Ergod. Th. & Dynam. Sys. (2019), **39**, 2128–2158 © *Cambridge University Press, 2017*
doi:10.1017/etds.2017.123

Purely infinite labeled graph C^*-algebras

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(*Received* 13 *March* 2017 *and accepted in revised form* 3 *October* 2017)

Abstract. In this paper, we consider pure infiniteness of generalized Cuntz–Krieger algebras associated to labeled spaces $(E, \mathcal{L}, \mathcal{E})$. It is shown that a C^*-algebra $C^*(E, \mathcal{L}, \mathcal{E})$ is purely infinite in the sense that every non-zero hereditary subalgebra contains an infinite projection (we call this property (IH)) if $(E, \mathcal{L}, \mathcal{E})$ is disagreeable and every vertex connects to a loop. We also prove that under the condition analogous to (K) for usual graphs, $C^*(E, \mathcal{L}, \mathcal{E}) = C^*(p_A, s_a)$ is purely infinite in the sense of Kirchberg and Rørdam if and only if every generating projection p_A, $A \in \mathcal{E}$, is properly infinite, and also if and only if every quotient of $C^*(E, \mathcal{L}, \mathcal{E})$ has property (IH).

1. *Introduction*

In the literature, one can find at least two definitions of pure infiniteness of C^*-algebras. The property of a C^*-algebra having infinite projections in each of its non-zero hereditary subalgebras was called 'purely infinite' in some articles (for example, [**1**, **6**, **10**, **25**, **28**]), and then the same term 'purely infinite' began to be used by Kirchberg and Rørdam [**27**] to mean another infiniteness of C^*-algebras. Their motivation was to extend to non-simple C^*-algebras the fact that a simple separable nuclear unital C^*-algebra A is purely infinite if and only if it is isomorphic to $A \otimes \mathcal{O}_\infty$ by finding the right definition of being purely infinite for non-simple C^*-algebras. While these two definitions are known to be equivalent for simple C^*-algebras, each is neither stronger nor weaker than the other in general, and thus to avoid any confusion we will say that a C^*-algebra has property (IH) if it has only infinite hereditary subalgebras, that is, every non-zero hereditary subalgebra contains an infinite projection.

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Pure infiniteness and property (IH) have been studied for various classes of C^*-algebras (a few examples are [**1**, **5**, **7**, **8**, **12**, **16**, **17**, **28**, **29**, **31**]), including graph C^*-algebras (see [**6**, **10**, **25**]). The purpose of the present paper is to do this sort of study for labeled graph C^*-algebras $C^*(E, \mathcal{L}, \mathcal{E})$ which are a generalization of the graph C^*-algebras.

The graph C^*-algebras $C^*(E)$ associated to (directed) graphs E have drawn much interest from many authors since they were introduced in [**26**] ([**13**] for finite graphs) and have been generalized in many ways over the past twenty years. A graph C^*-algebra $C^*(E)$ is generated by a universal Cuntz–Krieger E-family consisting of projections p_v and partial isometries s_e indexed by the vertices v and the edges e of E satisfying the relations determined by the graph E so that every C^*-algebra generated by a Cuntz–Krieger E-family is a quotient of $C^*(E)$. By virtue of [**10**], it is enough to consider row-finite graphs as far as the properties that are preserved under strong Morita equivalence are concerned. (IH) is one such property, and the following theorem is known for a row-finite graph E.

THEOREM 1.1. ([**6**, Proposition 5.3] and [**25**, Theorem 3.9]) *$C^*(E)$ has property (IH) if and only if E satisfies condition (L) and every vertex connects to a loop.*

Condition (L) means that every loop has an exit, and the above theorem actually holds true for arbitrary graphs [**10**, Corollary 2.14]. On the other hand, characterizations of purely infinite graph C^*-algebras were obtained in [**16**, **22**] for locally finite graphs with no sinks and in [**17**] for arbitrary graphs. One consequence of [**17**] that we are interested in as follows.

THEOREM 1.2. [**17**, Theorem 2.3] *For a graph E, the following are equivalent:*

(a) *every quotient of $C^*(E)$ has property (IH);*
(b) *$C^*(E) = C^*(p_v, s_e)$ is purely infinite;*
(c) *there are no breaking vertices, and every projection p_v is properly infinite.*

(If E is row-finite, there are no breaking verices.) In the same paper, it is also shown that if $C^*(E)$ is purely infinite, then E satisfies condition (K) (that is, every vertex lies on no loops or lies on at least two loops each of which is not an initial path of the other) and that E satisfies (K) if and only if every ideal of $C^*(E)$ is invariant for the gauge action of $\mathbb{T}$ which is induced by the universal property of $C^*(E)$.

In order to obtain characterizations for property (IH) or pure infiniteness of labeled graph C^*-algebras as in Theorem 1.1 or 1.2, we need labeled space analogues of the notions used in the above theorems. A graph with condition (L) was generalized to a *disagreeable* labeled space in [**5**], and this notion will play an important role throughout this paper together with the notion of *loop* introduced in [**18**]. Besides, we will make heavy use of the notion of *quotient labeled space* of [**21**] to deal with quotient algebras of labeled graph C^*-algebras. Before we state our results, recall that a labeled graph is a graph E equipped with a labeling map $\mathcal{L} : E^1 \to \mathcal{A}$ assigning a letter in the alphabet $\mathcal{A}$ to each edge of E. If $\mathcal{B}$ is an accommodating set, a certain collection of vertex subsets, then a triple $(E, \mathcal{L}, \mathcal{B})$ is called a *labeled space*. It is known from [**4**] that a C^*-algebra $C^*(E, \mathcal{L}, \mathcal{B}) = C^*(p_A, s_a)$, $A \in \mathcal{B}, a \in \mathcal{A}$, can be associated to $(E, \mathcal{L}, \mathcal{B})$ in a similar way to associating a graph C^*-algebra to a graph. A labeled graph C^*-algebra $C^*(E, \mathcal{L}, \mathcal{B})$, by

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definition, depends on the choice of an accommodating set $\mathcal{B}$, and we will consider mostly the smallest accommodating set, which we denote by $\mathcal{E}$. Throughout this paper, we deal only with labeled spaces $(E, \mathcal{L}, \mathcal{E})$ of graphs E with no sinks that satisfy some standard conditions. The class of labeled graph C^*-algebras strictly includes all the graph C^*-algebras; for example, it was recently shown [**19**] that there exists a family of unital simple labeled graph C^*-algebras which are $A\mathbb{T}$ algebras with non-zero K_1-groups, whereas any simple graph C^*-algebra is known to be either AF or purely infinite.

This paper is organized as follows. We review in §2 the definitions and known facts about (quotient) labeled graph C^*-algebras and purely infinite C^*-algebras, and then investigate in §3 the property (IH) for labeled graph C^*-algebras. Our main result in §3 is the following theorem.

THEOREM 1.3. (Theorem 3.14(a)) *If $(E, \mathcal{L}, \mathcal{E})$ is a disagreeable labeled space such that every vertex connects to a loop, then $C^*(E, \mathcal{L}, \mathcal{E})$ has property (IH).*

Note here that a loop in a labeled space is different from a cycle used in [**9**] while every cycle is a loop and they are the same in a (trivially labeled) graph. In fact, we provide an example of a labeled space $(E, \mathcal{L}, \mathcal{E}_\omega)$ (see Example 3.17) without any cycles but with loops for which the C^*-algebra $C^*(E, \mathcal{L}, \mathcal{E}_\omega)$ is simple and purely infinite. This example shows the sharp contrast between loops and cycles in a labeled space. The converse of the above theorem is also shown in Theorem 3.14(b) under some extra conditions.

In §4 we consider pure infiniteness of labeled graph C^*-algebras. We say that a labeled space $(E, \mathcal{L}, \mathcal{E})$ is *strongly disagreeable* if every quotient labeled space is disagreeable. Then we show in Proposition 4.4 that if $(E, \mathcal{L}, \mathcal{E})$ is strongly disagreeable, then every ideal of $C^*(E, \mathcal{L}, \mathcal{E})$ is gauge invariant. Thus the condition of being strongly disagreeable can be considered as an analogue of condition (K). The following theorem is our main result for pure infiniteness of $C^*(E, \mathcal{L}, \mathcal{E})$.

THEOREM 1.4. (Theorem 4.9) *Let $(E, \mathcal{L}, \mathcal{E})$ be a strongly disagreeable labeled space. Then the following are equivalent for $C^*(E, \mathcal{L}, \mathcal{E}) = C^*(s_a, p_A)$:*

(a) *every quotient of $C^*(E, \mathcal{L}, \mathcal{E})$ has property (IH);*
(b) *$C^*(E, \mathcal{L}, \mathcal{E})$ is purely infinite;*
(c) *every non-zero projection p_A is properly infinite for $A \in \mathcal{E}$.*

We believe that the condition of $(E, \mathcal{L}, \mathcal{E})$ being strongly disagreeable should be a necessary condition for its C^*-algebra to be purely infinite. As evidence, in Proposition 4.8 we show under some conditions that if $C^*(E, \mathcal{L}, \mathcal{E})$ is purely infinite, then the labeled space $(E, \mathcal{L}, \mathcal{E})$ is strongly disagreeable.

2. *Preliminaries*

2.1. *Directed graphs and labeled spaces.* A *directed graph* is a quadruple $E = (E^0, E^1, r, s)$ consisting of two countable sets of vertices E^0 and edges E^1, and the range and source maps $r, s : E^1 \to E^0$. If a vertex $v \in E^0$ emits (respectively, receives) no edges, it is called a *sink* (respectively, a *source*). A graph E is *row-finite* if every vertex emits only finitely many edges.

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For each $n \geq 1$, a finite path λ of *length* n, $|\lambda| = n$, is a sequence of edges $\lambda_i \in E^1$ such that $r(\lambda_i) = s(\lambda_{i+1})$ for $1 \leq i \leq n-1$. E^n denotes the set of all paths of length n, and the vertices in E^0 are regarded as finite paths of length zero. The maps r, s naturally extend to the set $E^* = \bigcup_{n\geq 0} E^n$ of all finite paths, especially with $r(v) = s(v) = v$ for $v \in E^0$. We write E^∞ for the set of all infinite paths $x = \lambda_1\lambda_2 \ldots$, $\lambda_i \in E^1$ with $r(\lambda_i) = s(\lambda_{i+1})$ for $i \geq 1$, and define $s(x) := s(\lambda_1)$. For $A, B \subset E^0$ and $n \geq 0$, set

$$AE^n := \{\lambda \in E^n : s(\lambda) \in A\}, \quad E^nB := \{\lambda \in E^n : r(\lambda) \in B\},$$

and $AE^nB := AE^n \cap E^nB$. We write E^nv for $E^n\{v\}$ and vE^n for $\{v\}E^n$, and we will use notation such as $AE^{\geq k}$ and vE^∞ the meaning of which should be obvious.

A finite path $\lambda \in E^{\geq 1}$ with $r(\lambda) = s(\lambda)$ is called a *loop*, and an *exit* of a loop λ is a path $\delta \in E^{\geq 1}$ such that $|\delta| \leq |\lambda|$, $s(\delta) = s(\lambda)$, and $\delta \neq \lambda_{[1,|\delta|]}$. A graph E is said to satisfy *condition* (L) if every loop has an exit and E is said to satisfy *condition* (K) if every vertex $v \in E^0$ lies on no loops, or if there are two loops α and β such that $s(\alpha) = s(\beta) = v$ and neither α nor β is an initial path of the other.

