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Morse cocycles and simple Lebesgue spectrum

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Abstract. It is well known that the spectrum of dynamical systems arising from generalized Morse sequences (M. Keane. Generalized Morse sequences. *Z. Wahr. Verw. Geb.* **10** (1968), 335–353.) is simple as soon as it is non-discrete. We give, in this paper, a necessary and sufficient condition for the existence of such a transformation with non-purely singular spectrum. From this, it follows that this problem is equivalent to an open problem of the existence of 'flat' polynomials on the Torus group. We show that this latter question can be given an affirmative answer on some other group, and this allows us to construct a countable abelian group action with simple spectrum whose spectral type is the sum of a discrete measure and of the Haar measure on the dual group.

1. *Introduction*

Let T be an ergodic translation on a compact abelian group X endowed with its Haar measure μ. We define as usual a $\mathbb{Z}$-cocycle ϕ by a single measurable function from X to $\{-1, 1\}$. The 2-point extension associated to ϕ is the measure-preserving transformation T_ϕ defined on $(X \times \{-1, 1\}, \mu \otimes (\delta_1 + \delta_{-1})/2)$ by

$$T_\phi(x, y) = (Tx, y\phi(x)).$$

We split $L^2(X \times \{-1, 1\})$ into the sum $L_1 \oplus L_2$, where $L_1 = \{f \otimes 1,\ f \in L^2(X)\}$ and $L_2 = \{f \otimes Id_{\{-1,1\}},\ f \in L^2(X)\}$ are both isomorphic to $L^2(X)$. These subspaces are closed and invariant under the unitary operator $U_\phi f = f \circ T_\phi$ associated to T_ϕ, and the restrictions of U_ϕ to these subspaces are unitarily equivalent to the operators U and V defined on $L^2(X)$ by

$$Uf = f \circ T \quad \text{and} \quad Vf = \phi . f \circ T.$$

As the spectrum of U is simple and discrete, the spectral study of U_ϕ restricts to the study of the operator V. Namely, if V admits a continuous, simple spectrum, then U_ϕ has simple spectrum, and its spectral type is the sum of the discrete spectral type of U and of the spectral type of V.

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We shall refer to the spectrum of V as the 'spectrum of the cocycle'. In [**5**], Helson shows that the spectral type of V has the following purity property: it is either discrete, or continuous and purely singular, or equivalent to the Lebesgue measure. Moreover, Helson and Parry in [**6**] construct, for any aperiodic measure-preserving transformation, some cocycles with ± 1 values admitting an absolutely continuous spectrum; but nothing is shown about their simplicity. The transformation arising from the Rudin–Shapiro sequence has multiplicity two, the smallest known in this case. More generally, the problem of the existence of a transformation with simple non-purely singular spectrum still remains an open issue.

In this paper we restrict our study to a class of cocycles which always have simple spectrum, and we shall then investigate conditions for spectral singularity or non-singularity. We consider the so-called 'Morse cocycles', which are defined as M-cocycles (see [**4**] for definition) with ± 1 values. Their associated 2-point extensions (which we call Morse extensions) correspond to the measure-preserving transformations arising from generalized Morse sequences described by Keane in [**10**]. In this case, the spectral simplicity of the operator V is known (see [**13**]). For the original Morse sequence, the spectral singularity was stated in [**9**]. In [**10**], Keane asks whether the spectral type is always purely singular for this class of transformations, but no general result seems to be known so far.

The first description of their spectral type is given by Keane in [**11**] in terms of g-measures. Besides, in the case of the original Morse sequence, Queffélec writes the spectral type as a Riesz product [**15**]. We shall follow the method used in [**3**] for the description of the spectral type of rank one transformations. This allows us to show that the spectral type of every Morse cocycle is given by a generalized Riesz product. We can then extend to Morse extensions a criterion of spectral singularity given by Bourgain [**2**] for rank one transformations which yields, in our case, further results.

