QUANTUM MEASUREMENTS AND ALGORITHMIC RANDOMNESS

TEJAS BHOJRAJ

Abstract. Nies and Scholz formalized the notion of an infinite qubitstring and referred to it as a 'state'. They defined 'quantum Martin-Löf randomness' for states. We give a notion of measurement of a state in a computable basis and introduce 'quantum measurement randomness', a randomness notion for states. A state is quantum measurement random if measuring it in any computable basis yields a Martin-Löf random bitstring with probability one. Our main result is that quantum Martin-Löf randomness strictly implies quantum measurement randomness. This uses the construction of a quantum measurement random state which is not quantum Martin-Löf random. We prove two general results on which this construction relies: The first concerns Martin-Löf randomness relative to computable measures and extends a result of V. Vovk. The second is a combinatorial result about Kronecker products.

§1. Introduction. Nies and Scholz formalized the concept of an infinite sequence of qubits and referred to it as a state [11]. We formalize a notion of measurement of a state in a computable basis. We define a state to be 'quantum measurement random' if it yields a Martin-Löf random bitstring with probability one when measured in any computable basis. Our main theorem is that there is a quantum measurement random state which is not quantum Martin-Löf random. Its proof goes through two general results. The first extends a result of V. Vovk on Martin-Löf randomness for computable measures while the second is a combinatorial lemma on Kronecker products.

1.1. Classical and quantum algorithmic randomness. Cantor space (2^ω) is the set of infinite sequences of bits equipped with a compact topology and the uniform (Lebesgue) measure. Some elements of 2^ω possess regularities or patterns while others are 'random': The infinite bitstring 101010... has a regularity, namely that the ones and zeroes alternate. By contrast, a bitstring obtained by tossing a fair coin infinitely often almost surely lacks any regularities. Algorithmic randomness quantifies the sense in which the latter bitstring is more 'random' than the former [5, 7, 8, 13]. An element of 2^ω is said to be random if it possesses no *effective* regularities.[1] A key idea in algorithmic randomness is that an element of 2^ω possesses an effective regularity if and only if it is contained in an effective null set.[2] Thus an

Received December 30, 2024.

2020 *Mathematics Subject Classification.* Primary 03D32, Secondary 81P45, 68Q30.

Key words and phrases. Martin-Löf randomness, Kolmogorov complexity, qubit, density matrix, projective measurement.

[1] Roughly speaking, 'effective' regularities are those which are 'describable' by a Turing machine.

[2] A set A is effectively null if there is a Turing machine which computes a sequence of effectively open sets, $(U_n)_n$ such that the uniform measure of U_n is at most 2^{-n} and $A \subseteq U_n$ for all n.

© The Author(s), 2026. Published by Cambridge University Press on behalf of The Association for Symbolic Logic. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (https://creativecommons.org/licenses/by/4.0/), which permits unrestricted re-use, distribution, and reproduction in any medium, provided the original work is properly cited.

0022-4812/00/0000-0000
DOI:10.1017/jsl.2026.10191

https://doi.org/10.1017/jsl.2026.10191 Published online by Cambridge University Press

element of 2^ω is random iff it is in no effective null set. Different levels of effectivity give different levels of randomness.

Martin-Löf randomness is arguably the most important notion of algorithmic randomness. Martin-Löf random bitstrings are incompressible and resist concise descriptions in that they have high initial segment prefix-free Kolmogorov complexity (K) [5, 7, 8, 13]. Schnorr randomness is a notion of randomness strictly weaker than Martin-Löf randomness. It has characterizations in terms of a version of K defined using computable measure machines. Quantum algorithmic randomness is a generalization of algorithmic randomness to the quantum setting [10]. While algorithmic randomness studies the randomness of infinite sequences of *bits*, quantum algorithmic randomness studies the randomness of infinite sequences of *qubits*. Nies and Scholz [10] formalized a notion of an infinite qubitstring (an infinite sequence of qubits) and referred to it as a *state*. They also defined quantum Martin-Löf randomness and quantum Solovay randomness for states. Quantum Schnorr randomness was defined in [3].

The theory of quantum algorithmic randomness has nice parallels with that of classical algorithmic randomness: Any $X \in 2^\omega$ gives a corresponding state ρ_X. Quantum Martin-Löf randomness and quantum Schnorr randomness generalize Martin-Löf randomness and Schnorr randomness, respectively, in the sense that X is Martin-Löf random iff ρ_X is quantum Martin-Löf random and X is Schnorr random iff ρ_X is quantum Schnorr random. Quantum Martin-Löf randomness is equivalent to quantum Solovay randomness and is strictly stronger than quantum Schnorr randomness [3]. A quantum version of the law of large numbers is satisfied by quantum Schnorr random states [3]. Weak quantum Solovay and quantum Schnorr states can be characterized using QK, a quantum analogue of K. These characterizations mirror the characterizations of Solovay random and Schnorr random bit sequences in terms of K [2]. The quantum algorithmic randomness of states is also related to the von-Neumann entropy of its initial segments [4]. This article introduces quantum measurement randomness, a new notion of quantum algorithmic randomness based on quantum measurement.

1.2. Motivation. An m dimensional quantum system is represented by a density matrix on $\mathbb{C}^m$ which is an $m \times m$ Hermitian, positive semidefinite matrix with trace one (see [6, Section 2.4]). A 'qubit' is a two dimensional quantum system and is the quantum analogue of the classical 'bit'. A system of n qubits is a 2^n dimensional quantum system and thus is given by a density matrix on $\mathbb{C}^{2^n}$. A state $\rho = (\rho_n)_{n=1}^\infty$ is a coherent sequence such that ρ_n is a density matrix on $\mathbb{C}^{2^n}$ and represents the first n qubits of ρ (see Definition 2.3 for the details).

Projective measurement (henceforth simply referred to as 'measurement') of a density matrix is one of the most important and fundamental operations in quantum mechanics (see [6, Section 2.2.5] and [15, Sections 3.4 and 3.5.5]). We formulate an analogous notion of measurement for a state which is a *sequence* of density matrices. This motivates our formalization of a notion of measurement of a state in Section 3.

Measuring a qubit yields 0 or 1 each with a certain probability. An obvious way to measure ρ is to measure its first qubit, followed by measuring its second qubit, followed by measuring its third qubit, and so on. That is, the n^{th} qubit of ρ is measured after the first $n-1$ qubits have been measured in order. As each

https://doi.org/10.1017/jsl.2026.10191 Published online by Cambridge University Press

measurement yields 0 or 1, measuring the first n qubits of ρ yields an element of $\{0,1\}^n$. Letting n tend to infinity thus yields an element of 2^ω.

As each measurement is probabilistic, measuring the first n qubits of ρ yields a probability distribution on $\{0,1\}^n$, where the probability of $\sigma \in \{0,1\}^n$ is the probability of obtaining σ when the first n qubits of ρ are measured. Letting n tend to infinity yields a distribution on 2^ω.

We elaborate on this idea: Let $B = (B_n)_n$ be an infinite sequence of orthonormal bases of $\mathbb{C}^2$. Measuring the first qubit of ρ using B_1 yields 0 or 1 thus inducing a distribution μ_1 on $\{0,1\}$. Measuring the second qubit of ρ using B_2 after the first qubit has been measured using B_1 similarly induces a distribution μ_2 on $\{0,1\}^2$. Continuing this, sequentially measuring the first n qubits of ρ induces a distribution μ_n on $\{0,1\}^n$. Moreover, the coherence of ρ_{n+1} and ρ_n (see Definition 2.3) implies that μ_{n+1} extends μ_n in the natural way. The state ρ thus induces an infinite sequence of distributions $(\mu_n)_n$ such that μ_{n+1} extends μ_n. This sequence induces a distribution on Cantor space.

The above intuitive idea motivates our formalization of the process of qubit-wise measuring ρ in Section 3.

If measuring ρ yields a non-random element of 2^ω with high probability, then the behavior of ρ on measurement exhibits a regularity. For example if measuring ρ yields 1010 ... with probability one, then the behavior of ρ on measurement is highly regular. By contrast, if measuring ρ yields a Martin-Löf random element of 2^ω with probability one, then the behavior of ρ on measurement is random and irregular. It is plausible to consider states of the former kind to be random while those of the latter variety to be non-random. This motivates our definition of quantum measurement randomness in Section 4. A state is defined to be quantum measurement random if measuring it yields a Martin-Löf random with probability one.[3]

1.3. Outline and main results. Section 2 introduces the relevant background from algorithmic randomness and quantum theory. Section 3 formalizes how measurement of a state in a computable basis induces a distribution on 2^ω. Section 4 introduces quantum measurement randomness. Section 4.1 gives an overview of the relationships between quantum measurement randomness and other quantum randomness notions. Section 4.3 shows that quantum measurement random states satisfy a quantum version of the law of large numbers. This is necessary for legitimizing quantum measurement randomness as a plausible randomness notion, as satisfying the law of large numbers is a weak form of randomness which any reasonable notion of randomness should imply.

Section 5 isolates two important results central to our main result in Section 6. These are general results which may be intrinsically mathematically interesting independently of their specific applications in Section 6. The first (Theorem 5.2) concerns Martin-Löf randomness for arbitrary computable measures. The second (Lemma 5.3) concerns Kronecker products of vectors in $\mathbb{C}^2$.

Section 6 contains the main result: There is a quantum measurement random state which is not quantum Martin-Löf random.

[3] Equivalently, a state is quantum measurement random if the distribution induced on 2^ω by measuring it assigns a probability one to the set of Martin-Löf random bit sequences.

https://doi.org/10.1017/jsl.2026.10191 Published online by Cambridge University Press

§2. Background. We state some preliminaries relevant to the article.

