A Appendix A Dirac Algebra, Spinors, Pauli and Gell-Mann Matrices

A.1 Dirac Algebra and Spinors

A.1.1 Gamma Matrices

The Dirac matrices γ_μ with $\mu = (0, 1, 2, 3)$ satisfy anticommutation relation

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu} \,, \tag{A.1}$$

with the metric tensor defined as $g_{\mu\mu} = g^{\mu\mu} \sim (+, -, -, -)$. This implies Dirac algebra with 16 elements

$$S = \mathbb{1}_{4\times 4}, \quad P = \gamma_5, \quad V = \gamma_\mu, \quad A = \gamma_\mu \gamma_5, \quad T = \sigma_{\mu\nu} \,, \tag{A.2}$$

where[1]

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5, \qquad \sigma_{\mu\nu} \equiv \frac{i}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu). \tag{A.3}$$

In (A.2) S stands for *scalar*, P for *pseudoscalar*, V for *vector*, A for *axial-vector*, and T for *tensor*.

The explicit expressions for γ_μ and γ_5 can be found in (1.83) and any textbook for relativistic quantum field theory like [13, 14], but we will not need them here. The relations given following are sufficient for our purposes.

The γ_μ can be chosen unitary, if Hermitian conjugation is defined as

$$(\gamma^\mu)^\dagger = \gamma_\mu \;\Rightarrow\; (\gamma^\mu)^\dagger = g_{\mu\nu}\gamma^\nu. \tag{A.4}$$

We have then a set of useful properties:

$$(\gamma^0)^\dagger = g_{0\nu}\gamma^\nu = +\gamma^0, \qquad (\gamma^i)^\dagger = g_{i\nu}\gamma^\nu = -\gamma^i \,, \tag{A.5}$$

where $i = (1, 2, 3)$. Moreover,

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0, \quad (\gamma_0)^2 = 1, \quad (\gamma_i)^2 = -1 \,, \tag{A.6}$$

and very important relations involving γ_5

$$\gamma_5\gamma_\mu = -\gamma_\mu\gamma_5, \qquad \gamma_5^\dagger = \gamma^5 = \gamma_5 \,. \tag{A.7}$$

[1] In the literature one finds sometimes $\sigma_{\mu\nu}$ defined without the i factor.

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Next come chirality projectors ($A = L,\ R$)

$$P_{L,R} \equiv \frac{1}{2}(1 \mp \gamma_5), \quad (P_A)^\dagger = P_A, \qquad P_A\gamma_\mu = \gamma_\mu P_{-A}\,. \tag{A.8}$$

Here the notation $-A$ means simply $-L = R$ and $-R = L$.

Of importance is also the charge conjugation matrix

$$C = i\gamma^2\gamma^0, \tag{A.9}$$

which has a number of properties

$$C^T = -C, \quad (C^{-1})^T = -C^{-1}, \quad C^\dagger = C^{-1}, \quad CC^\dagger = 1, \tag{A.10}$$

$$C(\gamma^\mu)^T C^{-1} = -\gamma^\mu, \quad C(\gamma_5)^T C^{-1} = +\gamma_5, \quad C(\gamma^0)^* C^\dagger = -\gamma^0, \quad C(P_A)^T = P_A C. \tag{A.11}$$

A.1.2 Dirac Spinors

Dirac adjoint spinor is given as usual by

$$\overline{\psi} = (\psi)^\dagger\gamma_0. \tag{A.12}$$

Spinors with chiral fields ($A = L, R$) are given by

$$\psi_A \equiv P_A\psi, \qquad\qquad \overline{\psi_A} = \overline{\psi}P_{-A} \tag{A.13}$$

and the charge-conjugated spinors by

$$\psi^c \equiv C\overline{\psi}^T, \qquad (\psi^c)^c = \psi, \tag{A.14}$$

with useful properties

$$\overline{\psi^c} = -\psi^T C^\dagger, \qquad \psi^c_A = P_{-A}\psi^c, \qquad \overline{\psi^c_A} = \overline{\psi^c}P_A. \tag{A.15}$$

Furthermore, Hermitian conjugation of spinors gives

$$\psi^\dagger = \overline{\psi}\gamma_0, \quad \overline{\psi}^\dagger = \gamma_0\psi, \quad (\psi^c)^\dagger = \overline{\psi^c}\gamma_0, \quad \overline{\psi^c}^\dagger = \gamma_0\psi^c\,. \tag{A.16}$$