THEOREM 1. *There exists a Morse cocycle with a simple Lebesgue spectrum, and thus a Morse extension admitting simple spectrum with a Lebesgue component if and only if there exists a sequence of trigonometric polynomials*

$$P_n(x) = \sum_{j=0}^{p_n-1} \varepsilon_j(n) e^{2i\pi jx} \quad \text{on } \mathbb{T},$$

where $p_n \geq 2$ *and* $\varepsilon_j(n) = \pm 1$ *for* $0 \leq j < p_n$, *such that* $\|P_n\|_1/\sqrt{p_n}$ *converge to one.*

Now the existence of sequences of trigonometric polynomials with coefficients of unit modulus admitting equivalent L^p and L^2 norms for a given $p \neq 2$ (the so called 'flat polynomials', or 'ultra flat polynomials' when $p = \infty$) is a classical problem in harmonic analysis. The construction of flat polynomials with respect to the L^1 norm was first set up in [**14**], and subsequently in [**1**] for L^p norms, $2 < p < \infty$. Kahane eventually showed the existence of ultra flat polynomials in [**7**]. Unfortunately, all of these constructions yield polynomials with complex coefficients and the problem with ± 1 coefficients still remains open.

Nevertheless we show here that this problem can be solved for another group: there exists some compact abelian group and a sequence of trigonometric polynomials with ± 1

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coefficients whose L^1 and L^2 norms are equivalent. Moreover, extending the definitions of Morse cocycles, we get the following.

THEOREM 2. *There exists a countable abelian group action which admits a simple spectrum, and whose spectral type is the sum of a discrete measure and of the Haar measure of the dual group.*

2. *Singularity conditions for Morse extensions*

We define Morse extensions as follows (cf. [**4**]).

Let $(p_n)_{n\geq 0}$ be a sequence of integers greater than or equal to two, and let (h_n) be defined by $h_{n+1} = p_n h_n$ and $h_0 = 1$. We consider the odometer $(X, \mathcal{B}, \mu, T)$, where X is the compact group of (h_n)-adic integers, $X = \{x = \sum_{n\geq 0} x_n h_n, 0 \leq x_n < p_n\}$, μ its Haar measure and $Tx = x + 1$. If $B_0 = X$ and $B_n = \{x \in X, x_0 = \cdots = x_{n-1} = 0\}$, then $(T^j B_n, 0 \leq j < h_n)$ yields a sequence of Rokhlin towers for T. A *Morse cocycle* on X (associated with T) is a cocycle ϕ with values in $\{-1, 1\}$ such that ϕ is constant on $T^j B_n$ for $0 \leq j < h_n - 1$ (ϕ is constant over all 'levels' of the tower except the last one). A *Morse extension* is the 2-point extension associated to a Morse cocycle.

Such a cocycle can be constructed inductively: suppose that ϕ is defined on the nth tower except on the last level. At the next step, each level of the nth tower is cut into p_n levels of the $(n + 1)$th tower, ϕ is then defined on the $(n + 1)$th tower except on the p_n levels $T^{jh_n-1} B_{n+1}$ for $0 < j \leq p_n$. If we choose arbitrarily the value of ϕ on $T^{jh_n-1} B_{n+1}$ for $0 < j \leq p_n - 1$, then ϕ is defined on the $(n + 1)$th tower except on the last level, ϕ is eventually defined everywhere except at one point.

Now we are concerned with the spectral type of the operator V defined on $L^2(X)$ by $Vf = \phi f \circ T$. We can write $V^j f = \phi^{(j)} f \circ T^j$ where

$$\phi^{(j)}(x) = \phi(x)\phi \circ T(x) \ldots \phi \circ T^{j-1}(x) \quad \text{for } j > 0,$$

$\phi^{(-j)}(x) = \phi^{(j)}(T^{-j}x)$, and $\phi^{(0)} = 1$.

We remark that a Morse cocycle on X is also a cocycle such that $\phi^{(j)}$ is constant on B_n for all $0 \leq j < h_n$. The values of $\phi^{(j)}$ on B_{n+1} for $0 \leq j < h_n$, are given by their value on B_n. At the $(n + 1)$th step in the construction of ϕ, as $\phi^{(j)}(x) = \phi^{(j-1)}(x)\phi(T^{j-1}x)$, it is then equivalent to choose the value of ϕ on $T^{h_n-1} B_{n+1}$ or the value of $\phi^{(h_n)}$ on B_{n+1}. Following this argument, it is easy to see that the values of ϕ at the $(n + 1)$th step are determined by the values $\varepsilon_n(j)$ of $\phi^{(jh_n)}$ on B_{n+1} for $0 < j < p_n$, and we can choose, arbitrarily, the sequence $(\varepsilon_n(j))_{0<j<p_n}$. We also let $\varepsilon_n(0) = \phi^{(0)} = 1$; we shall then say that ϕ is the Morse cocycle associated with the sequences (p_n) and $(\varepsilon_n(j))_{0\leq j<p_n}$.