2.1. Notation. Let 2^ω denote the collection of infinite sequences of bits, let 2^n denote the set of bit strings of length n, let $2^{<\omega} := \bigcup_n 2^n$ and let $2^{\leq\omega} := 2^{<\omega} \cup 2^\omega$. If $C \subseteq 2^{<\omega}$, let $[\![C]\!] \subseteq 2^\omega$ be the set of all $X \in 2^\omega$ such that an initial segment of X is in C. For all $n \in \omega$, let $X(n)$ denote the n^{th} bit of a bitstring $X \in 2^{\leq\omega}$. We index the positions in a string starting from 1 instead of from 0. We use 'distribution' and 'measure' interchangeably to mean a probability distribution. The uniform measure on 2^ω denoted by λ is the measure induced by letting the measure of $[\![\tau]\!]$ to be $2^{-|\tau|}$ for each $\tau \in 2^{<\omega}$. For any measure m on 2^ω, we will abuse notation by writing $m(\tau)$ instead of the more cumbersome $m([\![\tau]\!])$ for $\tau \in 2^{<\omega}$. By 'projection' we will always mean a Hermitian projection.

2.2. Quantum theory. As per the bra–ket notation, a vector $v \in \mathbb{C}^d$ is denoted as $|v\rangle$ and its dual is denoted by $\langle v|$. In a fixed basis of $\mathbb{C}^d$, $|v\rangle$ is a column vector while $\langle v|$ is a row vector which is equal to $((|v\rangle)^*)^T$ (see [6, Sections 2.1.1 and 2.1.4]). Let I denote the identity matrix on $\mathbb{C}^2$.

A density matrix is a complex, Hermitian, positive semidefinite matrix with trace one. A two-dimensional quantum system is called a qubit and is represented by a density matrix on $\mathbb{C}^2$. An n-qubit quantum system is a 2^n-dimensional quantum system lives in the tensor product $\bigotimes_{i=1}^n \mathbb{C}^2 = \mathbb{C}^{2^n}$. It is represented by a density matrix on $\mathbb{C}^{2^n}$.

Definition 2.1. The set $\{|0\rangle, |1\rangle\}$ is a fixed orthonormal basis for $\mathbb{C}^2$ (see [6, Section 2.2.1]). For any $n > 0$, the standard basis (or, the computational basis) of $\mathbb{C}^{2^n}$ is the orthonormal basis,

$$\left\{ \bigotimes_{i=1}^{n} |\sigma(i)\rangle : \sigma \in 2^n \right\}.$$

For convenience, we denote $\bigotimes_{i=1}^n |\sigma(i)\rangle$ simply by $|\sigma\rangle$.

All matrices, henceforth, are with respect to the standard basis.

Definition 2.2. Let $m > 0$. A projective measurement on $\mathbb{C}^m$ is a finite set $\{\Pi_j\}_j$, where each Π_j is a projection on $\mathbb{C}^m$ and $\sum_j \Pi_j$ is the identity operator. By 'measurement' we will always mean a projective measurement.

Suppose that d, a density matrix on $\mathbb{C}^m$, is measured using $\{\Pi_j\}_j$. The measurement obeys the following measurement axioms (see [15, p. 126]):

(1) The outcome j occurs with probability $\mathrm{Tr}(\Pi_j d)$.
(2) If the outcome j occurs, then the state *after* the measurement (the 'post-measurement' state) is described by the density matrix $\dfrac{\Pi_j d \Pi_j}{\mathrm{Tr}(\Pi_j d)}$.

We sometimes say that 'Π_i is the outcome of the measurement' when we really mean that the 'outcome of the measurement is the integer i which labels the operator Π_i in $\{\Pi_j\}_j$'.

2.3. Classical and quantum algorithmic randomness. The classical algorithmic randomness notions are from [7, 8, 13] and the quantum algorithmic randomness notions are from [3, 10].

https://doi.org/10.1017/jsl.2026.10191 Published online by Cambridge University Press

DEFINITION 2.3. A state, $\rho = (\rho_n)_{n=1}^{\infty}$ is an infinite sequence of density matrices such that $\rho_n \in \mathbb{C}^{2^n \times 2^n}$ and $\forall n > 1$, $PT_{\mathbb{C}^2}(\rho_n) = \rho_{n-1}$.

Here $PT_{\mathbb{C}^2}$ denotes the partial trace which 'traces out' the last qubit from $\mathbb{C}^{2^n}$ (see [6, Section 2.4.3]). The state ρ represents an infinite sequence of qubits whose first n qubits are described by ρ_n. The definition requires ρ to be coherent in the sense that for all n, ρ_n, when 'restricted' via the partial trace to its first $n-1$ qubits, has the same measurement statistics as the state on $n-1$ qubits given by ρ_{n-1}. For any $\sigma \in 2^{<\omega}$ let $|\sigma\rangle$ denote $\bigotimes_i |\sigma(i)\rangle$. Any element of 2^{ω} can be thought of as a state: If $X \in 2^{\omega}$, the state $\rho_X = (\rho_n)_n$ given by $\rho_n = |X \upharpoonright n\rangle\langle X \upharpoonright n|$ is the quantum analog of X.

DEFINITION 2.4. Let $\tau = (\tau_n)_{n=1}^{\infty}$ be the state given by $\tau_n = 2^{-n} \bigotimes_{i=1}^{n} I$.

Recall that a complex algebraic number is the root of a polynomial with rational coefficients.

DEFINITION 2.5. A *special projection* is a Hermitian projection matrix with complex algebraic entries.

By a result of Rabin, there is a 1–1 function from the complex algebraic numbers to ω such that the field operations in the image are computable. This function may be used to identify a special projection with a natural number (see [10, p. 8]). We define a sequence of special projections to be computable when the corresponding sequence of natural numbers is computable.[4] A state $\rho = (\rho_n)_n$ is computable if each ρ_n has complex algebraic entries and the sequence $(\rho_n)_n$ is computable.

DEFINITION 2.6. A quantum Σ_1^0 class (q-Σ_1^0) G is a computable sequence $G = (p_n)_{n=1}^{\infty}$ such that p_n is a special projection on $\mathbb{C}^{2^n}$ and $\text{range}(p_n \otimes I) \subseteq \text{range}(p_{n+1})$ for all n.

This is the quantum analogue of a Σ_1^0 class in the classical setting.

DEFINITION 2.7. A Σ_1^0 class $S \subseteq 2^{\omega}$ is any set of the form, $S = \bigcup_{\sigma \in C} [\![\sigma]\!]$, where C is a computably enumerable set of strings.[5]

DEFINITION 2.8. If $G = (p_n)_n$ is a q-Σ_1^0 class and $\rho = (\rho_n)_n$ is a state define $\rho(G) := \lim_n(\text{Tr}(\rho_n p_n))$.

DEFINITION 2.9. For a state $\rho = (\rho_n)_n$ and q-Σ_1^0 class $G = (p_n)_n$, $\rho(G) := \lim_n \text{Tr}(\rho_n p_n)$.

Recall the definition of a Martin-Löf test (ML-test).

DEFINITION 2.10. An ML-test is a computable sequence, $(S_m)_{m=1}^{\infty}$ of Σ_1^0 classes such that $\lambda(S_m) \leq 2^{-m}$ for all m. A $X \in 2^{\omega}$ fails $(S_m)_m$ if $X \in S_m$ for all m and is Martin-Löf random if it does not fail any ML-test. Let **ML** denote the set of Martin-Löf random elements of 2^{ω}.

Its quantum generalization is as follows.

---

[4] Recall that a sequence $(a_n)_n$ of natural numbers is computable if there is a total computable function φ such that $\varphi(n) = a_n$ for all n.

[5] An equivalent definition which will be used later is: A Σ_1^0 class $S \subseteq 2^{\omega}$ is any set of the form $S = \bigcup_{n=1}^{\infty} [\![A_n]\!]$, where $(A_n)_{n=1}^{\infty}$ is a computable sequence such that for all n, $A_n \subseteq 2^n$ and $[\![A_n]\!] \subseteq [\![A_{n+1}]\!]$.

https://doi.org/10.1017/jsl.2026.10191 Published online by Cambridge University Press

DEFINITION 2.11. A quantum Martin-Löf test (q-ML test) is a computable sequence $(G_m)_{m=1}^{\infty}$ of q-Σ_1^0 classes such that $\tau(G_m) \leq 2^{-m}$ for all m. A state ρ is said to *fail* the q-ML test $(G_m)_m$ at order δ if $\rho(G_m) > \delta$ for all m. A state ρ is said to pass the q-ML test $(G_m)_m$ at order δ if it does not fail the test at order δ. A state ρ is said to be quantum Martin-Löf random if it passes all q-ML tests at all $\delta > 0$.

§3. Measuring a state. We formalize the notion of qubit-wise measurement of a state which involves measuring each qubit of the state in order. We first define the measurement operators used in the measurement of a state.

3.1. The measurement system.

DEFINITION 3.1. A computable measurement system $B = ((b_0^n, b_1^n))_{n=1}^{\infty}$ ('system' for short) is a computable sequence of orthonormal bases of $\mathbb{C}^2$ such that b_i^n has complex algebraic entries for all n and i.

