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The direct image of a vector bundle

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1. *Introduction.* It is well known that if M is a vector bundle over a space Y, and $r: X \to Y$ is a map, then there is an 'induced' vector bundle r^*M over X. It is also known that a vector bundle L over X determines a 'direct image' vector bundle $r_* L$ over Y provided that r is a finite covering map ((1), § 1).

In algebraic and analytic geometry r_* can be defined for an arbitrary proper map $r: X \to Y$ by introducing coherent sheaves. It is then possible to reconstruct spaces from the sheaves which occur; the main examples are the linear spaces of Grauert ((3), § 3) and the fibred varieties of Grothendieck ((4), 1·7). These are like vector bundles, except that their dimension varies upper semicontinuously.

In this paper we consider only the category of topological spaces and maps, and work with spaces rather than sheaves. For this purpose we define 'normed linear bundles': these are similar to the linear spaces of Grauert except that fibre dimension may vary arbitrarily. The direct image functor r_* is defined for maps $r: X \to X/G$ which arise from the action of a compact group G, and reduces to the known definition in the case that G acts freely. In addition, a normed linear bundle M/G over X/G is associated to every normed G-linear bundle M over X.

In order to deal with compact, and not only finite, groups G it is necessary to consider linear bundles with infinite dimensional fibres, and this makes the theory clearer even when G is finite. In the finite case the linear bundle $r_* 1$ is used to give partial answers to the problem: for which vector bundles E over X/G is there an embedding $X \subset E$ compatible with projections onto X/G?

Throughout this paper K denotes either the real or complex field. A measure μ on a compact space X is a continuous linear functional on the space K^X of K-valued functions on X, and we denote the element $\langle f, \mu \rangle \in K$ by the traditional notation

$$\int_X f(x)\, d\mu.$$

The content of the remaining paragraphs is as follows: 2. Basic properties of linear bundles; 3. Definition of the linear bundles $r_* M$ and M/G; 4. Proof that r^* and r_* are adjoint functors and construction of related canonical isomorphisms; 5. The relation between the linear bundle $r_* 1$ and maps of X into vector bundles E over X/G.

2. *Linear bundles.* In this paragraph the definitions and basic properties of linear bundles are summarized. We use the words linear and quasi-linear in a sense similar to that of Grauert ((3), § 3).

2·1. Let X be a topological space. A *space* over X consists of a topological space V and a continuous function $p: V \to X$. The subspace $V_x = p^{-1}(x)$ of V is called the *fibre* of V at

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the point $x \in X$. A *section* of V is a continuous function $s: X \to V$ such that $s(x) \in V_x$ for all $x \in X$. If $p_1: V_1 \to X$ and $p_2: V_2 \to X$ are spaces over X define

$$V_1 \times_X V_2 = \{(v_1, v_2) \in V_1 \times V_2;\ p_1(v_1) = p_2(v_2)\} \subset V_1 \times V_2.$$

There are continuous functions

$$r_1: V_1 \times_X V_2 \to V_1 \text{ defined by } r_1(v_1, v_2) = v_1,$$

$$r_2: V_1 \times_X V_2 \to V_2 \text{ defined by } r_2(v_1, v_2) = v_2,$$

$$r: V_1 \times_X V_2 \to X \text{ defined by } r(v_1, v_2) = p_1(v_1) = p_2(v_2),$$

and so $V_1 \times_X V_2$ can be regarded as a space over V_1, V_2 or X. We have

$$(V_1 \times_X V_2)_x = (V_1)_x \times (V_2)_x$$

for all $x \in X$.

If $p: V \to X$ and $q: W \to X$ are spaces over X then a *map* $\phi: W \to V$ is a continuous function such that $p\phi = q$. If $\phi_1: W_1 \to V_1$ and $\phi_2: W_2 \to V_2$ are maps of spaces over X then there is a natural map $\phi_1 \times \phi_2: W_1 \times_X W_2 \to V_1 \times_X V_2$.

The space $p: X \times K \to X$, where p is the product projection, is denoted by 1; the space $p: X \times K^n \to X$ by $\mathbf{1}_n$. We refer to elements of K as *scalars*.

2·2. A *quasi-linear bundle* V over X is a space $p: V \to X$, together with a linear structure on each fibre V_x, such that

(1) the function $V \times_X V \to V$ defined by vector addition is continuous,

(2) the function $1 \times_X V \to V$ defined by scalar multiplication is continuous.

A *map* of quasi-linear bundles V, W over X is a map $\phi: W \to V$ of spaces over X which is linear on each fibre. Every topological vector space E over K defines a quasi-linear bundle $p: X \times E \to X$, where p is the product projection. This quasi-linear bundle is denoted by $\mathbf{E}$. If V_1, V_2 are quasi-linear bundles over X then $r: V_1 \times_X V_2 \to X$ is a quasi-linear bundle over X, which is denoted by $V_1 \oplus V_2$.

Let $p_1: X_1 \to X$ be a space over X, and $p_2: V \to X$ a quasi-linear bundle over X. Then $r_1: X_1 \times_X V \to X_1$ is a quasi-linear bundle over X_1 which is denoted by $p_1^* V$. A map $\phi: W \to V$ defines, as in 2·1, a map $p_1^*\phi = 1 \times \phi: p_1^* W \to p_1^* V$. If $i: Y \to X$ is the inclusion of a subspace Y of X then $i^*\phi$ is just the restriction of ϕ to Y; it is also denoted by $\phi \mid Y: W \mid Y \to V \mid Y$, or, by $\phi_x: W_x \to V_x$ if $Y = \{x\}$.

If ϕ_x is one-one for all $x \in X$, we call ϕ an *injection*; if also W has the induced topology we call W a *subbundle* of V. If ϕ_x is onto for all $x \in X$, we call ϕ a *surjection*; if also V has the identification topology we call V a *quotient bundle* of W. If ϕ_x is one-one and onto for all $x \in X$, we call ϕ_x a *bijection*; if also there exists a map $\psi: V \to W$ such that $\phi\psi$ and $\psi\phi$ are the identity, we call ϕ an *isomorphism*. Clearly if $\phi: W \to V$ is a map of quasi-linear bundles over X then the *kernel* $\ker \phi = \{w \in W;\ \phi(w) = 0 \in V_{q(w)}\}$ is a subbundle of W, and the *image* $\operatorname{im} \phi = \{v \in V;\ v = \phi(w) \text{ for some } w \in W\}$ is a subbundle of V.

2·3. A quasi-linear bundle V over X is a *linear bundle* if it is Hausdorff and if each point $x_0 \in X$ has a neighbourhood B_0 such that $V \mid B_0$ is isomorphic to a subbundle of $B_0 \times F_0$ for some topological vector space F_0 over K. We say that V is a *normed linear bundle* if each fibre V_x is equipped with a norm, and if neighbourhoods B_0 can be found as above so that the norm on $B_0 \times F_0$ is induced from a norm on F_0.

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The following properties of linear bundles are easily proved:

(1) the function $z\colon X \to V$ defined by $z(x) = 0 \in V_x$ is continuous (and called the *zero section* of V),

(2) $Z = z(X)$ is a closed subspace of V,

(3) if $\phi\colon W \to V$ is a map of linear bundles then $\ker \phi$ is a closed subbundle of W.

On the other hand the following example shows that $\operatorname{im} \phi$ need not be a closed subspace of W. Let $W = K \times K$ be the linear bundle over K given by the projection $p\colon K \times K \to K$ onto the first factor, and let $\phi\colon W \to W$ be defined by $(x, y) \longmapsto (x, xy)$. Then

$$\ker \phi = \{(x, y);\ x = 0 \text{ or } y = 0\},$$

$$\operatorname{im} \phi = \{(x, y);\ x \neq 0 \text{ or } x = y = 0\}.$$

LEMMA 2·3·1. *Let X be a Hausdorff space, E a finite-dimensional vector space, and W a closed subbundle of $X \times E$. Then*

(i) *the set $\{x \in X;\ W_x \neq 0\}$ is closed,*

(ii) *if F is a subspace of E, and $W|A \subset A \times F$ for some set $A \subset X$, there is an open set B containing A such that $W|B \subset B \times F$.*

Proof. (i) Let $S = \{e \in E;\ \|e\| = 1\}$ and suppose that $W_{x_0} = 0$. Then each point $(x_0, e_i) \in x_0 \times S$ has an open neighbourhood $B_i \times U_i \subset X \times E$ which does not intersect W. Since S is compact it is covered by a finite number of neighbourhoods $U_1, \ldots, U_n$. Then $B = B_1 \cap \ldots \cap B_n$ is an open neighbourhood of x_0 such that $W_x = 0$ for all $x \in B$.