Let $\mathcal{A}$ be a countable alphabet and let $\mathcal{A}^*$ (respectively, $\mathcal{A}^\infty$) denote the set of all finite words (respectively, infinite words) in symbols of $\mathcal{A}$. A *labeled graph* $(E, \mathcal{L})$ over $\mathcal{A}$ consists of a directed graph E and a *labeling map* $\mathcal{L} : E^1 \to \mathcal{A}$ which is always assumed to be onto. Given a graph E, one can define a so-called *trivial labeling* map $\mathcal{L}_{\mathrm{id}} := \mathrm{id} : E^1 \to E^1$ which is the identity map on E^1 with the alphabet E^1. To each finite path $\lambda = \lambda_1 \cdots \lambda_n \in E^n$ of a labeled graph $(E, \mathcal{L})$ over $\mathcal{A}$, there corresponds a finite labeled path $\mathcal{L}(\lambda) := \mathcal{L}(\lambda_1) \cdots \mathcal{L}(\lambda_n) \in \mathcal{L}(E^n) \subset \mathcal{A}^*$, and similarly an infinite labeled path $\mathcal{L}(x) := \mathcal{L}(\lambda_1)\mathcal{L}(\lambda_2) \cdots \in \mathcal{L}(E^\infty) \subset \mathcal{A}^\infty$ to each infinite path $x = \lambda_1\lambda_2 \cdots \in E^\infty$. For convenience we often call these labeled paths just paths if there is no risk of confusion, and use the notation $\mathcal{L}^*(E) := \mathcal{L}(E^{\geq 1})$, where $E^{\geq 1} = E^* \setminus E^0$. We also write $\mathcal{L}(v) := v$ for $v \in E^0$ and $\mathcal{L}(A) := A$ for $A \subset E^0$. A subpath $\alpha_i \cdots \alpha_j$ of $\alpha = \alpha_1\alpha_2 \cdots \alpha_{|\alpha|} \in \mathcal{L}^*(E)$ is denoted by $\alpha_{[i,j]}$ for $1 \leq i \leq j \leq |\alpha|$, and each $\alpha_{[1,j]}$, $1 \leq j \leq |\alpha|$, is called an *initial path* of α. The range and source of a path $\alpha \in \mathcal{L}^*(E)$ are defined by

$$r(\alpha) = \{r(\lambda) \in E^0 : \lambda \in E^{\geq 1},\ \mathcal{L}(\lambda) = \alpha\},$$
$$s(\alpha) = \{s(\lambda) \in E^0 : \lambda \in E^{\geq 1},\ \mathcal{L}(\lambda) = \alpha\},$$

and the *relative range of* $\alpha \in \mathcal{L}^*(E)$ *with respect to* $A \subset E^0$ is defined by

$$r(A, \alpha) = \{r(\lambda) : \lambda \in AE^{\geq 1},\ \mathcal{L}(\lambda) = \alpha\}.$$

A collection $\mathcal{B}$ of subsets of E^0 is said to be *closed under relative ranges* for $(E, \mathcal{L})$ if $r(A, \alpha) \in \mathcal{B}$ whenever $A \in \mathcal{B}$ and $\alpha \in \mathcal{L}^*(E)$. We call $\mathcal{B}$ an *accommodating set* for $(E, \mathcal{L})$ if it is closed under relative ranges, finite intersections and unions, and contains $r(\alpha)$ for all $\alpha \in \mathcal{L}^*(E)$. A set $A \in \mathcal{B}$ is called *minimal* (in $\mathcal{B}$) if $A \cap B$ is either A or $\emptyset$ for all $B \in \mathcal{B}$. By $\mathcal{B}_{\min}$, we denote the set of all *minimal* sets in $\mathcal{B}$, that is,

$$\mathcal{B}_{\min} := \{A \in \mathcal{B} : A \cap B \text{ is either } A \text{ or } \emptyset \text{ for all } B \in \mathcal{B}\}.$$

If $\mathcal{B}$ is accommodating for $(E, \mathcal{L})$, the triple $(E, \mathcal{L}, \mathcal{B})$ is called a *labeled space*. We say that a labeled space $(E, \mathcal{L}, \mathcal{B})$ is *set-finite* (respectively,*receiver set-finite*) if for every

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$A \in \mathcal{B}$ and $k \geq 1$ the set $\mathcal{L}(AE^k)$ (respectively, $\mathcal{L}(E^k A)$) is finite. A labeled space $(E, \mathcal{L}, \mathcal{B})$ is said to be *weakly left-resolving* if it satisfies

$$r(A, \alpha) \cap r(B, \alpha) = r(A \cap B, \alpha)$$

for all $A, B \in \mathcal{B}$ and $\alpha \in \mathcal{L}^*(E)$. If $\mathcal{B}$ is closed under relative complements, we call $(E, \mathcal{L}, \mathcal{B})$ a *normal* labeled space as in [**2**].

Assumption. Throughout this paper, we assume that graphs E have no sinks and that labeled spaces $(E, \mathcal{L}, \mathcal{B})$ are weakly left-resolving, set-finite, receiver set-finite, and normal unless stated otherwise.

By $\mathcal{E}$ we denote the smallest accommodating set for which $(E, \mathcal{L}, \mathcal{E})$ is a normal labeled space. Let $\Omega_0(E)$ be the set of all vertices that are not sources, and for each $l \geq 1$ define a relation $\sim_l$ on $\Omega_0(E)$ by $v \sim_l w$ if and only if $\mathcal{L}(E^{\leq l} v) = \mathcal{L}(E^{\leq l} w)$. Then $\sim_l$ is an equivalence relation, and the equivalence class $[v]_l$ of $v \in \Omega_0(E)$ is called a *generalized vertex* (or simply a vertex if there is no risk of confusion). If $k > l$, then $[v]_k \subset [v]_l$ is obvious and $[v]_l = \bigcup_{i=1}^m [v_i]_{l+1}$ for some vertices $v_1, \ldots, v_m \in [v]_l$ [**5**, Proposition 2.4]. The generalized vertices of a labeled graph play the same role as vertices in a graph. Moreover, we have

$$\mathcal{E} = \left\{ \bigcup_{i=1}^n [v_i]_l : v_i \in E^0,\ l \geq 1, n \geq 0 \right\}, \tag{1}$$

with the convention $\sum_{i=1}^0 [v_i]_l := \emptyset$ by [**18**, Proposition 2.3].

2.2. *C^*-algebras of labeled spaces.* We review the definition of C^*-algebras associated to labeled spaces from [**2**, **4**, **5**].

Definition 2.1. A *representation* of a labeled space $(E, \mathcal{L}, \mathcal{B})$ is a family of projections $\{p_A : A \in \mathcal{B}\}$ and partial isometries $\{s_a : a \in \mathcal{A}\}$ such that for $A, B \in \mathcal{B}$ and $a, b \in \mathcal{A}$:

(i) $p_\emptyset = 0$, $p_{A \cap B} = p_A p_B$, and $p_{A \cup B} = p_A + p_B - p_{A \cap B}$;
(ii) $p_A s_a = s_a p_{r(A,a)}$;
(iii) $s_a^* s_a = p_{r(a)}$ and $s_a^* s_b = 0$ unless $a = b$;
(iv) $p_A = \sum_{a \in \mathcal{L}(AE^1)} s_a p_{r(A,a)} s_a^*$.

It is known [**4**, Theorem 4.5] that, given a labeled space $(E, \mathcal{L}, \mathcal{B})$, there exists a C^*-algebra $C^*(E, \mathcal{L}, \mathcal{B})$ generated by a universal representation $\{s_a, p_A\}$ of $(E, \mathcal{L}, \mathcal{B})$. We call $C^*(E, \mathcal{L}, \mathcal{B})$ the *labeled graph C^*-algebra* of a labeled space $(E, \mathcal{L}, \mathcal{B})$ which is unique up to isomorphism, and simply write $C^*(E, \mathcal{L}, \mathcal{B}) = C^*(s_a, p_A)$ to indicate the generators. Note that $s_a \neq 0$ and $p_A \neq 0$ for $a \in \mathcal{A}$ and $A \in \mathcal{B}$, $A \neq \emptyset$. Every graph C^*-algebra is a labeled graph C^*-algebra of a trivial labeled space. The labeled graph C^*-algebra $C^*(E, \mathcal{L}, \mathcal{B})$ depends on the choice of an accommodating set $\mathcal{B}$, and we are mainly interested in C^*-algebras of labeled spaces $(E, \mathcal{L}, \mathcal{E})$ with the smallest normal accommodating sets $\mathcal{E}$ throughout this paper.

Remark 2.2. Let $(E, \mathcal{L}, \mathcal{B})$ be a labeled space with $C^*(E, \mathcal{L}, \mathcal{B}) = C^*(s_a, p_A)$. By ε we denote a symbol (not belonging to $\mathcal{L}^*(E)$) such that $r(\varepsilon) = E^0$ and $r(A, \varepsilon) = A$ for all $A \subset E^0$. We write $\mathcal{L}^\#(E)$ for the union $\mathcal{L}^*(E) \cup \{\varepsilon\}$. Let s_ε denote the unit of the multiplier algebra of $C^*(E, \mathcal{L}, \mathcal{B})$.

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(a) We have the equality

$$(s_\alpha p_A s_\beta^*)(s_\gamma p_B s_\delta^*) = \begin{cases} s_{\alpha\gamma'} p_{r(A,\gamma')\cap B} s_\delta^* & \text{if } \gamma = \beta\gamma', \\ s_\alpha p_{A\cap r(B,\beta')} s_{\delta\beta'}^* & \text{if } \beta = \gamma\beta', \\ s_\alpha p_{A\cap B} s_\delta^* & \text{if } \beta = \gamma, \\ 0 & \text{otherwise,} \end{cases}$$

for $\alpha, \beta, \gamma, \delta \in \mathcal{L}^{\#}(E)$ and $A, B \in \mathcal{B}$ (see [**4**, Lemma 4.4]). Since $s_\alpha p_A s_\beta^* \neq 0$ if and only if $A \cap r(\alpha) \cap r(\beta) \neq \emptyset$, it follows that

$$C^*(E, \mathcal{L}, \mathcal{B}) = \overline{\operatorname{span}}\{s_\alpha p_A s_\beta^* : \alpha, \beta \in \mathcal{L}^{\#}(E) \text{ and } A \subseteq r(\alpha) \cap r(\beta)\}. \tag{2}$$

(b) The universal property of $C^*(E, \mathcal{L}, \mathcal{B}) = C^*(s_a, p_A)$ defines a strongly continuous action $\gamma : \mathbb{T} \to \operatorname{Aut}(C^*(E, \mathcal{L}, \mathcal{B}))$, called the *gauge action*, such that

$$\gamma_z(s_a) = z s_a \quad \text{and} \quad \gamma_z(p_A) = p_A$$

for $a \in \mathcal{A}$ and $A \in \mathcal{B}$. Averaging over γ with respect to the normalized Haar measure of the compact group $\mathbb{T}$,

$$\Phi(a) := \int_{\mathbb{T}} \gamma_z(a)\, dz, \quad a \in C^*(E, \mathcal{L}, \mathcal{B}),$$

defines a conditional expectation $\Phi : C^*(E, \mathcal{L}, \mathcal{B}) \to C^*(E, \mathcal{L}, \mathcal{B})^\gamma$ onto the fixed point algebra which is also known to be faithful.

(c) It is shown in [**5**, Theorem 4.4] that $C^*(E, \mathcal{L}, \mathcal{B})^\gamma$ is an AF algebra isomorphic to $\overline{\bigcup_{k,l}(\oplus_{[v]_l} \mathcal{F}^k([v]_l))}$, where

$$\mathcal{F}^k([v]_l) = \overline{\operatorname{span}}\{s_\alpha p_{[v]_l} s_\beta^* : \alpha, \beta \in \mathcal{L}(E^k)\}. \tag{3}$$

2.3. *Cuntz–Krieger uniqueness theorem.* Recall that a *Cuntz–Krieger E-family* for a graph E is a representation of the labeled space $(E, \mathcal{L}_{\mathrm{id}}, \mathcal{B})$ with the trivial labeling, and *the Cuntz–Krieger uniqueness theorem* for graph C^*-algebras says that if E satisfies condition (L), then every Cuntz–Krieger E-family of non-zero operators generates the same C^*-algebra $C^*(E)$ up to isomorphism (for example, see [**6**, Theorem 3.1], [**10**, Corollary 2.12], and [**25**, Theorem 3.7]). A condition for labeled spaces corresponding to condition (L) for directed graphs was given in [**5**, Definition 5.2], and we briefly review it here.