Recall the notion of a measurement from Definition 2.2. Given a system $B = ((b_0^n, b_1^n))_n$, let $\Pi_i^B(n) = |b_i^n\rangle\langle b_i^n|$ for $i \in \{0,1\}$ and set

$$\Pi^B(n) := \{\Pi_0^B(n), \Pi_1^B(n)\}. \tag{3.1}$$

Note that $(\Pi^B(n))_n$ is a sequence of measurements on $\mathbb{C}^2$. Measuring ρ qubit-wise in the system B will correspond to measuring the first qubit of ρ using $\Pi^B(1)$, followed by measuring the second qubit using $\Pi^B(2)$, followed by measuring the third qubit using $\Pi^B(3)$, and so on for all n. As each measurement will have an outcome in $\{0,1\}$, this sequence of qubit-wise measurements will yield an element of 2^ω in the limit. Fix a $\tau \in 2^n$ and consider the projection $\bigotimes_{i=1}^n \left(|b^i_{\tau(i)}\rangle\langle b^i_{\tau(i)}|\right) = \bigotimes_{i=1}^n \Pi^B_{\tau(i)}(i)$. For notational simplicity, define

$$\Pi^B(\tau) := \bigotimes_{i=1}^{n} \Pi^B_{\tau(i)}(i). \tag{3.2}$$

(This notation should not cause any confusion with the object $\Pi(n)$ as n is a natural number while τ is a bitstring.) For each i, $|b^i_{\tau(i)}\rangle$ is an element of $\mathbb{C}^2$ (a column vector) and $\langle b^i_{\tau(i)}|$ is its dual (a row vector). Thus, $\Pi^B_{\tau(i)}(i) = |b^i_{\tau(i)}\rangle\langle b^i_{\tau(i)}|$ is the projection operator onto the one dimensional subspace span$(|b^i_{\tau(i)}\rangle)$. $\Pi^B(\tau)$ is thus the tensor product of n many projection operators such that the i^{th} projection in the product is $\Pi^B_{\tau(i)}(i)$. Note that $\{\Pi^B(\tau) : \tau \in 2^n\}$ is a measurement on $\mathbb{C}^{2^n}$ (Definition 2.2) since $\{|\bigotimes_{i=1}^n b^i_{\tau(i)}\rangle : \tau \in 2^n\}$ is an orthonormal basis of $\mathbb{C}^{2^n}$.

3.2. Premeasures. A (probability) premeasure p is a function from $2^{<\omega}$ to $[0,1]$ satisfying $p(\emptyset) = 1$ and $\forall \tau, p(\tau) = p(\tau 0) + p(\tau 1)$ (Section 6.12.1 in [7]). A premeasure p uniquely extends to a probability measure $\mathcal{D}$ on 2^ω such that $p(\tau) = \mathcal{D}([\![\tau]\!])$ for all $\tau \in 2^{<\omega}$. A random process whose outcome is in 2^ω induces a premeasure p where $p(\tau)$ is the probability that the outcome of the random process extends τ. For example, an infinite sequence of independent tosses of a fair coin induces a premeasure which maps $\tau \mapsto 2^{-|\tau|}$. An infinite sequence of independent tosses of a coin which has a 0.75 probability of turning up '1' induces a premeasure

https://doi.org/10.1017/jsl.2026.10191 Published online by Cambridge University Press

which maps

$$\tau \mapsto (0.75^{|\{i:\tau(i)=1\}|})(0.25^{|\{i:\tau(i)=0\}|}).$$

We will be interested in the premeasure induced by the random process of measuring a state.

3.3. Informal description. We first give an intuitive, informal description of the process of qubit-wise measuring a state in the system B. The reader may choose to skip this section and move directly to Section 3.4 without loss of continuity.

Consider the random process of tossing a fair coin independently, infinitely often. This is represented by a sequence $(Z_n)_n$ of $\{0,1\}$-valued, Bernoulli-1/2, i.i.d. random variables, where Z_n is the outcome of the n^{th} toss. This process induces a premeasure p_u, where $p_u(\sigma) = 2^{-|\sigma|}$ is the probability that $Z_i = \sigma(i)$ for all $i \leq |\sigma|$.

The random process of measuring a state is similarly represented by a sequence of random variables and the induced premeasure, which we now describe.

Fix a state $\rho = (\rho_n)_n$ and a system $B = (\Pi^B(n))_n$ (see Definition 3.1). Measuring ρ qubit-wise using B can be formalized by a sequence of $\{0,1\}$-valued random variables $(X_n)_n$, where X_n is the outcome of the measurement of the n^{th} qubit of ρ. The details are as follows. We drop the superscript B from Π^B as B is fixed henceforth.

Measure the first qubit ρ_1 of ρ using $\Pi(1)$ and let $X_1 \in \{0,1\}$ be the outcome of this measurement. This measurement causes ρ_2 to change to the post-measurement state represented by the density matrix $\hat{\rho}_2$ (see Definition 2.2). Now measure the second qubit of $\hat{\rho}_2$ using $\Pi(2)$ and let $X_2 \in \{0,1\}$ be the outcome of this second measurement. The first and second measurements cause ρ_3 to change to the post-measurement state represented by the density matrix $\hat{\rho}_3$ and so on. This process, referred to as the 'qubit-wise measurement of ρ using B', generates an infinite bitstring whose n^{th} bit is X_n.

We now describe the premeasure p^B_ρ (denoted simply by p) induced by this process. For any $\tau \in 2^{<\omega}$, $p(\tau)$ will be the probability that qubit-wise measuring the first $|\tau|$ many qubits of ρ using B gives τ.

Measure ρ_1 in the system $\Pi(1) = \{\Pi_0(1), \Pi_1(1)\}$ (recall the notation (3.1)) and define X_1 to be the $i \in \{0,1\}$ such that $\Pi_i(1)$ is the outcome of the measurement. Let $\hat{\rho}_2$ be the density matrix corresponding to the post-measurement state of ρ_2 given that ρ_2 yields $\Pi_{X_1}(1) \otimes I$ when measured in the system

$$\{\Pi_0(1) \otimes I, \Pi_1(1) \otimes I\},$$

which is the system for measuring the first qubit. The measurement axioms (recall Section 2.2) imply that

$$\hat{\rho}_2 = \frac{(\Pi_{X_1}(1) \otimes I)\rho_2(\Pi_{X_1}(1) \otimes I)}{\mathrm{Tr}(\Pi_{X_1}(1) \otimes I\rho_2)}.$$

To define X_2, measure $\hat{\rho}_2$ in the system

$$\{I \otimes \Pi_0(2), I \otimes \Pi_1(2)\},$$

which is the system for measuring the second qubit. Set $X_2 := i$, where $i \in \{0,1\}$ is such that $I \otimes \Pi_i(2)$ is obtained after the measurement. (The measurement when

https://doi.org/10.1017/jsl.2026.10191 Published online by Cambridge University Press

defining X_2 is on $\hat{\rho}_2$ and not on ρ_2 as the state after the first measurement is $\hat{\rho}_2$ and not ρ_2.)

The random variable X_n is defined similarly. The function p is: For any n and for any $\tau \in 2^n$, $p(\tau) = \mathbb{P}(X_1 = \tau(1), \dots, X_n = \tau(n))$. For any $\tau \in 2^n$ standard calculations show that

$$p(\tau) = \mathbb{P}(X_1 = \tau(1), \dots, X_n = \tau(n)) = \mathrm{Tr}\big[\rho_n \Pi(\tau)\big]. \tag{3.3}$$

We see that for any $\tau \in 2^{<\omega}$, $p(\tau)$ is the probability that $X_i = \tau(i)$ for all $i \leq |\tau|$. We show in Lemma 3.4 that $p = p_\rho^B$ is a premeasure and thus extends to a unique distribution μ_ρ^B on 2^ω.

Note that while the random variables $Z_1, Z_2, \dots$ are mutually independent, the random variables $X_1, X_2, \dots$ need not be independent as the density matrices $\rho_1, \rho_2, \dots$ could be entangled tensors in general.

REMARK 3.2. We have shown in Equation (3.3) that $p(\tau) = \mathrm{Tr}\big[\rho_n \Pi^B(\tau)\big]$ is the probability of obtaining $\tau \in 2^n$ when the first n many qubits of ρ are measured *sequentially in increasing order.* (That is, the first qubit is measured, followed by a measurement of the second qubit of the resulting post-measurement state, followed by a measurement of the third qubit on the post-measurement state after the first and second measurements and so on.) Obviously this probability should equal the probability of obtaining τ when the first n qubits of ρ are measured *simultaneously*. We verify this. Recall from Section 3.1 that $\Lambda_n := \{\Pi^B(\tau) : \tau \in 2^n\}$ is the measurement corresponding to simultaneously measuring the first n qubits using B. By the axioms in Section 2.2, the probability of obtaining τ when measuring ρ_n using Λ_n is equal to $\mathrm{Tr}\big[\rho_n \Pi^B(\tau)\big]$. This is equal to the value of $p(\tau)$ in Equation (3.3) which is the probability of obtaining τ when the qubits of ρ_n are measured sequentially in increasing order. This shows that for any n, the sequential process of qubit-wise measuring ρ_n in B is equivalent to measuring ρ_n using Λ_n.

3.4. Formal details. We formalize the above intuitive description of measurement of a state. Take a state ρ and a system B. We will first define a premeasure (see Definition 3.3) p_ρ^B depending on ρ and B. Letting μ_ρ^B be the unique extension of p_ρ^B to 2^ω we will show that for any measurable $S \subseteq 2^\omega$, $\mu_\rho^B(S)$ is the probability that the qubit-wise measurement of ρ in the system B yields an element from S.

DEFINITION 3.3. Let $\rho = (\rho_n)_n$ be a state and $B = ((b_0^n, b_1^n))_n$ be a system. The function p_ρ^B on $2^{<\omega}$ is defined as: $p_\rho^B(\emptyset) = 1$. For all $n > 0$ and all $\tau \in 2^n$,

$$p_\rho^B(\tau) = \mathrm{Tr}[\rho_n \Pi^B(\tau)]. \tag{3.4}$$

As explained in Remark 3.2, $p_\rho^B(\tau)$ is the probability of obtaining τ when the first $|\tau|$ many qubits of ρ are measured in the system B, where the measurement can be thought of as a simultaneous measurement of all n qubits or as a sequential, qubit-wise measurement. We now show the following lemma.