(ii) Let $E \to F$ be a projection operator and consider the composite map

$$\phi\colon W \to X \times E \to X \times F.$$

Then $\ker \phi$ is a closed subspace of $X \times E$ and $(\ker \phi)_x = 0$ for $x \in A$. By (i) there is an open set B containing A such that $(\ker \phi)_x = 0$ for $x \in B$.

COROLLARY 2·3·2. *Let E, W, X be as above. If $\dim W_x$ is constant then W is a vector bundle.*

Proof. For each $x \in X$ apply (ii) with $A = \{x\}$ and $F = W_x$.

COROLLARY 2·3·3. *Let E, W, X be as above. Then $\dim W_x$ is upper semicontinuous (i.e. the set $X_i = \{x \in X;\ \dim W_x < i\}$ is open).*

Proof. For each $x \in X_i$ apply (ii) with $A = \{x\}$.

COROLLARY 2·3·4. *Let E, X be as above, and $\phi\colon \mathbf{E} \to V$ a map of linear bundles over X. Then $\dim (\ker \phi)_x$ is upper semicontinuous and $\dim (\operatorname{im} \phi)_x$ is lower semicontinuous.*

2·4. Let X be a compact Hausdorff space, and $p\colon V \to X$ a linear bundle over X. Then there is a finite open covering $\{X_i\}_{i=1,\ldots,n}$ of X, and injections

$$\psi_i\colon V|X_i \to X_i \times F_i,$$

where F_i is a topological vector space. Let $\{\lambda_i\}$ be a continuous partition of unity for $\{X_i\}$ with

$$\lambda_i(x) \geqslant 0 \quad \text{and} \quad \sum_{i=1}^{n} \lambda_i(x) = 1,$$

and consider the map $\qquad \psi\colon V \to X \times (F_1 \oplus \ldots \oplus F_n)$

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defined by $\psi(v) = p(v) \times (\lambda_1(p(v))\,\psi_1(v), \ldots, \lambda_n(p(v))\,\psi_n(v))$.

Clearly ψ is an injection $V \to \mathbf{F}$ where $F = F_1 \oplus \ldots \oplus F_n$.

If V is a normed linear bundle then the covering $\{X_i\}$ can be chosen so that each F_i is a normed vector space, and so that the norm on $V|X_i$ is induced by the norm on F_i. Then F is a normed vector space with norm

$$\|(f_1, \ldots, f_n)\| = \sum_{i=1}^{n} \|f_i\|$$

and therefore

$$\|\psi(v)\| = \sum_{i=1}^{n} |\lambda_i(p(v))| \,.\, \|\psi_i(v)\| = \sum_{i=1}^{n} \|v\| \,.\, \lambda_i(p(v)) = \|v\|.$$

Thus we may without loss of generality assume that, if V is a normed linear bundle, there is an injection $\psi\colon V \to \mathbf{F}$ such that the norm of V is induced by the norm of F.

Now consider the following standard construction. Let F be a normed vector space, and let

$$\|\phi\| = \sup\{|\phi(f)|\,;\, \|f\| = 1\}$$

be the norm on the topological dual F' of F. Let E be the set of continuous functions $e\colon X \to F'$. Then E is a vector space and, since X is compact, E has a norm

$$\|e\| = \sup\{\|e(x)\|\,;\, x \in X\} = \sup\{|e(x)(f)|\,;\, x \in X, \|f\| = 1\}.$$

The topological dual D of E is again a normed vector space with

$$\begin{aligned} \|d\| &= \sup\{|d(e)|\,;\, e \in E, \|e\| = 1\} \\ &= \sup\{|d(e)|\,;\, e \in E, |e(x)(f)| \leqslant \|f\| \text{ for all } x \in X\}. \end{aligned}$$

Consider the function $\theta\colon \mathbf{F} \to \mathbf{D}$ defined by $\theta(x,f)(e) = e(x)(f)$. Clearly θ is one-one. Moreover

$$\|\theta(x,f)\| = \sup\{|e(x)(f)|\,;\, e \in E, |e(x)(f)| \leqslant \|f\|\} = \|f\|$$

and therefore the norm on $\mathbf{F}$ is induced by the norm on $\mathbf{D}$. An open neighbourhood in $\mathbf{D}$ of the form $X_0 \times D_0$ (where X_0 is open in X and $D_0 = \{d \in D;\, \|d\| < \lambda\}$) has counter-image

$$\theta^{-1}(X_0 \times D_0) = \{(x,f) \in \mathbf{F};\, x \in X_0, \|f\| < \lambda\}$$

and therefore θ is continuous. The injection $\theta\colon \mathbf{F} \to \mathbf{D}$ has the property:

LEMMA 2·4·1. *Let μ be a measure on a closed subspace Y of X, and consider a section $y \longmapsto y, s(y)$ of $\mathbf{F}|Y$. Suppose that $\theta(y, s(y)) = (y, t(y)) \in \mathbf{D}$, and $\int_Y t(y)\,d\mu = 0$. Then the function $s\colon Y \to F$ is zero on the support of μ.*

Proof. If

$$\int_Y t(y)\,d\mu = 0,$$

then by (2), Chap. III, § 3 it follows that

$$\int_Y \langle e(y), s(y)\rangle\,d\mu = 0$$

for all $e \in E$. In particular consider elements $e \in E$ of the form $e(x) = \lambda(x).\phi$ where $\lambda\colon X \to K$ is continuous and $\phi \in F'$. Then

$$\int_Y \lambda(y) \,.\, \langle \phi, s(y)\rangle\,d\mu = \int_Y \langle \lambda(y)\,\phi, s(y)\rangle\,d\mu = 0$$

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for all $\mu\colon X \to K$ and $\phi \in F'$. It follows from (2), Chap. III, § 2, that $\langle \phi, s(y)\rangle = 0$ for all $\phi \in F'$ and $y \in \operatorname{supp}\mu$, and hence that $s(y) = 0$ for all $y \in \operatorname{supp}\mu$.

The results of this section can be summarized in

THEOREM 2·4·2. *Let X be a compact Hausdorff space, and $p\colon V \to X$ a normed linear bundle over X. Then there is a normed vector space F, and an injection*

$$\psi\colon V \to \mathbf{F}$$

such that

(i) *the norm of V is induced by ψ from the norm of F;*

(ii) *if μ is a measure on a closed subspace Y of X and $s\colon Y \to V|Y$ is a section such that*

$$\psi(s(y)) = (y, t(y)) \in Y \times F \quad \text{and} \quad \int_Y t(y)\, d\mu = 0,$$

then s is zero on the support of μ.

Example. Suppose $V = 1$. Then F can be chosen as the topological dual of K^X; that is, the vector space of measures on X.

2·5. If V is a linear bundle over X, the set $\Gamma(V)$ of all sections of V (see 2·1) has the structure of a module over the ring K^X of continuous K-valued functions on X given by

$$(f_1 s_1 + f_2 s_2)(x) = f_1(x)\, s_1(x) + f_2(x)\, s_2(x),$$

where $f_1, f_2 \in K^X$, $s_1, s_2 \in \Gamma(V)$ and $x \in X$. Similarly, the set $\Gamma(W, V)$ of all maps $\phi\colon W \to V$ is a K^X-module. There is a natural isomorphism $\Gamma(V) \cong \Gamma(1, V)$ under which a section s corresponds to the map $(x, k) \longmapsto ks(x)$.

Now let X be a compact Hausdorff space, and $p\colon V \to X$ a normed linear bundle over X. Let $E = \Gamma(V)$ be the space of sections of V with the norm

$$\|s\| = \sup\{\|s(x)\|;\, x \in X\}.$$

There is a natural map $\rho\colon \mathbf{E} \to V$ defined by $\rho(x, s) = s(x)$. An open set

$$\{v \in V;\, p(v) \in X_0, \|v\| < \lambda\},$$

where X_0 is open in X, has counter-image $\{(x, s);\, x \in X_0, \|s(x)\| < \lambda\}$. This is an open set in $X \times E$ and therefore ρ is continuous.

We say that the normed linear bundle V *has sufficient sections* if ρ is a surjection. Clearly V has sufficient sections if and only if there exists a topological vector space D and a surjection $\rho\colon \mathbf{D} \to V$.