A labeled path $\alpha \in \mathcal{L}^*(E)$ with $s(\alpha) \cap [v]_l \neq \emptyset$ is called *agreeable* for $[v]_l$ if $\alpha = \beta\alpha' = \alpha'\gamma$ for some $\alpha', \beta, \gamma \in \mathcal{L}^*(E)$ with $|\beta| = |\gamma| \leq l$. Otherwise α is called *disagreeable*. (It is useful to note that any path α agreeable for $[v]_l$ must be of the form $\alpha = \beta^k\beta'$ for some $\beta \in E^{\leq l}$, $k \geq 0$, and an initial path β' of β.) We say that $[v]_l$ is *disagreeable* if there is an $N \geq 1$ such that for all $n > N$ there is an $\alpha \in \mathcal{L}(E^{\geq n})$ which is disagreeable for $[v]_l$. (It can be easily seen that $[v]_l$ is not disagreeable if and only if there is an $N \geq 1$ such that every path in $\mathcal{L}([v]_l E^{\geq N})$ is agreeable for $[v]_l$.) A labeled space $(E, \mathcal{L}, \mathcal{B})$ is said to be *disagreeable* if for every $v \in E^0$, there is an $L_v \geq 1$ such that every $[v]_l$ is disagreeable for all $l \geq L_v$. Then it is shown in [**5**, Lemma 5.3] that for a graph E, the labeled space $(E, \mathcal{L}_{\mathrm{id}}, \mathcal{E})$ with the trivial labeling is disagreeable if and only if E satisfies condition (L).

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In [**9**, Definition 9.5], the notion of cycle was introduced to define condition $(L_{\mathcal{B}})$ for a labeled space $(E, \mathcal{L}, \mathcal{B})$ (more generally for Boolean dynamical systems) which can be regarded as another condition equivalent to condition (L) for usual directed graphs. For a path $\alpha \in \mathcal{L}^*(E)$ and a set $\emptyset \neq A \in \mathcal{E}$, the pair (α, A) is called a *cycle* if $B = r(B, \alpha^k)$ holds for all $k \geq 0$ and non-empty subsets $B \in \mathcal{E}$ of A (see [**9**, Definition 9.5]). We say that a labeled space $(E, \mathcal{L}, \mathcal{E})$ satisfies condition $(L_{\mathcal{E}})$ if there is no cycle without an exit. We will see in Proposition 3.7 that every disagreeable labeled space satisfies condition $(L_{\mathcal{E}})$. The following is the *Cuntz–Krieger uniqueness theorem* for labeled graph C^*-algebras.

THEOREM 2.3. ([**4**, Theorem 5.5], [**9**, Theorem 9.9]) *Let $\{t_a, q_A\}$ be a representation of a labeled space $(E, \mathcal{L}, \mathcal{E})$ such that $q_A \neq 0$ for all non-empty $A \in \mathcal{E}$. If $(E, \mathcal{L}, \mathcal{E})$ satisfies condition $(L_{\mathcal{E}})$, in particular if $(E, \mathcal{L}, \mathcal{E})$ is disagreeable, then the canonical homomorphism $\phi : C^*(E, \mathcal{L}, \mathcal{E}) = C^*(s_a, p_A) \to C^*(t_a, q_A)$ such that $\phi(s_a) = t_a$ and $\phi(p_A) = q_A$ is an isomorphism.*

2.4. *The ideal structure of labeled graph C^*-algebras.* We first review definitions of quotient labeled spaces and their C^*-algebras which were introduced in [**21**] to study the ideal structure of labeled graph C^*-algebras.

Let $(E, \mathcal{L}, \mathcal{E})$ be a labeled space and $\sim_R$ an equivalence relation on $\mathcal{E}$. Denote the equivalence class of $A \in \mathcal{E}$ by $[A]$ (rather than $[A]_R$) and set

$$\begin{aligned} \mathcal{A}_R &:= \{a \in \mathcal{A} : [r(a)] \neq [\emptyset]\}, \\ \mathcal{L}_R(E^n) &:= \{\alpha \in \mathcal{L}(E^n) : [r(\alpha)] \neq [\emptyset]\}, \quad \text{for } n \geq 1, \\ \mathcal{L}_R^*(E) &:= \{\alpha \in \mathcal{L}^*(E) : [r(\alpha)] \neq [\emptyset]\}, \end{aligned}$$

and so forth. If the operations

$$[A] \cup [B] := [A \cup B], \quad [A] \cap [B] := [A \cap B], \quad [A] \setminus [B] := [A \setminus B]$$

are well defined on $\mathcal{E}/R := \{[A] : A \in \mathcal{E}\}$ and if the relative range

$$r([A], \alpha) := [r(A, \alpha)]$$

is well defined for $[A] \in \mathcal{E}/R$ and $\alpha \in \mathcal{L}_R^*(E)$ so that $r([A], \alpha) = [\emptyset]$ for all $\alpha \in \mathcal{L}_R^*(E)$ implies $[A] = [\emptyset]$, we call $(E, \mathcal{L}, \mathcal{E}/R)$ a *quotient labeled space* of $(E, \mathcal{L}, \mathcal{E})$. We also use the notation $\mathcal{L}_R([A]E^{\geq N}) := \bigcup_{n \geq N} \mathcal{L}_R([A]E^n)$, where

$$\mathcal{L}_R([A]E^n) := \{\alpha \in \mathcal{L}([A]E^n) : r([A], \alpha) \neq [\emptyset]\}.$$

We say that a quotient labeled space $(E, \mathcal{L}, \mathcal{E}/R)$ is *weakly left-resolving* if for $[A], [B] \in \mathcal{E}/R$ and $\alpha \in \mathcal{L}_R^*(E)$, the following holds:

$$r([A], \alpha) \cap r([B], \alpha) = r([A] \cap [B], \alpha).$$

We write $[A] \subseteq [B]$ if $[A] \cap [B] = [A]$, and $[A] \subsetneq [B]$ if $[A] \subseteq [B]$ and $[A] \neq [B]$.

Definition 2.4. [**21**, Definition 3.3] A *representation* of a quotient labeled space $(E, \mathcal{L}, \mathcal{E}/R)$ (always assumed weakly left-resolving) is a family of projections $\{p_{[A]} : [A] \in \mathcal{E}/R\}$ and partial isometries $\{s_a : a \in \mathcal{A}_R\}$ such that:

(i) $p_{[\emptyset]} = 0$, $p_{[A]\cap[B]} = p_{[A]}p_{[B]}$, and $p_{[A]\cup[B]} = p_{[A]} + p_{[B]} - p_{[A]\cap[B]}$;

(ii) $p_{[A]}s_a = s_a p_{r([A],a)}$;

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(iii) $s_a^* s_a = p_{[r(a)]}$ and $s_a^* s_b = 0$ unless $a = b$;
(iv) $p_{[A]} = \sum_{a \in \mathcal{L}_R([A]E^1)} s_a p_{r([A],a)} s_a^*$ if $\mathcal{L}_R([A]E^1) \neq \emptyset$.

It is known [**21**, Theorem 3.10] that if $(E, \mathcal{L}, \mathcal{E}/R)$ is a quotient labeled space, there exists a C^*-algebra $C^*(E, \mathcal{L}, \mathcal{E}/R)$, called the *quotient labeled graph C^*-algebra* of $(E, \mathcal{L}, \mathcal{E}/R)$, generated by a universal representation of $(E, \mathcal{L}, \mathcal{E}/R)$. By [**21**, Lemma 3.8], we have

$$C^*(E, \mathcal{L}, \mathcal{E}/R) = \overline{\operatorname{span}}\{s_\alpha p_{[A]} s_\beta^* : \alpha, \beta \in \mathcal{L}_R^*(E),\ [A] \neq [\emptyset]\}. \tag{4}$$

The equivalence relation on $\mathcal{E}$ which we are mostly interested in is related with hereditary saturated subsets H of $\mathcal{E}$. Recall from [**21**, Definition 3.4] that a subset H of $\mathcal{E}$ is *hereditary* if it is closed under finite unions, relative ranges, and subsets in $\mathcal{E}$. A hereditary set H is *saturated* if $A \in H$ whenever $A \in \mathcal{E}$ satisfies $r(A, \alpha) \in H$ for all $\alpha \in \mathcal{L}^*(E)$. Every hereditary saturated subset $H \subset \mathcal{E}$ defines an equivalence relation $\sim_H$ on $\mathcal{E}$; for $A, B \in \mathcal{E}$,

$$A \sim_H B \iff A \cup W = B \cup W \quad \text{for some } W \in H.$$

For a hereditary set H, let $\overline{H}$ be the smallest hereditary saturated set containing H. Then the ideal I_H of $C^*(E, \mathcal{L}, \mathcal{E})$ generated by the projections $\{p_A : A \in H\}$ is gauge invariant [**21**, Lemma 3.7] and

$$I_H = \overline{\operatorname{span}}\{s_\mu p_A s_\nu^* : \mu, \nu \in \mathcal{L}^{\#}(E),\ A \in \overline{H}\}.$$

Remark 2.5. Let $(E, \mathcal{L}, \mathcal{E})$ be a labeled space.

(a) $H \mapsto I_H$ is an inclusion-preserving bijection between the non-empty hereditary saturated subsets of $\mathcal{E}$ and the non-zero gauge-invariant ideals of the C^*-algebra $C^*(E, \mathcal{L}, \mathcal{E})$. Moreover, the quotient algebra $C^*(E, \mathcal{L}, \mathcal{E})/I_H$ is isomorphic to $C^*(E, \mathcal{L}, \mathcal{E}/H)$ [**21**, Theorem 5.2].

(b) Every quotient labeled graph C^*-algebra $C^*(E, \mathcal{L}, \mathcal{E}/R) = C^*(s_a, p_{[A]})$ admits the *gauge action* γ of $\mathbb{T}$,

$$\gamma_z(s_a) = z s_a \quad \text{and} \quad \gamma_z(p_{[A]}) = p_{[A]},$$

for $a \in \mathcal{A}_R$ and $[A] \in \mathcal{E}/R$. It can be seen as in [**5**, Theorem 4.4] that the fixed point algebra

$$C^*(E, \mathcal{L}, \mathcal{E}/R)^\gamma = \overline{\operatorname{span}}\{s_\alpha p_{[A]} s_\beta^* : \alpha, \beta \in \mathcal{L}_R(E^n),\ n \geq 1,\ [A] \in \mathcal{E}/R\}$$

is an AF algebra (see the proof of [**21**, Theorem 4.2]). Also, one sees as in Remark 2.2(b) that there exists a faithful conditional expectation Φ of $C^*(E, \mathcal{L}, \mathcal{E}/R)$ onto the fixed point algebra $C^*(E, \mathcal{L}, \mathcal{E}/R)^\gamma$.