LEMMA 3.4. *The function p_ρ^B is a premeasure.*

PROOF. As B will be fixed henceforth, we drop the B superscript to keep the notation readable. We show that $\forall \tau \in 2^{<\omega}$, $p_\rho(\tau) = p_\rho(\tau 0) + p_\rho(\tau 1)$. If $\tau = \emptyset$, then

https://doi.org/10.1017/jsl.2026.10191 Published online by Cambridge University Press

it can be easily checked that $p_\rho(0) + p_\rho(1) = 1$. Let $n > 0$ be arbitrary and fix a $\tau \in 2^n$. Recall the notation in (3.2). Note that $\Pi(\tau j) = \Pi(\tau) \otimes \Pi_j(n+1)$ for $j \in \{0,1\}$. So

$$p_\rho(\tau 0) + p_\rho(\tau 1) = \mathrm{Tr}\big[\rho_{n+1}(\Pi(\tau) \otimes \Pi_0(n+1))\big] + \mathrm{Tr}\big[\rho_{n+1}(\Pi(\tau) \otimes \Pi_1(n+1))\big].$$

By the linearity of Tr, this equals

$$\mathrm{Tr}\big[\rho_{n+1}\big(\Pi(\tau) \otimes \{\Pi_0(n+1) + \Pi_1(n+1)\}\big)\big] = \mathrm{Tr}\big[\rho_{n+1}(\Pi(\tau) \otimes I)\big].$$

By the definition of the partial trace, this equals

$$\mathrm{Tr}\big[\mathrm{Tr}_{\mathbb{C}^2}(\rho_{n+1})\Pi(\tau)\big] = \mathrm{Tr}[\rho_n \Pi(\tau)] = p_\rho(\tau). \qquad \dashv$$

As p_ρ^B is a premeasure, we can make the following definition.

DEFINITION 3.5. Let μ_ρ^B be the probability measure on 2^ω which is the unique extension of p_ρ^B.

As $p_\rho^B(\tau)$ is the probability of obtaining $\tau \in 2^{<\omega}$ when the first $|\tau|$ many qubits of ρ are measured in B, μ_ρ^B is the probability distribution induced on 2^ω by the qubit-wise measurement of ρ in B. That is, for any measurable $S \subseteq 2^\omega$, $\mu_\rho^B(S)$ is the probability that qubit-wise measurement of ρ in B yields an infinite bitstring in S.

§4. Quantum measurement randomness. Recall that $\mathbf{ML} \subseteq 2^\omega$ is the set of Martin-Löf random bitstrings. If ρ is a state and B is a system, $\mu_\rho^B(\mathbf{ML})$ is the probability of getting a Martin-Löf random bitstring by a qubit-wise measurement of ρ in B as described in Section 3.

DEFINITION 4.1. A state ρ is quantum measurement random if $\mu_\rho^B(\mathbf{ML}) = 1$ for any B.

Nies and Stephan define a measure μ on 2^ω to be Martin-Löf absolutely continuous if $\mu(\mathbf{ML}) = 1$ (see [11, Definition 2.3]). So a state ρ is quantum measurement random if μ_ρ^B is Martin-Löf absolutely continuous for any B. Any quantum measurement random state hence yields a countably infinite collection of Martin-Löf absolutely continuous measures.

4.1. Relationship with other randomness notions. We give an overview of the relationships of quantum measurement randomness with other notions of randomness which will be established in this article. We first study how quantum measurement randomness relates to Martin-Löf randomness. Recall the state ρ_X from Section 2.3. We now show that quantum measurement randomness agrees with Martin-Löf randomness when restricted to states of the form ρ_X for $X \in 2^\omega$.

LEMMA 4.2. *$X \in 2^\omega$ is Martin-Löf random iff ρ_X is quantum measurement random.*

PROOF. ($\Longrightarrow$): Let X be Martin-Löf random. Then ρ_X is quantum Martin-Löf random by [10, Theorem 3.9]. By Theorem 4.3, ρ_X is quantum measurement random.

($\Longleftarrow$) Let X not be Martin-Löf random and let $\rho := \rho_X$. Let $B = ((b_0^n, b_1^n))_n$ be the system such that $b_i^n = |i\rangle$ for all n and i.

https://doi.org/10.1017/jsl.2026.10191 Published online by Cambridge University Press

For any n and any $\tau \in 2^n$, $\mu_\rho^B(\tau) = \mathrm{Tr}(|X \upharpoonright n\rangle\langle X \upharpoonright n|\Pi^B(\tau)) = |\langle \tau | X \upharpoonright n\rangle|^2$ which is 1 if $\tau = X \upharpoonright n$ and is 0 otherwise. So, $\mu_\rho^B(X) = 1$. As X is not Martin-Löf random this shows that ρ is not quantum measurement random. ⊣

The result [10, Theorem 3.9] (which says that $X \in 2^\omega$ is Martin-Löf random iff ρ_X is quantum Martin-Löf random) together with the above result imply that quantum measurement randomness is equivalent to quantum Martin-Löf randomness for states of the form ρ_X for $X \in 2^\omega$. Also since both quantum measurement randomness and quantum Martin-Löf randomness are equivalent to Martin-Löf randomness when restricted to states of the form ρ_X for $X \in 2^\omega$, quantum measurement randomness and quantum Martin-Löf randomness may be thought of as generalizations of Martin-Löf randomness from 2^ω to the quantum realm. We note that quantum Schnorr randomness does not imply and is not implied by quantum measurement randomness: Let $X \in 2^\omega$ be Schnorr random and not Martin-Löf random. By [3, Lemma 3.9] ρ_X is quantum Schnorr random and by the above it is not quantum measurement random. By Lemma 6.7 there is a quantum measurement random state which is not quantum Schnorr random. This shows that quantum Schnorr randomness and quantum measurement randomness are incomparable notions. Both quantum Schnorr randomness and quantum measurement randomness imply a quantum version of the law of large numbers by [3, Theorem 4.2] and Section 4.3. We will show later that quantum Martin-Löf randomness strictly implies quantum measurement randomness (Lemma 4.3 and Theorem 6). Let q-MLR, q-MR, and q-SR denote quantum Martin-Löf random, quantum measurement random, and quantum Schnorr random, respectively. Let q-LLN denote the property of satisfying the quantum law of large numbers. Then the following diagram summarizes all implications existing between these four notions. All implications are strict. There are no implications between q-MR and q-SR.

$$\begin{array}{ccc} \text{q-MLR} & \xrightarrow{[3]} & \text{q-SR} \\ \downarrow{\scriptstyle Lem.4.3} & & \downarrow{\scriptstyle [3]} \\ \text{q-MR} & \xrightarrow{Thm.4.4} & \text{q-LLN.} \end{array}$$

4.2. Quantum Martin-Löf randomness implies quantum measurement randomness.

LEMMA 4.3. *If a state is quantum Martin-Löf random, then it is quantum measurement random.*

PROOF. Let $\rho = (\rho_n)_n$ be quantum Martin-Löf random. Suppose towards a contradiction that there is a $\delta > 0$ and a system $B = ((b_0^n, b_1^n))_n$ such that $\mu_\rho^B(2^\omega \backslash \mathbf{ML}) > \delta$.

The idea is to derive a contradiction by using the universal ML-test to construct a q-ML-test that ρ fails at order δ. Let $(S^m)_m$ be the universal ML-test [8] and let for all m,

$$S^m = \bigcup_{i=m}^{\infty} [\![A_i^m]\!], \tag{4.1}$$

https://doi.org/10.1017/jsl.2026.10191 Published online by Cambridge University Press

where the A_i^ms satisfy the conditions of Definition 2.7. Since $\lambda(S^m) \leq 2^{-m}$, we have

$$A_i^m = \{\tau_1^{m,i}, \ldots, \tau_{k^{m,i}}^{m,i}\} \subseteq 2^i$$

for some $0 \leq k^{m,i} \leq 2^{i-m}$ for all m and $i \geq m$.

For all m and all $i \geq m$, define the special projection:

$$p_i^m = \sum_{a \leq k^{m,i}} \Pi^B(\tau_a^{m,i}). \tag{4.2}$$

Let $P^m := (p_i^m)_{i=m}^{\infty}$. It is easy to see that $(P^m)_m$ is a q-ML-test.[6] We now show that ρ fails $(P^m)_m$.

As $(S^m)_m$ is the universal ML-test, we have for all m that $\mu_\rho^B(S^m) \geq \mu_\rho^B(2^\omega \backslash \mathbf{ML}) > \delta$. Since (4.1) is an increasing union, for all m there exists an $i(m) > m$ such that

$$\mu_\rho^B([\![A_{i(m)}^m]\!]) > \delta. \tag{4.3}$$

Fix an m and $i = i(m)$. Let p_i^m be as in (4.2). By (3.4) we have that

$$\delta < \mu_\rho^B([\![A_{i(m)}^m]\!]) = \sum_{a \leq k^{m,i}} p_\rho^B(\tau_a) = \sum_{a \leq k^{m,i}} \mathrm{Tr}\big[\rho_i \Pi^B(\tau_a)\big] = \mathrm{Tr}\big[\rho_i p_i^m\big] \leq \rho(P^m). \tag{4.4}$$

As m was arbitrary, we have shown that ρ fails the q-ML test $(P^m)_m$ at order δ thus contradicting that ρ is quantum Martin-Löf random. ⊣

Theorem 6.4 shows that the implication in Lemma 4.3 is strict.