Now we show that the spectral type of ϕ is given by a generalized Riesz product. Let $f_n = 1_{B_n}/\sqrt{\mu(B_n)}$. We follow the proof given in [**3**] for rank one transformations, which is based on two properties of the sequence of Rohklin towers $(T^j B_n)_{0\leq j<h_n}$ associated with a rank one transformation T; here, let $Vf = f \circ T$.

- $L^2(X)$ is the closure of the union of the cyclic subspaces $[V, f_n]$ spanned by f_n under V.
- For all $n \geq 0$ we have an induction relation $f_n = P_n(V)(f_{n+1})$ where P_n is a trigonometric polynomial on $\mathbb{T}$.

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Both properties hold in our case. Indeed, since $\phi^{(-j)}(T^j B_n) = \phi^{(j)}(B_n)$, we can write for all $0 \leq j < h_n$,

$$V^{-j} f_n = \frac{1}{\sqrt{\mu(B_n)}} \phi^{(-j)} 1_{T^j B_n} = \frac{1}{\sqrt{\mu(B_n)}} \phi^{(j)}(B_n) 1_{T^j B_n}.$$

Therefore $[V, f_n]$ contains the characteristic functions $1_{T^j B_n}$ for $0 \leq j < h_n$. Since the Rohklin towers $(T^j B_n)_{0 \leq j < h_n}$ span the σ-algebra, the first property follows immediately. From $B_n = \cup_{0 \leq j < p_n} T^{jh_n} B_{n+1}$ and $\varepsilon_n(j) = \phi^{(jh_n)}(B_{n+1})$, we also get the induction relation:

$$f_n = \frac{1}{\sqrt{p_n}} \sum_{j=0}^{p_n - 1} \varepsilon_n(j) V^{-jh_n}(f_{n+1}),$$

and we obtain the second property. The corresponding trigonometric polynomial on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is

$$P_n(x) = \frac{1}{\sqrt{p_n}} \sum_{j=0}^{p_n - 1} \varepsilon_n(j) e^{-2i\pi j h_n x}. \tag{1}$$

Let $J_n = \{jh_n, 0 \leq j < p_n\}$ the set of the frequencies of P_n. Each integer can be written in at most one way as a finite sum $\sum_n j_n$ where $j_n \in J_n$: such a sequence (P_n) of trigonometric polynomials with $\|P_n\|_2 = 1$, is called a normed 1-dissociated sequence, and for all $J \subset \mathbb{N}$, we have $\|\prod_J P_j\|_2 = 1$.

With these properties, the rest of the proof is the same as in [**3**]. It follows that the spectral type of V is given by the weak* limit of the products $\prod_0^n |P_j|^2 \lambda$. The existence of this limit is a consequence of the 1-dissociation property; this measure is called a generalized Riesz product and is denoted by $\prod_0^\infty |P_n|^2 \lambda$.

LEMMA 1. *The spectral type of a Morse extension defined by the Morse cocycle ϕ is equivalent, up to a discrete measure, to the generalized Riesz product $\prod_0^\infty |P_n|^2 \lambda$ where (P_n) is the sequence of trigonometric polynomials on $\mathbb{T}$ defined by (1), with $\varepsilon_n(j) = \phi^{(jh_n)}(B_{n+1})$ for $0 \leq j < p_n$.*

The criterion of singularity given in [**2**] by Bourgain for rank one transformations, as stated more precisely in [**12**], holds without change in our case.

