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Non-normality and its consequences in active control of thermoacoustic instabilities

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Non-normality can cause transient growth of perturbations in thermoacoustic systems with stable eigenvalues. This can cause low-amplitude perturbations to grow to amplitudes high enough to make nonlinear effects significant, and the system can become nonlinearly unstable, even though it is stable under classical linear stability. In this paper, we have demonstrated that this feature can lead to the failure of the traditional controllers that were designed on the basis of classical linear stability analysis. We have also shown in a simple model that it is possible to prevent 'nonlinear driving' by controlling transient growth, using linear controllers. The analysis is performed in the context of a horizontal Rijke tube.

Key words: aeroacoustics, instability, nonlinear instability

1. Introduction

Combustion instabilities are excited by a feedback loop between the combustion process and the acoustic modes of the combustor. The combustion process adds energy to the acoustic modes if heat is added in phase with the pressure oscillations (Rayleigh 1878). In general, the occurrence of combustion instabilities is problematic because the high-amplitude oscillations result in structural damage to the system, wearing at part interfaces, flame flashback or blow-off, enhanced heat transfer to liners and overall performance degradation. Hence, it is important to control combustion instabilities for which adequate knowledge of the combustion process and the acoustic field and that of their interaction is required.

In general, there are two methods to control the oscillations: passive and active control. Passive control involves either redesigning the system, such as changing the flame anchoring point, or changing the operating conditions, such as installing baffles and acoustic dampers. A review of the implementation of this control method can be found in Richards & Straub (2005). Even though the passive control approach is simple, it is effective only over a limited range of operating conditions and is inadequate to account for the dynamically changing system conditions. On the other hand, active control provides a high degree of flexibility and a wide range of operation. Active control can take corrective actions against dynamically changing conditions and make the system stable.

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In active control, the output from the combustor is continuously monitored. If the output is undesirable, it is then fed back to the input through a controller. Hence, according to the output sensed, the controller will modify the input continuously. Active control can be implemented in many ways; two examples are (i) modulating the acoustic field (acoustic velocity or pressure) and (ii) modulating the heat release by incorporating secondary fuel injection techniques. Dowling & Morgans (2005) provided a review of the practical methods to implement feedback control techniques both on the laboratory scale and on large-scale combustors.

Experimental investigations of active control of thermoacoustic oscillations using the principle of 'anti-sound' were performed by Lang, Poinsot & Candel (1987). In their study, they noticed that if the unstable dominant mode alone is controlled, modes that were not originally excited then become unstable. Eliminating this 'pop-up' mode is a major challenge for active control. The physical reason and the control method to overcome this challenge are not mentioned in this study.

A detailed review of active control of combustion instabilities was provided by McManus, Poinsot & Candel (1993). In their work, the classical n–τ model of Crocco (1952) was used to calculate the heat release fluctuations. A simple pole placement technique was used to derive a condition for making an unstable mode stable. An interesting result called the 'overlap problem' showing an overall decrease in the stability margin was highlighted by the authors. This linear control may fail if an initially unexcited mode becomes unstable when the controller is added. The authors have not given any physical explanation for this phenomenon. Other authors have referred to this phenomenon as 'secondary peaks' in fast Fourier transforms (FFTs) of the evolution of the acoustic pressure or the 'peak-splitting' phenomenon as described in the following paragraphs.

The significance of the linear coupling between the acoustic modes was demonstrated by Annaswamy *et al.* (1997) for certain sensor actuator configurations. They showed that a controller designed to suppress pressure oscillations may fail because of the coupling present between acoustic modes. Fleifil *et al.* (1998) investigated the behaviour of controllers used in experiments. They explained secondary peaks as a resonance between various mechanisms in the combustor and in the controller. However, the authors did not discuss the physics which leads to this behaviour.

The peak-splitting phenomenon was observed in the practical implementation of active control in a lean premixed single-nozzle combustor by Cohen *et al.* (1998). They showed that the reduction in the combustor pressure oscillations is limited by the splitting of the spectral peak into two smaller peaks. The same phenomenon was also observed in the study of the application of active control developed for a three-nozzle combustor with a phase shift controller (Hibshman, Cohen & Banaszuk 1999). The explanation of the peak-splitting phenomenon with the linear and nonlinear analyses of the controlled combustion process was given by Banaszuk *et al.* (1999*a*,*b*). This phenomenon is explained with the help of the Bode integral. The performance of a feedback controller is limited by the open-loop dynamics of the system. A mathematical characterization of one of the limitations was given by Bode, showing that if the open-loop system $L(z)$ is stable, the sensitivity function $S(z) = 1/(1 + L(z))$ must satisfy the relation $\int_0^{\pi} \mathrm{d}\omega \ln |S\,\mathrm{e}^{\mathrm{i}\omega}| = 0$. In other words, if the sensitivity function is brought down over a certain bandwidth, it will increase for another frequency range. However, the design of the controller to avoid this effect was not discussed by Banaszuk *et al.* (1990*a*,*b*). An adaptive controller for controlling combustion instability was presented by Koshigoe, Komatsuzaki & Yang (1999). The authors

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described the instability phenomenon with the help of a nonlinear inhomogeneous wave equation. The adaptive controller used an online system identification for robustness with respect to transient operating states, modelling uncertainties, external disturbances and additional modes of instability that may arise during the course of control. They showed that the adaptive controller performed better than a previously developed proportional-plus-integral controller in robustness.

A swirl-stabilized spray combustor was used to demonstrate the effectiveness of an extremum-seeking adaptive controller by Murugappan *et al.* (2000). The adaptive controller algorithm required the addition of the continuous perturbation to the pressure sensor signal. The authors mentioned that the stability margins were determined by varying the operational parameters including the perturbation signal amplitude and the frequency. A review of research on combustion instability and its control was presented by Candel (2002). The author discussed the recent research trends in combustion dynamics and its active control. It is mentioned in that paper that the multidimensional simulation of the active control allows the complete study of the control concepts. The actuator effectiveness constitutes the most critical issue and introduces a complicated trade-off between the bandwidth and the power. The critical issue is to control the equivalence ratio in order to keep NO_x emission low. As suggested by the author, further requirements are necessary in the control algorithm for increased performance and reduced sensitivity to noise and parametric variability.

An experimental study of a lean premixed pre-vaporized swirl-stabilized combustor was reported by Bernier *et al.* (2003). The authors used a secondary fuel injector as an actuator to control combustion instability. They studied the combined transfer function of the burner and the actuation system. The adaptive control algorithm was implemented to control the unbounded oscillations in the system by the authors. As explained by them, two techniques are used for the transfer function measurement. The first method is the direct measurement of pressure between the command and the sensor, and the second method is through the finite impulse response filter. The authors mentioned that the results obtained by both the methods are in good agreement.

The traditional linear controllers designed for a thermoacoustic system have focused on the unstable eigenmodes of the system (McManus *et al.* 1993). The objective of these controllers is to make the system stable under classical linear stability, i.e. to make all the eigenmodes decay. However, controlling the dominant eigenmode alone causes the excitation of another mode which was originally stable. This observation indicates that the modes of a thermoacoustic system are not orthogonal. Recently, Balasubramanian & Sujith (2008*a*,*b*) showed that thermoacoustic systems are non-normal and nonlinear. They showed that transient growth occurs even when all the eigenvalues of a system indicate system stability. If the amplitude of transient growth is high enough, then the system may enter into the region in which nonlinear effects are significant, and the system may grow unstable. Nicoud *et al.* (2007) showed that the eigenvectors of a thermoacoustic system are non-orthogonal in the presence of unsteady heat release or in the presence of general complex impedance boundary conditions.