2.5. *Simplicity of labeled graph C^*-algebras.* Let $(E, \mathcal{L}, \mathcal{E})$ be a labeled space and let

$$\overline{\mathcal{L}(E^\infty)} := \{x \in \mathcal{A}^{\mathbb{N}} \mid x_{[1,n]} \in \mathcal{L}(E^n) \text{ for all } n \geq 1\}$$

be the set of all infinite sequences x such that every finite subpath of x occurs as a labeled path in $(E, \mathcal{L})$. Clearly $\mathcal{L}(E^\infty) \subset \overline{\mathcal{L}(E^\infty)}$, and in fact, $\overline{\mathcal{L}(E^\infty)}$ is the closure of $\mathcal{L}(E^\infty)$

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in the totally disconnected perfect space $\mathcal{A}^{\mathbb{N}}$ which has the topology with a countable basis of open-closed cylinder sets $Z(\alpha) := \{x \in \mathcal{A}^{\mathbb{N}} : x_{[1,n]} = \alpha\}$, $\alpha \in \mathcal{A}^n$, $n \geq 1$ (see [**24**, §7.2]). We say that a labeled space $(E, \mathcal{L}, \mathcal{E})$ is *strongly cofinal* if for each $x \in \overline{\mathcal{L}(E^\infty)}$ and $[v]_l \in \mathcal{E}$, there exist an $N \geq 1$ and a finite number of paths $\lambda_1, \ldots, \lambda_m \in \mathcal{L}^*(E)$ such that

$$r(x_{[1,N]}) \subset \bigcup_{i=1}^{m} r([v]_l, \lambda_i).$$

Remarks 2.6. For simplicity of $C^*(E, \mathcal{L}, \mathcal{E})$, we note the following.

(a) If $(E, \mathcal{L}, \mathcal{E})$ is disagreeable and strongly cofinal, then $C^*(E, \mathcal{L}, \mathcal{E})$ is simple (see [**5**, Theorem 6.4], [**20**, Theorem 3.16], and [**19**, Remark 3.8]). A condition of being disagreeable and cofinal assumed in [**5**, Theorem 6.4] is in fact not sufficient for the C^*-algebra $C^*(E, \mathcal{L}, \mathcal{E})$ to be simple, but by modifying its proof for a disagreeable and strongly cofinal labeled space one can obtain the simplicity result.

(b) We see from [**9**, Theorem 9.16] that $C^*(E, \mathcal{L}, \mathcal{E})$ is simple if and only if $(E, \mathcal{L}, \mathcal{E})$ satisfies condition $(L_{\mathcal{E}})$ and the only hereditary and saturated subsets of $\mathcal{E}$ are $\emptyset$ and $\mathcal{E}$.

2.6. *Purely infinite C^*-algebras.* For positive elements a, b in a C^*-algebra A, we write $a \precsim b$ if there exists a sequence $\{x_k\}_{k=1}^{\infty}$ in A such that $x_k^* b x_k \to a$. More generally, for positive elements $a \in M_n(A)$ and $b \in M_m(A)$, we write $a \precsim b$ if there exists a sequence $\{x_k\}_{k=1}^{\infty}$ in $M_{m,n}(A)$ such that $x_k^* b x_k \to a$. A positive element $a \in A$ is said to be *infinite* if there exists a non-zero positive element $b \in A$ such that $a \oplus b \precsim a$, and *properly infinite* if $a \oplus a \precsim a$. Every properly infinite a is obviously infinite. Because the relation $a \oplus a \precsim a$ is preserved under $*$-homomorphisms, if $a \in A$ is properly infinite, then for any ideal I of A the element $a + I$ is either zero or properly infinite in the quotient A/I of A.

A C^*-algebra A is said to be *purely infinite* if there are no characters on A and if for every pair of positive elements $a, b \in A$, $a \precsim b$ if and only if $a \in \overline{AbA}$ (see [**27**, Definition 4.1]). Note that any C^*-algebra of type I is not purely infinite [**27**, Definition 4.4]. For example, $C(\mathbb{T}) \otimes M_n$ is not purely infinite for $n \geq 1$.

Remarks 2.7. Let A be a C^*-algebra.

(a) A is purely infinite if and only if every non-zero positive element in A is properly infinite [**27**, Theorem 4.16]. For projections p, q in A, we write $p \preceq q$ if p is Murray–von Neumann equivalent to a subprojection of q, namely $p \sim q \leq p$. A projection $p \in A$ is infinite if and only if $p \sim q \lneq p$ for some subprojection q of p, and is properly infinite if and only if p has mutually orthogonal subprojections p_1 and p_2 such that $p_1 \sim p_2 \sim p$.

(b) A is purely infinite if every non-zero hereditary subalgebra in every quotient of A contains an infinite projection, namely every quotient of A has property (IH) [**27**, Proposition 4.7].

(c) Every non-zero hereditary subalgebra and every quotient algebra of a purely infinite C^*-algebra is always purely infinite [**27**, Propositions 4.17 and 4.3].

(d) If A is simple, then A is purely infinite if and only if it has property (IH) (see [**27**, Propositions 4.6 and 5.4]). It should be noted that for non-simple C^*-algebras, these

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two properties is neither weaker nor stronger than each other in general (for example, see [**27**, Example 4.6]).

3. *Labeled graph C^*-algebras with property (IH)*

3.1. *Infinite C^*-algebras of disagreeable labeled spaces.* Because the original definition of a disagreeable labeled space seems a bit complicated, we list its equivalent but simpler conditions in Proposition 3.2 after we observe the following lemma which will also be frequently used.

LEMMA 3.1. *Let α and β be paths in $\mathcal{L}^*(E)$ (or words in $\mathcal{A}^*$) with $|\alpha| \le |\beta|$. If $\alpha^m = \beta^n$ for some $m, n \ge 1$, then $\alpha, \beta \in \{\alpha_0^k : k \ge 1\}$ for an initial path α_0 of α.*

Proof. If $|\alpha| = 1$ or $n = 1$, there is nothing to prove. Thus we assume $|\alpha| > 1$ and $n > 1$. Since $|\alpha| \le |\beta|$ and the path $\alpha^m = \beta^n$ begins with both of α and β, the longer path β must have α as an initial path. If $\beta = \alpha^{n_1}$ for some $n_1 \ge 1$, we are done. Otherwise, we can write $\beta = \alpha^{n_1}\alpha^{(1)}$ for some $n_1 \ge 1$ and an initial path $\alpha^{(1)}$ of α with $1 \le |\alpha^{(1)}| < |\alpha|$. Then by canceling α^{n_1} from the left of both sides of $\alpha^m = \beta^n$, we obtain $\alpha^{m-n_1} = \alpha^{(1)}\beta^{n-1}$. Thus, with a subpath $\alpha^{(2)}$ of α such that $\alpha = \alpha^{(1)}\alpha^{(2)}$, it follows that

$$\alpha^{(1)}\alpha^{(2)} \cdots \alpha^{(1)}\alpha^{(2)} = \alpha^{(1)}\alpha^{n_1}\alpha^{(1)} \cdots \alpha^{n_1}\alpha^{(1)}$$

in which the right-hand side is equal to $\alpha^{(1)}(\alpha^{(1)}\alpha^{(2)})^{n_1}\alpha^{(1)} \cdots \alpha^{n_1}\alpha^{(1)}$, hence comparing initial parts of both sides we see that $\alpha^{(2)}\alpha^{(1)} = \alpha^{(1)}\alpha^{(2)}$. Thus the lemma reduces to the following claim.

Claim. If $\alpha, \beta \in \mathcal{L}^*(E)$ and $\alpha\beta = \beta\alpha$, then $\alpha, \beta \in \{\delta^k : k \ge 1\}$ for some $\delta \in \mathcal{L}^*(E)$.

Again we may assume that $|\alpha| < |\beta|$ and $\beta = \alpha^m\alpha^{(1)}$ for some $m \ge 1$ and an initial path $\alpha^{(1)}$ of α with $1 \le |\alpha^{(1)}| < |\alpha|$. Then $\alpha\beta = \beta\alpha$ gives $\alpha\alpha^{(1)} = \alpha^{(1)}\alpha$. If α is a repetition of $\alpha^{(1)}$, that is, $\alpha = (\alpha^{(1)})^n$ for some $n \ge 1$, the claim follows. If not, with $\alpha = \alpha^{(1)}\alpha^{(2)}$, we have from $\alpha\alpha^{(1)} = \alpha^{(1)}\alpha$ that $\alpha^{(1)}\alpha^{(2)} = \alpha^{(2)}\alpha^{(1)}$. If one of $\alpha^{(1)}$ or $\alpha^{(2)}$ is a repetition of the other, for example if $\alpha^{(1)} = (\alpha^{(2)})^s$ for some $s \ge 1$, we can take $\alpha^{(2)}$ for δ. If not, we continue the process to obtain

$$\alpha^{(k)}\alpha^{(k+1)} = \alpha^{(k+1)}\alpha^{(k)} \tag{5}$$

for initial paths $\alpha^{(k)}$, $k \ge 1$, of α. We can check if one of $\alpha^{(k)}$ or $\alpha^{(k+1)}$ is a repetition of the other whenever we arrive at step (5). If it is the case, then we stop and obtain the claim. If not, again we can go on to the next step but with paths having length smaller than the previous step. Obviously this process should stop after a finite number of steps, and when it stops we obtain an initial path δ of α satisfying the claim. □

PROPOSITION 3.2. *For a labeled space $(E, \mathcal{L}, \mathcal{B})$, the following are equivalent:*

(a) *$(E, \mathcal{L}, \mathcal{B})$ is disagreeable;*
(b) *$[v]_l$ is disagreeable for all $v \in E^0$ and $l \ge 1$;*
(c) *for each non-empty $A \in \mathcal{B}$ and a path $\beta \in \mathcal{L}^*(E)$, there is an $n \ge 1$ such that $\mathcal{L}(AE^{|\beta|n}) \ne \{\beta^n\}$.*

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Proof. The first two are known to be equivalent from [**20**, Proposition 3.9(iii)].

To show (b) $\Rightarrow$ (c), suppose there are a non-empty set $A \in \mathcal{E}$ and a path $\beta \in \mathcal{L}^*(E)$ such that $\mathcal{L}(AE^{|\beta|n}) = \{\beta^n\}$ for all $n \geq 1$. Then for a vertex $\emptyset \neq [v]_l \subset A$ with $l \geq |\beta|$, $\mathcal{L}([v]_l E^{|\beta|n}) = \{\beta^n\}$ for all $n \geq 1$, and $[v]_l$ is not disagreeable.

For (c) $\Rightarrow$ (b), suppose $[v]_l$ is not disagreeable for some $v \in E^0$ and $l \geq 1$. Then, by [**20**, Proposition 3.9(i)], there is an $N \geq l$ such that every path in $\mathcal{L}([v]_l E^{\geq N})$ is agreeable for $[v]_l$. Observe here that for any $\delta \in \mathcal{L}^*(E)$, every path in $\mathcal{L}(r([v]_l, \delta^N)E^{\geq 1})$ is agreeable for $[v]_l$, which is immediate from the inclusion relation $\mathcal{L}(r([v]_l, \delta^N)E^{\geq 1}) \subset \mathcal{L}([v]_l E^{\geq N})$. One can choose a (finite) set $\{\delta_1, \ldots, \delta_m\}$ of paths $\delta \in \mathcal{L}^*(E)$ with $|\delta| \leq l$ such that every path in $\mathcal{L}([v]_l E^{\geq N})$ is of the form

$$\delta_i^n \delta_i' \tag{6}$$

for some $n \geq 1$, $1 \leq i \leq m$, and δ_i' an initial path of δ_i. We may assume that

$$\text{if } i \neq j \text{ then } |\delta_i^{n_i}| = |\delta_j^{n_j}| \text{ and } \delta_i^{n_i} \neq \delta_j^{n_j} \text{ for some } n_i, n_j \geq 1. \tag{7}$$

In fact, if $\delta_1^{n_1} = \delta_2^{n_2}$ whenever $|\delta_1^{n_1}| = |\delta_2^{n_2}|$, then by Lemma 3.1 we can find an initial path δ of δ_1 such that δ_1 and δ_2 are equal to some repetitions of δ. Then we remove δ_1, δ_2 from the set $\{\delta_1, \ldots, \delta_m\}$, and include δ instead. Pick $L \geq N$ large enough so that the path δ_1^L cannot be of the form in (6) for $i \neq 1$ (this is possible by (7)). Then for $A := r([v]_l, \delta_1^L) \in \mathcal{E}$, every path in $\mathcal{L}(AE^{\geq 1})$ is, as already observed, agreeable for $[v]_l$, hence must be of the form $\delta_1^n \delta_1'$. But this contradicts (c). □

We can naturally extend the notion of 'disagreeable labeled space' to 'disagreeable quotient labeled space' as in the next definition. The Cuntz–Krieger uniqueness theorem for disagreeable quotient labeled spaces will be given in §4.