4.3. The quantum law of large numbers. A quantum version of the law of large numbers (q-LLN) has been formulated ([3, Definition 4.1] and [9, Section 6.6]). Satisfying the law of large numbers is a weak form of randomness and thus ought to be implied by any reasonable notion of classical randomness [7].[7] Similarly, quantum measurement randomness can qualify as a plausible notion of quantum randomness only if it implies satisfaction of the q-LLN.[8] Theorem 4.4 shows this.

THEOREM 4.4. *All quantum measurement random states satisfy the q-LLN.*

The proof of Theorem 4.4 is almost identical to the proof of [3, Theorem 4.2] and is omitted.

§5. General computable measures and Kronecker products. We prove two general mathematical results that may be inherently interesting beyond their specific

[6]For each m, the sequence $(p_i^m)_{i=m}^{\infty}$ is computable since the system B and $(A_i^m)_{i=m}^{\infty}$ are computable. Condition (3) in Definition 2.7 implies that for all i, $\mathrm{range}(p_i^m) \subseteq \mathrm{range}(p_{i+1}^m)$. So, P^m is a q-Σ_0^1 class for all m. Also, $k^{m,i} \leq 2^{i-m}$ for all m, i implies that $\tau(P^m) \leq 2^{-m}$ for all m. Lastly, $(P^m)_m$ is a computable sequence since $(S^m)_m$ is an ML-test.

[7]For any sense of the word 'random', one would expect a 'random' bitstring to have an equal numbers of ones and zeroes in the limit.

[8]Quantum Schnorr random states, and hence quantum Martin-Löf random states, satisfy the q-LLN [3].

https://doi.org/10.1017/jsl.2026.10191 Published online by Cambridge University Press

applications in Section 6. The first (Theorem 5.2) concerns Martin-Löf randomness for general computable measures and is invoked in the proof of Lemma 6.5. The second (Lemma 5.3) is a multi-linear algebraic result characterizing the entries of vectors which are Kronecker products of vectors in $\mathbb{C}^2$. It is crucially used in the proof of Lemma 6.3.

DEFINITION 5.1. Let μ be any computable measure on 2^ω. A μ-ML-test is a computable sequence $(S_m)_{m=1}^\infty$ of Σ_1^0 classes such that $\mu(S_m) \leq 2^{-m}$ for all m. A $X \in 2^\omega$ fails $(S_m)_m$ if $X \in S_m$ for all m and X is μ-Martin-Löf random if it does not fail any μ-ML-test.[9] Let $\mathbf{ML}(\mu)$ denote the set of infinite bitstrings Martin-Löf random with respect to μ.

Note that $\mathbf{ML}(\lambda) = \mathbf{ML}$ using Definition 2.10. Theorem 5.2 which concerns the inclusion between $\mathbf{ML}(\nu)$ and $\mathbf{ML}(\pi)$ when ν and π are related in a specific way extends a result of V. Vovk [14]. We use the notation: Let f be a strictly positive computable function on ω and let $\beta \in \omega - \{0\}$. Any $X \in 2^\omega$ may be written as a concatenation of bitstrings of lengths $f(\beta), f(\beta+1), f(\beta+2), \ldots$ in order from left to right. Formally, X can be written as $X = \sigma_\beta \sigma_{\beta+1} \sigma_{\beta+2} \ldots$, where $\sigma_i \in 2^{f(i)}$ for $i \geq \beta$.

THEOREM 5.2. *Let f be a strictly positive computable function on ω and let β be a positive natural number. Let $(\nu_i)_{i=\beta}^\infty$ and $(\pi_i)_{i=\beta}^\infty$ be uniformly computable (in i) sequences such that ν_i and π_i are measures on $2^{f(i)}$ such that $2\nu_i(\tau) - \pi_i(\tau) > 0$ for all $\tau \in 2^{f(i)}$. Let $\nu = \prod_{i=\beta}^\infty \nu_i$ and $\pi = \prod_{i=\beta}^\infty \pi_i$ be the corresponding product measures on 2^ω. Suppose that there is a constant C such that for all $X \in \boldsymbol{ML}(\nu)$ written as a concatenation $X = \sigma_\beta \sigma_{\beta+1} \sigma_{\beta+2} \ldots \sigma_i \ldots$ as above*

$$\prod_{i=\beta}^{\infty} \left(1 - \frac{(\nu_i(\sigma_i) - \pi_i(\sigma_i))^2}{\nu_i(\sigma_i)^2}\right) = C.$$

Then $\boldsymbol{ML}(\nu) \subseteq \boldsymbol{ML}(\pi)$.

The proof of Theorem 5.2 borrows ideas used in the proof of [13, Theorem 196(a)] which is as follows.

THEOREM. *Let $\varepsilon > 0$ and $(p_i)_i$ and $(q_i)_i$ be computable sequences of reals in $(\varepsilon, 1-\varepsilon)$. Let ν_i and π_i be the measures on $\{0,1\}$ associated with the Bernoulli(p_i) distribution and Bernoulli(q_i) distribution, respectively. Let $\nu = \prod_i \mu_i$ and let $\pi = \prod_i \pi_i$ be the corresponding product measures on 2^ω. Then $\boldsymbol{ML}(\nu) = \boldsymbol{ML}(\pi)$ if $\sum_i (p_i - q_i)^2 < \infty$.*[10]

We use the notions of prefix-free complexity (K) and monotone complexity (KM) for strings. The definitions of K and KM, which can be found in [13, Chapter 4 and Section 5.5], respectively, are not relevant to my arguments. The symbols $\leq^+$ and $\leq^\times$ will denote an inequality which holds up to an additive constant and up to a multiplicative constant, respectively.

[9] See Section 6.12.1 and Definition 6.12.3 in [7] for details.
[10] As mentioned in [13], this result is originally due to V.Vovk [14]

https://doi.org/10.1017/jsl.2026.10191 Published online by Cambridge University Press

PROOF. Let $X \in \mathbf{ML}(\nu)$. We aim to show that $X \in \mathbf{ML}(\pi)$. Recall the notation from above which writes X as a concatenation of bitstrings $\sigma_i \in 2^{f(i)}$. For $n \geq \beta$, let $S_n := \sigma_\beta \sigma_{\beta+1} \ldots \sigma_n$. For all i, let $r_i := \nu_i(\sigma_i)$ and $\delta_i := \pi_i(\sigma_i) - r_i$. We outline the main idea of the proof before giving the details. ⊣

PROOF SKETCH. Note that ν and π are computable. As $X \in \mathbf{ML}(\nu)$ and ν is computable, the Levin–Schnorr theorem (see, e.g.,[13, Theorem 90, Section 5.6]) implies that

$$\forall n \geq \beta, -\log(\nu(S_n)) \leq^+ KM(S_n). \tag{5.1}$$

Showing that $\nu(S_n) \leq^\times \pi(S_n)$ will imply that $-\log(\pi(S_n)) \leq^+ -\log(\nu(S_n)) \leq^+ KM(S_n)$. This shows that $-\log(\pi(S_n)) - KM(S_n) \leq O(1)$. Theorem 93 from [13] then implies that $X \in MLR(\pi)$, as required.

We hence desire to show that $\nu(S_n) \leq^\times \pi(S_n)$ or equivalently that $\prod_{i=\beta}^n r_i \leq^\times \prod_{i=\beta}^n (r_i + \delta_i)$. We observe that

$$\prod_{i=\beta}^n (r_i - \delta_i) \prod_{i=\beta}^n (r_i + \delta_i) = \prod_{i=\beta}^n r_i^2 \prod_{i=\beta}^n \left(1 - \frac{\delta_i^2}{r_i^2}\right) = C \prod_{i=\beta}^n r_i^2$$

since by assumption we have $\prod_{i=\beta}^n \left(1 - (\delta_i^2/r_i^2)\right) = C$.

The desired result (namely that $\prod_{i=\beta}^n r_i \leq^\times \prod_{i=\beta}^n (r_i + \delta_i)$) can hence be proven by showing that $\forall n \geq \beta$, we have

$$\prod_{i=\beta}^n (r_i - \delta_i) \leq^\times \prod_{i=\beta}^n r_i.$$

We show this (see (5.3)) by defining a computable measure, ν' such that $\nu'(S_n) = \prod_{i=\beta}^n r_i - \delta_i$ and then relating it to ν. *Proof details:* Define a family, $(\nu_i')_{i=\beta}^\infty$, where ν_i' is a function on $2^{f(i)}$. For all $i \geq \beta$, define ν_i' on 2^i by $\nu_i'(\tau) := 2\nu_i(\tau) - \pi_i(\tau)$. Informally, ν_i' is a 'reflection of π_i around ν_i'. Each ν_i' is a measure on $2^{f(i)}$ since $\sum_{\tau \in 2^i} \nu_i'(\tau) = 1$ and as $2\nu_i(\tau) - \pi_i(\tau) \geq 0$ for all $\tau \in 2^{f(i)}$ by assumption. Let ν' be the product measure, $\Pi_{i=\beta}^\infty \nu_i'$. In particular, for all $n \geq \beta$ we have that

$$\nu'(S_n) = \prod_{i=\beta}^n \nu_i'(\sigma_i). \tag{5.2}$$

Note that $\nu_i'(\sigma_i) = \nu_i(\sigma_i) - (\pi_i(\sigma_i) - \nu_i(\sigma_i)) = r_i - \delta_i$ for all i. So, for all $n \geq \beta$ we have that

$$\nu'(S_n) = \prod_{i=\beta}^n r_i - \delta_i.$$

As ν' is computable (as ν and π are), Theorem 89 in Section 5.6 in [13] gives that for all $n \geq \beta$,

$$KM(S_n) \leq^+ -\log(\nu'(S_n)).$$

https://doi.org/10.1017/jsl.2026.10191 Published online by Cambridge University Press