An operator is said to be non-normal if it does not commute with its adjoint (Schmid & Henningson 2001). In a normal system, the eigenvectors are orthogonal. The dynamics of normal systems is determined entirely by the time evolution of the eigenvectors whose growth or decay is determined by the individual eigenvalues. Figure 1(*a*) shows that for a normal system, the resultant decreases monotonically if the amplitudes of the individual eigenvectors themselves decay. On the contrary,

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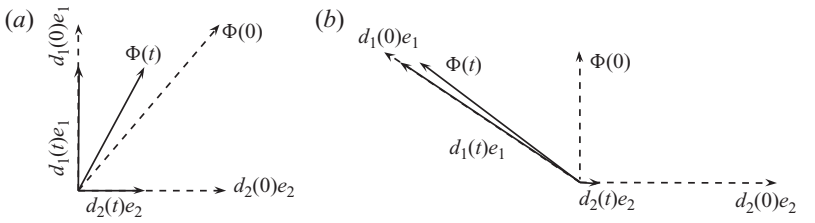


FIGURE 1. (*a*) Monotonic decay of a normal system. (*b*) Transient growth of a non-normal system. The initial state is $\Phi(0) = d_1(0)e_1 + d_2(0)e_2$, and the final state is $\Phi(t) = d_1(t)e_1 + d_2(t)e_2$. The dashed lines denote the vectors at $t = 0$, and the solid lines denote the vectors at some time t.

for a non-normal system shown in figure 1(*b*), the resultant increases even when the amplitudes of individual eigenvectors decay. For such systems, the eigenvectors are not orthogonal, showing growth in the energy of the system before eventually reaching the equilibrium state. Studies of the effect of non-normality on the evolution dynamics have been performed in the field of turbulence (Trefethen *et al.* 1993; Baggett, Driscoll & Trefethen 1995; Trefethen & Embree 2005; Farrrell & Ioannou 1996). The effect of non-normality on active control has been discussed by Collies *et al.* (2004) and Zhao & Bau (2006). They showed that the linear controllers designed on the basis of the eigenvalue analysis may fail for non-normal systems. The initial disturbances which get amplified because of transient growth are referred to as 'dangerous disturbances'. Even if a system is normal to begin with, the addition of a controller can make the system non-normal, which will completely change the evolution dynamics of the system (Handel 2004). Control of transient growth in a non-normal system has been discussed by Whidborne & McKernan (2007). The convex optimization technique was used by them to minimize the maximum transient growth in the system. A static gain controller design for transient growth control was presented in their study.

As a thermoacoustic system is non-normal even before the addition of a controller, non-normality can be expected to play an important role in the active control of combustion instabilities. Therefore, transient dynamics needs to be accounted for while designing the controller. It was shown by Balasubramanian & Sujith (2008*a*,*b*) that transient growth can make the nonlinear effects significant, and the system can become nonlinearly unstable. The objective of the current analysis is to point out that 'triggering' of initially low-amplitude perturbations that lead to nonlinear driving can be prevented by controlling transient growth. The dynamics is far too complicated if the perturbations have high amplitudes to begin with, and our analysis does not apply to such cases. However in most practical situations, controlling low-amplitude perturbations is sufficient to prevent the failure of the system. In the present study, a linear controller which can control the algebraic transient growth as well as the exponential asymptotic growth of the acoustic energy is designed using the conditions derived by Whidborne & McKernan (2007). The analysis is performed in the framework of a model problem similar to that described by McManus *et al.* (1993). The designed controller is then adapted to a horizontal Rijke tube, and the success of the model is demonstrated.

2. Active control and non-normality

A system implementing active control consists of a microphone which is a sensor, acoustic drivers (actuators), time delay generators and amplifiers. The sensor measures

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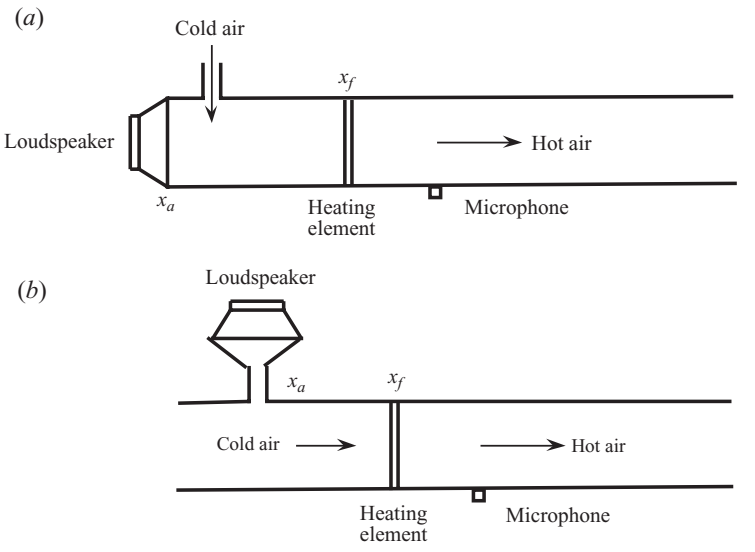


FIGURE 2. (*a*) End-mounted loudspeaker. (*b*) Side-mounted loudspeaker.

the acoustic pressure near the heat source location, and this signal is modified (amplified and phase shifted) through the controller. The input to the system is modified according to the signal measured by the sensor. Adding a controller to the system will change the system parameters. We will provide a short description of the actuators before we present a mathematical analysis of the effect of adding a controller to the system.

2.1. *Actuators*

The active control of combustion instability is effected by modulating the flow. The actuators in the system respond to the signal obtained from the controller and modify the input to the system. The modulation in the flow is obtained by either implementing secondary fuel injectors (altering the mass flow rate) or using loudspeakers (altering the acoustic field in the duct). Although practical actuators employed in large-scale combustors are typically fuel flow modulators, we have chosen loudspeakers as the actuators in our system following the analysis of Fleifil *et al.* (1998) to keep the analysis simple and tractable so that we can concentrate more on the physics of the problem. In the present paper, more than one actuator will be used to implement the control. The position of the ith actuator is denoted by x_{a_i}, and the loudspeakers could be end-mounted or side-mounted on the duct (figure 2*a*,*b*). The diaphragm motion of the loudspeakers can be modelled as a piston. The diaphragm motion is parallel to the flow direction when the loudspeaker is end-mounted, whereas it is normal to the flow when the loudspeaker is side-mounted. The schematic diagram of the loudspeaker used as an actuator (adapted from Fleifil *et al.* 1998) is shown in figure 3. The diaphragm velocity, displacement and cross-section area are given by v_{c_i}, y_i and A_{s_i} respectively. The width of the duct is given by w, and D_0 is the height of the duct considered. It is assumed that the width of the loudspeaker, Δx, is relatively small compared with the smallest wavelength of the acoustic mode (Fleifil *et al.* 1998), and the diaphragm velocity is small compared with the speed of sound. Further, it is also assumed that y/D_0 is small. The ratio of the cross-sectional area

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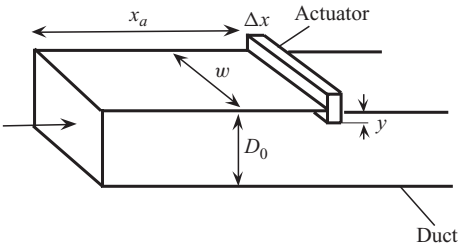


FIGURE 3. Schematic diagram of the combustor with the actuator modelled as a movable piston.

of the actuator (A_{s_i}) to the cross-sectional area of the duct (A) is denoted by α_{r_i} ($= A_{s_i}/A$).