Let V be a normed linear bundle with sufficient sections. If $\rho\colon \mathbf{D} \to V$ is a surjection consider the subbundle

$$V' = \{(x, \theta);\, x \in X, \theta \in D', \theta(d) = 0 \text{ for all } d \in \ker\rho\}$$

of $\mathbf{D}'$. Here D' denotes the topological dual of D. The norm $\|\theta\| = \sup\{|\theta(d)|;\, \|d\| = 1\}$ induces a norm for V'. If $\psi\colon V \to \mathbf{F}$ is an injection the transpose of $\psi\rho\colon \mathbf{D} \to \mathbf{F}$ defines a map $\mathbf{F}' \to \mathbf{D}'$ with image V'. Thus there is a surjection $\psi'\colon \mathbf{F}' \to V'$ and an injection $\rho'\colon V' \to \mathbf{D}'$.

Now let V and W be normed vector bundles with sufficient sections, and $\phi\colon W \to V$ a map. If $\rho\colon \mathbf{D} \to V$ and $\sigma\colon \mathbf{E} \to W$ are surjections, and if $\psi\colon V \to \mathbf{F}$ is an injection,

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then the transpose of $\psi\phi\sigma\colon \mathbf{E} \to \mathbf{F}$ defines a map $\mathbf{F}' \to \mathbf{E}'$ with image W' which factors through V'. In this way we obtain a transpose map $\phi'\colon V' \to W'$. In particular, if ϕ is an isomorphism then so is ϕ'. Therefore the definition of the normed linear bundle V' does not depend on the choice of D. We call V' the linear bundle *dual* to V.

The definition implies that, if $p_1\colon X_1 \to X$ is a map then $(p_1^*V)' = p_1^*(V')$. In particular if Y is a subspace of X then $(V|Y)' = V'|Y$.

2.6. If L_1, L_2 are linear bundles over X then the *tensor product* $L_1 \otimes L_2$ can be defined in a natural way: if $L_1|U \subset I \times F_1$ and $L_2|U \subset U \times F_2$ for some open set $U \subset X$ then $(L_1 \otimes L_2)|U$ is the subspace of $U \times (F_1 \otimes F_2)$ generated by elements of the form $(x, v_1 \otimes v_2)$, where $(x, v_1) \in L_1$ and $(x, v_2) \in L_2$. We have immediately

LEMMA 2·6·1. *If $f\colon X' \to X$ is a map then $f^*(L_1 \otimes L_2) = f^*L_1 \otimes f^*L_2$.*

COROLLARY 2·6·2. *If $Y \subset X$ is a subspace then*

$$(L_1 \otimes L_2)|Y = (L_1|Y) \otimes (L_2|Y).$$

3. *Linear bundles over G-spaces.* From now on we assume that G is a compact topological group with identity element $e \in G$, and that X is a compact Hausdorff G-space. That is, there is a continuous action

$$G \times X \to X, \quad (g, x) \longmapsto gx$$

such that $(g_1 g_2)x = g_1(g_2 x)$ and $ex = x$. Let $r\colon X \to X/G$ denote the identification map.

3·1. A *G-quasi-linear bundle* over X is a quasi-linear bundle $p\colon V \to X$, together with a continuous action of G on V which is linear and compatible with the action on X. Thus an element $g \in G$ induces a linear map $g\colon V_x \to V_{gx}$ for each $x \in X$. Let G_x be the stability subgroup $\{g \in G;\ gx = x\}$ at x, and consider the quasi-linear subbundle

$$\tilde{V} = \{v \in V;\ gv = v \text{ for all } g \in G_{p(v)}\}$$

of V. The induced map $g\colon \tilde{V}_x \to \tilde{V}_{gx}$ is the identity for $g \in G_x$. For general $g \in G$ it is an isomorphism with inverse $g^{-1}\colon \tilde{V}_{gx} \to \tilde{V}_x$. Therefore the quotient is a quasi-linear space over X/G which we denote by V/G.

LEMMA 3·1·1. *If $r\colon X \to X/G$ is the identification map then $r^*(V/G) = \tilde{V}$.*

COROLLARY 3·1·2. *If G acts freely on X then $\tilde{V} = V$ and hence $r^*(V/G) = V$.*

LEMMA 3·1·3. *If $\phi\colon W \to V$ is a G-map there is an induced map $\phi/G\colon W/G \to V/G$.*

COROLLARY 3·1·4. *If $Y \subset X$ is a G-subspace then $(V/G)|Y = (V|Y)/G$.*

3·2. Now suppose that $q\colon L \to X$ is a Hausdorff quasi-linear bundle over X. We shall show how to associate to L a G-quasi-linear bundle $V(L)$ over X. Let L^G denote the space of continuous maps $v\colon G \to L$ with the compact open topology. Consider the subspace

$$V(L) = \{v\colon G \to L;\ qv\colon G \to X \text{ is a } G\text{-map}\} \subset L^G.$$

Define a map $p\colon V(L) \to X$ by $p(v) = q(v(e))$ where $e \in G$ is the identity. If $v_1, v_2 \in V(L)_x$ then $v_1(g), v_2(g) \in L_{gx}$ for all $g \in G$ and therefore $\lambda_1 v_1 + \lambda_2 v_2$ is a well-defined element of $V(L)_x$ for $\lambda_1, \lambda_2 \in K$. Thus $V(L)_x$ admits a linear structure for each $x \in X$.

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The function $V(L) \times_X V(L) \to V(L)$ defined by vector addition is continuous because the corresponding function $L^G \times L^G \to L^G$ is continuous. The function

$$1 \times_X V(L) \to V(L)$$

defined by scalar multiplication is continuous because the corresponding function $K \times L^G \to L^G$ is continuous. Therefore $V(L)$ is a Hausdorff quasi-linear space over X. Define an action of G on $V(L)$ by

$$hv(g) = v(gh) \quad (g, h \in G).$$

Then $k(hw)(g) = hw(gk) = w(gkh) = (kh)w(g)$ so that $k(hw) = (kh)w$ for $h, k \in G$. Moreover, if $g, h \in G$ then

$$p(hv) = q(hv(e)) = q(v(h)) = hq(v(e)) = hp(v)$$

and so $V(L)$ is a G-quasi-linear bundle over X.

LEMMA 3·2·1. $$V(\mathbf{F}) = X \times F^G.$$

Proof. Any map $G \to X \times F$ in $V(\mathbf{F})$ has the form $g \longmapsto gx, v(g)$ where $x \in X$ and $v \in F^G$. This defines a map $V(\mathbf{F}) \to X \times F^G$ which is clearly an isomorphism.

LEMMA 3·2·2. $$V(L_1 \oplus L_2) = V(L_1) \oplus V(L_2).$$

Proof. Any map $G \to (L_1 \oplus L_2)$ in $V(L_1 \oplus L_2)$ must have the form $g \longmapsto v_1(g) \oplus v_2(g)$ where $v_1 : G \to L_1$, $v_2 : G \to L_2$ are elements of $V(L_1)$, $V(L_2)$ respectively. This defines a map $V(L_1 \oplus L_2) \to V(L_1) \oplus V(L_2)$ which is clearly an isomorphism.

LEMMA 3·2·3. *If X' is a Hausdorff G-space and $j : X' \to X$ is a G-map then*

$$j^* V(L) = V(j^* L).$$

Proof. An element of $V(j^*L)$ is a map $v' : G \to X' \times L$ such that $g \longmapsto gx', v(g)$ for some $x' \in X'$ and $v \in V(L)$ with $v(e) = j(x')$. This defines a map $V(j^*L) \to j^*V(L)$ which is clearly an isomorphism.

COROLLARY 3·2·4. *If $X' \subset X$ is a G-subspace of X then $V(L|X') = V(L)|X'$.*

LEMMA 3·2·5. *A map $\phi : L' \to L$ defines a G-map $V(L') \to V(L)$.*

Proof. Consider composite maps $G \to L' \to L$.

LEMMA 3·2·6. *If G is a finite group then $V(L) = \bigoplus_{g \in G} g^* L$, with the action of G defined by permutation of summands.*

Proof. An element of $V(L)$ is a continuous map $v : G \to L$ such that $qv(g) = gqv(e)$. If G is finite the set of such maps with $qv(e) = x$ is the vector space $\bigoplus_{g \in G} L_{gx}$. This defines a map $V(L) \to \bigoplus_{g \in G} g^* L$ which is clearly an isomorphism.

