Notes

102.37 Markov numbers

Studies of Riemann surfaces often involve number-theoretic considerations. The tribute to Maryam Mirzakhani [1] was accompanied by a photograph of her in action at Oxford and on the board behind her, the name Selberg can be identified. Selberg's *trace formula* combines number theory with the geometry of a compact Riemannian manifold, specifically the hyperbolic plane, (see [2]). Riemann surfaces constituted the core of Mirzakhani's work as the Fields Medal citation affirms and studying the behaviour of geodesics on such surfaces had number-theoretic ramifications.

A paper [3] drew my attention to Markov numbers, defined as integer triples (x, y, z) which are solutions of the Diophantine equations

$$x^2 + y^2 + z^2 - 3xyz = 0, \qquad \text{(D)}$$

which arose in yet another study of hyperbolic geometry. How the Markov equation arises in the context of geometry is clearly described in the beautiful book [4]. Note that a slight variant of (D) is given there, namely, $x^2 + y^2 + z^2 - xyz = 0$; readers can easily show how solutions to this equation can be derived from those of (D).

We continue by concentrating on (D), firstly to give an iterative formula generating all solutions, but our main interest lies in two infinite subsets of this general solution set.

Fix x, y; then we have a quadratic in z, $z^2 - 3xyz + x^2 + y^2 = 0$. The two solutions for z, z_1 and z_2 say, are given by $z_1 = z$ and $z_2 = 3xy - z$. Thus $x^2 + y^2 + z^2 - 3xyz$ is invariant under the mapping $z \to 3xy - z$ since $x^2 + y^2 + (3xy - z)^2 - 3xy(3xy - z) = x^2 + y^2 + z^2 - 3yxz$. Indeed, starting with the solution $(1, 1, 1)$, the mappings $(x, y, z) \to (z, x, y)$ and $(x, y, z) \to (x, y, 3xy - z)$ generate all solutions of (D). (This is stated, but not proved, in [3].) We give a few examples:-

$$(1, 1, 2), (2, 1, 5), (5, 2, 29), (1, 5, 13), (29, 5, 2), (2, 29, 169),$$
$$(29, 5, 433), (5, 194, 13), (5, 2897, 194), (29, 37666, 433), \ldots .$$

We now turn to two particular cases based on fixing $x = 1$ and then $x = 2$.

The case $x = 1$:

(D) reduces to $1 + y^2 + z^2 = 3yz$. We shall prove that $F_{2n+1}^2 + F_{2n-1}^2 - 3F_{2n+1}F_{2n-1} = -1$, $n \geqslant 1$, where the F_i are members of the standard Fibonacci sequence:

F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	F_{15}	$F_{16}\ldots$
1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987…

Now,

$$F_{2n-1} = F_{2n+1} - F_{2n} = F_{2n+1} - (F_{2n+2} - F_{2n+1})$$
$$= 2F_{2n+1} - F_{2n+2}$$

https://doi.org/10.1017/mag.2018.117 Published online by Cambridge University Press

$$= 2F_{2n+1} - (F_{2n+3} - F_{2n+1})$$

$$\text{Thus, } F_{2n-1} = 3F_{2n+1} - F_{2n+3}. \quad (1)$$

Inductively, if the result is valid for $n = k$, then

$$F_{2k+1}^2 + F_{2k-1}^2 - 3F_{2k+1}F_{2k-1} = -1$$

so by (1), $$F_{2k+1}^2 + (3F_{2k+1} - F_{2k+3})(-F_{2k+3}) = -1.$$

Hence $$F_{2k+1}^2 + F_{2k+3}^2 - 3F_{2k+3}F_{2k+1} = -1,$$

and the induction goes through.