2.2. *Thermoacoustic model with actuators*

We will present a simple mathematical model of a thermoacoustic system with controllers. The gas is assumed to be perfect, inviscid and non-heat-conducting. The Mach number of the flow is assumed to be small. We will assume that the heating element causes a localized heat addition, and there is a jump in the mean temperature across the heating element. The gradient in mean pressure can be neglected when the Mach number is small. The non-dimensional acoustic equations of such a thermoacoustic system, in the presence of K actuators, can be written as (Fleifil *et al.* 1998; Nicoud *et al.* 2007; Balasubramanian & Sujith 2008*a*,*b*; Nicoud & Wieczorek 2009)

$$\gamma M_1 \frac{\bar{\rho}(x)}{\bar{\rho}_1}\frac{\partial u'}{\partial t} + \frac{\partial p'}{\partial x} = 0, \tag{2.1}$$

$$\frac{\partial p'}{\partial t} + \gamma M_1 \frac{\partial u'}{\partial x} = (\gamma - 1)\dot{Q}'_f \delta(x - x_f) + \gamma \sum_{m=1}^{K} \alpha_{r_m} v_{c_m} \delta(x - x_{a_m}), \tag{2.2}$$

where γ is the ratio of the specific heats and M_1 is the mean flow Mach number of the unheated gas. The temperature, pressure and density of the hot gas are related to those of the cold gas through the following steady-state equations:

$$\bar{p}_1 = \bar{p}_2 \Rightarrow \frac{\bar{\rho}_1}{\bar{\rho}_2} = \frac{\bar{T}_2}{\bar{T}_1}, \tag{2.3}$$

$$\frac{\bar{u}_1}{\bar{u}_2} = \frac{\bar{\rho}_2}{\bar{\rho}_1} \Rightarrow \frac{\bar{M}_1}{\bar{M}_2} = \sqrt{\frac{\bar{T}_1}{\bar{T}_2}}, \tag{2.4}$$

$$\frac{\bar{T}_2}{\bar{T}_1} = 1 + \frac{\bar{Q}_f}{\gamma C_v \bar{T}_1}, \tag{2.5}$$

where $(\,)_1$ denotes some attribute of the cold gas and $(\,)_2$ denotes some attribute of the hot gas. In (2.1), the density of mean flow, $\bar{\rho}(x)$, can be written as $\bar{\rho}_1 + (\bar{\rho}_2 - \bar{\rho}_1)\,\Theta(x - x_f)$, where Θ is a unit step function. The non-dimensional variables in (2.1) and (2.2) are obtained as follows: $\tilde{u}' = \bar{u}_1 u'$, $\tilde{p}' = (\bar{\rho}_1 \bar{c}_1^2/\gamma)p'$, $\tilde{\dot{Q}}'_f = \dot{Q}'_f \bar{\rho}_1 \bar{c}_1^3/L_d$, $\tilde{x} = L_d x$ and $\tilde{t} = t L_d/\bar{c}_1$. The variables with the tilde are the dimensional quantities; the quantities with the overbar are the mean quantities; $\bar{c}_1$ is the speed of the sound; and L_d is the length of the duct.

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In this paper, we will study the effect of a controller on the thermoacoustic behaviour of a horizontal Rijke tube (Balasubramanian & Sujith 2008*b*). The non-dimensional heat release rate $\dot{Q}'_f$ is modelled by a modified form of King's law given by (Heckl 1990)

$$\dot{Q}'_f = \frac{2L_w(T_w - \bar{T}_1)}{A\bar{c}_1^3\bar{\rho}_1}\sqrt{\pi\lambda C_v\bar{\rho}_1\bar{u}_1\frac{d_w}{2}}\left[\sqrt{\left|\frac{1}{3}+u'(x_f,t-\tau)\right|}-\sqrt{\frac{1}{3}}\right], \tag{2.6}$$

where L_w is the length of the heating wire; T_w is the temperature of the wire; d_w is the diameter of the wire; A is the cross-sectional area of the duct; λ is the thermal conductivity of the wire; C_v is the specific heat of the gas; and τ is the time lag. Note that, in the low-amplitude limit, this model reduces to the n–τ model of Crocco (1952).

The diaphragm velocity v_{c_m} is modelled as $v_{c_m} = R_m u' + (S_m/\gamma M_1)p'$ (Fleifil *et al.* 1998), where R_m and S_m are control parameters. The actuators are added to the system in order to modify the acoustic field in the duct. The modification of the acoustic field is done in such a way that the Rayleigh integral is negative. This change is effected by the additional source term in the acoustic energy equation due to the actuator. Further, the addition of actuators to a system with no heat release rate (i.e. a classical acoustic system which is normal) makes that system non-normal. This can be understood as follows.

In the absence of heat release rate, (2.1) and (2.2) simplify to

$$\begin{bmatrix} \dfrac{\bar{\rho}(x)}{\bar{\rho}_1}\dfrac{\partial}{\partial t} & \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial x}-\displaystyle\sum_{m=1}^{K}\frac{\gamma\alpha_{r_m}R_m}{\gamma M_1}\delta(x-x_{a_m}) & \dfrac{\partial}{\partial t}-\displaystyle\sum_{m=1}^{K}\frac{\gamma\alpha_{r_m}S_m}{\gamma M_1}\delta(x-x_{a_m}) \end{bmatrix}\begin{bmatrix}\gamma M_1 u' \\ p'\end{bmatrix} = 0. \tag{2.7}$$

The matrix of operators in the above equation does not commute with its adjoint. This implies that the above system is non-normal, which shows that the addition of controllers to a normal system can make the system non-normal.

The partial differential equations in (2.1) and (2.2) can be reduced to a system of ordinary differential equations through the Galerkin technique (Balasubramanian & Sujith 2008*a*,*b*). Figure 2(*b*) shows a schematic of the set-up used in this paper. The dependent variables are expanded as a linear combination of basis functions, which are chosen to satisfy the boundary conditions. The pressure inside a duct with open ends can be written as a superposition of the duct's natural modes as follows:

$$p'(x,t) = -\gamma M_1\sum_{j=1}^{N}\frac{\sin(j\pi x)}{j\pi}\dot{\eta}_j(t). \tag{2.8}$$

When this expansion for pressure is substituted in (2.1), the following expression for acoustic velocity can be obtained:

$$u'(x,t) = \sum_{j=1}^{N}\frac{\bar{\rho}_1}{\bar{\rho}(x)}\cos(j\pi x)\eta_j(t). \tag{2.9}$$

Substituting the above expansions for acoustic pressure and velocity in (2.1) and (2.2), and projecting along the basis functions, the following evolution equations are

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obtained:

$$\frac{\mathrm{d}\eta_j}{\mathrm{d}t} = j\pi\left(\frac{\dot{\eta}_j}{j\pi}\right), \tag{2.10}$$

$$\frac{\mathrm{d}(\dot{\eta}_j/j\pi)}{\mathrm{d}t} + \sum_{i=1}^{N} \mathscr{W}_{ji}\eta_i = -2(\gamma-1)\frac{k}{\gamma M_1}\sin(j\pi x_f)\left[\sqrt{\left|\frac{1}{3}+u'(x_f,t-\tau)\right|}-\sqrt{\frac{1}{3}}\right] - \frac{2\gamma}{\gamma M_1}\sum_{m=1}^{K}\alpha_{r_m}\left[R_m u'(x_{a_m},t)+\frac{S_m}{\gamma M_1}p'(x_{a_m},t)\right]\sin(j\pi x_{a_m}), \tag{2.11}$$

where $k = L_w(T_w-\bar{T}_1)\sqrt{2\pi\hat{\lambda}C_v\bar{\rho}_1\bar{u}_1 d_w}/(A\bar{c}_1^3\bar{\rho}_1)$ and

$$\mathscr{W}_{ji} = -2\int_0^1 \mathrm{d}x\,\sin(j\pi x)\frac{\partial}{\partial x}\left(\frac{\bar{\rho}_1}{\bar{\rho}(x)}\cos(i\pi x)\right). \tag{2.12}$$