THEOREM 3. ([**2, 12**]) *Let ρ be the generalized Riesz product $\prod_0^\infty |P_n|^2 \lambda$, where $(P_n)_{n \in \mathbb{N}}$ is a normed, 1-dissociated sequence of trigonometric polynomials on $\mathbb{T}$. Then ρ is purely singular if and only if $\|\prod_0^n P_j\|_1$ converges to zero. A sufficient condition is: there exists a subsequence $(n_k)_{k \in \mathbb{Z}}$ such that*

$$\lim_{k \to +\infty} \left\| \prod_0^k P_{n_j} \right\|_1 = 0.$$

COROLLARY 1. *A Morse extension associated to the sequence of trigonometric polynomials $(P_n)_{n \in \mathbb{N}}$ has purely singular spectrum if and only if there exists a subsequence $(n_k)_{k \in \mathbb{Z}}$ such that $\lim_{k \to +\infty} \|\prod_0^k P_{n_j}\|_1 = 0$.*

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We give in the following lemma a condition for non-singularity of the generalized Riesz product. We denote by $V(P) = \|P\|_2^2 - \|P\|_1^2$, the variance of $|P|$, for $P \in L^2(\mathbb{T})$.

LEMMA 2. *Let $(P_n)_{n\in\mathbb{N}}$ a normed, 1-dissociated sequence of trigonometric polynomials on $\mathbb{T}$. If $\sum_0^\infty V(P_n)^{1/2} < +\infty$ then the generalized Riesz product $\prod_0^\infty |P_n|^2\lambda$ admits an absolutely continuous component.*

Proof. Since $\|P_n\|_2 = 1$, we have $V(P_n) = 1 - \|P_n\|_1^2$, and the hypothesis is equivalent to $\sum_0^\infty (1 - \|P_n\|_1)^{1/2} < +\infty$. Hence $\prod_0^\infty \|P_n\|_1 > 0$.

For all functions P and Q in $L^2(\mathbb{T})$, one has

$$|\|PQ\|_1 - \|P\|_1\|Q\|_1| \le V(Q)^{1/2}V(P)^{1/2}.$$

Let n_0 be an integer and $n > n_0$, taking $P = P_n$ and $Q = \prod_{n_0}^{n-1} P_j$, we have

$$V(P) = 1 - \|P\|_1^2 \quad \text{and} \quad V(Q) \le \|Q\|_2^2 = 1.$$

Hence, by induction:

$$\left| \left\| \prod_{j=n_0}^{n} P_j \right\|_1 - \prod_{j=n_0}^{n} \|P_j\|_1 \right| \le \sum_{j=n_0}^{n} V(P_j)^{1/2},$$

whence

$$\limsup_{n\to+\infty} \left\| \prod_{j=n_0}^{n} P_j \right\|_1 \ge \prod_{j=n_0}^{+\infty} \|P_j\|_1 - \sum_{j=n_0}^{+\infty} V(P_j)^{1/2}.$$

We can choose n_0 such that $\sum_{n_0}^\infty V(P_j)^{1/2} < \prod_{n_0}^\infty \|P_j\|_1$. According to Theorem 3, it follows that the generalized Riesz product $\prod_{n_0}^\infty |P_n|^2\lambda$ is not purely singular. As $\prod_0^{n_0-1} |P_n|^2$ has only finitely many zeros, $\prod_0^\infty |P_n|^2\lambda$ also admits an absolutely continuous component. □

Recall that the spectrum of any Morse cocycle is simple and either purely singular or equivalent to the Lebesgue measure. Therefore, to show that there exists a Morse extension admitting a simple spectrum with a Lebesgue component, it is sufficient to find a Morse cocycle having a non-purely singular spectrum.

Proof of Theorem 1. To any sequence of trigonometric polynomials P'_n defined by

$$P'_n(x) = \sum_0^{p_n-1} \varepsilon_n(j) e^{2i\pi jx}, \quad \text{where } p_n > 1 \text{ and } \varepsilon_n(j) = \pm 1 \text{ for } 0 \le j < p_n \qquad (2)$$

we associate the Morse extension arising from the sequences (p_n) and $(\varepsilon_n(j))_{0\le j<p_n}$ (or $(-\varepsilon_n(j))_{0\le j<p_n}$ if $\varepsilon_n(0) = -1$).