Definition 3.3. Let $(E, \mathcal{L}, \mathcal{E}/R)$ be a quotient labeled space and $[[v]_l] \neq [\emptyset]$ in $\mathcal{E}/R$. A path $\alpha \in \mathcal{L}_R^*(E)$ with $s(\alpha) \cap [[v]_l] \neq \emptyset$ is said to be *agreeable* for $[[v]_l]$ if

$$[r([v]_l, \alpha)] = [\emptyset] \quad \text{or} \quad \alpha = \beta\alpha' = \alpha'\gamma$$

for some $\alpha', \beta, \gamma \in \mathcal{L}_R^*(E)$ with $|\beta| = |\gamma| \leq l$. Otherwise α is *disagreeable* for $[[v]_l]$. Also $[[v]_l]$ is called *disagreeable* if there is an $N \geq 1$ such that for all $n > N$, there is a path in $\mathcal{L}_R([[v]_l]E^{\geq n})$ which is disagreeable for $[[v]_l]$. The quotient labeled space $(E, \mathcal{L}, \mathcal{E}/R)$ is said to be *disagreeable* if every $[[v]_l] \neq [\emptyset]$ is disagreeable for $l \geq 1$ and $v \in E^0$.

Remark 3.4. As in Proposition 3.2, one can prove that $(E, \mathcal{L}, \mathcal{E}/R)$ is disagreeable if and only if for each $[A] \neq [\emptyset]$ in $\mathcal{E}/R$ and a path $\beta \in \mathcal{L}_R^*(E)$, there is an $n \geq 1$ such that $\mathcal{L}_R([A]E^{|\beta|n}) \neq \{\beta^n\}$.

It is easy to see that if a graph E has a loop with an exit, the C^*-algebra $C^*(E)$ contains an infinite projection. In order to see whether this is true for labeled graph C^*-algebras, we will use the following definitions of loop and exit in a quotient labeled space (see [**18**, Definition 3.2]).

Definition 3.5. Let $(E, \mathcal{L}, \mathcal{E}/R)$ be a quotient labeled space. For a path $\alpha \in \mathcal{L}_R^*(E)$ and a set $[\emptyset] \neq [A] \in \mathcal{E}/R$, we call $(\alpha, [A])$ (or α) a *loop* at $[A]$ if $[A] \subseteq r([A], \alpha)$. We say that a loop $(\alpha, [A])$ has an *exit* if one of the following holds:

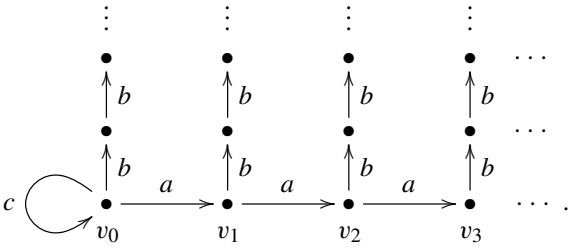
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(i) there exists a path $\beta \in \mathcal{L}_R([A]E^{|\alpha|})$ such that $\beta \neq \alpha$;
(ii) $[A] \subsetneq r([A], \alpha)$.

Clearly, every cycle is a loop, and an exit of a cycle must be of type (i).

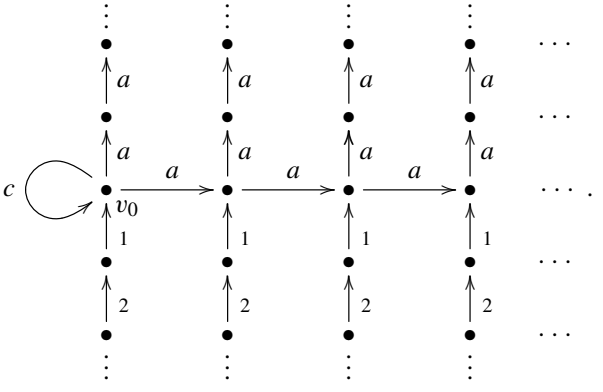
Example 3.6. We give examples of labeled spaces which have loops with exits.

(i) For the labeled graph $(E, \mathcal{L})$ given below, let H be the smallest hereditary saturated subset of $\mathcal{E}$ containing $r(c) = \{v_0\} \in \mathcal{E}$.



Then $\{v\} \in H$ for each $v \in E^0$. Consider the quotient labeled space $(E, \mathcal{L}, \mathcal{E}/H)$. Since $r([r(a)], a) = [r(a^2)] = [r(a)]$, we see that $(a, [r(a)])$ is a loop. On the other hand, $b \in \mathcal{L}_H([r(a)]E^1)$ and it follows from $r([r(a)], b) = [r(ab)] \neq [\emptyset]$ that the loop $(a, [r(a)])$ has an exit $b(\neq a)$ of type (i) of Definition 3.5.

(ii) Let H be the smallest hereditary saturated subset containing $r(c) = \{v_0\}$ in the labeled space of the following graph.



Then $(a, [r(1)])$ is a loop in the quotient labeled space $(E, \mathcal{L}, \mathcal{E}/H)$. Since $[r(1)] \subsetneq r([r(1)], a) = [(r(1a)]$, it has an exit of type (ii) of Definition 3.5.

If a directed graph E has a loop with an exit, the exit gives rise to an infinite projection in the graph C^*-algebra $C^*(E)$. The same is true for labeled graph C^*-algebras [**18**, Proposition 3.5], and more generally for quotient labeled graph C^*-algebras, as we see in the following proposition.

PROPOSITION 3.7. *Let $(E, \mathcal{L}, \mathcal{E}/R)$ be a quotient labeled space. If $(\alpha, [A])$ is a loop with an exit, then $p_{[A]}$ is an infinite projection in $C^*(E, \mathcal{L}, \mathcal{E}/R)$. In particular, if $(E, \mathcal{L}, \mathcal{E}/R)$ is disagreeable, every loop $(\alpha, [A])$ has an exit and $p_{[A]}$ is an infinite projection.*

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Proof. If $(\alpha, [A])$ has an exit of type (i) with a path $\beta \in \mathcal{L}_R([A]E^{|\alpha|})$ such that $\beta \neq \alpha$ and $r([A], \beta) \neq [\emptyset]$, then the projection $p_{[A]}$ is infinite since

$$p_{r([A],\alpha)} \geq p_{[A]} = \sum_{|\delta|=|\alpha|} s_\delta p_{r([A],\delta)} s_\delta^* > s_\alpha p_{r([A],\alpha)} s_\alpha^* \sim p_{r([A],\alpha)}.$$

If $(\alpha, [A])$ has an exit of type (ii), that is $[A] \subsetneq r([A], \alpha)$, then

$$p_{r([A],\alpha)} \sim s_\alpha p_{r([A],\alpha)} s_\alpha^* \leq p_{[A]} \lneq p_{r([A],\alpha)},$$

which shows that $p_{[A]}$ is infinite.

To prove the second assertion, let $(E, \mathcal{L}, \mathcal{E}/R)$ be a disagreeable labeled space. If $[A] \subsetneq r([A], \alpha)$, α has an exit of type (ii). Assume that $[A] = r([A], \alpha)$. Then by Proposition 3.2 (see Remark 3.4), there is an $n \geq 1$ such that $\mathcal{L}_R([A]E^{|\alpha|n}) \neq \{\alpha^n\}$. Let m be the smallest integer among such n satisfying $\mathcal{L}_R([A]E^{|\alpha|n}) \neq \{\alpha^n\}$. Choose $\beta \in \mathcal{L}_R([A]E^{|\alpha|m})$ with $\beta \neq \alpha^m$. If $m = 1$, or $m > 1$ and $\beta_{[1,|\alpha|]} \neq \alpha$, then $\beta_{[1,|\alpha|]}$ is an exit of $(\alpha, [A])$. If $m > 1$ and $\beta = \alpha\beta''$ for some $\beta''(\neq \alpha^{m-1})$, then from $[A] = r([A], \alpha)$ we have $\beta'' \in \mathcal{L}_R([A]E^{|\alpha|(m-1)})$. Hence $\mathcal{L}_R([A]E^{|\alpha|(m-1)}) \neq \{\alpha^{m-1}\}$ follows, but this contradicts the choice of m. □

3.2. *Labeled graph C^*-algebras with property (IH).* As mentioned in the Introduction, it is well known that for a graph E satisfying condition (L), the C^*-algebra $C^*(E)$ has property (IH) if and only if every vertex connects to a loop. We will prove in Theorem 3.14 that for a disagreeable labeled space $(E, \mathcal{L}, \mathcal{E})$, if every vertex connects to a loop in the sense of Definition 3.10 below, then $C^*(E, \mathcal{L}, \mathcal{E})$ has property (IH). Its converse will be shown to be true under an additional condition. To prove Theorem 3.14, we need the following useful proposition which can be obtained by slightly modifying the proofs of [**5**, Theorem 6.9] and [**6**, Proposition 5.3]. We provide a detailed proof for the reader's convenience.

PROPOSITION 3.8. *Let $(E, \mathcal{L}, \mathcal{E})$ be a disagreeable labeled space. Then every non-zero hereditary C^*-subalgebra of $C^*(E, \mathcal{L}, \mathcal{E})$ contains a non-zero projection p such that $0 \neq s_\mu p_A s_\mu^* \preceq p$ for some $\mu \in \mathcal{L}^*(E)$ and $A \in \mathcal{E}$ with $A \subset r(\mu)$.*

Proof. Let B be a non-zero hereditary C^*-subalgebra of $C^*(E, \mathcal{L}, \mathcal{E})$ and fix a positive element $a \in B$ with $\|\Phi(a)\| = 1$. Choose a positive element $b \in \operatorname{span}\{s_\alpha p_A s_\beta^* : \alpha, \beta \in \mathcal{L}^*(E) \text{ and } A \subseteq r(\alpha) \cap r(\beta)\}$ so that $\|a - b\| < \frac{1}{4}$. From [**5**, Proposition 2.4(ii) and (iii)], we may write $b = \sum_{(\alpha,[w]_l,\beta)\in F} c_{(\alpha,[w]_l,\beta)} s_\alpha p_{[w]_l} s_\beta^*$, where F is a finite subset of $\mathcal{L}^*(E) \times \Omega_l \times \mathcal{L}^*(E)$ for some $l \geq 1$. Let $b_0 = \Phi(b) \geq 0$. Since Φ is norm-decreasing, we have

$$|1 - \|b_0\|| = |\|\Phi(a)\| - \|\Phi(b)\|| \leq \|\Phi(a - b)\| \leq \|a - b\| < \tfrac{1}{4},$$

and hence $\|b_0\| \geq \frac{3}{4}$. Let $k = \max\{|\alpha|, |\beta| : (\alpha, [w]_l, \beta) \in F\}$. Applying Definition 2.1(iv) and changing F (if necessary), we can choose a $k \in \mathbb{N}$ so that $\min\{|\alpha|, |\beta|\} = k$ for every $(\alpha, [w]_l, \beta) \in F$. Let $M = \max\{|\alpha|, |\beta| : (\alpha, [w]_l, \beta) \in F\}$. Applying [**5**, Proposition 2.4(iii)] again, we may choose $m \geq M$ large enough so that

$$b_0 \in \oplus_{\{w:(\alpha,[w]_l,\beta)\in F\}} \mathcal{F}^k([w]_m)$$

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(see (3) for $\mathcal{F}^k([w]_m)$). Now $\|b_0\|$ must be attained in some summand $\mathcal{F}^k([v]_m)$. Let b_1 be the component of b_0 in $\mathcal{F}^k([v]_m)$ so that $\|b_0\| = \|b_1\|$ and note that $b_1 \geq 0$. Then we can choose a projection $r \in C^*(b_1) \subseteq \mathcal{F}^k([v]_m)$ such that $rb_1r = \|b_1\|r$. Since b_1 is a finite sum of $s_\alpha p_{[v]_m} s_\beta^*$, we can write r as a sum $\sum c_{\alpha\beta} s_\alpha p_{[v]_m} s_\beta^*$ over all pairs of paths in

$$G = \{\alpha \in \mathcal{L}(E^k) : \text{ either } (\alpha, [v]_m, \beta) \in F \text{ or } (\beta, [v]_m, \alpha) \in F\}.$$

Note that $rb_0r = rb_1r$ and the $G \times G$ matrix $(c_{\alpha\beta})$ is also a projection in a finite-dimensional matrix algebra $\mathcal{F}^k([v]_m) = \text{span}\{s_\alpha p_{[v]_m} s_\beta^* : \alpha, \beta \in G\}$.