A.2 Pauli and Gell-Mann Matrices

The Pauli matrices σ^a are given as follows

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.17}$$

The Gell-Mann matrices λ^a satisfying

$$\mathrm{Tr}(\lambda^a\lambda^b) = 2\delta^{ab} \tag{A.18}$$

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are given as follows

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{A.19}$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \tag{A.20}$$

$$\lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{A.21}$$

A.3 Fierz Identities

Fierz identities [1307], often called also Fierz relations or Fierz transformations, allow to transfer a given chain of spinors into a another one of which matrix elements between external states are easier to evaluate than of the original chain. It should be stressed that the usual Fierz identities are only valid in $D = 4$ dimensions. If the calculations are performed in $D \neq 4$ dimensions, these identities have to involve evanescent operators. We refer to [744] for details warning the reader that there $\sigma_{\mu\nu} \equiv [\gamma_\mu, \gamma_\nu]/2$, which implies additional $i^2 = -1$ in the terms involving contractions of two $\sigma_{\mu\nu}$. Here we stick to $D = 4$ and $\sigma_{\mu\nu}$ as defined in (A.3).

In what follows we will list three classes of Fierz identities that take into account the fact that we are dealing with Grassmann-valued operators. This means that they include additional minus signs whenever the fermion fields are anticommuted odd times. This is the case of classes A and B but not C, where the fields are anticommuted twice. In the literature such identities are usually given for spinors, but it is more useful to have them for fermion fields. The results presented here use the information presented in [744, 1308, 1309] and have also been derived by Christoph Bobeth. Useful information can also be found in [1310]. In all these formulas P_A and P_B stand for projectors in (A.8) but in a given relation $P_A \neq P_B$. This means that $P_B = P_{-A}$ or more explicitly if $P_A = P_L$, then $P_B = P_R$ and vice versa.

Class A

These are the original Fierz identities of type $(12)(34) \to (14)(32)$ in which the exchange of fermion fields $2 \leftrightarrow 4$ (or equivalently $1 \leftrightarrow 3$) takes place. We have then

$$(\bar{\psi}_1 P_A \psi_2)(\bar{\psi}_3 P_A \psi_4) = -\frac{1}{2}(\bar{\psi}_1 P_A \psi_4)(\bar{\psi}_3 P_A \psi_2) - \frac{1}{8}(\bar{\psi}_1 \sigma_{\mu\nu} P_A \psi_4)(\bar{\psi}_3 \sigma^{\mu\nu} P_A \psi_2), \tag{A.22}$$

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$$(\bar{\psi}_1 P_A \psi_2)(\bar{\psi}_3 P_B \psi_4) = -\frac{1}{2}(\bar{\psi}_1 \gamma_\mu P_B \psi_4)(\bar{\psi}_3 \gamma^\mu P_A \psi_2), \tag{A.23}$$

$$(\bar{\psi}_1 \gamma_\mu P_A \psi_2)(\bar{\psi}_3 \gamma^\mu P_A \psi_4) = (\bar{\psi}_1 \gamma_\mu P_A \psi_4)(\bar{\psi}_3 \gamma^\mu P_A \psi_2), \tag{A.24}$$

$$(\bar{\psi}_1 \gamma_\mu P_A \psi_2)(\bar{\psi}_3 \gamma^\mu P_B \psi_4) = -2(\bar{\psi}_1 P_B \psi_4)(\bar{\psi}_3 P_A \psi_2), \tag{A.25}$$

$$(\bar{\psi}_1 \sigma_{\mu\nu} P_A \psi_2)(\bar{\psi}_3 \sigma^{\mu\nu} P_A \psi_4) = -6(\bar{\psi}_1 P_A \psi_4)(\bar{\psi}_3 P_A \psi_2) + \frac{1}{2}(\bar{\psi}_1 \sigma_{\mu\nu} P_A \psi_4)(\bar{\psi}_3 \sigma^{\mu\nu} P_A \psi_2), \tag{A.26}$$

$$(\bar{\psi}_1 \sigma_{\mu\nu} P_A \psi_2)(\bar{\psi}_3 \sigma^{\mu\nu} P_B \psi_4) = 0\,. \tag{A.27}$$