Suppose first that there exists such a sequence (P'_n) such that $\|P'_n\|_1/\sqrt{p_n}$ converges to 1. Since $V(P'_n/\sqrt{p_n}) = 1 - \|P'_n\|_1^2/p_n \to 0$, by taking a subsequence, we can suppose that $\sum_0^\infty V(P'_n/\sqrt{p_n})^{1/2} < \infty$. Consider now the associated Morse cocycle. Its spectral type is equivalent, up to a discrete measure, to the generalized Riesz product $\prod_0^\infty |P_n|^2\lambda$, where $P_n(x) = P'_n(h_n x)/\sqrt{p_n}$. We have $V(P'_n/\sqrt{p_n}) = V(P_n)$, and it follows from

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Lemma 2 that the generalized Riesz product $\prod_0^\infty |P_n|^2\lambda$ is not purely singular. The Morse cocycle then has a Lebesgue spectrum.

Conversely, if for any sequence (P'_n) of the form (2), we have $\|P'_n\|_1/\sqrt{p_n} \not\to 1$, we have to prove that any Morse extension has purely singular spectrum. Consider a Morse extension and the corresponding trigonometric polynomials (P_n) given by (1). According to Theorem 3, it is sufficient to show that there exists a subsequence (n_k) such that $\|\prod_{j<k} P_{n_j}\|_1 \to 0$. We can write $P_n(x) = P'_n(h_n x)/\sqrt{p_n}$, where the P'_n satisfy (2), and thus there exists a subsequence (n_j) and $\delta \in [0, 1[$ such that $\|P'_{n_j}\|_1/\sqrt{p_{n_j}} \to \delta$.

For any continuous function g, we have

$$\int_0^1 g(x)\frac{1}{\sqrt{p_{n_j}}}|P'_{n_j}|(h_{n_j}x)\,dx \underset{j\to+\infty}{\longrightarrow} \delta\int_0^1 g(x)\,dx.$$

Indeed, this property holds when g is a trigonometric polynomial, hence for any continuous function g. So we can find a further subsequence still denoted by (n_j) such that for all j,

$$\left|\int_{\mathbb{T}}\prod_{k=0}^{j}|P_{n_k}|\,d\lambda - \delta\int_{\mathbb{T}}\prod_{k=0}^{j-1}|P_{n_k}|\,d\lambda\right| < 1/j!$$

If $\delta = 0$ it follows immediately that $\|\prod_{k=0}^j P_{n_k}\|_1$ converges to zero. Otherwise by summation we get

$$\int_{\mathbb{T}}\prod_{k=0}^{j}|P_{n_k}|\,d\lambda \le \delta^j e^{1/\delta} \underset{j\to\infty}{\longrightarrow} 0. \qquad \square$$

Remark. The analogous problem concerning rank one transformations is more complex: we are then lacking both the purity property and the good arithmetic repartition of the frequencies of the trigonometric polynomials.

With the second part of the proof of Theorem 1, we also get the following.

PROPOSITION 1. *Let (P'_n) be a sequence of trigonometric polynomials of the form (2), such that $\|P'_n\|_1$ does not converge to one. Then the Morse extension associated with (P'_n) admits a purely singular spectrum.*

A direct consequence is: if the sequence (p_n) does not go to infinity, then the spectrum of any Morse extension of the corresponding odometer is purely singular. The same result holds under the weaker condition $\sum_0^\infty (1/p_n)^2 = +\infty$, by small changes in the proof of the main theorem of [**12**].

3. *A countable abelian group action with simple Haar component*

As mentioned in the introduction, the existence problem which appears in Theorem 1 is open for trigonometric polynomials on the Torus group, but investigating the same problem in another compact abelian group, we shall obtain such a sequence. We shall then relate these polynomials to an ergodic group action.

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3.1. *A sequence of trigonometric polynomials with equivalent L^1 and L^2 norms.* In [**14**], Newman exhibits a sequence of trigonometric polynomials with equivalent L^1 and L^2 norms, with coefficients of unit modulus related to Gauss sums. We make use of such sums to construct a sequence of trigonometric polynomials with coefficients in $\{-1, 1\}$, on a product group.