The above system of equations can be written as follows:

$$\frac{\mathrm{d}\Psi_J}{\mathrm{d}t} + \sum_{I=1}^{2N}\Omega_{JI}\Psi_I = \mathscr{Q}_J + \mathscr{C}_J, \tag{2.13}$$

where $I, J \in 1, 2, \ldots, 2N-1, 2N$ and $\Psi = [\eta_1, \dot{\eta}_1/\pi, \eta_2, \dot{\eta}_2/(2\pi), \ldots, \eta_N, \dot{\eta}_N/(N\pi)]^T$. The even elements of the vectors $\mathscr{Q}$ and $\mathscr{C}$ are given by

$$\mathscr{Q}_{2j} = -2(\gamma-1)\frac{k}{\gamma M_1}\sin(j\pi x_f)\left[\sqrt{\left|\frac{1}{3}+u'(x_f,t-\tau)\right|}-\sqrt{\frac{1}{3}}\right], \quad \text{for} \quad j \in 1,\ldots,N, \tag{2.14}$$

$$\mathscr{C}_{2j} = -\frac{2\gamma}{\gamma M_1}\sum_{m=1}^{K}\alpha_{r_m}\left[R_m u'(x_{a_m},t)+\frac{S_m}{\gamma M_1}p'(x_{a_m},t)\right]\sin(j\pi x_{a_m}), \quad \text{for} \quad j \in 1,\ldots,N, \tag{2.15}$$

while the odd elements vanish, i.e.

$$\mathscr{Q}_{2j-1} = 0, \ \mathscr{C}_{2j-1} = 0, \quad \text{for} \quad j \in 1,\ldots,N. \tag{2.16}$$

The elements of the $2N \times 2N$ matrix $\boldsymbol{\Omega}$ are

$$\Omega_{I(2j-1)} = 0, \quad \text{for} \quad I \neq 2j \quad \text{and} \quad \Omega_{2j(2j-1)} = -j\pi, \quad \text{for} \quad j \in 1,\ldots,N, \tag{2.17}$$

$$\Omega_{(2i-1)2j} = \mathscr{W}_{ij}, \quad \Omega_{2i2j} = 0, \quad \text{for} \quad i, j \in 1,\ldots,N. \tag{2.18}$$

The above system of equations is both nonlinear and non-normal. As discussed before, non-normality can lead to transient growth, which can result in low-amplitude perturbations becoming nonlinearly unstable. Hence, it is not sufficient to design the controllers on the basis of eigenvalues only. The traditional controllers used for controlling thermoacoustic instabilities do not control the transient growth in the system.

We will study the linearized system in the next section and also present a condition for this system to not exhibit transient growth.

2.3. *Transient growth, energy and condition for unity growth factor*

In this section, we will assume (for convenience) that the time lag is small compared with the acoustic time scale, i.e. τ is small (Kopitz & Polifke 2005). The following

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equation is obtained by linearizing (2.13) and expanding to first order in τ:

$$\frac{\mathrm{d}\Psi_J}{\mathrm{d}t} + \sum_{I=1}^{2N} [B_{NN}]_{JI}\, \Psi_I = 0. \tag{2.19}$$

In the above equation B_{NN} is given by

$$[B_{NN}]_{JI} = \Omega_{JI} + \frac{\sqrt{3}k(\gamma-1)}{\gamma M_1} \mathscr{F}_J(x_f) \cdot \left(\mathscr{U}_I(x_f) - \tau \mathscr{D}_I(x_f)\right) + \sum_{m=1}^{K} \frac{2\alpha_{r_m}}{M_1} \mathscr{F}_J(x_{a_m}) \cdot \left(R_m \mathscr{U}_I(x_{a_m}) - \frac{S_m}{\gamma M_1} \mathscr{P}_I(x_{a_m})\right), \tag{2.20}$$

where

$$\mathscr{U}_I(x) = \frac{1}{2}\left(\cos\left(\frac{(\mathscr{I}+1)\pi x}{2}\right) + (-1)^{(\mathscr{I}+1)} \cos\left(\frac{(\mathscr{I}+1)\pi x}{2}\right)\right), \tag{2.21}$$

$$\mathscr{D}_I(x) = \frac{1}{2}\left(\cos\left(\frac{\mathscr{I}\pi x}{2}\right) + (-1)^{\mathscr{I}} \cos\left(\frac{\mathscr{I}\pi x}{2}\right)\right), \tag{2.22}$$

$$\mathscr{P}_I(x) = \frac{-\gamma M_1}{2}\left(\sin\left(\frac{\mathscr{I}\pi x}{2}\right) + (-1)^{\mathscr{I}} \sin\left(\frac{\mathscr{I}\pi x}{2}\right)\right) \text{ and } \mathscr{F}_I(x) = -\frac{\mathscr{P}_I(x)}{\gamma M_1}. \tag{2.23}$$

Controllers designed on the basis of classical linear stability analysis make the real parts of all eigenvalues B_{NN} positive. Such controllers make the system only asymptotically stable and do not control transient growth. Recently, Whidborne & McKernan (2007) derived a condition for a linear system to not exhibit transient growth. We will use this condition to design a controller for the thermoaoustic system considered in § 2.2.

Transient growth is quantified by evaluating the norm of the evolution of matrix. Furthermore, Ψ and B_{NN} can be transformed to make the definition of the square of the norm of the state vector the same as acoustic energy. Non-dimensional acoustic energy is given by

$$E(t) = \int_0^1 \mathrm{d}x \left(p'^2(x,t) + \frac{\bar{\rho}(x)}{\bar{\rho}_1} u'^2(x,t)\right) = \Psi^T(t) \cdot \mathscr{M} \cdot \Psi(t), \tag{2.24}$$

where energy is non-dimensionalized by $\bar{\rho}_1 \bar{c}_1^2/(2\gamma^2)$ and

$$\mathscr{M}_{2j2j} = 1, \quad \text{for} \quad j \in 1, \ldots, N, \tag{2.25}$$

$$\mathscr{M}_{(2i-1)(2j-1)} = \int_0^1 \mathrm{d}x \left(\frac{\bar{\rho}(x)}{\bar{\rho}_1}\right)^3 \cos(i\pi x)\cos(j\pi x), \quad \text{for} \quad i, j \in 1, \ldots, N, \tag{2.26}$$

$$\mathscr{M}_{\mathscr{I}\mathscr{J}} = 0, \quad \text{otherwise}. \tag{2.27}$$

Note that $\mathscr{M}$ is symmetric. We will now define the state vector as $\chi = \mathscr{M}^{1/2}\Psi$. This definition makes the square of the norm of the state vector the same as non-dimensional acoustic energy. Equation (2.19) can now be rewritten as

$$\frac{\mathrm{d}\chi}{\mathrm{d}t} + [\mathscr{M}^{1/2} B_{NN} \mathscr{M}^{-1/2}]\chi = 0. \tag{2.28}$$

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In the above equation, $L = [\mathscr{M}^{1/2} B_{NN} \mathscr{M}^{-1/2}]$ is the stability matrix. The growth factor is defined as follows:

$$G(t) = \max_{\chi(0)} \frac{\| \chi(t) \|^2}{\| \chi(0) \|^2} = \| \exp(-Lt) \|^2. \tag{2.29}$$

Note that $G(t)$ is the ratio of energy at some instant of time to the initial energy. There is growth in the energy if $G(t) > 1$ for some time t. Further, note that the growth factor is infinite as time tends to infinity if the system is linearly unstable, i.e. if the real parts of the eigenvalues of L are negative. We can define the maximum growth factor as follows:

$$G_{max} = \max_{t} G(t). \tag{2.30}$$

There is increase in the energy of system when $G_{max} > 1$, and it is unity when there is no transient growth in the energy. The condition for the linear system in (2.28) to have unity G_{max} is given by (Whidborne & McKernan 2007)

$$L + L^T \geqslant 0, \tag{2.31}$$

i.e. $L + L^T$ is positive definite. The above condition ensures that the rate of change of energy at any instant of time is not positive. This is also the condition for $\mathrm{d}E/\mathrm{d}t \leqslant 0$ for all initial conditions.