Combining this with (5.1) and taking exponents, we see that for all $n \geq \beta$,

$$v'(S_n) \leq^{\times} v(S_n),$$

and hence that, for all $n \geq \beta$,

$$\prod_{i=\beta}^{n} r_i - \delta_i \leq^{\times} \prod_{i=\beta}^{n} r_i. \tag{5.3}$$

Now, for all $n \geq \beta$ we have that

$$\prod_{i=\beta}^{n}(r_i - \delta_i)\prod_{i=\beta}^{n}(r_i + \delta_i) = \prod_{i=\beta}^{n} r_i^2 - \delta_i^2 = \prod_{i=\beta}^{n} r_i^2 \prod_{i=\beta}^{n}\left(1 - \frac{\delta_i^2}{r_i^2}\right) = C\prod_{i=\beta}^{n} r_i^2. \tag{5.4}$$

The last equality followed by noting that $\delta_i^2 = (v_i(\sigma_i) - \pi_i(\sigma_i))^2$ and then using the assumption. This gives that for all $n \geq \beta$,

$$\prod_{i=\beta}^{n} r_i^2 \leq^{\times} \prod_{i=\beta}^{n}(r_i - \delta_i)\prod_{i=\beta}^{n}(r_i + \delta_i). \tag{5.5}$$

Divide the right side by $\prod_{i=\beta}^{n}(r_i - \delta_i)$[11] and the left side by $\prod_{i=\beta}^{n} r_i$ and use (5.3) to see that for all $n \geq \beta$,

$$\prod_{i=\beta}^{n} r_i \leq^{\times} \prod_{i=\beta}^{n} r_i + \delta_i.$$

Recalling the definitions of r_i and δ_i we have that for all $n \geq \beta$,

$$v(S_n) \leq^{\times} \pi(S_n).$$

Together with (5.1), this gives that for all $n \geq \beta$,

$$-\log(\pi(S_n)) \leq^{+} -\log(v(S_n)) \leq^{+} KM(S_n),$$

and thus that for all $n \geq \beta$,

$$-\log(\pi(S_n)) - KM(S_n) \leq O(1).$$

As $|S_n|$ is increasing in n, this shows that $-\log(\pi(S_n)) - KM(S_n)$ is bounded from above over all n. That is, there is a uniform bound on $-\log(\pi(x)) - KM(x)$ for infinitely many prefixes x of X. [13, Theorem 93, Section 5.6] then implies that $X \in \mathbf{ML}(\pi)$. As X was an arbitrary element of $\mathbf{ML}(v)$, Theorem 5.2 is proved. ⊣

The following combinatorial lemma concerns the entries of vectors which are Kronecker products of vectors in $\mathbb{C}^2$.

LEMMA 5.3. *Let* $\{[a_i, b_i]^T\}_{i=1}^{n}$ *be a set of column vectors in* $\mathbb{C}^2$. *Let* $V = \bigotimes_{i=1}^{n}[a_i, b_i]^T$ *be their Kronecker product. If* $V = [v_1, v_2, \ldots, v_{2^n}]^T$, *then for all*

[11] This is positive since $(r_i - \delta_i) > 0$ for all i by assumption of the theorem.

https://doi.org/10.1017/jsl.2026.10191 Published online by Cambridge University Press

$1 \leq k \leq 2^{n-1}$, *we have that*

$$|v_k||v_{2^n-k+1}| = \prod_{i=1}^{n} |a_i||b_i|.$$

This lemma can be proved easily using induction and we only give a sketch the proof. We use the following convention for the Kronecker product [12]:

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \otimes \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 a_2 \\ b_1 a_2 \\ a_1 b_2 \\ b_1 b_2 \end{bmatrix}.$$

For natural numbers u and q, let $[u]_q$ denote the remainder obtained by dividing u by q. For any $k \leq 2^{n-1}$, v_k has the form $v_k = \prod_{i=1}^{n} c_i^k$, for some $c_i^k \in \{a_i, b_i\}$ and v_{2^n-k+1} has the form $v_{2^n-k+1} = \prod_{i=1}^{n} e_i^k$, for some $e_i^k \in \{a_i, b_i\}$. Note that $c_1^k = a_1$ iff k is odd iff $e_1^k = b_1$. Similarly, we have that $c_2^k = a_2$ iff $[k]_{2^2} \in \{1, 2\}$ iff $e_2^k = b_2$. In general, for $i \leq n$, for all $k \leq 2^{n-1}$,

$$c_i^k = a_i \iff [k]_{2^i} \in \{1, \ldots, 2^{i-1}\} \iff e_i^k = b_i.$$

§6. A quantum measurement random state that is not quantum Martin-Löf random. The state will be constructed as an infinite tensor product of the matrices d_n defined below.

DEFINITION 6.1. For any $n > 0$, let $r_n := \lfloor 2^n/n \rfloor$ and let d_n be a 2^n by 2^n matrix with 2^{-n} along the diagonal and r_n many 2^{-n}'s on the extreme ends of the anti-diagonal. That is, d_n is the symmetric matrix such that its non-zero entries are given as:

(1) For first r_n rows: For $i \leq r_n$, $d_n(i, j) = 2^{-n}$ if $j = i$ or $j = 2^n - i + 1$.
(2) For the rows from $r_n + 1$ to $2^n - r_n - 1$: For $r_n < i < 2^n - r_n$, $d_n(i, j) = 2^{-n}$ if $j = i$.
(3) For the last r_n many rows: For $i \geq 2^n - r_n$, $d_n(i, j) = 2^{-n}$ if $j = i$ or $j = 2^n - i + 1$.

We illustrate the case $n = 3$ by an example. We have $r_3 = 2$ and

$$d_3 = \begin{bmatrix} 2^{-3} & 0 & 0 & 0 & 0 & 0 & 0 & 2^{-3} \\ 0 & 2^{-3} & 0 & 0 & 0 & 0 & 2^{-3} & 0 \\ 0 & 0 & 2^{-3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2^{-3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2^{-3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2^{-3} & 0 & 0 \\ 0 & 2^{-3} & 0 & 0 & 0 & 0 & 2^{-3} & 0 \\ 2^{-3} & 0 & 0 & 0 & 0 & 0 & 0 & 2^{-3} \end{bmatrix}.$$

Define $\theta_4 := 0$ and for natural numbers $i \geq 5$, define

$$\theta_i := \sum_{j=5}^{i} j. \tag{6.1}$$

https://doi.org/10.1017/jsl.2026.10191 Published online by Cambridge University Press

DEFINITION 6.2. For all $N \geq 5$, let $S_N := \bigotimes_{n=5}^{N} d_n$. Note that S_N is a density matrix on θ_N qubits. Let $\rho := (\rho_n)_n$ be the state such that $\rho_{\theta_N} = S_N$ for all N.

Informally, ρ is the state: '$\bigotimes_{n=5}^{\infty} d_n$'. Formally, ρ is the state such that for all N, its finite initial segment of length θ_N is given by the density matrix S_N. The reader may be curious about the reason for the choice of 5 when constructing ρ^e. This reason will become clear in the proof of Lemma 6.5 (see Remark 6.6). We first establish a property of d_n which is vital to showing that ρ is quantum measurement random.

LEMMA 6.3. *Let $n > 0$. Let $\{[a_i, b_i]^T\}_{i=1}^n$ be such that for all i, $[a_i, b_i]^T$ is a unit vector in $\mathbb{C}^2$ and let $W = \bigotimes_{i=1}^n [a_i, b_i]^T$. Then $|\langle W|d_n|W\rangle| \in [2^{-n}(1 - 2n^{-1}), 2^{-n}(1 + 2n^{-1})]$.*

The lemma roughly says that for all n, if W is a tensor product of n many unit norm vectors in $\mathbb{C}$, then $|\langle W|d_n|W\rangle|$ is 'close to' 2^{-n}. This will be used to later show that μ_ρ^B is 'close' to λ for any B. The proof crucially relies on Lemma 5.3.

PROOF. Fix n and W as in the statement and write d_n as a block matrix with each block of size 2^{n-1} by 2^{n-1}:

$$d_n = \begin{bmatrix} A & B \\ B^T & A \end{bmatrix}.$$

Let $V = \bigotimes_{i=1}^{n-1} [a_i, b_i]^T$. Then in block matrix form,

$$W = [a_n V^T, b_n V^T]^T.$$

Let $V = [v_1, v_2, \ldots, v_{2^{n-1}}]^T$. It is easily checked that

$$\langle W|d_n|W\rangle = 2^{-n} + a_n^* b_n V^\dagger B V + a_n b_n^* V^\dagger B^T V.$$

By the definition of d_n, the block B has 2^{-n} at the r_n many top-right entries along the antidiagonal and has zeroes everywhere else. This form of B implies that

$$\begin{aligned} V^\dagger B V &= 2^{-n}[v_1^*, v_2^*, \ldots, v_{2^{n-1}}^*][v_{2^{n-1}}, v_{2^{n-1}-1}, \ldots, v_{2^{n-1}-r_n+1}, 0, \ldots, 0]^T. \\ &= 2^{-n} \sum_{k=1}^{r_n} v_k^* v_{2^{n-1}-k+1}. \end{aligned}$$

By Lemma 5.3,

$$|V^\dagger B V| \leq 2^{-n} \sum_{k=1}^{r_n} |v_k||v_{2^{n-1}-k+1}| = 2^{-n} r_n \prod_{i=1}^{n-1} |a_i||b_i| = 2^{-n} r_n \prod_{i=1}^{n-1} |a_i|\sqrt{1 - |a_i|^2}.$$

Since the maximum value of $x\sqrt{1-x^2}$ on $x \in [0, 1]$ is $1/2$ and recalling the definition of r_n we see that

$$|V^\dagger B V| \leq 2^{-n} \frac{1}{2^{n-1}} \frac{2^n}{n} = \frac{2^{1-n}}{n}.$$

Similarly, $|V^\dagger B^T V| \leq \dfrac{2^{1-n}}{n}$. Noting that $|a_n^* b_n|, |a_n b_n^*| \leq 1/2$,

$$|\langle W|d_n|W\rangle| \leq 2^{-n} + |a_n^* b_n V^\dagger B V| + |a_n b_n^* V^\dagger B^T V| \leq 2^{-n} + \frac{2^{1-n}}{n},$$

https://doi.org/10.1017/jsl.2026.10191 Published online by Cambridge University Press

and

$$|\langle W|d_n|W\rangle| \geq 2^{-n} - |a_n^* b_n V^\dagger B V| - |a_n b_n^* V^\dagger B^T V| \geq 2^{-n} - \frac{2^{1-n}}{n}. \quad \dashv$$

The following is our central result. Recall the state ρ from Definition 6.2.