3·3. We may combine the constructions of 3·1 and 3·2 to define, for every Hausdorff quasi-linear space L over X, the quasi-linear space

$$r_* L = V(L)/G$$

over X/G. We call $r_* L$ the *direct image* of L. The properties of V/G (see 3·1) and $V(L)$ (see 3·2), imply

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LEMMA 3·3·1. *$r_* \mathbf{F}$ consists of pairs (x, v) such that $x \in X$, $v \in F^G$ and $v(gh) = v(g)$ for all $g \in G$, $h \in G_x$, and subject to the equivalence relation*

$$(x, g \longmapsto v(g)) \sim (hx, g \longmapsto v(gh)) \text{ for } g, h \in G.$$

In particular $(r_* \mathbf{F})_{r(x)}$ is the space F^{G/G_x} of maps from G/G_x to F.

LEMMA 3·3·2. $$f_*(L_1 \oplus L_2) = f_* L_1 \oplus f_* L_2.$$

LEMMA 3·3·3. *If X' is a Hausdorff G-space and $j: X' \to X$ is a G-map there is a commutative diagram*

$$\begin{array}{ccc} X' & \xrightarrow{\ j\ } & X \\ \downarrow{\scriptstyle r'} & & \downarrow{\scriptstyle r} \\ X'/G & \xrightarrow{\ j/G\ } & X/G \end{array}$$

such that $(j/G)^* r_* L = r'_* j^* L$.

COROLLARY 3·3·4. *If $X' \subset X$ is a G-subspace then $r'_*(L|X') = (r_* L)|X'/G$.*

LEMMA 3·3·5. *A map $\phi: L' \to L$ induces a map $r_* \phi: r_* L' \to r_* L' \to r_* L$.*

LEMMA 3·3·6. *If G is a finite group then $r_* L$ is the quotient under the action of G of the quasi-linear space*

$$r^* r_* L = \{(l_1, l_2, \ldots) \in L_{g_1 x} \oplus L_{g_2 x} \oplus \ldots; \text{ if } g_i x = g_j x \text{ then } l_i = l_j\}.$$

If also G acts freely then

$$r^* r_* L = \bigoplus_{g \in G} g^* L,$$

$$(r_* L)_{r(x)} = \bigoplus_{z \in r^{-1} r(x)} L_z.$$

3·4. In this section we consider the construction of 3·3 when L is a normed linear bundle over X. Then there is a Banach space F and an injection

$$\psi: L \to \mathbf{F}$$

which satisfies conditions (i) and (ii) of Theorem 2·4·2. By 3·2·1 and 3·2·5 there is an injection $V(L) \to X \times F^G$ which commutes with the action of G. Moreover, $V(L)$ is a normed linear bundle over X with norm

$$\|v\| = \sup \{\|v(g)\|; g \in G\}$$

induced by the same norm on F^G. Now consider the composite map

$$V(L) \to X \times F^G \to X \times F$$

induced by the map $F^G \to F$ such that $v \longmapsto \int_G v(g)\, d\mu$. Here μ denotes the Haar measure on G such that $\int_G d\mu = 1$, so that the map can be regarded as a projection operator onto the subspace

$$F = \{v \in F^G; v(g) = v(gh) \text{ for all } h \in G\}$$

of F^G. The action of G on the corresponding subbundle $\mathbf{F}$ of $X \times F^G$ is trivial.

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Consider the linear subbundle

$$W = \{v \in V(L);\ v(gh) = v(g) \text{ for } g \in G,\ h \in G_{p(v)}\}.$$

An element of the fibre W_x of W over $x \in X$ is a section of L over the orbit G/G_x of x. The Haar measure on G induces a measure on the compact space G/G_x. By Theorem 2·4·2 (ii), the composite map

$$W \to V(L) \to X \times F^G \to \mathbf{F}$$

is an injection. By definition $W/G = V(L)/G$. On the other hand $\mathbf{F}/G = \mathbf{F}$. Therefore there is an injection

$$V(L)/G \to \mathbf{F}.$$

In particular, we deduce

THEOREM 3·4·1. *Let $r: X \to X/G$ be the identification map, and L a normed linear bundle over X. Then $r_* L$ is a normed linear bundle over X/G.*

Now suppose that $\phi: L \to M$ is a map of normed linear bundles over X. Then there are natural maps

$$V(\phi): V(L) \to V(M),$$

$$r_* \phi: r_* L \to r_* M.$$

LEMMA 3·4·2. *There is an isomorphism* $\ker r_* \phi \to r_*(\ker \phi)$.

Proof. By definition an element of $\ker r_* \phi$ is represented by a map $v: G \to L$ such that $\phi(v(g)) = 0$ for all $g \in G$. Therefore v maps G to $\ker \phi$ and determines an element of $r_*(\ker \phi)$. The converse is immediate.

COROLLARY 3·4·3. *If ϕ is an injection, so is $r_* \phi$.*

LEMMA 3·4·4. *There is an injection* $\operatorname{im} r_* \phi \to r_*(\operatorname{im} \phi)$, *which is an isomorphism if G is finite and M is finite dimensional.*

Proof. An element of $\operatorname{im} r_* \phi$ is represented by a map $w: G \to M$ which factors through maps $G \to L \to M$. Therefore w can be regarded as a map $G \to \operatorname{im} \phi$ and determines an element of $r_*(\operatorname{im} \phi)$. This map is clearly an injection; in the case that G is finite and M is finite dimensional we have $\dim(\operatorname{im} r_* \phi) = \dim r_*(\operatorname{im} \phi)$ and therefore obtain an isomorphism.

THEOREM 3·4·5. *Let L be a normed G-linear bundle over X. Then L/G is a normed linear bundle over X/G.*

Proof. There is a natural map $\phi: L \to V(L)$

defined by $\phi(l) = [g \longmapsto gl]$. Clearly $\phi(hl) = [g \longmapsto ghl] = h\phi(l)$ so that ϕ is a G-map. Also ϕ is an injection. Therefore the induced map

$$\phi/G: L/G \to V(L)/G$$

is also an injection. Since $V(L)/G$ is a normed linear bundle, so is L/G.

Remark. Even if V is a closed linear subspace of $\mathbf{F}$ there is no reason in general why V/G should be, even locally, closed. For example, let $V = K \times K$ be as in 2·3, and let $G = Z_2$ act by $(x, y) \longmapsto (-x, -y)$. Then

$$\tilde{V} = \{(x, y);\ \text{either } x \neq 0 \text{ or } x = y = 0\} \subset V,$$

$$X/G = \{x;\ x \geqslant 0\},$$

$$V/G = \{(x, y);\ \text{either } x > 0 \text{ or } x = y = 0\}.$$

Thus V/G is a linear subspace of $(X/G) \times K$ but not closed.

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3·5. Let L_1, L_2 be normed linear bundles L_1, L_2 over X. The tensor product $L_1 \otimes L_2$ defined in 2·6 has a norm induced by the norm

$$\|f\| = \inf\{\sum_i \|f_{1,i}\| \cdot \|f_{2,i}\| ; f = \sum_i f_{1,i} \otimes f_{2,i}\}$$

on $F_1 \otimes F_2$. In contrast to the corresponding property for r^*, the linear bundles $r_*(L_1 \otimes L_2)$ and $r_* L_1 \otimes r_* L_2$ are in general quite different.

There is a natural map

$$V(L_1) \otimes V(L_2) \to V(L_1 \otimes L_2)$$

defined on generators by $v_1 \otimes v_2 \to [g \longmapsto v_1(g) \otimes v_2(g)]$ which induces a natural map

$$r_* L_1 \otimes r_* L_2 \to r_*(L_1 \otimes L_2).$$

Clearly each of these maps is a surjection.