It is also desirable to minimize the actuators' effort (power) for practical applications. However, this requires sophisticated control techniques, and it is beyond the scope of this paper.

The objective of this paper is to show that it is possible to reduce the maximum growth factors so as to prevent nonlinear triggering. In the next section, we will design a linear controller which implements the condition in (2.31) for a six-dimensional system (with constant mean temperature). We will further demonstrate that this linear controller is effective in preventing nonlinear triggering in the Rijke tube model described at the beginning of this section and also demonstrate the failure of the traditional controllers in preventing it. We will also investigate the importance of temperature jump on the dynamics of this system.

3. Results and discussion

3.1. *Linear controller for a horizontal Rijke tube without mean temperature gradient*

In this section, we will compare the effectiveness of a classical pole placement controller (linear) and a 'transient growth controller' in preventing nonlinear triggering of acoustic oscillations in a horizontal model of Rijke tube described in the previous section. Annaswamy *et al.* (1997) noted that three terms in the Galerkin expansion were needed to model the system dynamics for active control. Following this we have chosen three Galerkin modes to get a six-dimensional system. We will neglect the temperature gradient in this subsection, but its importance will be discussed in a later subsection.

A transient growth controller minimizes the transient growth by choosing the locations and response functions of the actuators so as to satisfy the condition in (2.31). Classical linear controllers are designed to make the real parts of the eigenvalues of L positive, and this ensures only asymptotic exponential decay and cannot restrict transient growth in general.

We will now study the linear and nonlinear behaviours of a six-dimensional system satisfying (2.10) and (2.11), for two different sets of control parameters: (i) satisfying

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the classical stability criterion (which does not attempt to control the transient growth) and (ii) satisfying the condition for unity G_{max} in (2.31). In this analysis, the control system consists of six actuators (located sufficiently far from the heating element), and the control parameters are chosen such that the assumption of linear response holds good. This may not be the minimum number of actuators required to make $G_{max} = 1$, and an analysis which optimizes the number of actuators and the effort required to control is left for future work.

The ordinary differential equations in (2.10) and (2.11) are numerically integrated using the fourth-order Runge–Kutta technique to study the evolution of the system from the optimum initial condition (the condition for which the growth factor is maximum). The following system parameters were used for this numerical study: $C_v = 719.0\,\mathrm{kJ\ (kg\ K)^{-1}}$, $\gamma = 1.4$, $x_f = 0.75$, $T_1 = T_2 = 300\ \mathrm{K}$, $M = 0.003$, $S = 1.56 \times 10^{-3}\,\mathrm{m^2}$, $L_w = 2.0\,\mathrm{m}$, $\rho = 1.205\,\mathrm{kg\,m^{-3}}$, $\lambda = 0.0328\,\mathrm{W\,mK}$, $\tau = 0.01$ and $T_w = 700\,\mathrm{K}$. The eigenvalues of the uncontrolled system are

$$\{-0.01517 \pm 9.5054i, \pm 6.28319i, 0.0049 \pm 3.0605i\}. \tag{3.1}$$

The following locations of actuators are chosen for both the traditional and the transient growth controller:

$$x_a = \{0.2582, 0.3651, 0.4472, 0.5163, 0.5773, 0.6324\}. \tag{3.2}$$

Figure 4(a) shows the linear evolution of acoustic velocity at the heating element when the first controller is used. The initial condition is a vector pointing along the optimum initial condition. The actuator response functions for the traditional controller are

$$R = \{0.0573, -0.0495, 0.0425, -0.1055, 0.1918, -0.2098\}, \tag{3.3}$$

$$S = \{-0.0652, 0.2560, -0.4198, 0.3774, -0.0902, -0.0300\}. \tag{3.4}$$

Note that the actuators' response are small enough for linear response assumption to hold good. The eigenvalues of the stability matrix L are

$$\{0.09817 \pm 8.6127i, 0.2058 \pm 5.1859i, 0.1084 \pm 4.6548i\}. \tag{3.5}$$

Since all the real parts of the eigenvalues are positive, this system is asymptotically stable. Figure 4(a) indicates that there is a transient growth of oscillations, even though they decay exponentially at large times. This leads to nonlinear effects which can be seen from the nonlinear evolution of acoustic velocity shown in figure 4(b). The oscillations do not decay asymptotically when nonlinear effects are present. This example clearly shows that a traditional linear controller can fail when nonlinear effects are present even if the initial amplitude of the disturbance is small. Figures 4(c) and 4(d) show that nonlinear triggering can be prevented by controlling transient growth. Note that the initial condition in this system has the same norm (or acoustic energy) as the previous system (with the classical controller), and it points along the optimum initial direction which is also sometimes referred to as the most 'dangerous' initial condition. The system experiences maximum transient growth from this initial condition. This indicates that there is no transient growth for any other initial direction. The optimal initial condition can be obtained by performing a singular value decomposition of the evolution operator e^{-Lt} (Nagaraja, Kedia & Sujith 2009). The square of the principal singular value gives the maximum amplification of the energy. The corresponding right singular vector gives the most sensitive or the optimal initial condition for the maximum energy growth. The control parameters

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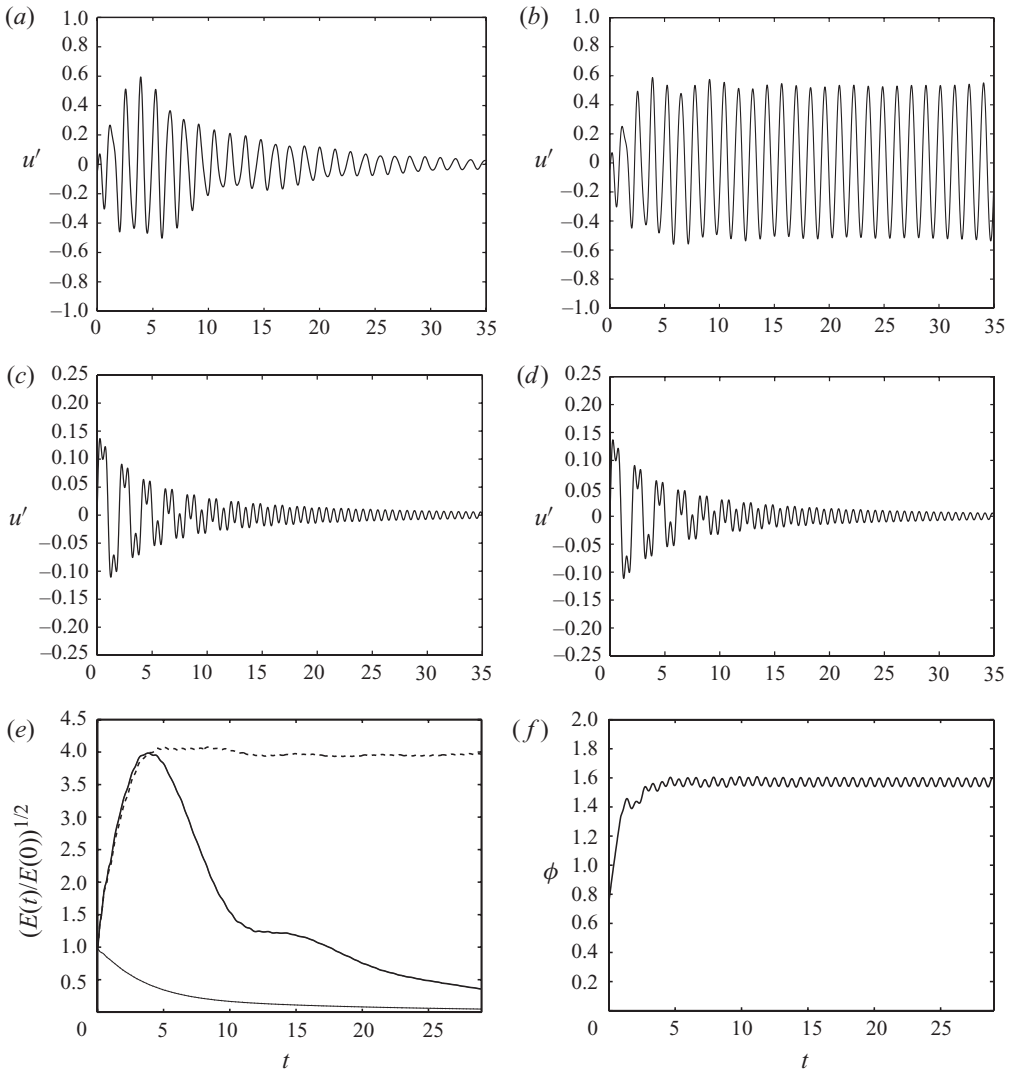