Class B

These are Fierz transformations *with* charge-conjugated fermion fields in which after Fierz transformation charged-conjugated fields are absent, which simplifies the calculations. The transformation is given by $(12^c)(3^c4) \rightarrow (13)(24)$ so that the fields ψ_1 and ψ_4 do not move, and ψ_2 and ψ_3 are not only interchanged but also charged conjugation on them is removed. We have then

$$(\bar{\psi}_1 P_A \psi_2^c)(\bar{\psi}_3^c P_A \psi_4) = -\frac{1}{2}(\bar{\psi}_1 P_A \psi_3)(\psi_2 P_A \psi_4) + \frac{1}{8}(\bar{\psi}_1 \sigma_{\mu\nu} P_A \psi_3)(\bar{\psi}_2 \sigma^{\mu\nu} P_A \psi_4), \tag{A.28}$$

$$(\bar{\psi}_1 P_A \psi_2^c)(\bar{\psi}_3^c P_B \psi_4) = \frac{1}{2}(\bar{\psi}_1 \gamma_\mu P_B \psi_3)(\bar{\psi}_2 \gamma^\mu P_B \psi_4), \tag{A.29}$$

$$(\bar{\psi}_1 \gamma_\mu P_A \psi_2^c)(\bar{\psi}_3^c \gamma^\mu P_A \psi_4) = 2(\bar{\psi}_1 P_B \psi_3)(\bar{\psi}_2 P_A \psi_4), \tag{A.30}$$

$$(\bar{\psi}_1 \gamma_\mu P_A \psi_2^c)(\bar{\psi}_3^c \gamma^\mu P_B \psi_4) = (\bar{\psi}_1 P_A \psi_3)(\bar{\psi}_2 P_B \psi_4), \tag{A.31}$$

$$(\bar{\psi}_1 \sigma_{\mu\nu} P_A \psi_2^c)(\bar{\psi}_3^c \sigma^{\mu\nu} P_A \psi_4) = 6(\bar{\psi}_1 P_A \psi_3)(\bar{\psi}_2 P_A \psi_4) + \frac{1}{2}(\bar{\psi}_1 \sigma_{\mu\nu} P_A \psi_3)(\bar{\psi}_2 \sigma^{\mu\nu} P_A \psi_4)\,. \tag{A.32}$$

Class C

In some leptoquark models (for example, S_3) with charge-conjugated fields the transformation $(12^c)(3^c4) \rightarrow (14)(23)$ is useful. We have then

$$(\bar{\psi}_1 P_A \psi_2^c)(\bar{\psi}_3^c P_A \psi_4) = -\frac{1}{2}(\bar{\psi}_1 P_A \psi_4)(\bar{\psi}_2 P_A \psi_3) + \frac{1}{8}(\bar{\psi}_1 \sigma_{\mu\nu} P_A \psi_4)(\bar{\psi}_2 \sigma^{\mu\nu} P_A \psi_3), \tag{A.33}$$

$$(\bar{\psi}_1 P_A \psi_2^c)(\bar{\psi}_3^c P_B \psi_4) = \frac{1}{2}(\bar{\psi}_1 \gamma_\mu P_B \psi_4)(\bar{\psi}_2 \gamma^\mu P_B \psi_3), \tag{A.34}$$

$$(\bar{\psi}_1 \gamma_\mu P_A \psi_2^c)(\bar{\psi}_3^c \gamma^\mu P_A \psi_4) = -(\bar{\psi}_1 P_A \psi_4)(\bar{\psi}_2 P_B \psi_3), \tag{A.35}$$

$$(\bar{\psi}_1 \gamma_\mu P_A \psi_2^c)(\bar{\psi}_3^c \gamma^\mu P_B \psi_4) = -2(\bar{\psi}_1 P_B \psi_4)(\bar{\psi}_2 P_A \psi_3), \tag{A.36}$$

$$(\bar{\psi}_1 \sigma_{\mu\nu} P_A \psi_2^c)(\bar{\psi}_3^c \sigma^{\mu\nu} P_A \psi_4) = 6(\bar{\psi}_1 P_A \psi_4)(\bar{\psi}_2 P_A \psi_3) + \frac{1}{2}(\bar{\psi}_1 \sigma_{\mu\nu} P_A \psi_4)(\bar{\psi}_2 \sigma^{\mu\nu} P_A \psi_3)\,. \tag{A.37}$$

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