LEMMA 3·5·1. *Let L_1 be a linear bundle over X, and L_2 a linear subbundle of $X \times F_2$ where F_2 is a finite dimensional vector space. Then $r_* L_1 \otimes r_* L_2 = r_*(L_1 \otimes L_2) \otimes r_* 1$.*

Proof. Let L_1 be a linear subbundle of $X \times F_1$ and consider the map

$$F_1^G \otimes F_2^G \to (F_1 \otimes F_2)^G \otimes K^G$$

defined by

$$v_1 \otimes v_2 \longmapsto \sum_{i=1}^{n} (v_1 \otimes e_i) \otimes p_i(v_2).$$

Here v_1 is a map $G \to F_1$, v_2 is a map $G \to F_2$, $\{e_1, \ldots, e_n\}$ is a basis for F_2 and

$$v_2 = \sum_{i=1}^{n} p_i(v_2)\, e_i.$$

This map is clearly continuous and defines a map

$$r_* \mathbf{F}_1 \otimes r_* \mathbf{F}_2 \to r_*(\mathbf{F}_1 \otimes \mathbf{F}_2) \otimes r_* 1$$

which is an isomorphism. This isomorphism defines, by restriction, an isomorphism

$$r_* L_1 \otimes r_* L_2 \to r_*(L_1 \otimes L_2) \otimes r_* 1.$$

LEMMA 3·5·2. *Let L be a linear bundle over X, and M a linear subbundle of $(X/G) \times F_2$, where F_2 is a finite dimensional vector space. Then $M \otimes r_* L = r_*(r^* M \otimes L)$.*

Proof. Let L be a linear subbundle of $X \times F_1$ and consider the map

$$F_2 \otimes F_1^G \to (F_1 \otimes F_2)^G,$$

$$m \otimes v_1 \longmapsto [g \longmapsto m \otimes v_1(g)].$$

This map is an isomorphism for F_2 finite dimensional. It induces a map

$$\mathbf{F}_2 \otimes r_* \mathbf{F}_1 \to r_*(\mathbf{F}_2 \otimes \mathbf{F}_1)$$

which is also an isomorphism, and which defines by restriction an isomorphism

$$M \otimes r_* L \to r_*(r^* M \otimes L).$$

3·6. Consider a normed vector space F.

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LEMMA 3·6·1. *The normed linear bundle $r_* \mathbf{F}$ over X/G has sufficient sections.*

Proof. A section of $r_* \mathbf{F}$ is a map $X \to (X/G) \times F$ such that $x \longmapsto (r(x), f(x))$ where $f\colon X \to F$ is a function. At $r(x)$ this section determines the element

$$g \longmapsto (gx, f(gx))$$

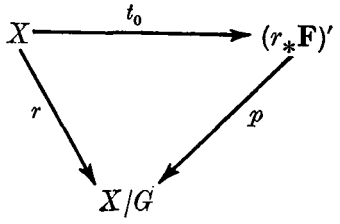
of $r_* \mathbf{F}$, and hence a map $\tilde{f}\colon (G/G_x) \to F$ of the orbit through x into F. Conversely, since X is normal, any function $\tilde{f}\colon (G/G_x) \to F$ can be extended to a function $f\colon X \to F$, and therefore arises from a section of $r_* \mathbf{F}$.

Since $r_* \mathbf{F}$ has sufficient sections, the dual bundle $(r_* \mathbf{F})'$ is well defined as in 2·5. In fact if $E = (F^X)'$ is the topological dual of F^X then

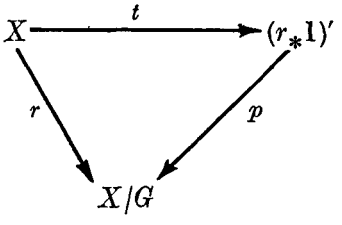
$$(r_* \mathbf{F})' = \{(r(x), \phi);\ x \in X, \phi \in E, \phi(f) = 0 \text{ for all } f\colon X \to F \text{ such that } f(gx) = 0 \text{ for all } g\}.$$

Now let $\phi_0 \in F'$ be a fixed element and consider the map $t_0\colon X \to (r_* \mathbf{F})'$ defined by $t_0(x) = (r(x), f \longmapsto \phi_0(f(x)))$ for $f\colon X \to F$. Clearly this is an embedding for $\phi_0 \neq 0$ and we have

LEMMA 3·6·2. *For each non-zero element $\phi_0 \in F'$ there is a commutative diagram*



where t_0 is an embedding and p is the projection. In particular when $F = K$ and $\phi_0 = 1$ there is a commutative diagram



where t is an embedding.

LEMMA 3·6·3. *Let μ be the Haar measure on G and let F be a Banach space. Then there is an injection $\psi\colon r_*(\mathbf{F}') \to (r_* \mathbf{F}')$ under which*

$$[r(x), g \longmapsto gx, v(g)] \longmapsto [r(x), f \longmapsto \int_G \langle v(g), f(gx) \rangle \, d\mu].$$

Proof. Since F is complete, an element $g \longmapsto gx, v(g)$ of $r_*(\mathbf{F}')$ is mapped by ψ to the element $r(x), v(g)\,\mu$ of $X \times (F^X)'$. Here $v(g)\,\mu$ is a measure with support contained in the orbit Gx. Now

$$\|v(g)\| < \lambda \Rightarrow \left| \int_G \langle v(g), f(gx) \rangle \, d\mu \right| < \lambda \left\| \int_G f(gx) \, d\mu \right\|$$
$$\Rightarrow \|v(g)\,\mu\| < \lambda$$

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and therefore the counter-image of an open set $r(X_0) \times E_0$, where X_0 is open in X and $E_0 = \{\phi \in (F^X)'; \|\phi\| < \lambda\}$, contains the set $\{[rx, g \longmapsto gx, v(g)]; x \in X_0, \|v(g)\| < \lambda\}$ which is open in $r_*(\mathbf{F})'$. Therefore ψ is continuous. It is then clear that ψ is an injection.

As a corollary we have

LEMMA 3·6·4. *If G is finite and F is finite dimensional then $r_*(\mathbf{F}') = (r_* \mathbf{F})'$.*

4. *Canonical maps.* Throughout this paragraph G is a compact topological group, X a compact Hausdorff G-space, $r: X \to X/G$ the identification map, $p: L \to X$ a normed linear bundle over X, and $q: M \to X/G$ a normed linear bundle over X/G. We seek relations between r_* (defined in 3·3) and r^* (defined in 2·2).

4·1. Consider the map

$$\alpha: M \to r_* r^* M$$

defined by $\alpha(m): g \longmapsto (gx, m)$ where $m \in M$ and $x \in X$ is some point such that

$$r(x) = q(m).$$

This is well defined since $g \longmapsto (gx, m)$ and $g \longmapsto (ghx, m)$ determine the same element of $r_* r^* M$. If $M \subset \mathbf{F}$ there is a commutative diagram

$$\begin{array}{ccc} M & \subset & \mathbf{F} \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ r_* r^* M & \subset & r_* \mathbf{F} \end{array}$$

where $\alpha: \mathbf{F} \to r_* \mathbf{F}$ maps $(r(x), f)$ to $g \longmapsto gx, f$. An open neighbourhood in $r_* \mathbf{F}$ of the form

$$\{g \longmapsto gx, f(g); x \in X_0, \|f(g)\| < \lambda\},$$

where X_0 is open in X, has counter-image

$$\{(r(x), f); r(x) \in X_0, \|f\| < \lambda\} \subset (X/G) \times F$$

and therefore α is continuous.

Now consider the normed linear bundle $r^* r_* L$. An element of $r^* r_* L$ can be represented by a pair (x, v) where $x \in X$, $v \in L^G$, $p(v(e)) = x$, and $v(gh) = v(g)$ for $h \in G_x$. Here $e \in G$ is the identity element. We can then define a map

$$\beta: r^* r_* L \to L$$

by $\beta(x, v) = v(e)$. This does not depend on the representation chosen because $v(h) = v(e)$ for $h \in G_x$. If $L \subset \mathbf{F}$ there is a commutative diagram

$$\begin{array}{ccc} r^* r_* L & \subset & r^* r_* \mathbf{F} \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\beta} \\ L & \subset & \mathbf{F} \end{array}$$

where $\beta: r^* r_* \mathbf{F} \to \mathbf{F}$ maps $(x, g \longmapsto gx, f(g))$ to $(x, f(e))$. An open neighbourhood in $\mathbf{F}$ of the form

$$\{(x, f); x \in X_0, \|f\| < \lambda\},$$

where X_0 is open in X, has counter-image

$$\{(x, g \longmapsto gx, f(g)); x \in X_0, \|f(e)\| < \lambda\} \subset r^* r_* \mathbf{F}$$

and therefore β is continuous.

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The maps α and β can be used to construct homomorphisms

$$A\colon \Gamma(r^*M, L) \to \Gamma(M, r_* L) \quad \text{and} \quad B\colon \Gamma(M, r_* L) \to \Gamma(r^*M, L)$$

where $\Gamma(W, V)$ is defined as in 1·3:

If $\phi\colon r^*M \to L$ is a map then $A(\phi)$ is the composite map $M \xrightarrow{\alpha} r_* r^*M \xrightarrow{r_*\phi} r_* L$.