FIGURE 4. (*a*) Linear and (*b*) nonlinear evolution of acoustic velocity at the heating element in a system with a traditional controller. The initial condition is $\Psi = (0.0019, 0.1149, 0.0650, -0.1418, -0.0653, 0.1126)$. (*c*) Linear and (*d*) nonlinear evolution of acoustic velocity at the heating element in a system with a transient growth controller. The initial condition is $\Psi = (0, -0.2251, -0.0107, -0.0075, -0.0014, 0.0600)$. (*e*) The thick solid line and the dashed line are the time evolution of acoustic energy in the systems in (*a*) and (*b*). The thin solid line is the linear as well as the nonlinear evolution of acoustic energy in the system with a transient growth controller. (*f*) Phase evolves to $\pi/2$.

were chosen such that all the eigenvalues of $L + L^T$ are positive. The following values of the actuator response functions make $L + L^T$ positive:

$$R = \{0.0170, -0.1100, 0.3397, -0.5813, 0.5470, -0.2321\}, \tag{3.6}$$

$$S = \{0.01857, -0.0521, 0.1191, -0.1288, 0.0636, 0.0\}. \tag{3.7}$$

The above values for R and S are obtained by solving $L + L^T = \mathscr{P}$, where $\mathscr{P}$ is some positive definite matrix. This matrix equation yields a set of linear equations in R_i and

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S_i. Note that this procedure of obtaining the control parameters need not optimize the actuators' effort. The eigenvalues of the stability matrix L are

$$\{0.05000 \pm 9.4247i, 0.2500 \pm 6.2781i, 0.2500 \pm 3.1316i\}. \tag{3.8}$$

The nonlinear evolution of acoustic velocity coincides with the linear evolution in this case. Figure 4(e) shows the evolution of acoustic energy (calculated after locally weighted regression smoothing) of these two systems. Note that for a system with the first controller, the linear evolution of acoustic energy coincides with the nonlinear evolution for a short time before nonlinear triggering occurs. Further, it can be seen that the acoustic energy saturates after nonlinearities get triggered leading to a limit cycle. This can be explained as follows.

It is known from the Rayleigh criterion that the acoustic oscillations are driven if this 'driving' is in phase with acoustic pressure. The 'driving' (F') for acoustics is the net effect of heat release and the response of the controller. Hence, acoustic oscillations are driven if

$$\cos\phi(t) = \frac{\displaystyle\int_0^t \mathrm{d}t' \int_0^1 p'(x,t')F'(x,t')\,\mathrm{d}x}{\|p'\| \cdot \|F'\|} > 0, \tag{3.9}$$

where $F' = \dot{Q}'_f \delta(x - x_f) + \gamma/(\gamma - 1)\sum_{m=1}^{K} \alpha_{r_m} v_{c_m} \delta(x - x_{a_m})$ is the acoustic driving. Note that the above expression for phase difference (ϕ) is the correlation between the 'acoustic driving' and the acoustic pressure. Figure 4(f) shows that this phase difference evolves to 90°, causing the acoustic energy to saturate. This is just a sufficient condition for the energy to saturate and not a necessary condition. Figure 5(a) shows the projection of the limit cycle on the Q'_f–u'_f plane. This figure also describes the acoustic response to the heat release rate. Note that the origin is a stable fixed point for sufficiently small-amplitude disturbances as they decay to zero amplitude when $t \to \infty$. However for certain initial conditions, transient growth can lead to nonlinear effects which can sustain the oscillations. This results in a limit cycle even though the origin is a stable fixed point. This is a key difference between this limit cycle and a van der Pol limit cycle, where small-amplitude disturbances are linearly unstable. Further, the limit cycle may not be stable for all values of the control parameters and might disintegrate at some critical values of the control parameters. Figure 5(b) shows the sensitivity of the limit cycle to R_6 (the sixth element of R). Note that $R_6 = -0.2098$ for which the limit cycle behaviour occurs. It is clear that the limit cycle is not stable to a small change in R_6 (denoted by ΔR_6). Note that for $\Delta R_6 > 0$ the oscillations decay. When $\Delta R_6 < 0$, the system becomes linearly unstable (see figure 5c), while for $\Delta R_6 > 0$, the nonlinear effects do not result in the saturation or growth of the oscillations, leading to the decay of the oscillations. This example illustrates that the torus is not stable for all values of R. It would be interesting to explore the torus disintegration route to chaos in the future.

Transient growth associated with non-normality of the system can amplify certain initial disturbances to a large value. However, in a system that is stable according to classical linear stability, this growth cannot be sustained by the linear process alone. High enough amplitude attained because of transient growth may cause the system to enter into a region in which nonlinearity is significant. In the nonlinear regime, the eigenmodes of the system interact directly, causing energy transfer between them. Bagget *et al.* (1995) proposed (in the context of subcritical transition to turbulence) that the role of nonlinearities is to transform a part of the output during this linear

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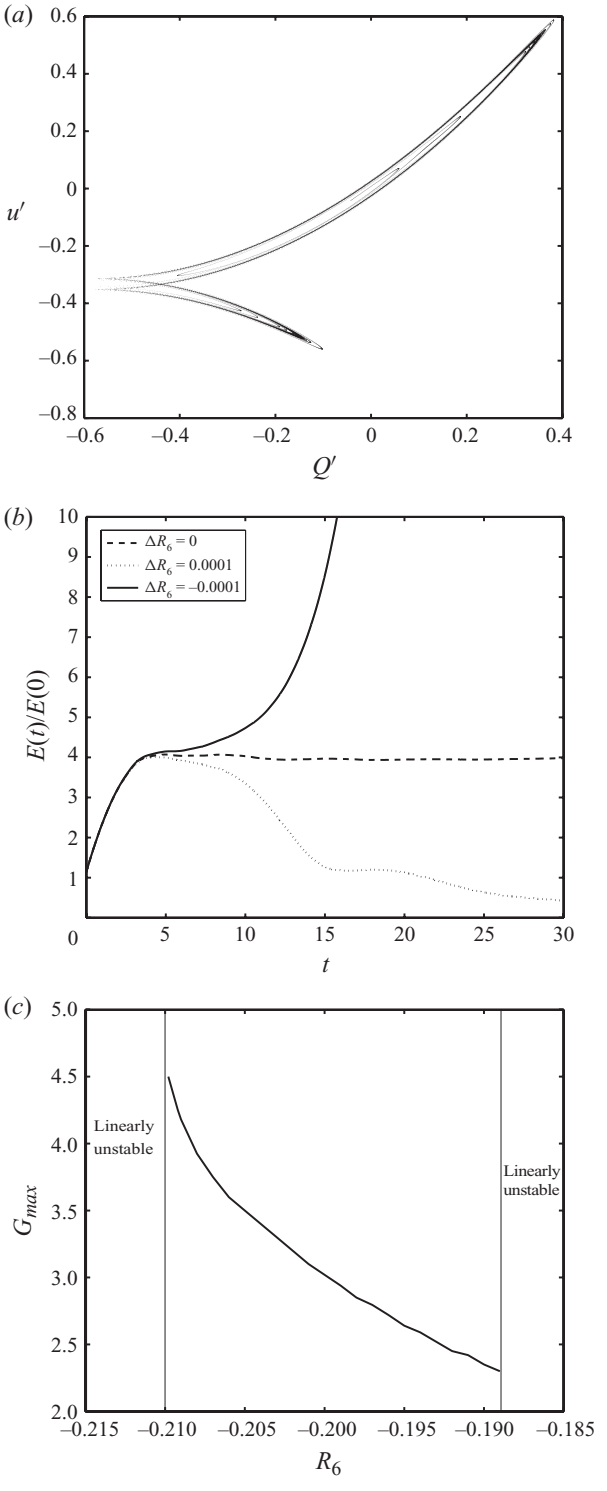