THEOREM 6.4. *The state ρ is quantum measurement random and not quantum Martin-Löf random.*

PROOF. It is not hard to show that ρ is not quantum Martin-Löf random (Lemma 6.7). Showing that ρ is quantum measurement random (Lemma 6.5) is more involved and crucially relies on two general mathematical results from Section 5. The first (Theorem 5.2) extends a result of V.Vovk concerning Martin-Löf randomness for computable measures. The second (Lemma 5.3) is a combinatorial result about Kronecker products and gets used here through Lemma 6.3, a property of the d_n's proved earlier.

LEMMA 6.5. *The state ρ is quantum measurement random.*

PROOF. Let $B = ((b_0^n, b_1^n))_{n=1}^\infty$ be an arbitrary computable measurement system. We need to show that $\mu_\rho^B(\mathbf{ML}) = 1$. $\dashv$

PROOF SKETCH. We prove that $\mathbf{ML}(\mu_\rho^B) \subseteq \mathbf{ML}$. Since $\mu_\rho^B[\mathbf{ML}(\mu_\rho^B)] = 1$, this implies that $\mu_\rho^B(\mathbf{ML}) = 1$ as desired.

To simplify notation, denote μ_ρ^B by μ. To show $\mathbf{ML}(\mu) \subseteq \mathbf{ML}$, we show that μ and λ satisfy the conditions of Theorem 5.2 under the replacements: $\nu \mapsto \mu$, $\pi \mapsto \lambda$, $\beta \mapsto 5$, and $f \mapsto$ the identity function. Theorem 5.2 then implies that $\mathbf{ML}(\mu) \subseteq \mathbf{ML}(\lambda) = \mathbf{ML}$ as required.

Proof details: Recall the definition of θ_i (6.1). Let μ_i and λ_i be functions on 2^i such that for all $\tau \in 2^i$,

$$\mu_i(\tau) := \mathrm{Tr}\Big[d_i\big(|\bigotimes_{q=1}^{i} b_{\tau(q)}^{q+\theta_{i-1}}\rangle\langle\bigotimes_{q=1}^{i} b_{\tau(q)}^{q+\theta_{i-1}}|\big)\Big] = \langle \bigotimes_{q=1}^{i} b_{\tau(q)}^{q+\theta_{i-1}}|d_i|\bigotimes_{q=1}^{i} b_{\tau(q)}^{q+\theta_{i-1}}\rangle,$$

and

$$\lambda_i(\tau) := 2^{-i}.$$

Note that $\sum_{\tau\in 2^i} \mu_i(\tau) = 1$ and hence that μ_i is a measure on 2^i for each i. μ_i is the measure on 2^i induced by measuring the i^{th} block of ρ (namely, d_i) when ρ is measured in $B = ((b_0^n, b_1^n))_{n=1}^\infty$. This measurement hence uses the i^{th} block of the system B, when B is divided into a sequence (indexed by $i = 5, 6, \ldots$) of blocks with the i^{th} block in the sequence having length i. The indexing of the b's in the definition of μ_i starts from $1 + \theta_{i-1}$ as the measurement of the i^{th} block of ρ uses the bases $B = ((b_0^n, b_1^n))_{n=1+\theta_{i-1}}^{\theta_i}$. By the cyclicity of the trace and by Lemma 6.3,

$$\mu_i(\tau) = \langle \bigotimes_{q=1}^{i} b_{\tau(q)}^{q+\theta_{i-1}}|d_i|\bigotimes_{q=1}^{i} b_{\tau(q)}^{q+\theta_{i-1}}\rangle \in [2^{-i}(1-2i^{-1}), 2^{-i}(1+2i^{-1})].$$

By (3.3) and by the form of ρ we see that μ is the product measure $\Pi_{i=5}^\infty \mu_i$.

https://doi.org/10.1017/jsl.2026.10191 Published online by Cambridge University Press

Note that $(\mu_i)_{i=5}^{\infty}$ and $(\lambda_i)_{i=5}^{\infty}$ are uniformly computable (in i) sequences where μ_i and λ_i are measures on $2^{f(i)}$ where f is the strictly positive function on $\omega - \{0\}$ given by $f(i) = i$.

Fix a $\tau \in 2^i$. By Lemma 6.3, $2\mu_i(\tau) - \lambda_i(\tau) \geq 2^{1-i} - 2^{2-i}i^{-1} - 2^{-i}$. As $i \geq 5$, this is strictly greater than $2^{1-i} - 2^{2-i}4^{-1} - 2^{-i} = 2^{1-i} > 0$. This shows that $2\mu_i(\tau) - \lambda_i(\tau) > 0$ for all $\tau \in 2^i$.[12]

Let X be an arbitrary element of $\mathbf{ML}(\mu)$. Write X as a concatenation of finite bitstrings:

$$X = \sigma_5\sigma_6 \dots \sigma_i \dots,$$

where $\sigma_i \in 2^i$ for all $i \in \omega$. Let $S_n := \sigma_5\sigma_6 \dots \sigma_n$ be the concatenation up to n. In particular, for all n,

$$\mu(S_n) = \prod_{i=5}^{n} \mu_i(\sigma_i). \tag{6.2}$$

As λ is the uniform measure, for all n,

$$\lambda(S_n) = \prod_{i=5}^{n} \lambda_i(\sigma_i). \tag{6.3}$$

For all i, let $r_i := \mu_i(\sigma_i)$ and $\delta_i := \lambda_i(\sigma_i) - r_i = 2^{-i} - r_i$. By Lemma 6.3 and by recalling the definition of μ_i, we have that $r_i \in [2^{-i}(1 - 2i^{-1}), 2^{-i}(1 + 2i^{-1})]$ for all i.[13] So $|\delta_i| = |r_i - 2^{-i}| \in [0, 2^{-i+1}i^{-1}]$. Hence,

$$\frac{|\delta_i|}{r_i} \leq \frac{2^{-i+1}i^{-1}}{2^{-i}(1 - 2i^{-1})} = 2(i-2)^{-1}.$$

For all $n \geq 5$, we have that

$$1 \geq \prod_{i=5}^{n} \left(1 - \frac{\delta_i^2}{r_i^2}\right) \geq \prod_{i=5}^{\infty} \left(1 - \frac{\delta_i^2}{r_i^2}\right) \geq \prod_{i=5}^{\infty} \left(1 - 4(i-2)^{-2}\right) = O(1) > 0.$$

This shows that there is a constant $C > 0$ independent of X such that

$$\prod_{i=5}^{\infty} \left(1 - \frac{(\mu_i(\sigma_i) - \lambda_i(\sigma_i))^2}{\mu_i(\sigma_i)^2}\right) = C.$$

We can thus apply Theorem 5.2 with the following replacements: $\nu \mapsto \mu$, $\pi \mapsto \lambda$, $\beta \mapsto 5$, and $f \mapsto$ the identity function to conclude that $\mathbf{ML}(\mu) \subseteq \mathbf{ML}$. As $\mu[\mathbf{ML}(\mu)] = 1$, this implies that $\mu[\mathbf{ML}] = 1$. This shows that ρ is measurement random since the B used to define μ was an arbitrary measurement system. ⊣

REMARK 6.6. A reason for using $i \geq 5$ in constructing ρ is to allow for the existence of $C > 0$ as above. As $(1 - 4(4-2)^{-2}) = 0$,

[12]This is a reason we use $i \geq 5$ instead of $i \geq 1$ in our constructions (see Remark 6.6 for another reason).

[13]This is another place where Lemma 6.3 is crucially used.

https://doi.org/10.1017/jsl.2026.10191 Published online by Cambridge University Press

$$\prod_{i=1}^{\infty}\left(1-\frac{(\mu_i(\sigma_i)-\lambda_i(\sigma_i))^2}{\mu_i(\sigma_i)^2}\right)=0$$

and the constant $C>0$ as above thus does not exist if $i\geq 1$. Another reason, as mentioned above, was to ensure $2\mu_i(\tau)-\lambda_i(\tau)\geq 2^{1-i}-2^{2-i}i^{-1}-2^{-i}>2^{1-i}-2^{2-i}4^{-1}-2^{-i}=2^{1-i}>0$.

It remains to show that the state ρ is not quantum Martin-Löf random. We prove this by showing the stronger result that ρ is quantum Schnorr random (recall that quantum Martin-Löf randomness implies quantum Schnorr randomness [3]). A quantum Schnorr test is a computable sequence of special projections $(S^m)_{m=1}^{\infty}$ such that $\sum_m \tau(S^m)$ is a computable real number [3]. A state ρ fails the quantum Schnorr test $(S^m)_m$ at order $\delta>0$ if $\rho(S^m)>\delta$ for infinitely many m. A state ρ passes the test $(S^m)_m$ at order $\delta>0$ if it does not fail it at δ. A state is quantum Schnorr random if it passes all quantum Schnorr tests at all $\delta>0$.