If $\psi\colon M \to r_* L$ is a map then $B(\psi)$ is the composite map $r^*M \xrightarrow{r^*\psi} r^* r_* L \xrightarrow{\beta} L$.

LEMMA 4·1·1. *$BA\colon \Gamma(r^*M, L) \to \Gamma(r^*M, L)$ is the identity homomorphism.*

Proof. If $\phi\colon r^*M \to L$ is a map then $BA(\phi)$ is the composite map

$$r^*M \xrightarrow{r^*\alpha} r^* r_* r^*M \xrightarrow{r^* r_*\phi} r^* r_* L \xrightarrow{\beta} L$$

under which

$$(x, m) \longmapsto (x, g \longmapsto (gx, m)) \longmapsto (x, g \longmapsto \phi(gx, m)) \longmapsto \phi(x, m).$$

LEMMA 4·1·2. *$AB\colon \Gamma(M, r_* L) \to \Gamma(M, r_* L)$ is the identity homomorphism.*

Proof. If $\psi\colon M \to r_* L$ is a map then $AB(\psi)$ is the composite map

$$M \xrightarrow{\alpha} r_* r^*M \xrightarrow{r_* r^*\psi} r_* r^* r_* L \xrightarrow{r_*\beta} r_* L$$

under which

$$m \longmapsto (g \longmapsto (gx, m)) \longmapsto (g \longmapsto (gx, \psi(m))) \longmapsto (g \longmapsto \psi(m)(g)).$$

Here the explanation of the final arrow is that $\psi(m) \in r_* L$ is itself an equivalence class of maps $g \longmapsto l(g)$; the map β picks out an element $l(g)$ of L which lies in the fibre over gx, and this element is precisely $\psi(m)(g)$.

COROLLARY 4·1·3. *If $L = r^*M$ then $B(\alpha)$ is the identity map*

$$r^*M \xrightarrow{r^*\alpha} r^* r_* r^*M \xrightarrow{\beta} r^*M.$$

In particular $r^*\alpha$ is an injection, and β is a surjection when $L = r^*M$.

Proof. By Lemma 4·1·1 with ϕ the identity map.

COROLLARY 4·1·4. *If $M = r_* L$ then $A(\beta)$ is the identity map*

$$r_* L \xrightarrow{\alpha} r_* r^* r_* L \xrightarrow{r_*\beta} r_* L.$$

In particular $r_*\beta$ is a surjection, and α is an injection when $M = r_* L$.

Proof. By Lemma 4·1·2 with ψ the identity map.

As a corollary to 4·1·1 and 4·1·2 we see that r_* and r^* are adjoint functors:

LEMMA 4·1·5. $\Gamma(r^*M, L) \cong \Gamma(M, r_* L)$. *In particular* $\Gamma(L) \cong \Gamma(r_* L)$.

4·2. Recall that an element of $r_* r^*M$ is a map $g \longmapsto gx,\ m(g)$, subject to the equivalence relation $(g \longmapsto gx, m(g)) \sim (g \longmapsto ghx, m(gh))$.

Let μ be the Haar measure on G such that $\int_G d\mu = 1$, and assume that M is a Banach linear bundle over X/G (that is, it is a normed linear bundle in which every fibre is complete). Then there is another canonical map

$$\gamma\colon r_* r^*M \to M$$

defined by

$$\gamma(g \longmapsto gx, m(g)) = \int_G m(g)\, d\mu.$$

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LEMMA 4·2·1. *The composite map* $M \xrightarrow{\alpha} r_* r^*M \xrightarrow{\gamma} M$ *is the identity. In particular* γ *is a surjection, and there is a direct sum decomposition* $r_* r^*M = M \oplus (r_* r^*M)_0$.

Proof. If $m \in M$ then

$$m \longmapsto (g \longmapsto gx, m) \longmapsto \int_G m \, d\mu = m.$$

Note that in general the maps β: $r^*r_* r^*M \to r^*M$ and $r^*\gamma$: $r^*r_* r^*M \to r^*M$ are not the same, although Lemma 4·2·1 and Corollary 4·1·3 imply that they are the same on the subspace r^*M of $r^*r_* r^*M$.

The map γ can be used to define a homomorphism C: $\Gamma(L, r^*M) \to \Gamma(r_* L, M)$. If ϕ: $L \to r^*M$ is a map then $C(\phi)$ is the composite map

$$r_* L \xrightarrow{r_*\phi} r_* r^*M \xrightarrow{\gamma} M.$$

The special case $L = 1$ gives a homomorphism C: $\Gamma(r^*M) \to \Gamma(r_* 1, M)$. An element of $\Gamma(r^*M)$ is a section $x \longmapsto (x, s(x)), s(x) \in M$; the corresponding element of $\Gamma(r_* 1, M)$ maps (x, v), $x \in X$, $v \in K^G$, to

$$\int_G v(g)\, s(gx)\, d\mu \in M.$$

In this case we have

LEMMA 4·2·2. *The homomorphism* C: $\Gamma(r^*M) \to \Gamma(r_* 1, M)$ *is a monomorphism.*

Proof. Suppose that, for all (x, v), $x \in X$, $v \in K^G$ such that $v(g) = v(gh)$ for $h \in G_x$, we have

$$\int_G v(g)\, s(gx)\, d\mu = 0.$$

Then by Theorem 2·4·2, $s(x) = 0$ for all $x \in X$.

5. *The bundle* $r_* 1$. We continue to assume that G is a compact topological group, X a compact Hausdorff G-space, r: $X \to X/G$ the identification map, and 1 the trivial bundle $X \times K$ over X. It was shown in 3·4·1 and 3·6·1 that $r_* 1$ is a normed linear bundle over X/G with sufficient sections.

5·1. Let $(r_* 1)'$ be the linear bundle dual to $r_* 1$. Let $M = (K^X)'$ be the topological dual of K^X. Then by 3·6,

$$(r_* 1)' = \{(r(x), \mu);\ x \in X,\ \mu \in M,\ \mu(f) = 0 \text{ for all } f\text{: } X \to K \text{ such that } f(gx) = 0 \text{ for all } g\}$$

is a subbundle of $(X/G) \times M$. Here M is the vector space of measures on X. The definition implies that

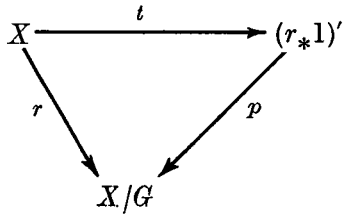
$$(r(x), \mu) \in (r_* 1)' \Rightarrow \mu(f) = 0 \text{ for any function } f \text{ with compact support in the complement of the orbit } Gx$$
$$\Rightarrow \operatorname{supp} \mu \subset \text{orbit } Gx.$$

Conversely, by Proposition 8 of (2), III, § 2 we have that $\operatorname{supp} \mu \subset$ orbit Gx implies $(r(x), \mu) \in (r_* 1)'$. Thus

$$(r_* 1)' = \{(r(x), \mu);\ x \in X, \mu \in M, \operatorname{supp} \mu \subset \text{orbit } Gx\}.$$

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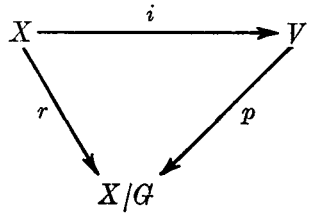
Let ϵ_x be the Dirac measure at x defined by $\epsilon_x(f) = f(x)$. Then by 3·6·2 there is a continuous map $t\colon X \to (r_*1)'$ such that $t(x) = (r(x), \epsilon_x)$ and such that the diagram



commutes. We obtain

THEOREM 5·1·1. *If $p\colon V \to X/G$ is a linear bundle over X/G, every map $\phi\colon (r_*1)' \to V$ induces a map $\phi t\colon X \to V$ such that $p\phi t = r$.*

We now seek a converse statement. Suppose that there is a commutative diagram of maps



and that, in addition, V is a Banach linear bundle: that is, it is a normed linear bundle with complete fibres. Let F be a normed vector space such that $V \subset \mathbf{F}$ and consider the map $j\colon X \to F$ such that $i(x) = (x, j(x))$. There is a map

$$(r_*1)' \to X/G \times F,$$

$$(r(x), \mu) \longmapsto \left(r(x), \quad \int j(x)\, d\mu\right).$$

Since $\operatorname{supp}\mu \subset$ orbit Gx and V is complete, the image is contained in V. In this way we obtain