Since $[v]_m$ is disagreeable, we may choose a path $\lambda \in \mathcal{L}^*(E)$ with $|\lambda| > M$ so that λ has no factorization $\lambda = \lambda'\lambda'' = \lambda''\delta$ for some $|\lambda'|, |\delta| \leq m$. Then because $\text{span}\{s_{\alpha\lambda} p_{r([v]_m,\lambda)} s_{\beta\lambda}^* : \alpha, \beta \in G\}$ is also a finite-dimensional matrix algebra generated by the family of non-zero matrix units $\{s_{\alpha\lambda} p_{r([v]_m,\lambda)} s_{\beta\lambda}^* : \alpha, \beta \in G\}$,

$$Q = \sum_{\alpha,\beta\in G} c_{\alpha\beta} s_{\alpha\lambda} p_{r([v]_m,\lambda)} s_{\beta\lambda}^*$$

is a projection satisfying

$$r = \sum c_{\alpha\beta} s_\alpha p_{[v]_m} s_\beta^* = \sum c_{\alpha\beta} s_\alpha (s_\lambda p_{r([v]_m,\lambda)} s_\lambda^* + (p_{[v]_m} - s_\lambda p_{r([v]_m,\lambda)} s_\lambda^*)) s_\beta^* \geq Q.$$

We claim that for $(\mu, [v]_m, \nu) \in F$,

$$Q s_\mu p_{[v]_m} s_\nu^* Q = 0 \text{ unless } |\mu| = |\nu| = k \quad \text{and} \quad [v]_m \subseteq r(\mu) \cap r(\nu).$$

Suppose that $(\mu, [v]_m, \nu) \in F$ with $|\mu| \neq |\nu|$. We may assume $|\mu| = k$ because either μ or ν has length k. Since $s_{\beta\lambda}^* s_\mu \neq 0$ if and only if $\beta = \mu$, we have

$$\begin{aligned} Q(s_\mu p_{[v]_m} s_\nu^*)Q &= \Big(\sum_{\alpha',\beta'\in G} c_{\alpha'\beta'} s_{\alpha'\lambda} p_{r([v]_m,\lambda)} s_{\beta'\lambda}^*\Big)(s_\mu p_{[v]_m} s_\nu^*)\Big(\sum_{\alpha,\beta\in G} c_{\alpha\beta} s_{\alpha\lambda} p_{r([v]_m,\lambda)} s_{\beta\lambda}^*\Big) \\ &= \Big(\sum_{\alpha'\in G} c_{\alpha'\mu} s_{\alpha'\lambda} p_{r([v]_m,\lambda)} s_{\mu\lambda}^* s_\mu p_{[v]_m} s_\nu^*\Big)\Big(\sum_{\alpha,\beta\in G} c_{\alpha\beta} s_{\alpha\lambda} p_{r([v]_m,\lambda)} s_{\beta\lambda}^*\Big) \\ &= \sum_{\alpha,\beta\in G} c_{\alpha\beta}\Big(\sum_{\alpha'\in G} c_{\alpha'\mu} s_{\alpha'\lambda} p_{r([v]_m,\lambda)} s_{\nu\lambda}^*\Big) s_{\alpha\lambda} p_{r([v]_m,\lambda)} s_{\beta\lambda}^*. \end{aligned}$$

For $s_{\nu\lambda}^* s_{\alpha\lambda}$ to be non-zero, $\nu\lambda$ must extend $\alpha\lambda$, so that $\nu\lambda = \alpha\lambda\delta$ for some $\delta \in \mathcal{L}^*(E)$. Then $\nu = \alpha\lambda'$ and $\lambda = \lambda'\lambda''$ for some $\lambda', \lambda'' \in \mathcal{L}^*(E)$ since $|\nu| > |\alpha| = k$. As $\alpha\lambda'\lambda = \nu\lambda = \alpha\lambda\delta = \alpha\lambda'\lambda''\delta$, we have

$$\lambda = \lambda'\lambda'' = \lambda''\delta$$

with $|\lambda'| = |\delta|$. Because $|\nu| = |\alpha\lambda'| \leq M$ with $|\alpha| = k$, we know $|\lambda'| \leq M - k \leq m$, which contradicts the fact that λ is disagreeable for $[v]_m$.

Thus, we see that

$$QbQ = Qb_1Q = Qrb_1rQ = \|b_1\|rQ = \|b_0\|Q \geq \tfrac{3}{4}Q.$$

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Since $\|a-b\| < \frac{1}{4}$, we have $QaQ \geq QbQ - \frac{1}{4}Q \geq \frac{1}{2}Q$. This implies that QaQ is invertible in $QC^*(E, \mathcal{L}, \mathcal{E})Q$. Let c be the inverse of QaQ in $QC^*(E, \mathcal{L}, \mathcal{E})Q$ and put $v = c^{1/2}Qa^{1/2}$. Then $v^*v = a^{1/2}QcQa^{1/2} \leq \|c\|a$, and hence $v^*v \in B$. Since

$$v^*v \sim vv^* = c^{1/2}QaQc^{1/2} = Q,$$

the hereditary C^*-subalgebra B contains a non-zero projection equivalent to Q. Note that Q belongs to the finite-dimensional subalgebra $C := \operatorname{span}\{s_{\alpha\lambda}p_{r([v]_m,\lambda)}s^*_{\beta\lambda} : \alpha, \beta \in G\}$ for which the elements $\{s_{\alpha\lambda}p_{r([v]_m,\lambda)}s^*_{\beta\lambda}\}$ form a matrix unit. This means that Q dominates a minimal projection in C. Since every minimal projection in C is equivalent to a minimal projection of the form $s_{\alpha\lambda}p_{r([v]_m,\lambda)}s^*_{\alpha\lambda}$, the hereditary subalgebra B also contains a projection equivalent to the desired form. □

COROLLARY 3.9. *Let $(E, \mathcal{L}, \mathcal{E})$ be a disagreeable and strongly cofinal labeled space. Then the following are equivalent:*

(a) *$C^*(E, \mathcal{L}, \mathcal{E})$ is purely infinite;*

(b) *every non-zero projection p_A is infinite for $A \in \mathcal{E}$.*

Proof. We only need to show that (*b*) implies (a). By Proposition 3.8, every non-zero hereditary subalgebra contains a projection p such that $p_A \preceq p$ for a non-empty set $A \in \mathcal{E}$. Thus p should be infinite, and $C^*(E, \mathcal{L}, \mathcal{E})$ has property (IH). Since $C^*(E, \mathcal{L}, \mathcal{E})$ is simple, it is purely infinite. □

Definition 3.10. Let $(E, \mathcal{L}, \mathcal{E})$ be a labeled space. We say that *every vertex connects to a loop* in $(E, \mathcal{L}, \mathcal{E})$ if for each $[v]_l$, there exist a loop (α, A) and a finitely many paths $\delta_1, \ldots, \delta_m \in \mathcal{L}^*(E)$ such that δ_i is not an initial path of δ_j for $j \neq i$ and

$$A \subseteq \bigcup_{i=1}^{m} r([v]_l, \delta_i). \tag{8}$$

This definition naturally extends to quotient labeled spaces $(E, \mathcal{L}, \mathcal{E}/R)$.

Note that the condition in the above definition for a path δ_i to be an initial path of none of the rest of the paths implies that $s^*_{\delta_i}s_{\delta_j} = 0$ for $i \neq j$ in $C^*(E, \mathcal{L}, \mathcal{E})$. This is the case if δ_i's are distinct paths with the same length.

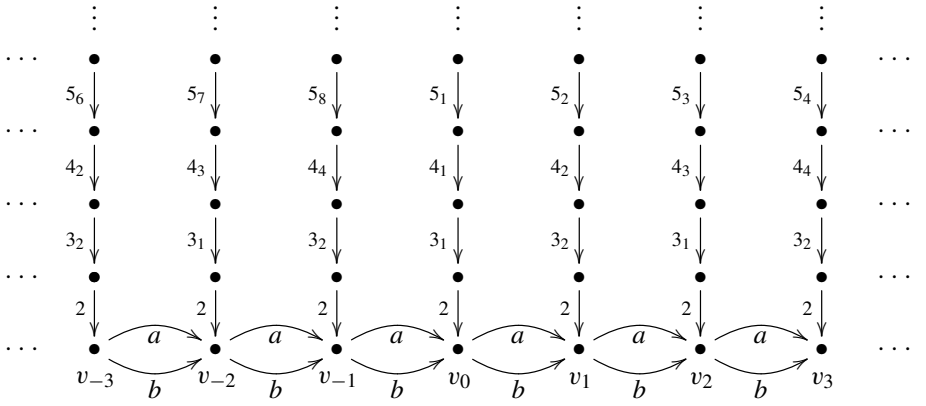
Remark 3.11. In [**5**, Definition 6.6], it is said that *every vertex connects to a repeatable path* if for each $[v]_m$, there exist $w \in E^0$, $L \geq 1$, and $\alpha, \delta \in \mathcal{L}^*(E)$ such that

$$[w]_L \subseteq r([v]_m, \delta) \quad \text{and} \quad (\alpha, [w]_l) \text{ is a loop for } l \geq L. \tag{9}$$

This property is obviously stronger than the one (every vertex connects to a loop) given in Definition 3.10, and actually strictly stronger, as we see in Example 3.12.

Example 3.12. In the following labeled space $(E, \mathcal{L}, \mathcal{E})$, if $(\alpha, [w]_l)$ is a loop, then $w = v_k$ and $\alpha \in \{a, b\}^n$ for some $k, n \geq 1$.

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But for a fixed n, if l' is large enough, then $(\alpha, [v_k]_{l'})$ fails to be a loop, and accordingly there is no vertex that connects to a repeatable path in the sense of (9). On the other hand, every vertex connects to a loop $([v_j]_l, a^n)$ for some $j, l, n \geq 1$. Since $(E, \mathcal{L}, \mathcal{E})$ is disagreeable and strongly cofinal, the C^*-algebra $C^*(E, \mathcal{L}, \mathcal{E})$ is simple, and we will see that it is purely infinite by Theorem 3.14 below.

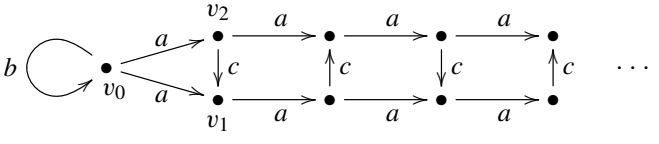
Recall that a graph C^*-algebra $C^*(E)$ has property (IH) if and only if E satisfies condition (L) and every vertex connects to a loop. For labeled graph C^*-algebras, it is known from [**5**, Theorem 6.9] that if a labeled space $(E, \mathcal{L}, \mathcal{E})$ is disagreeable, strongly cofinal, and every vertex connects to a repeatable path, then the simple C^*-algebra $C^*(E, \mathcal{L}, \mathcal{E})$ has property (IH). We will extend this result to disagreeable labeled spaces in which every vertex connects to a loop (see Theorem 3.14(a)). But for the converse, we need additional assumptions for $(E, \mathcal{L}, \mathcal{E})$.