Let p be a prime integer, and $\left(\frac{j}{p}\right)$ be the Legendre symbol (taking value 1 if j is a square mod p, and -1 otherwise). We consider the trigonometric polynomial on $\mathbb{Z}/p\mathbb{Z}$ defined by

$$P(x) = \sum_{j=1}^{p-1} \left(\frac{j}{p}\right) e^{2i\pi jx/p}.$$

It is related to the ordinary Gauss sum by $P(x) = \sum_0^{p-1} e^{2i\pi j^2 x/p} - \sum_0^{p-1} e^{2i\pi jx/p}$, and it follows that $P(0) = 0$ and $|P(x)| = \sqrt{p}$ if $x \neq 0$. Let λ be the normalized Haar measure on $\mathbb{Z}/p\mathbb{Z}$. We have $\int |P|\, d\lambda = (p-1)/\sqrt{p}$. If $\widetilde{P} = P + 1$, then we obtain a trigonometric polynomial with coefficients in $\{-1, 1\}$, such that $\|\widetilde{P}\|_1/\|\widetilde{P}\|_2 \geq 1 - 2/\sqrt{p}$.

Now let $(p_n)_{n\in\mathbb{N}}$ be a increasing sequence of prime numbers such that

$$\sum_0^\infty 1/\sqrt{p_n} < +\infty.$$

Let $G = \oplus_{n\geq 0}\mathbb{Z}/p_n\mathbb{Z}$, and let $(e_n)_{n\in\mathbb{N}}$ be the natural basis of G. The dual group $\widehat{G}$ is the product group $\prod_{n\geq 0}\mathbb{Z}/p_n\mathbb{Z}$, where the duality is defined by $\langle e_n, t\rangle = e^{2i\pi t_n/p_n}$ for $t = (t_n)_{n\geq 0} \in \widehat{G}$. Its normalized Haar measure λ is the product measure of the normalized Haar measures λ_n on $\mathbb{Z}/p_n\mathbb{Z}$. Let P_n be the trigonometric polynomial defined on $\mathbb{Z}/p_n\mathbb{Z}$ by

$$P_n(t) = \frac{1}{\sqrt{p_n}}\left(1 + \sum_{j=1}^{p_n-1} \left(\frac{j}{p_n}\right) e^{-2i\pi jt/p_n}\right).$$

Let R_n be the trigonometric polynomial defined on $\widehat{G}$ by $R_n(t) = P_n(t_n)$, then it is normalized in $L^2(\widehat{G})$ and $\|R_n\|_1 = \|P_n\|_1$ converges to one. Here the generalized Riesz product $\prod_0^\infty |R_n|^2\lambda$ is nothing but the product measure $\prod_0^{+\infty}(|P_n|^2\lambda_n)$. Since

$$\prod_{j=0}^{+\infty} \int_{\mathbb{Z}/p_j\mathbb{Z}} |P_j|\, d\lambda_j \geq \prod_{j=0}^{+\infty}(1 - 2/\sqrt{p_j}) > 0,$$

according to the criterion of Kakutani [**8**], the product measure $\prod_0^\infty(|P_n|^2\lambda_n)$ is absolutely continuous with respect to λ.

3.2. *Construction of the corresponding group action.* We shall build, by a generalization of Morse extensions, an action of G whose continuous part of the spectrum is given by the above product measure. Let $X = \widehat{G} = \prod_{n\geq 0}\mathbb{Z}/p_n\mathbb{Z}$ be endowed with its Haar measure $\mu = \lambda$. We may consider G as a subgroup of X and then we have a natural action $T = (T_g)_{g\in G}$ of G on X defined by $T_g x = x + g$. Since G is dense in X this action

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is ergodic. A cocycle associated with T is a measurable function from $X \times G$ to $\{-1, 1\}$, which satisfies $\phi(x, g + g') = \phi(x, g)\phi(T_g x, g')$ for all $(x, g, g') \in X \times G^2$. We shall also use the notation $\phi^{(g)}(x) = \phi(x, g)$.