FIGURE 5. (*a*) Acoustic response to the heat release rate. (*b*) Sensitivity to a control parameter R_6. (*c*) Variation of the maximum growth factor as a function of R_6.

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process back into an input to complete a feedback loop and sustain the growth. If the initial condition vector is in the same direction as the one that excites the transient growth (the principal singular vector), small-amplitude perturbations get amplified by several orders of magnitude. The nonlinearity in the system has the effect of transferring some of the energy back to the original direction. Hence, every amplification causes change in the controlled state of the system. If the magnitude of the transformed state is higher than the magnitude of the initial disturbances, then further amplification of the energy occurs. In other words, the target state always gets modified along the transient growth in the energy. Another important point to note is that the redistribution of the energy does not depend upon the precise nature of the nonlinearity (Trefethen *et al.* 1993).

Hence, non-normality can accelerate the nonlinear breakdown of a traditional linear controller by reducing the amplitude of tolerable disturbance. This can be prevented by controlling the transient growth of the system. A transient growth controller prevents a system from entering the nonlinear regime.

3.2. *Secondary modes and linear breakdown of traditional controllers*

A small disturbance in velocity can alter the acoustic driving which in turn creates acoustic perturbations. The driving of the acoustic mode depends upon the phase difference between the acoustic mode and the acoustic driving. In general, the phase evolves with the time. Hence, the phase difference at some instant depends on the phase difference at an earlier instant. The acoustic modes that are driven at these two instants are different. Since the phase difference between the driving force and a particular mode depends on their phase difference at an earlier time, the interaction would depend on that mode which was in phase with acoustic driving at an earlier time. Hence the mode which is driven at a particular instant of time depends on which mode got driven at an earlier time. This can cause frequencies that were initially unexcited to get excited. These frequencies appear as secondary peaks in the Fourier transform of acoustic velocity. Balsubramanian & Sujith (2008*a*,*b*) discussed the nonlinear growth of secondary peaks in the context of bootstrapping. In such a situation, the secondary peaks are sustained by nonlinear interactions, and hence they do not disappear even asymptotically. Note that this happens even when the system is linearly stable.

We will now examine a case in which the secondary peaks appear to be innocuous even when nonlinearities are present, but their presence is responsible for nonlinear triggering. The system with a traditional controller studied in the previous subsection exhibits such a feature. The control parameters are the same as those chosen for figure 4. The number of samples used for the FFT is 35 000 for all time intervals, and the sampling frequency is the ratio of the number of samples to the length of the time interval. Note that there is not much change in the results when the sampling frequency is the same for all time intervals. Figures 6(*a*)–6(*d*) show the FFTs of acoustic velocity obtained in the linearized system for the following time intervals: (*a*) $0 < t < 2$, (*b*) $2 < t < 5$, (*c*) $5 < t < 8$ and (*d*) $8 < t < 30$. It is clear that the frequencies that are initially not dominant begin to grow. These are called 'secondary peaks'. Note that the amplitudes of the peaks initially increase and then decay eventually. This is because the system is asymptotically stable. When nonlinearities are present, some energy from the secondary modes is transferred to the direction of maximum growth (optimum initial direction), causing continuous growth in the system. This redistribution of energy also changes the phase between the driving force and the acoustic pressure. The energy saturates when the phase between the driving force and

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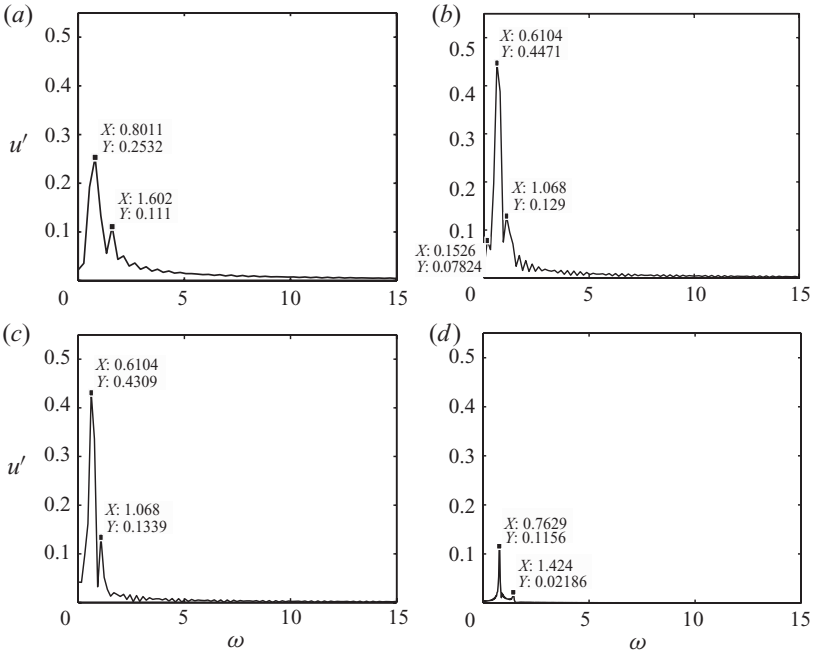


FIGURE 6. FFTs of acoustic velocity at the heating element obtained in the linearized system (with a traditional controller) for (*a*) $0 < t < 2$, (*b*) $2 < t < 5$, (*c*) $5 < t < 8$ and (*d*) $8 < t < 30$.

the acoustic pressure becomes 90° as explained earlier. This can be inferred from figures 7(*a*)–7(*d*), which show the FFTs of acoustic velocity in the full nonlinear system for the same time intervals as those considered above. Note that the secondary peaks decay eventually, but they transfer energy to the optimum initial direction before they decay. Earlier studies on the significance of the secondary modes in a thermoacoustic system emphasized on the fact that their presence can make a system asymptotically unstable, even though the dominant mode is stable (Annaswamy *et al.* 1997). The present study indicates that an asymptotically stable linear system can become nonlinearly unstable even if the secondary modes disappear asymptotically. The presence of secondary peaks during the transients can cause transfer of energy to the dominant mode, leading to nonlinear failure of the traditional controllers.

In the case of a system with a transient growth controller, the secondary peaks (if at all they are present) cannot transfer enough energy to the dominant direction, as the nonlinear effects are not triggered.

The above study indicates that the presence of secondary modes become important only if nonlinear effects are present when the system is asymptotically stable. However, instead of a single initial disturbance, if continuous noise is present in the system, the controller may fail even in the linear regime (Handel 2004). The continuous noise always has a non-zero projection along the growing eigenvector of the system. This continuous departure of the target state from the controlled state because of the transient growth may lead to the failure of a traditional linear controller. Note that this linear breakdown cannot happen if there is no transient growth. Hence, a transient growth controller cannot fail in the linear regime.