Obvious solutions are $(1, 1, 2), (1, 2, 5), (1, 5, 13), (1, 13, 34), \ldots$

The case $x = 2$:

To help solve (D) in this case, we first consider integer pairs that satisfy the Pellian equation $u^2 - 2v^2 = -1$. If (m, n) is a solution then the next solution is $(3m + 4n, 2m + 3n)$ since

$$9m^2 + 24mn + 16m^2 - 2(4m^2 + 12mn + 9n^2) = m^2 - 2n^2 = -1.$$

We show now that the general solution of (D) in this case is given by $(x, y, z) = (2, n, 2m + 3n)$ subject to $m^2 - 2n^2 = -1$. For

$$x^2 + y^2 + z^2 - 3xyz = 4 + n^2 + 4m^2 + 12mn + 9n^2 - 6n(2m + 3n)$$

$$= 4(1 + m^2 - 2n^2)$$

$$= 0, \text{ as required.}$$

We list a few solutions of the Pellian equation, namely,

$$(1, 1), (7, 5), (41, 29), (239, 169), (1393, 985), \ldots$$

and the corresponding solutions of (D) are given by

$$(2, 1, 5), (2, 5, 29), (2, 29, 169), (2, 169, 985), (2, 985, 5741), \ldots .$$

The ubiquity of Fibonacci numbers in mathematics is well documented and at the highest level, too. Hilbert's tenth problem, to find an algorithm for deciding whether any given Diophantine equation has a solution, was finally solved negatively by Yuri Matijasevič, who exploited a particular property of Fibonacci numbers. To find Fibonacci *and* Pellian numbers associated with the set of Markov numbers was an unexpected bonus.

Acknowledgement

The author wishes to thank the anonymous referee for helpful suggestions.

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https://doi.org/10.1017/mag.2018.117 Published online by Cambridge University Press

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10.1017/mag.2018.117

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102.38 A very elementary short proof of Conway's little theorem

In this short Note, we give a very elementary, brief and self-contained proof of the following interesting theorem.

Theorem (Conway's Little Theorem): A triangle is equilateral if, and only if, the ratio between the length of any two sides is rational and the ratio between any two angles is also rational.

This theorem first appeared in [1, last comment], and recently with some added witty remarks (e.g., why should only Fermat have a little theorem? .....) has re-appeared in [2], and Conway this time calls it his little theorem.

We should admit that the re-publication of the theorem in [2], with those added witty remarks, made us read the theorem and its proof in [2] more carefully. Since the theorem is a nice characterisation of equilateral triangles (perhaps, an unusual characterisation in the classical sense of Euclidean geometry), one is tempted to present a geometric proof which applies tools as elementary as possible. Before proceeding with our promised proof, let us recall that as a consequence of two rather sophisticated theorems in [3, Theorems 3.9 and 3.11], it is shown in [3, Corollary 3.12] that the only rational values of the trigonometric functions of θ are given by: $\sin\theta$, $\cos\theta = 0, \pm 1, \pm\frac{1}{2}$, $\sec\theta$, $\operatorname{cosec}\theta = \pm 1, \pm 2$, $\tan\theta$, $\cot\theta = 0, \pm 1$, where $\theta = r\pi$, with r rational.

In what follows, we first give a quick self-contained proof of this interesting fact; see also [4, p. 358]. It is clear that we may, without loss, prove this only for the cosine function since, for example,

$$\sin\theta = \cos\left(\tfrac{\pi}{2} - \theta\right) \qquad \text{and} \qquad \tan\theta = \frac{\cos\left(\frac{\pi}{2} - \theta\right)}{\cos\theta}.$$

Let us now suppose that $\cos\theta$ is rational and $r = \frac{a}{b}$, where a, b are positive coprime integers. Note that we may, without loss, assume that r is positive. For any positive integer m we have $m = bq + t$, where q, t are non-negative integers with $0 \leqslant t < b$. Hence $\cos m\theta = \pm\cos t\theta$, where $0 \leqslant t < b$. This implies that for all positive integers m, $\cos m\theta$ belongs to a fixed finite set of rational numbers. Note that, for $0 \leqslant t < b$, $\cos t\theta$ takes only a finite number of rational values.

https://doi.org/10.1017/mag.2018.117 Published online by Cambridge University Press