1)/(p-1)}\, t^{(\mu p+1)/(p-1)}\, dt \right)^{(p-1)/p}.$$

The second factor is less than the corresponding expression with the integral taken from 0 to y. But this is equal to Cy^λ (because $\lambda > \mu + 1/p$ is equivalent to

$$p(\lambda - \mu - 1)/(p-1) > -1$$

and $\mu > -1$ is equivalent to $(\mu p + 1)/(p-1) > -1$). Hence the result.

4. Let $\partial_\alpha(t)$, the (C, α) transform of $f(t)$, be defined, as usual, by

$$\partial_\alpha(t) = \begin{cases} f(t) & (\alpha = 0) \\ \dfrac{\alpha}{t^\alpha} \displaystyle\int_0^t (t-u)^{\alpha-1} f(u)\, du & (\alpha > 0). \end{cases}$$

https://doi.org/10.1017/S0305004100056103 Published online by Cambridge University Press

If $\alpha \geqslant 0$, $p \geqslant 1$, we say that $f(t)$ is summable $|C,\alpha|_p$ (absolutely summable (C,α) with index p) if

$$\int_0^\infty t^{p-1}\left|\frac{d}{dt}\partial_\alpha(t)\right|^p dt < \infty. \tag{4·1}$$

This is the analogue for functions of a definition for sequences given by Flett (2). In any results involving $|C,\alpha|_p$ for values of $\alpha < 1$, we restrict ourselves to the case in which $f(x)$ is an indefinite Lebesgue integral of a function $\phi(x)$, say; this ensures that the derivative $(d/dt)\,\partial_\alpha(t)$ exists almost everywhere. Such a restriction is not, however, needed when $\alpha \geqslant 1$. We remark that it is easily verified that, for $\alpha \geqslant 0$, $\delta > 0$,

$$t\frac{d}{dt}\partial_{\alpha+\delta}(t) = \frac{\Gamma(\alpha+\delta+1)}{\Gamma(\alpha+1)\,\Gamma(\delta)}\frac{1}{t^{\alpha+\delta}}\int_0^t (t-u)^{\delta-1}u^{\alpha+1}\frac{d}{du}\partial_\alpha(u)\,du. \tag{4·2}$$

Using (4·2) in place of (2), (2·1·2), an argument precisely similar to the special case $r = k$ of (2), Theorem 1 yields the following result.

THEOREM 5. *Let $\beta > \alpha \geqslant 0$, $p \geqslant 1$. If $f(x)$ is summable $|C,\alpha|_p$, then it is summable $|C,\beta|_p$.*

Now, we want to establish the relative strength between $|C,\alpha|_p$ and $|D,\lambda|_p$.

THEOREM 6. *Let $p \geqslant 1$, $\alpha \geqslant 0$, $\lambda > 0$. If $f(x)$ is summable $|C,\alpha|_p$ and if (1·2) converges then $f(x)$ is summable $|D,\lambda|_p$. If $\alpha \leqslant 1$, then the convergence of (1·2) is implied by $|C,\alpha|_p$ summability of $f(x)$, and so this hypothesis may be omitted.*

Proof. In virtue of Theorem 5, it is enough to prove the result if α is an integer. We have, by (1·6),

$$\frac{d}{dy}g^{(\lambda)}(y) = \lambda\int_0^\infty \frac{x^{\lambda-1}}{(x+y)^{\lambda+2}}(x-\lambda y)f(x)\,dx. \tag{4·3}$$

For any $p > 0$, we write $f_p(x)$ for the 'pth integral',

$$f_p(x) = \frac{1}{\Gamma(p)}\int_0^x (x-t)^{p-1}f(t)\,dt.$$

As was pointed out in (4), equation (15), the convergence of (1·2) implies that

$$f_p(x) = o(x^{p+1}). \tag{4·4}$$

Now, integrating (4·3) by parts α times, we deduce with the help of (4·4) that

$$\frac{d}{dy}g^{(\lambda)}(y) = (-1)^\alpha\lambda\int_0^\infty \left\{\frac{\partial^\alpha}{\partial x^\alpha}\left[\frac{x^{\lambda-1}(x-\lambda y)}{(x+y)^{\lambda+2}}\right]\right\}f_\alpha(x)\,dx. \tag{4·5}$$

It is easily verified that the expression in curly brackets in (4·5) is

$$O\left(\frac{x^{\lambda-\alpha-1}}{(x+y)^{\lambda+1}}\right) \tag{4·6}$$

uniformly in $x \geqslant 1$, $y > 0$. By (4·5), we have

$$\frac{d}{dy}g^{(\lambda)}(y) = \frac{(-1)^\alpha\lambda}{\Gamma(\alpha+1)}\int_0^\infty x^\alpha\partial_\alpha(x)\frac{\partial^\alpha}{\partial x^\alpha}\left[\frac{x^{\lambda-1}(x-\lambda y)}{(x+y)^{\lambda+2}}\right]dx. \tag{4·7}$$

https://doi.org/10.1017/S0305004100056103 Published online by Cambridge University Press

Let
$$R(x,y) = \int_0^x t^\alpha \frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{t^{\lambda-1}(t-\lambda y)}{(t+y)^{\lambda+2}}\right) dt.$$