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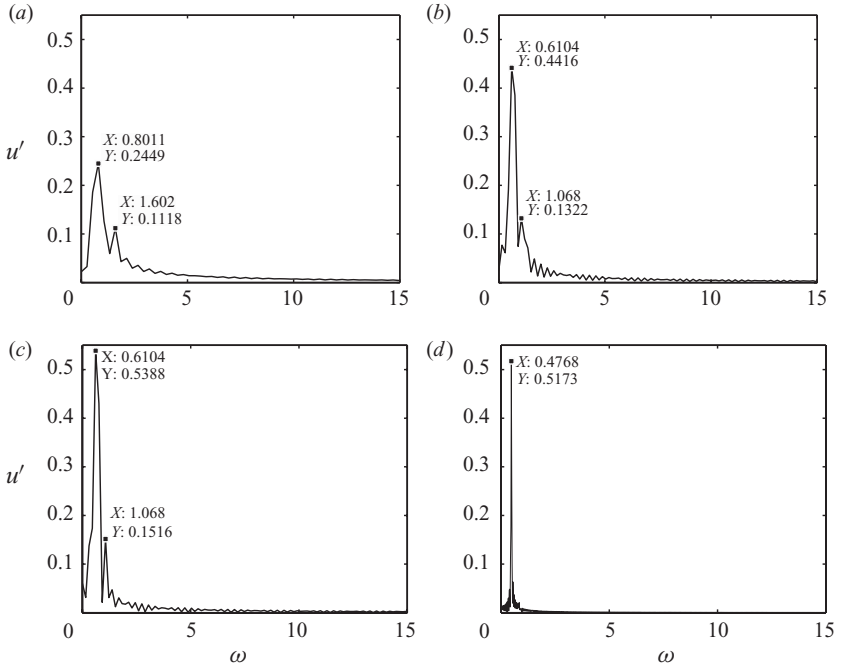


FIGURE 7. FFTs of acoustic velocity at the heating element obtained in the nonlinear system (with a traditional controller) for (*a*) $0 < t < 2$, (*b*) $2 < t < 5$, (*c*) $5 < t < 8$ and (*d*) $8 < t < 30$.

3.3. *Effect of mean temperature jump*

In this subsection we will study the effect of temperature jump on the dynamics of a system with a transient growth controller. Using (2.5) we will first calculate the ratio of temperatures for a system with the following system parameters: $C_v = 719.0\,\mathrm{kJ\ (kg\ K)^{-1}}$, $\gamma = 1.4$, $x_f = 0.75$, $T_1 = 300\,\mathrm{K}$, $M_1 = 0.003$, $S = 1.56 \times 10^{-3}\,\mathrm{m^2}$, $L_w = 2.0\,\mathrm{m}$, $\rho_1 = 1.205\,\mathrm{kg\,m^{-3}}$, $\lambda = 0.0328\,\mathrm{W\ mK^{-1}}$, $\tau = 0.01$ and $T_w = 1200\,\mathrm{K}$. Note that this system has the same cold flow parameters as the system considered in the previous subsections except for the wire temperature. The ratio of the temperature of hot flow to the temperature of cold flow is

$$\frac{\bar{T}_2}{\bar{T}_1} \sim 1.6 \implies T_2 \sim 480\,\mathrm{K}. \tag{3.10}$$

The mean temperature jump could be higher in practical combustors as discussed in Kaufmann *et al.* (2002), $T_2/T_1 = 5$; however, this temperature ratio is quite high for a Rijke tube. We will now demonstrate that the transient growth controller designed using the constant mean temperature (T_1) assumption can restrict transient growth of this system as well. Figure 8 shows the difference in the behaviour of linear and nonlinear evolution (from the 'optimum' initial direction) of acoustic energy in this system with a transient growth controller when the mean temperature gradient is taken into account. The actuator locations are the same as those mentioned in § 3.1. The following values for the actuator response functions were chosen for both the cases (with and without the temperature gradient):

$$R = \{0.01702, -0.1100, 0.3397, -0.5813, 0.5470, -0.2321\}, \tag{3.11}$$

$$S = \{0.01857, -0.0521, 0.1191, -0.1288, 0.0636, 0.0\}. \tag{3.12}$$

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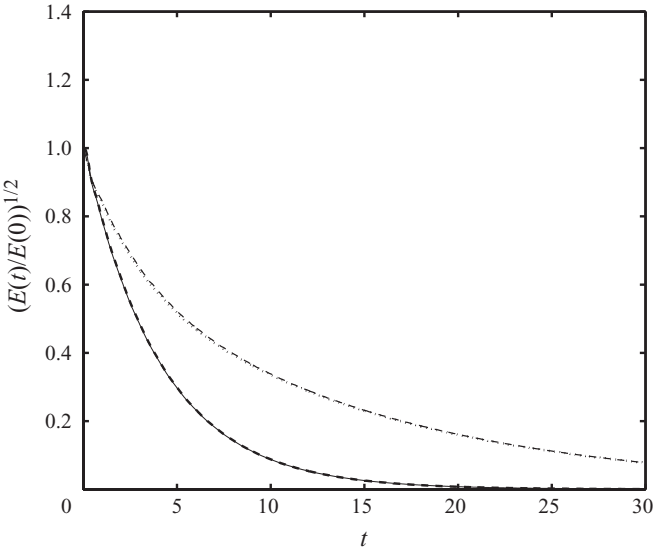


FIGURE 8. Linear and nonlinear evolution of acoustic energy in a system with (dashed line) and without (solid line) mean temperature jump. The nonlinear evolution coincides with the linear evolution for both cases.

Note that the control designed on the basis of the constant temperature assumption can prevent nonlinear driving even when there is a temperature jump. This shows that there is no significant change in the dynamics of a system with a transient growth controller in the presence of the temperature gradient.

4. Conclusion

Non-normality of a thermoacoustic system causes transient growth of acoustic energy due to the non-orthogonal eigenmodes interacting with the base flow in the linear regime. The non-normal nature of the system causes transient growth, even though all the eigenvectors are decaying. The transient growth in acoustic energy can cause the amplitude of acoustic oscillations to grow to values high enough to make nonlinear effects significant. This can lead to the failure of a linear controller designed on the basis of the individual eigenvalues of the system. Classical stability analysis focuses on the asymptotic nature of the system, while it does not give any idea about the transient dynamics of the system. Hence, controlling transient growth is essential to ensure that the system is completely stable.

We have analysed the role of non-normality in the active control of thermoacoustic instabilities in the context of a horizontal Rijke tube. We have reduced the system of partial differential equations to a system of coupled ordinary differential equations. Following Annaswamy *et al.* (1997), three acoustic modes have been used to study the effect of actuators on the this simple model of a Rijke tube. There is appearance of secondary peaks in the linearized system which eventually decay, as the system is asymptotically stable. However, these secondary peaks can be sustained by nonlinear effects. Further, a traditional linear controller might fail even in the linear regime if there is continuous noise in the system. We have shown that 'nonlinear driving' of low-amplitude disturbance in a horizontal Rijke tube can be prevented by controlling transient growth. It has also been shown that a control designed on the basis of

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the constant temperature assumption works even when there is a mean temperature jump.

In this paper, the actuator parameters do not correspond to the optimum value for which the transfer function norm is minimized. Systematic feedback control methods (such as linear–quadratic–Gaussian/H_∞) can be used to minimize the actuator norm. Further, the use of the three-mode Galerkin expansion may not be suitable for control of a general thermoacoustic system. In fact, there are known examples of systems in fluid control in which the effect of non-normality cannot be completely removed. In such cases, the controllers must be designed to minimize transient growth and prevent the onset of nonlinear effects. Design of controllers that incorporate the effect of non-normality for more sophisticated physical models is left for future work.

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