THEOREM 5·1·2. *If $p\colon V \to X/G$ is a Banach linear bundle over X/G, every map $i\colon X \to V$ such that $pi = r$ induces a map $\phi\colon (r_*1)' \to V$.*

The definitions imply immediately that the constructions of Theorems 5·1·1 and 5·1·2 are inverses. Thus, if we denote by $\operatorname{map}_{X/G}(X, V)$ the set of maps of (X/G)-spaces $X \to V$ and by $\Gamma((r_*1)', V)$ the set of maps of linear bundles $(r_*1)' \to V$, we have

THEOREM 5·1·3. *If V is a Banach linear bundle then there is an isomorphism*

$$D\colon \operatorname{map}_{X/G}(X, V) \to \Gamma((r_*1)', V).$$

Let V be a Banach linear bundle over X/G and let $i\colon X \to V$ be a map such that $pi = r$. The image $i(X) \subset V$ is said to be *linearly independent* if for all $x \in X$ and maps $\lambda\colon G \to K$

$$\int_G \lambda(g)\, i(gx)\, d\mu = 0 \Rightarrow \lambda = 0.$$

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The image $i(X)$ is said to be a *linear span* if for all $x \in X$ and $v \in p^{-1}(r(x))$ there exists a map $\lambda: G \to K$ such that

$$\int_G \lambda(g)\, i(gx)\, d\mu = v.$$

Here μ denotes the Haar measure on G. These definitions become the usual definitions when G is finite.

Now consider the isomorphism

$$D: \operatorname{map}_{X/G}(X, V) \xrightarrow{\cong} \Gamma((r_* 1)', V)$$

given by 5·1·3. Let $t: X \to (r_* 1)'$ be the embedding of 3·6·2, and $\phi: r_* 1 \to (r_* 1)'$ the injection of 3·6·3. Then $D(i)\,\phi: r_* 1 \to V$ is the map

$$[g \longmapsto gx_0,\ \lambda(g)] \longmapsto \int_G \lambda(g)\, i(gx_0)\, d\mu.$$

The definitions now imply immediately

THEOREM 5·1·4. *The map $D(i)\,\phi$ is an injection if and only if the image $i(X)$ is linearly independent. The map $D(i)\,\phi$ is a surjection if and only if the image $i(X)$ is a linear span.*

A similar result holds if, as is more natural in topological applications, we regard the fibres of V as affine spaces rather than vector spaces. Then the image $i(X)$ is said to be *affine independent* if for all $x \in X$ and maps $\lambda: G \to K$

$$\int_G \lambda(g)\, i(gx)\, d\mu = \int_G \lambda(g)\, d\mu = 0 \Rightarrow \lambda = 0.$$

The image $i(X)$ is an *affine span* if for all $x \in X$ and $v \in p^{-1}(r(x))$ there exists a map $\lambda: G \to K$ such that

$$\int_G \lambda(g)\, i(gx)\, d\mu = v, \quad \int_G \lambda(g)\, d\mu = 0.$$

By $r_* 1 = 1 \oplus (r_* 1)_0$, where $(r_* 1)_0$ is the subbundle defined by functions $\lambda: G \to K$ such that

$$\int_G \lambda(g)\, d\mu = 0.$$

There are induced maps $(D(i)\,\phi)_0: (r_* 1)_0 \to V$ for each map $i: X \to V$ and the definitions imply immediately

THEOREM 5·1·5. *The map $(D(i)\,\phi)_0$ is an injection if and only if the image $i(X)$ is affine independent. The map $(D(i)\,\phi)_0$ is a surjection if and only if $i(X)$ is an affine span.*

5·2. In this section we discuss the relation between the maps constructed in 5·1 and the canonical maps of 4·1 and 4·2. We recall that a map $i: X \to V$ of (X/G)-spaces determines a map

$$X \times K \to r^* V \subset X \times V,$$

$$(x, \lambda) \longmapsto (x, \lambda i(x))$$

of linear bundles over X. Conversely every map $1 \to r^* V$ arises in this way from a map $i: X \to V$. Therefore Theorem 5·1·3 implies the existence of isomorphisms

$$P: \Gamma(1, r^* V) \to \Gamma((r_* 1)', V),$$

$$Q: \Gamma((r_* 1)', V) \to \Gamma(1, r^* V).$$

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If $\phi\colon 1 \to r^*V$ is a map then $P(\phi)$ is the map

$$(r(x), \mu) \longmapsto \int_X \phi(x, 1)\, d\mu.$$

If $\psi\colon (r_* 1)' \to V$ is a map then $Q(\psi)$ is the map

$$(x, \lambda) \longmapsto (x, \psi(\epsilon_x)).$$

The particular case $V = 1$ defines a map $P(id) = \pi\colon (r_* 1)' \to 1$, such that

$$\pi(r(x), \mu) = \left(r(x), \quad \int_X d\mu\right).$$

The particular case $V = (r_* 1)'$ defines a map $Q(id) = \tilde{\omega}\colon 1 \to r^*(r_* 1)'$ such that

$$\tilde{\omega}(x, \lambda) = (x, (r(x), \lambda\epsilon_x)).$$

LEMMA 5·2·1. *The map π is the transpose of the map $\alpha\colon 1 \to r_* 1$ defined by*

$$(r(x), \lambda) \longmapsto (r(x), g \longmapsto \lambda).$$

The map $\tilde{\omega}$ is the transpose of the map $\beta\colon r^* r_* 1 \to 1$ defined by $(x, g \longmapsto v(g)) \longmapsto (x, v(e))$.

Proof. An element $(r(x), \mu)$ of $(r_* 1)'$ is mapped by α' to the function

$$\lambda \longmapsto \int \lambda\, d\mu = \lambda \int d\mu$$

and so $\alpha' = \pi$. An element (x, λ) of 1 is mapped by β' to the function

$$(x, g \longmapsto v(g)) \longmapsto (x, \lambda v(e)) = (x, \lambda\epsilon_x(v))$$

and so $\beta' = \tilde{\omega}$.

It follows that $Q(\psi)$ is the composite map

$$1 \xrightarrow{\beta'} r^*(r_* 1)' \xrightarrow{r^*\psi} r^* V.$$

In particular, if $V = W'$ and $\psi = \phi'$ where $\phi\colon W \to r_* 1$ is a map then $Q(\psi)$ is the transpose of $B(\phi)$. In other words, if T denotes transpose, the diagram

$$\begin{array}{ccc} \Gamma(r^*W, 1) & \xrightarrow{T} & \Gamma(1, r^*W') \\ \uparrow B & & \uparrow Q \\ \Gamma(W, r_* 1) & \xrightarrow{T} & \Gamma((r_* 1)', W') \end{array}$$

commutes. Similarly the inverse diagram

$$\begin{array}{ccc} \Gamma(r^*W, 1) & \xrightarrow{T} & \Gamma(1, r^*W') \\ A \downarrow & & \downarrow P \\ \Gamma(W, r_* 1) & \xrightarrow{T} & \Gamma((r_* 1)', W') \end{array}$$

commutes.

Finally, there is a composite map

$$\Gamma((r_* 1)', V) \xrightarrow{Q} \Gamma(1, r^*V) \xrightarrow{C} \Gamma(r_* 1, V).$$
$$\xrightarrow[R]{\qquad\qquad}$$

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If $\psi\colon (r_*\,1)' \to V$ is a map then $R(\psi)$ is the composite map

$$r_*\,1 \xrightarrow{r_*\beta'} r_*\,r^*(r_*\,1)' \xrightarrow{r_*r^*\psi} r_*\,r^*V \xrightarrow{\gamma} V$$

$$(r(x), g \longmapsto \lambda(g)) \longmapsto (r(x), g \longmapsto \lambda(g)\epsilon_{gx}) \longmapsto$$

$$(r(x), g \longmapsto \lambda(g)\,\psi(\epsilon_{gx})) \longmapsto \int_G \lambda(g)\psi(\epsilon_{gx})\,d\mu$$

where μ is the Haar measure on G. But

$$\int_G \lambda(g)\,\psi(\epsilon_{gx})\,d\mu = \psi\left(\int_G \lambda(g)\,\epsilon_{gx}\,d\mu\right)$$

and

$$\left\langle\left(\int_G \lambda(g)\,\epsilon_{gx}\,d\mu\right), f\right\rangle = \int_G \langle\lambda(g)\,\epsilon_{gx}, f\rangle\,d\mu = \int_G \lambda(g)f(gx)\,d\mu.$$

Therefore $R(\psi)$ is the composite of ψ and the injection

$$\phi\colon r_*\,1 \to (r_*\,1)',$$

$$(r(x), g \longmapsto \lambda(g)) \longmapsto (r(x), f \longmapsto \int_G \lambda(g)f(gx)\,d\mu)$$

defined in 3·6.