In fact, for fixed $\lambda > 0, \alpha \geqslant 0$, we have uniformly in $x > 0, y > 0$

$$R(x,y) = O\left(\frac{x^\lambda}{(x+y)^{\lambda+1}}\right). \tag{4·8}$$

This may be proved by induction on α. If $\alpha = 0$,

$$R(x,y) = \int_0^x \frac{t^{\lambda-1}(t-\lambda y)}{(t+y)^{\lambda+2}} dt = \frac{-x^\lambda}{(x+y)^{\lambda+1}}$$

whence the result is evident. Suppose that $\alpha \geqslant 1$, and assume the result true for $\alpha - 1$. Integrating by parts

$$R(x,y) = x^\alpha \frac{\partial^{\alpha-1}}{\partial x^{\alpha-1}}\left(\frac{x^{\lambda-1}(x-\lambda y)}{(x+y)^{\lambda+2}}\right) - \alpha \int_0^x t^{\alpha-1} \frac{\partial^{\alpha-1}}{\partial t^{\alpha-1}}\left(\frac{t^{\lambda-1}(t-\lambda y)}{(t+y)^{\lambda+2}}\right) dt.$$

The first term is of the required order by (4·6) (with α replaced by $\alpha - 1$) and the second by the induction hypothesis. Now, integrating (4·7) by parts, we have

$$\frac{d}{dy} g^{(\lambda)}(y) = C \int_0^\infty R(x,y)\left(\frac{d}{dx}\partial_\alpha(x)\right) dx,$$

since the integrated term tends to 0 as $\partial_\alpha(x)$ is bounded and $R(x,y) \to 0$ as $x \to 0$ and $x \to \infty$. By Hölder's inequality when $p > 1$, and trivially when $p = 1$, we deduce that,

$$\left|\frac{d}{dy} g^{(\lambda)}(y)\right|^p \leqslant C\left(\int_0^\infty |R(x,y)|\, x^{p-1}\left|\frac{d}{dx}\partial_\alpha(x)\right|^p dx\right)\left\{\int_0^\infty \frac{|R(x,y)|}{x} dx\right\}^{p-1}.$$

Using (4·8) and putting $x = ty$, we see that the expression in curly brackets

$$\leqslant C \int_0^\infty \frac{x^{\lambda-1}}{(x+y)^{\lambda+1}} dx = \frac{C}{y}\int_0^\infty \frac{t^{\lambda-1}}{(1+t)^{\lambda+1}} dt = \frac{C}{y}$$

(since the integral converges). Hence

$$\int_0^\infty y^{p-1}\left|\frac{d}{dy} g^{(\lambda)}(y)\right|^p dy \leqslant C \int_0^\infty dy \int_0^\infty |R(x,y)|\, x^{p-1}\left|\frac{d}{dx}\partial_\alpha(x)\right|^p dx$$
$$= C \int_0^\infty x^{p-1}\left|\frac{d}{dx}\partial_\alpha(x)\right|^p dx \int_0^\infty |R(x,y)|\, dy.$$

Again using (4·8), the inner integral

$$\leqslant C x^\lambda \int_0^\infty \frac{dy}{(x+y)^{\lambda+1}} = C.$$

Hence the result. Since we are supposing that α is an integer, we have $\alpha = 1$ in the last clause of the theorem. Now

$$\int_1^X \frac{f(x)}{x^2} dx = \frac{\partial_1(X)}{X} - \partial_1(1) + 2\int_1^X \frac{\partial_1(x)}{x^2} dx. \tag{4·9}$$

But we have
$$\int_1^\infty x^{p-1}\left|\frac{d}{dx}\partial_1(x)\right|^p dx < \infty.$$

https://doi.org/10.1017/S0305004100056103 Published online by Cambridge University Press

Also
$$\partial_1(X) = \partial_1(1) + \int_1^X \left(\frac{d}{dx}\partial_1(x)\right) dx. \tag{4·10}$$

By Hölder's inequality with indices p and p', we have,

$$\left|\int_1^X \left(\frac{d}{dx}\partial_1(x)\right) dx\right| \leqslant \left(\int_1^X x^{p-1}\left|\frac{d}{dx}\partial_1(x)\right|^p dx\right)^{1/p} \left(\int_1^X \frac{1}{x} dx\right)^{1/p'}$$
$$= O(\log X)^{1/p'}. \tag{4·11}$$

From (4·9), (4·10) and (4·11) we see that

$$\int_1^\infty \frac{f(x)}{x^2} dx$$

is convergent. We may compare this with the result for strong summability. Assuming that (1·2) converges, $[C,\alpha]_p$ implies $[D,\lambda]_{p'}$ for $\alpha \geqslant 0$, $\lambda > 0$ and any $p \geqslant 1$, $p' \geqslant 1$; we need not have $p' = p$. This follows at once from the chain of implications

$$[C,\alpha]_p \Rightarrow (C,\alpha) \Rightarrow (D,\lambda+1) \Rightarrow [D,\lambda]_p.$$

We express our deepest sense of gratitude to Professor B. Kuttner for his criticisms and suggestions for improvements in the paper. We also feel greatly indebted to the referee for his valuable suggestions.

REFERENCES

(1) BORWEIN, D. On a scale of Abel type summability methods. *Proc. Cambridge Philos. Soc.* **53** (1957), 318–322.

(2) FLETT, T. M. On an extension of absolute summability and some theorems of Littlewood and Paley. *Proc. London Math. Soc.* (3) **7** (1957), 113–141.

(3) HYSLOP, J. M. Note on the strong summability of series. *Proc. Glasgow Math. Assoc.* **1** (1951/53), 16–20.

(4) KUTTNER, B. On 'translated quasi-Cesàro' summability. *Proc. Cambridge Philos. Soc.* **62** (1966), 705–712.

(5) MISHRA, B. P. Strong summability of infinite series on a scale of Abel type summability methods. *Proc. Cambridge Philos. Soc.* **63** (1967), 119–127.

(6) MISHRA, B. P. Absolute summability of infinite series on a scale of Abel type summability methods. *Proc. Cambridge Philos. Soc.* **64** (1968), 377–387.

https://doi.org/10.1017/S0305004100056103 Published online by Cambridge University Press