Definition 3.13. We say that a labeled space $(E, \mathcal{L}, \mathcal{E})$ has *sufficiently many minimal sets* if it satisfies the following:

(i) every non-empty $A \in \mathcal{E}$ contains a non-empty minimal subset;

(ii) if $A \in \mathcal{E}_{\min}$, then $r(A, \alpha)$ is a finite union of minimal sets for any $\alpha \in \mathcal{L}^*(E)$.

Note that the notion of having sufficiently many minimal sets also makes sense even for quotient labeled spaces $(E, \mathcal{L}, \mathcal{E}/R)$ (with $\mathcal{E}/R$ and $\mathcal{L}_R^*(E)$ in place of $\mathcal{E}$ and $\mathcal{L}^*(E)$, respectively). Even if $(E, \mathcal{L}, \mathcal{E})$ has sufficiently many minimal sets, not every set in $\mathcal{E}$ is a finite union of minimal sets. For example, in the following labeled space,



each vertex set $\{v\} \in \mathcal{E}$, $v \in E^0$, is a minimal set in $\mathcal{E}$, hence every non-empty set $A \in \mathcal{E}$ contains a minimal set and obviously every relative range with respect to a minimal set is a finite union of minimal sets. But not every $A \in \mathcal{E}$, like $A = r(a)$, is a finite union of minimal sets.

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THEOREM 3.14. *For a disagreeable labeled space* $(E, \mathcal{L}, \mathcal{E})$*, we have the following statements.*

(a) *If every vertex connects to a loop, then* $C^*(E, \mathcal{L}, \mathcal{E})$ *has property (IH). Moreover, every non-zero hereditary* C^**-subalgebra of* $C^*(E, \mathcal{L}, \mathcal{E})$ *contains an infinite projection equivalent to a projection* p_A *for some* $A \in \mathcal{E}$*.*

(b) *Assume further that* $(E, \mathcal{L}, \mathcal{E})$ *has sufficiently many minimal sets. If* $C^*(E, \mathcal{L}, \mathcal{E})$ *has property (IH), then every vertex connects to a loop.*

Proof. (a) Let $C^*(E, \mathcal{L}, \mathcal{E}) = C^*(s_a, p_A)$. We first show that the projections $p_{[v]_l}$ of vertices $[v]_l$ are all infinite. Given a vertex $[v]_l$, choose a path $\gamma \in \mathcal{L}^l(E)$ such that $[v]_l \subset r(\gamma)$. Since $[v]_l$ connects to a loop by our assumption, there exist a loop (α, A) and finitely many paths $\delta_1, \ldots, \delta_m$ such that $s_{\delta_i}^* s_{\delta_j} = 0$ whenever $i \neq j$ and $A \subseteq \bigcup_{i=1}^m r([v]_l, \delta_i)$. Set

$$A_i := A \cap r([v]_l, \delta_i) \subset A \cap r(\gamma\delta_i) \quad \text{and} \quad B_i := A_i \Big\backslash \bigcup_{j=1}^{i-1} A_j \ (B_1 := A_1)$$

for $i = 1, \ldots, m$. Then the projections p_{B_i} are mutually orthogonal and

$$\begin{aligned}
p_A &= \sum_i p_{B_i} \preceq \sum_i p_{B_i} s_{\gamma\delta_i}^* s_{\gamma\delta_i} p_{B_i} \\
&= \Big(\sum_i p_{B_i} s_{\gamma\delta_i}^*\Big)\Big(\sum_i p_{B_i} s_{\gamma\delta_i}^*\Big)^* \sim \Big(\sum_i p_{B_i} s_{\gamma\delta_i}^*\Big)^*\Big(\sum_i p_{B_i} s_{\gamma\delta_i}^*\Big) \\
&= \sum_i s_{\gamma\delta_i} p_{B_i} s_{\gamma\delta_i}^* = s_\gamma \cdot \sum_i s_{\delta_i} p_{B_i} s_{\delta_i}^* \cdot s_\gamma^* \\
&\leq s_\gamma \cdot \sum_i s_{\delta_i} p_{r([v]_l, \delta_i)} s_{\delta_i}^* \cdot s_\gamma^* \\
&\leq s_\gamma p_{[v]_l} s_\gamma^* \sim p_{[v]_l}.
\end{aligned}$$

Thus the projection $p_{[v]_l}$ is infinite since p_A is infinite by Proposition 3.7. On the other hand, every non-empty set in $\mathcal{E}$ is a finite union of vertices by (1), hence it follows that every projection p_B, $\emptyset \neq B \in \mathcal{E}$, is infinite. By Proposition 3.8, any non-zero hereditary subalgebra of $C^*(E, \mathcal{L}, \mathcal{E})$ contains a non-zero projection p such that $0 \neq s_\mu p_B s_\mu^* \preceq p$ for some $\mu \in \mathcal{L}^*(E)$ and $B \in \mathcal{E}$ with $B \subset r(\mu)$. But the projection $s_\mu p_B s_\mu^* \sim p_B$ is infinite, and thus p is infinite.

(b) We claim that if there is a vertex $[v]_l$ which does not connect to any loop, then $C^*(E, \mathcal{L}, \mathcal{E})$ has an AF ideal.

By our assumption, we may assume that $[v]_l$ is minimal in $\mathcal{E}$. Let

$$\{B_n : n \geq 1\} := \{B \in \mathcal{E}_{\min} : B \subset r([v]_l, \alpha) \text{ for some } \alpha \in \mathcal{L}^\#(E)\}.$$

(There could be only finitely many B_n, but the arguments given below work even in this case.) Note that each B_n does not connect to any loop because $[v]_l$ would connect to a loop otherwise. Hence for each $n \geq 1$ and $\alpha \in \mathcal{L}(B_n E^{\geq 1})$, we have

$$r(B_n, \alpha) \cap B_n = \emptyset \tag{10}$$

since minimality of B_n implies $B_n \subset r(B_n, \alpha)$ whenever $r(B_n, \alpha) \cap B_n \neq \emptyset$. If $B_n \cap r(B_m, \lambda) \neq \emptyset$ for some $\lambda \in \mathcal{L}^*(E)$ and $n \neq m$ (hence, if $B_n \subseteq r(B_m, \lambda)$), we write

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$B_m > B_n$. Note that $B_n \not> B_n$ for all $n \geq 1$, and once $B_m > B_n$ holds for $n \neq m$, then $B_n \not> B_m$ because of (10). The set of all finite unions of B_n which we denote by H is readily seen to be a hereditary subset of $\mathcal{E}$ by our assumption. With $p_n := p_{B_n}$, $n \geq 1$, the ideal I_H generated by the projections $\{p_B : B \in H\}$ can be easily seen to be

$$I_H = \overline{\text{span}}\{s_\alpha p_n s_\beta^* : \alpha, \beta \in \mathcal{L}^*(E),\ n \geq 1\}.$$

Set $J_n := \overline{\text{span}}\{s_\alpha p_n s_\beta^* : \alpha, \beta \in \mathcal{L}^*(E)\}$ for each $n \geq 1$. Consider the product $X := (s_\alpha p_n s_\beta^*)(s_\mu p_m s_\nu^*)$ of two elements $s_\alpha p_n s_\beta^*$, and $s_\mu p_m s_\nu^*$ in I_H. Using the properties of the minimal sets B_n mentioned above, one can check that

$$\begin{cases} X \in J_n & \text{if } n = m, \\ X \in J_n & \text{if } B_m > B_n, \\ X \in J_m & \text{if } B_n > B_m, \\ X = 0 & \text{otherwise.} \end{cases} \tag{11}$$

It then follows that both J_n and

$$I_n := \overline{\text{span}}\{s_\alpha p_k s_\beta^* : \alpha, \beta \in \mathcal{L}^*(E),\ 1 \leq k \leq n\}$$

are C^*-subalgebras of I_H, and, moreover, that $I_H = \overline{\bigcup_{n\geq 1} I_n}$ holds. Thus it is enough to show that each I_n is an AF algebra to obtain a contradiction.

Fix $n \geq 1$ and consider the C^*-subalgebra I_n. For each $k \in \{1, \ldots, n\}$, J_k is easily seen to be isomorphic to the C^*-algebra $\mathcal{K}(\ell^2(\mathcal{L}(E^{\geq 1} B_k)))$ of the compact operators (see [**25**, Corollary 2.2]). Let $\{k_1, \ldots, k_{n_k}\} \subset \{1, \ldots, n\}$ be the set of all such k that $B_k \not> B_j$ for all $1 \leq j \leq n$. Then J_k is an ideal of I_n for each $k \in \{k_1, \ldots, k_{n_k}\}$, and by (11), $J_{k_i} J_{k_j} = \{0\}$ for $i \neq j$. Thus I_n contains an AF ideal $\oplus_{k\in\{k_1,\ldots,k_n\}} J_k$. For the quotient map $\pi : I_n \to I_n / \oplus_k J_k$, the image $\pi(J_i)$ of J_i is either zero or isomorphic to itself since each J_i is a simple C^*-algebra. Then one can apply the above argument to $\pi(I_n)$ which has the subalgebras $\pi(J_i)(\cong J_i)$, and find AF ideals and the quotient algebra that is generated by subalgebras each of which is isomorphic to the compact operators. This process should stop when a quotient algebra consists of ideals which are all isomorphic to the compact operators and mutually orthogonal, which proves that I_n is an AF algebra. □

The following corollary generalizes [**5**, Theorem 6.9]. In fact, as mentioned earlier, the disagreeable and strongly cofinal labeled space $(E, \mathcal{L}, \mathcal{E})$ in Example 3.12 satisfies the condition of the following corollary but not the condition of [**5**, Theorem 6.9].

COROLLARY 3.15. *Let $(E, \mathcal{L}, \mathcal{E})$ be a strongly cofinal labeled space. If there exist a vertex $w \in E^0$, a strictly increasing sequence $\{l_i\}_i$ of integers, and paths $\{\beta_i\}_i$ such that $(\beta_i, [w]_{l_i})$ is a loop for each $i \geq 1$, then every vertex connects to a loop $(\beta_i, [w]_{l_i})$ for some $i \geq 1$.*

If, in addition, $(E, \mathcal{L}, \mathcal{E})$ is disagreeable, then the simple C^*-algebra $C^*(E, \mathcal{L}, \mathcal{E})$ is purely infinite. In particular, if $(E, \mathcal{L}, \mathcal{E})$ is a strongly cofinal and disagreeable labeled space with a cycle, then $C^*(E, \mathcal{L}, \mathcal{E})$ is purely infinite and simple.

Proof. Given a generalized vertex $[v]_l$, since $(E, \mathcal{L}, \mathcal{E})$ is strongly cofinal, one can find $N \geq 1$ and paths δ_j, $1 \leq j \leq k$, such that

$$[w]_{l_1} \subset r([w]_{l_1}, \beta_1^N) \subset r(\beta_1^N) \subset \bigcup_{j=1}^{k} r([v]_l, \delta_j).$$

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We may assume $w \in [w]_{l_1} \cap r([v]_l, \delta_1)$, and then for all large l_i we have $[w]_{l_i} \subset r([v]_l, \delta_1)$ by (1). Thus $[v]_l$ connects to a loop $(\beta_{l_i}, [w]_{l_i})$.

If $(E, \mathcal{L}, \mathcal{E})$ is disagreeable, by Theorem 3.14(a) we see that the simple C^*-algebra $C^*(E, \mathcal{L}, \mathcal{E})$ is purely infinite. □

Remark 3.16. Let $(E, \mathcal{L}, \mathcal{E})$ be a labeled space.