Then the extension of T associated with ϕ is the action T_ϕ defined on $(X \times \{-1, 1\}, \mu \otimes (\delta_1 + \delta_{-1})/2)$ by

$$(T_\phi)_g(x, y) = (T_g x, \phi_g(x)y).$$

Let $U_\phi = ((U_\phi)_g)$ be the group of the unitary operators on $L^2(X \times \{-1, 1\})$ given by $(U_\phi)_g f = f \circ (T_\phi)_g$. We consider the same decomposition of $L^2(X)$ into $L_1 \oplus L_2$, as in the case of a $\mathbb{Z}$-action. Both subspaces are invariant under U_ϕ, and the spectrum of the restriction of U_ϕ on L_1 is simple and discrete. The restriction of U_ϕ on L_2, is unitarily equivalent to the group $V = (V_g)_{g \in G}$ of the unitary operators on $L^2(X)$ defined by

$$V_g f = \phi^{(g)} f \circ T_g \quad \text{for } g \in G.$$

We still have that U_ϕ has multiplicity one as soon as the spectrum of V is simple and continuous.

There is a natural sequence of nested partitions $\mathcal{P}_n = (T_g B_n)_{g \in G_n}$ where $B_0 = X$, $G_0 = \{0\}$ and for $n > 0$, $B_n = \{x \in X, x_0 = \cdots = x_{n-1} = 0\}$ and $G_n = \oplus_{k<n} \mathbb{Z}/p_k\mathbb{Z}$. These partitions converge to the point partition, and will take place of the Rokhlin towers of the odometer. We define the analogue of a Morse cocycle as a cocycle such that $\phi^{(g)}$ is constant on B_n for all $g \in G_n$. In this case, such a cocycle may be again defined by an arbitrary sequence $(\varepsilon_n(j))_{0 \le j < p_n, 0 \le n}$ with ± 1 values and $\varepsilon_n(0) = 1$, by the relation

$$\phi(x, g) = \prod_{n=0}^{+\infty} \varepsilon_n(x_n)\varepsilon_n(x_n + g_n) \quad \text{for all } (x, g) \in X \times G. \tag{3}$$

Indeed if ϕ is defined by (3), we easily check that ϕ is a cocycle such that $\phi^{(g)}$ is constant on B_n for $g \in G_n$.

Conversely, let ϕ be a cocycle such that for $x \in B_n$ $\phi^{(g)}(x) = \phi^{(g)}(0)$, for all $g \in G_n$ and let $\varepsilon_n(j) = \phi^{(je_n)}(0)$ for $0 \le j < p_n$. For all $g \in G_n$, since $T_{g_{n-1}e_{n-1}} 0 \in B_{n-1}$ and $g - g_{n-1}e_{n-1} \in G_{n-1}$, we have $\phi(0, g) = \phi(0, g_{n-1}e_{n-1})\phi(0, g - g_{n-1}e_{n-1})$, and by induction

$$\phi(0, g) = \prod_{k=0}^{n-1} \phi(0, g_k e_k) = \prod_{k=0}^{n-1} \varepsilon_k(g_k).$$

For all $x \in X$, $g \in G_n$, if $x^{(n)} = \sum_0^{n-1} x_k e_k$, we have

$$\phi(x - x^{(n)}, g + x^{(n)}) = \phi(x - x^{(n)}, x^{(n)})\phi(x, g).$$

Since $x - x^{(n)} \in B_n$, we also get:

$$\phi(x, g) = \phi(0, g + x^{(n)})\phi(0, x^{(n)}) = \prod_{k=0}^{n-1} \varepsilon_k(x_k + g_k)\varepsilon_k(x_k).$$

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Now let $(\varepsilon_n(k))_{0\le k<p_n}$ be a sequence of ± 1, let ϕ be the cocycle defined by (3), and let $f_n = 1_{B_n}/\sqrt{\mu(B_n)}$. Since $\phi^{(g)}$ is constant on B_n for $g \in G_n$, we have:

$$V_{-g} f_n = \frac{1}{\sqrt{\mu(B_n)}} \phi^{(-g)} 1_{T_g B_n} = \frac{1}{\sqrt{\mu(B_n)}} \phi^{(g)}(B_n) 1_{T_g B_n}.$$