LEMMA 6.7. *The state ρ is not quantum Schnorr random.*

PROOF. We denote S_N^1 (Definition 6.2) by S_N and $\theta_i(1)$ (6.1) by θ_1. It is easy to see that d_n has the eigenvalues zero with multiplicity r_n, 2^{-n} with multiplicity 2^n-2r_n and 2^{-n+1} with multiplicity r_n. So d_n has $q_n:=2^n-r_n$ many non-zero eigenvalues.

So the eigenpairs of d_n can be listed as $\{\alpha_i^n,v_i^n\}_{i=1}^{2^n}$, where $\alpha_i^n=0$ for exactly those i's in the range $q_n+1\leq i\leq 2^n$ and $(v_i^n)_{i=1}^{2^n}$ is an orthonormal basis of $\mathbb{C}^{2^n}$. That is, the list puts the r_n many eigenvectors with zero eigenvalue at the last r_n places in the list.

Fix an $N>5$. Recall that $\theta_N=\sum_{n=5}^N n$ for any $N>5$. By properties of the Kronecker product, S_N is a density matrix on $\mathbb{C}^{2^{\theta_N}}$ and has an orthonormal basis of eigenvectors:

$$\{\bigotimes_{n=5}^{N} v_{l(n)}^n : (l(n))_{n=5}^N \text{ is a sequence such that } \forall n, l(n)\in\{1,\dots 2^n\}\}\subseteq\mathbb{C}^{2^{\theta_N}}.$$

The eigenvector $\bigotimes_{n=5}^N v_{l(n)}^n$ has eigenvalue $\prod_{n=5}^N \alpha_{l(n)}^n$. Let M_N be the elements of the above eigenbasis having non-zero eigenvalues. Then the order of listing the eigenpairs implies that

$$M_N=\{\bigotimes_{n=5}^{N} v_{l(n)}^n : (l(n))_{n=5}^N \text{ is a sequence such that for all } n, l(n)\leq q_n\}\subseteq\mathbb{C}^{2^{\theta_N}}. \tag{6.4}$$

By the definition of q_n,

$$\begin{aligned}|M_N|&=\prod_{n=5}^N 2^n-\lfloor 2^n/n\rfloor\leq\prod_{n=5}^N 2^n-(2^n/n)+1=\prod_{n=5}^N 2^n(1-n^{-1}+2^{-n})\\&=\prod_{n=5}^N 2^n\prod_{n=5}^N(1-n^{-1}+2^{-n}).\end{aligned}$$

https://doi.org/10.1017/jsl.2026.10191 Published online by Cambridge University Press

Note that $\prod_{n=5}^{\infty}(1-n^{-1}+2^{-n})=0$. Define a quantum Schnorr test, $(T_m)_m$ as follows. Given m, we describe the construction of T_m. Find $N=N(m)$ such that $\prod_{n=5}^{N}(1-n^{-1}+2^{-n})<2^{-m}$. The idea is to use the elements of M_N to define a special projection. As M_N contains all the eigenvectors of ρ_{θ_N} with non-zero eigenvalues, ρ_{θ_N} will have a 'large projection' onto span(M_N) (see (6.5)).

Let

$$T_m=\sum_{v\in M_{N(m)}}|v\rangle\langle v|.$$

This completes the construction of T_m for a fixed m. As ρ is computable, T_m can be computed uniformly in m. Note that T_m is a projection on $\mathbb{C}^{2^{\theta_{N(m)}}}$ having rank equal to $|M_{N(m)}|$. The choice of $N(m)$ implies that $|M_{N(m)}|<2^{\theta_{N(m)}}2^{-m}$ and hence that $\tau(T_m)<2^{-m}$. So, $\sum_m\tau(T_m)$ is a computable real number. This shows that $(T_m)_m$ is a quantum Schnorr test. This test demonstrates that ρ is not quantum Schnorr random as follows. Fix an m. Recalling that $M_{N(m)}$ is the set consisting of all eigenvectors of $S_{N(m)}$ with non-zero eigenvalue, we have that

$$\rho(T_m)=\mathrm{Tr}(\rho_{\theta_{N(m)}}T_m)=\mathrm{Tr}(S_{N(m)}T_m)=\mathrm{Tr}(S_{N(m)})=1. \tag{6.5}$$

As m was arbitrary, ρ fails the quantum Schnorr test $(T_m)_m$ at order 1. ⊣

Given $S\in 2^\omega$, let $\mathbf{ML}^S\subset 2^\omega$ be the set of Martin-Löf randoms with respect to the oracle S.[14] Observe that the state ρ (from Theorem 6.4) is such that $\mu_\rho^B(\mathbf{ML}^S)=1$ for any $S\in 2^\omega$ and any computable measurement system B. We can see this as follows. Fix an $S\in 2^\omega$ and a computable measurement system B. Relativizing the proof of Lemma 6.5 to S shows that $\mathbf{ML}^S(\mu_\rho^B)\subseteq\mathbf{ML}^S$ (Take an $X\in\mathbf{ML}^S(\mu_\rho^B)$. Relativizing [13, Theorems 85 and 90] and [8, Proposition 3.2.14] to S and noting that $KM^S(.)\leq KM(.)$ and following the proof of Lemma 6.5 shows that $X\in\mathbf{ML}^S$. So $\mathbf{ML}^S(\mu_\rho^B)\subseteq\mathbf{ML}^S$). This shows that $\mu_\rho^B(\mathbf{ML}^S)\geq\mu_\rho^B(\mathbf{ML}^S(\mu_\rho^B))=1$.

Remark 6.8. The state ρ (from Theorem 6.4) is such that μ_ρ^B is absolutely continuous with respect to λ for any computable measurement system B. To see this, let B be an arbitrary computable measurement system and let $N\subset 2^\omega$ be such that $\lambda(N)=0$. It is easy to see that there is some oracle $S\in 2^\omega$ such that $\mathbf{ML}^S\subseteq(2^\omega\backslash N)$. Together with the previous observation, this implies that $\mu_\rho^B(2^\omega\backslash N)\geq\mu_\rho^B(\mathbf{ML}^S)=1$. This shows that $\mu_\rho^B(N)=0$.[15]

Acknowledgments. The article was written during my PhD studies under Joe Miller at the Department of Mathematics, University of Wisconsin-Madison and during my post-doctoral fellowship at the School of Computer Science, University of Auckland under André Nies.[16] J.M. and Peter Cholak (independently) asked if a state can be 'measured'. I thank J.M. and A.N. for reading the article in detail and suggesting key improvements.

[14]Relativizing Martin-Löf randomness to oracles is a standard notion in algorithmic randomness [7, 8].

[15]I thank an anonymous referee for making this remark.

[16]The full PhD thesis is available at [1].

https://doi.org/10.1017/jsl.2026.10191 Published online by Cambridge University Press

Funding. The post-doctoral fellowship at the School of Computer Science, University of Auckland was funded by the Marsden grant UOA1931 awarded by the Royal Society of New Zealand to A.N.

REFERENCES

[1] T. Bhojraj, *Algorithmic randomness and Kolmogorov complexity for qubits*, PhD thesis, University of Wisconsin-Madison, Madison, WI, 2021.

[2] ———, *Prefix-free quantum Kolmogorov complexity*. ***Theoretical Computer Science***, vol. 875 (2021), pp. 65–80.

[3] ———, *Quantum algorithmic randomness*. ***Journal of Mathematical Physics***, vol. 62 (2021), no. 2, p. 22202.

[4] ———, *von Neumann entropy and quantum algorithmic randomness*. ***Theoretical Computer Science***, vol. 1042 (2025), p. 115235.

[5] C. Calude, ***Information and Randomness—An Algorithmic Perspective***, Texts in Theoretical Computer Science. An EATCS Series, Springer, 2002. https://doi.org/10.1007/978-3-662-04978-5.

[6] I. Chuang and M. Nielsen, ***Quantum Computation and Quantum Information: 10th Anniversary Edition***, Cambridge University Press, New York, United States of America, 2010. https://books.google.co.nz/books?id=-s4DEy7o-a0C.

[7] R. Downey and D. Hirschfeldt, ***Algorithmic Randomness and Complexity***, Springer New York, NY, USA, 2010.

[8] A. Nies, ***Computability and Randomness***, Oxford University Press, Oxford, UK, 2009.

[9] ———, *Logic blog 2017*, preprint, 2017, arXiv:1804.05331.

[10] A. Nies and V. Scholz, *Martin-Löf random quantum states*. ***Journal of Mathematical Physics***, vol. 60 (2019), no. 9, p. 92201.

[11] A. Nies and F. Stephan, *Randomness and initial segment complexity for measures*. ***Theoretical Computer Science***, vol. 900 (2022), pp. 1–19.

[12] P. Regalia and S. Mitra, *Kronecker products, unitary matrices, and signal processing applications*. ***SIAM Review***, vol. 31 (1989), no. 4, pp. 586–613.

[13] A. Shen, V. A. Uspensky, and N. Vereshchagin, *Kolmogorov complexity and algorithmic randomnes*. ***Mathematical Surveys and Monographs***, vol. 220 (2017), pp. 130–147.

[14] V. Vovk, *On a randomness criterion*. ***Soviet Mathematics Doklady***, vol. 35 (1987), pp. 656–660.

[15] M. Wilde, ***Quantum Information Theory,*** Cambridge University Press, New York, United States of America, 2017. https://markwilde.com/qit-notes.pdf.

SCHOOL OF COMPUTER SCIENCE
UNIVERSITY OF AUCKLAND
AUCKLAND, NEW ZEALAND
E-mail: tejasbhojrajpnp@gmail.com

https://doi.org/10.1017/jsl.2026.10191 Published online by Cambridge University Press