5·3. In this section we consider the case in which G is a finite group of order n. Let $Y = X/G$ and let $r\colon X \to Y$ be the identification map. Then $r_*\,1$ is a finite dimensional linear bundle: at a point $r(x) \in Y$ the fibre dimension is the index of the stability group G_x in G.

If G acts freely then $r\colon X \to Y$ is a covering map, and $r_*\,1$ is a vector bundle. Moreover $r_*\,1 = 1 \oplus (r_*\,1)_0$, where $(r_*\,1)_0$ is a vector bundle with fibre dimension $n-1$ (compare Lemma 4·2·1).

THEOREM 5·3·1. *Let $p\colon E \to Y$ be a vector bundle with fibre dimension $n-1$, and suppose there is a map $i\colon X \to E$ such that*

(i) $pi = r$,

(ii) *for all $y \in Y$ the points $ir^{-1}(y)$ span the affine space $p^{-1}(y)$.*

Then $E = (r_*\,1)_0$.

Proof. Since E has fibre dimension $n-1$, condition (ii) implies that the image $i(X) \subset E$ is affine independent. The result then follows from 5·1·4 and the fact that G acts freely.

The vector bundles $r_*\,1$ and $(r_*\,1)_0$ can also be constructed directly from the covering $r\colon X \to Y$ by means of representations. Regard $r\colon X \to Y$ as a G-bundle ξ over Y, and let ρ be the regular representation $\rho\colon G \to GL(n, K)$. Then $\rho_*(\xi)$ is a principal $GL(n, K)$-bundle, and is associated to the vector bundle $r_*\,1$. There is also a decomposition $\rho = 1 \oplus \rho_0$ and then $\rho_{0*}(\xi)$ is associated to the vector bundle $(r_*\,1)_0$.

For example, suppose that $G = \mathbf{Z}_n$ is cyclic, and $K = \mathbf{C}$. Then

$$\rho = 1 \oplus \sigma \oplus \ldots \oplus \sigma^{n-1},$$

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where σ is the representation as nth roots of unity. The covering $r\colon X \to Y$ determines an element $\xi \in H^1(X, \mathbf{Z}_n)$. The exact sequence $0 \to \mathbf{Z} \xrightarrow{n} \mathbf{Z} \to \mathbf{Z}_n \to 0$ has an exact cohomology sequence

$$\dots \to H^1(Y, \mathbf{Z}_n) \xrightarrow{\delta} H^2(Y, \mathbf{Z}) \xrightarrow{n} H^2(Y, \mathbf{Z}) \to \dots$$

Here the coboundary δ is such that $\delta(\xi)$ is the first Chern class of the $GL(1, \mathbf{C})$-bundle $\sigma_*(\xi)$. Thus $\sigma_*(\xi)$ is associated to a complex line bundle H such that $c_1(H) = \delta(\xi)$ and $H^n = 1$. It follows that

$$r_* 1 = 1 \oplus H \oplus \dots \oplus H^{n-1},$$

$$(r_* 1)_0 = H \oplus \dots \oplus H^{n-1}.$$

Note that the complex conjugate of H^s is H^{n-s}. Therefore

$$(r_* 1)_0 = \begin{cases} H^{\frac{1}{2}n} \oplus (H^{\frac{1}{2}(n+2)} \oplus H^{\frac{1}{2}(n-2)}) \oplus \dots \oplus (H^{n-1} \oplus H), & n \text{ even}, \\ (H^{\frac{1}{2}(n+1)} \oplus H^{\frac{1}{2}(n-1)}) \oplus \dots \oplus (H^{n-1} \oplus H), & n \text{ odd}, \end{cases}$$

is an expression of $(r_* 1)_0$ as a sum of vector bundles each of which is the complexification of a real vector bundle. When $K = \mathbf{R}$ we have

$$(r_* 1)_0 = \begin{cases} L \oplus [H^{\frac{1}{2}(n-2)}] \oplus \dots \oplus [H], & n \text{ even} \\ [H^{\frac{1}{2}(n-1)}] \oplus \dots \oplus [H], & n \text{ odd}. \end{cases}$$

Here $[H^r]$, $r \geqslant 1$, is the real 2-plane bundle underlying the complex line bundle H^r, and L is the real line bundle associated to the Z_2-bundle $\tau_*(\xi)$ where $\tau\colon \mathbf{Z}_n \to \mathbf{Z}_2$ sends the generator to the generator.

Such explicit information about the vector bundle $(r_* 1)_0$ can then be combined with Theorem 5·3·1 to obtain specific results. Needless to say such results can often also be obtained directly without the apparatus of this paper. For example we have

COROLLARY 5·3·2. *Let $r\colon X \to Y$ be a covering of order 2, let $\xi \in H^1(Y, \mathbf{Z}_2)$ be the corresponding cohomology class, L the line bundle with $w_1(L) = \xi$ and $p\colon E \to Y$ a real vector bundle over Y. There is an embedding $i\colon X \to E$ such that $r = pi$ if and only if there is a never zero section of $E \otimes L$.*

Similarly we have

THEOREM 5·3·3. *Let $r\colon X \to Y$ be a cyclic covering of order n, let $\xi \in H^1(Y, \mathbf{Z}_n)$ be the corresponding cohomology class, H the complex line bundle with $c_1(H) = \delta(\xi)$, and $p\colon E \to Y$ a complex line bundle over Y. There is a map $i\colon X \to E$, such that $r = pi$ and $i(X)$ is an affine span of E, if and only if there is a never zero section of $E \otimes (H \oplus \dots \oplus H^{n-1})$.*

Proof. By Theorem 5·1·4 a map $i\colon X \to E$ exists if and only if there is a surjection $(r_* 1)'_0 \to E$. Since E is a line bundle this is equivalent to the existence of a never zero map $(H \oplus \dots \oplus H^{n-1})' \to E$, that is, a never zero section of $E \otimes (H \oplus \dots \oplus H^{n-1})$.

Conversely it is possible to deduce information about the linear bundle $r_* 1$ from information about vector bundles in which X can be embedded. For example, if G acts freely we have

THEOREM 5·3·4. *Suppose that G has order n, that $p\colon L \to Y$ is a complex line bundle, and that $j\colon X \to L$ is an embedding such that $pj = r$. Then $(r_* 1)' = 1 \oplus L \oplus \dots \oplus L^{n-1}$. If also $j(x)$ is never zero then $r^*L = 1$ and $L^n = 1$.*

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Proof. Consider, for each $s \geqslant 0$, the map $j^s\colon X \to L^s$ defined by

$$j^s(x) = j(x) \otimes \ldots \otimes j(x).$$

These maps are not necessarily embeddings; however, they induce an embedding

$$X \xrightarrow{i} 1 \oplus L \oplus \ldots \oplus L^{n-1},$$

$$x \longmapsto (1, j(x), \ldots, j^{n-1}(x)).$$

Over an open set U of Y such that $L|U$ is trivial we may choose coordinates so that i is defined by

$$x \longmapsto r(x),\ (1, f(x), \ldots, f(x)^{n-1}) \in U \times K^n$$

and the embedding j by $\qquad x \longmapsto r(x), f(x) \in U \times K.$

Thus the n points $f(gx)$, $g \in G$ are all distinct; hence the n points $(1, f(gx), \ldots, f(gx)^{n-1})$, $g \in G$ are linearly independent (recall that n distinct points on a rational normal curve in P^{n-1} are always linearly independent). It follows from 5·1·4 that there is an injection, and hence an isomorphism

$$(r_* 1)' \to 1 \oplus L \oplus \ldots \oplus L^{n-1}.$$

If also $j(x)$ is never zero then there is a never zero section $x \longmapsto x,\ j(x)$ of r^*L and therefore $r^*L = 1$. It follows from 3·5·2 that

$$r_* 1 = r_* r^* L = r_*(r^* L \otimes 1) = L \otimes r_* 1.$$

Since $r_* 1$ is a vector bundle with fibre dimension n

$$\det(r_* 1) = L^n \otimes \det(r_* 1)$$

and therefore $L^n = 1$.

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