So the cyclic subspace $[V, f_n]$ contains $1_{T_g B_n}$ for every $g \in G_n$, therefore $L^2(X)$ is the closure of the union of the $[V, f_n]$. This also implies $1_{B_{n-1}} \in [V, f_n]$, hence $([V, f_n])_{n\ge 0}$ is an increasing sequence of cyclic subspaces and it follows that V admits a simple spectrum. We have here $B_n = \cup_{k=0}^{p_n-1} T_{ke_n} B_{n+1}$, which yields the induction relation

$$f_n = \frac{1}{\sqrt{p_n}} \sum_{k=0}^{p_n-1} \varepsilon_n(k) V_{-ke_n} f_{n+1}.$$

Recall that the spectral measure of a function $f \in L^2(X)$ under V is the measure ρ_f on $\widehat{G}$ where its Fourier transform $\widehat{\rho}_f$ is given by $\widehat{\rho}_f(g) = \langle V_g f, f\rangle$ for all $g \in G$.

LEMMA 3. *The spectral measure ρ_0 of the function $f_0 = 1$ is the product measure $\prod_0^{+\infty}(|P_n|^2\lambda_n)$ where P_n is defined on $\mathbb{Z}/p_n\mathbb{Z}$ by*

$$P_n(x) = 1/\sqrt{p_n} \sum_{k=0}^{p_n-1} \varepsilon_n(k) e^{-2i\pi kx/p_n}. \tag{4}$$

Proof. We will prove the equality of Fourier transform on G. For all $g \in G$, we have $\widehat{\rho}_0(g) = \langle V_g 1, 1\rangle = \int_X \phi(x, g)\, d\mu(x)$. On the other hand

$$\begin{aligned}
\left(\prod_{n=0}^{+\infty} |P_n|^2\lambda_n\right)\widehat{\ }(g) &= \prod_{n=0}^{+\infty} \int_{\mathbb{Z}/p_n\mathbb{Z}} |P_n|^2(x_n) e^{2i\pi x_n g_n/p_n}\, d\lambda_n(x_n) \\
&= \prod_{n=0}^{+\infty} \int_{\mathbb{Z}/p_n\mathbb{Z}} \frac{1}{p_n} \sum_{j=0}^{p_n-1} \sum_{j'=0}^{p_n-1} \varepsilon_n(j)\varepsilon_n(j') e^{2i\pi x_n(g_n+j-j')/p_n}\, d\lambda_n(x_n) \\
&= \prod_{n=0}^{+\infty} \left(\frac{1}{p_n} \sum_{j+g_n=j'} \varepsilon_n(j)\varepsilon_n(j')\right) \\
&= \prod_{n=0}^{+\infty} \int_{\mathbb{Z}/p_n\mathbb{Z}} \varepsilon_n(j+g_n)\varepsilon_n(j)\, d\lambda_n(j)
\end{aligned}$$

Since ϕ satisfies (3), it follows

$$\left(\prod_{n=0}^{+\infty} |P_n|^2\lambda_n\right)\widehat{\ }(g) = \int_X \phi(x, g)\, d\lambda(x) = \widehat{\rho}_0(g). \qquad \square$$

Proof of Theorem 2. Since the action T is ergodic, the purity property still holds for the spectral type ρ of V: it is either purely singular or equivalent to the Haar measure [**5**]. So, as soon as ρ_0 is absolutely continuous, ρ must be equivalent to the Haar measure of the product group.

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Now let $\varepsilon_n(k) = \binom{k}{p_n}$ for $0 < k < p_n$ and $\varepsilon_n(0) = 1$, we consider the associated cocycle ϕ defined by the relation (3):

$$\phi(x, g) = \prod_{n=0}^{+\infty} \varepsilon_n(x_n)\varepsilon_n(x_n + g_n).$$

We have proved that the corresponding group of the operators $(V_g)_{g\in G}$ admits a simple spectrum, and that the spectral measure ρ_0 is given by the product measure $\prod_0^{+\infty} |P_n|^2\lambda_n$, where the trigonometric polynomials P_n are defined by (4), and are equal to the polynomials P_n of §3.1. The product measure is then absolutely continuous and thus the spectral type ρ of V is equivalent to the Haar measure of the dual group $\widehat{G}$. It follows that U_ϕ also has multiplicity one, and its spectral type is the sum of a discrete measure and of the Haar measure. □

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