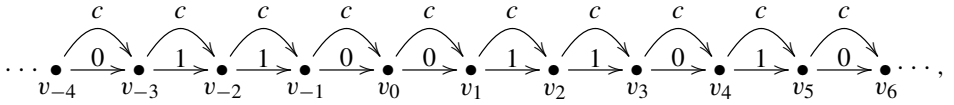
(a) One can modify the proof of Proposition 3.8 to obtain the same criterion for quotient labeled spaces. Let $(E, \mathcal{L}, \mathcal{E}/R)$ be a disagreeable quotient labeled space. Then every non-zero hereditary C^*-subalgebra of $C^*(E, \mathcal{L}, \mathcal{E}/R)$ contains a non-zero projection p such that $s_\mu p_{[A]} s_\mu^* \preceq p$ for some $\mu \in \mathcal{L}_R^*(E)$ and $[A] \in \mathcal{E}/R$ with $s_\mu p_{[A]} s_\mu^* \neq 0$ and $[A] \subset [r(\mu)]$.

(b) An argument similar to the proof of Theorem 3.14(a) shows that if a quotient labeled space $(E, \mathcal{L}, \mathcal{E}/R)$ in which every vertex connects to a loop is disagreeable, then $C^*(E, \mathcal{L}, \mathcal{E}/R)$ has property (IH).

3.3. *An example: a purely infinite simple C^*-algebra of a labeled space $(E, \mathcal{L}, \mathcal{E}_\omega)$ with no cycles.* It is well known that for a simple graph C^*-algebra $C^*(E)$, it is infinite if and only if it is purely infinite, and this is exactly when there is a cycle (or a loop) in the graph E.

For a simple labeled graph C^*-algebra $C^*(E, \mathcal{L}, \mathcal{E})$ associated to a disagreeable and strongly cofinal labeled space $(E, \mathcal{L}, \mathcal{E})$, we know that the existence of a cycle implies pure infiniteness of $C^*(E, \mathcal{L}, \mathcal{E})$ by Corollary 3.15. But the converse may not be true. In fact, our purpose in this subsection is to provide an example of a labeled space $(E, \mathcal{L}, \mathcal{E}_\omega)$ with no cycle but with a loop for which $C^*(E, \mathcal{L}, \mathcal{E}_\omega)$ is purely infinite and simple.

Example 3.17. In the following labeled graph $(E, \mathcal{L})$



the $\{0, 1\}$ sequence is the bi-infinite Thue–Morse sequence which is an infinite $\{0, 1\}$ sequence with no finite subpaths of the form $\alpha\alpha\alpha_1$ for any $\alpha = \alpha_1 \cdots \alpha_{|\alpha|} \in \{0, 1\}^*$ (see [**15**], [**19**, Examples 2.5 and 2.7], for instance). Let $e_i \in E^1$, $i \in \mathbb{Z}$, be the edge of E with $s(e_i) = v_i$ and $\mathcal{L}(e_i) \in \{0, 1\}$. For convenience, we write the Thue–Morse sequence as a bi-infinite labeled path

$$\omega = \cdots \omega_{-2}\omega_{-1}\omega_0\omega_1\omega_2\omega_3 \cdots,$$

where $\omega_i = \mathcal{L}(e_i) \in \{0, 1\}$, $i \in \mathbb{Z}$. It is well known that ω is not periodic but *almost periodic* (or uniformly recurrent) in the sense that every finite subpath occurs with a bounded gap. We write the accommodating set $\mathcal{E}$ as $\mathcal{E}_\omega$. Note that $(E, \mathcal{L}, \mathcal{E}_\omega)$ is a disagreeable labeled space in which (c^k, E^0) is a loop for each $k \geq 1$. Thus $C^*(E, \mathcal{L}, \mathcal{E}_\omega)$ is an infinite C^*-algebra by Proposition 3.7; actually we have

$$1 = p_{r(c)} = s_c^* s_c \sim s_c s_c^* < s_c s_c^* + s_0 s_0^* + s_1 s_1^* = 1.$$

PROPOSITION 3.18. *Let $(E, \mathcal{L}, \mathcal{E}_\omega)$ be the disagreeable labeled space of Example 3.17. Then:*

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(a) *if (α, A) is a loop, then $\alpha = c^k$ for some $k \geq 1$;*
(b) *there exist no cycles;*
(c) *$C^*(E, \mathcal{L}, \mathcal{E}_\omega)$ is purely infinite and simple.*

Proof. It is helpful to note that in the labeled space $(E, \mathcal{L}, \mathcal{E}_\omega)$, every vertex $[v]_l$ coincides with the range of a $\{0, 1\}$ path of length l.

(a) Suppose (α, A) is a loop with $k := |\alpha|$. Then α^r occurs in $\mathcal{L}^*(E)$ for every $r \geq 1$ since $A \subset r(A, \alpha) \subset r(A, \alpha^2) \subset \cdots$. If α has a $\{0, 1\}$ subpath, say α' (which we may assume ends with 0), then $r(\alpha')$ must contain a finite periodic set of arbitrary cardinality. But this is not possible by Gelfond's result [**14**] saying that for any $a, b \in \mathbb{N}$,

$$|\{n < N : \omega_{an+b} = 0\}| = \frac{N}{2} + O(N^\lambda).$$

Thus we have $\alpha = c^k$.

(b) Suppose (α, A) is a cycle in $(E, \mathcal{L}, \mathcal{E}_\omega)$. Then $\alpha = c^k$ for some $k \geq 1$ by (a), and $B = r(B, c^k)$ holds for any non-empty subset $B \subset A$. Especially for $B = r(\beta) \subset A$ where β is a $\{0, 1\}$ path of length $|\beta| > 3k$, we have

$$r(\beta) = B = r(B, c^k) = r(r(\beta), c^k) = r(\beta c^k) = \bigcup_{\delta \in \{0,1\}^k} r(\beta\delta).$$

Choose any vertex $v \in r(\beta)$. Then $v \in r(\beta\delta)$ for some $\{0, 1\}$ path δ with $|\delta| = k$. This means that there must be a path β' such that $\beta'\beta = \beta\delta$. Since $|\beta| > 3|\delta|$, we obtain that β ends with δ^3, which is not possible since β is a subpath of the Thue–Morse sequence ω.

(c) We first show that $C^*(E, \mathcal{L}, \mathcal{E}_\omega)$ has property (IH). Let $[v]_l$ be a generalized vertex. Then there is a finite $\{0, 1\}$ path $\beta \in \mathcal{L}(E^l)$ such that $[v]_l = r(\beta)$. Since β is a subpath of the almost periodic sequence ω, it occurs with a bounded gap, say L_β. Thus we can find finitely many paths δ_i of the form $\delta_i := \alpha_i c$ for some $\alpha_i \in \{0, 1\}^*$ such that $s_{\delta_i}^* s_{\delta_j} = 0$ for $i \neq j$ and

$$E^0 \subset \bigcup_i r([v]_l, \delta_i) = \bigcup_i r(\beta\delta_i).$$

This is possible since if $w \in E^0$ is a vertex, then $w \in r(\beta\alpha c)$ for some $\{0, 1\}$-path α (possibly $\alpha = \epsilon$) such that $|\alpha| \leq L_\beta$. Thus $[v]_l$ connects to a loop (c, E^0), and by Theorem 3.14(a), $C^*(E, \mathcal{L}, \mathcal{E}_\omega)$ has property (IH).

To see that $C^*(E, \mathcal{L}, \mathcal{E}_\omega)$ is simple, we only need to show by Remark 2.6 that $(E, \mathcal{L}, \mathcal{E}_\omega)$ is strongly cofinal. Let $[v]_l = r(\beta)$, $\beta \in \{0, 1\}^l$, be a generalized vertex as above and let $x \in \overline{\mathcal{L}(E^\infty)}$. We already know that $E^0 \subset \bigcup_i r([v]_l, \delta_i)$ for finitely many paths δ_i, and thus $r(x_{[1,N]}) \subset E^0 = \bigcup_i r([v]_l, \delta_i)$ for any $N \geq 1$, which shows that $(E, \mathcal{L}, \mathcal{E}_\omega)$ is strongly cofinal as desired. □

Remark 3.19. Pure infiniteness of $C^*(E, \mathcal{L}, \mathcal{E}_\omega)$ can also be obtained if one can prove that the standard action of the inverse semigroup

$$\mathcal{S} := \{s_\alpha p_{r(\delta)} s_\beta^* : \alpha, \beta, \delta \in \mathcal{L}^*(E), |\delta| \geq \max\{|\alpha|, |\beta|\}\}$$

associated to the labeled space $(E, \mathcal{L}, \mathcal{E}_\omega)$ on the tight spectrum $\hat{\mathcal{E}}(\mathcal{S})_{\text{tight}}$ of $\mathcal{E}(\mathcal{S})$ is locally contracting in the sense of [**12**, Definition 6.2] (although this does not seem to work easily),

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because then by [**12**, Proposition 6.3] the tight groupoid $\mathcal{G}_{tight}(\mathcal{S})$ is locally contracting, and hence by [**1**, Proposition 2.4] one sees that the C^*-algebra $C^*(E, \mathcal{L}, \mathcal{E}_\omega)$ is purely infinite.

For this inverse semigroup $\mathcal{S}$ of $C^*(E, \mathcal{L}, \mathcal{E}_\omega)$, the standard action is locally contracting if and only if $\mathcal{S}$ is locally contracting in the sense of [**12**, Definition 6.4] by [**12**, Theorem 6.5] since every tight filter is an ultra-filter, which we briefly show here. First recall that a filter ξ is an *ultra-filter* if and only if ξ contains every idempotent f such that $fe \neq 0$ for every $e \in \xi$. Let η be a tight filter and $f = s_\sigma p_{r(\beta)} s_\sigma^* \in \mathcal{E}(\mathcal{S})$ be an idempotent such that $fe \neq 0$ for every $e \in \eta$. There is a unique tight character ϕ on $\mathcal{E}(\mathcal{S})$ such that $\eta = \eta_\phi := \{e \in \mathcal{E}(\mathcal{S}) : \phi(e) = 1\}$. Pick $e = s_\mu p_{r(\alpha)} s_\mu^* \in \eta_\phi$. By [**11**, Proposition 1.8] we have for every finite cover $Z \subset \mathcal{E}(\mathcal{S})$ for e,

$$\bigvee_{z \in Z} \phi(z) \geq \phi(e) = 1,$$

which implies that if $e = \sum_{i=1}^n e_i$ is a finite sum of mutually orthogonal idempotents $e_i \in \mathcal{E}(\mathcal{S})$, for example, one can write e as

$$e = \sum_{\nu \in \mathcal{L}^k(E)} s_{\mu\nu} p_{r(\alpha\nu)} s_{\mu\nu}^* \quad \text{or} \quad e = \sum_{\alpha_i \in \{0,1\}^l} s_\mu p_{r(\alpha_i\alpha)} s_\mu^*$$

for $k, l \geq 1$, then $\phi(e_i) = 1$ for one and only one e_i because $\{e_1, \ldots, e_n\}$ is a cover for e and ϕ is a multiplicative function with $\phi(0) = 0$. Also the identity $r(\alpha) = \bigcup_{\substack{\alpha_i \in \{0,1\}^{|\alpha|} \\ r(\alpha_i) = r(\alpha)}} r(\alpha_i)$ allows us to assume that α is a $\{0, 1\}$ path. Thus we may pick $e \in \eta_\phi$ such that $|\mu| \geq |\sigma|$ and α is a $\{0, 1\}$ path with $|\alpha| \geq |\beta| + |\mu| - |\sigma|$. Then

$$\begin{aligned}
0 \neq fe &= s_\sigma p_{r(\beta)} s_\sigma^* \cdot s_\mu p_{r(\alpha)} s_\mu^* \\
&= \sum_{|\sigma'| = |\mu| - |\sigma|} s_{\sigma\sigma'} p_{r(\beta\sigma')} s_{\sigma\sigma'}^* \cdot s_\mu p_{r(\alpha)} s_\mu^* \\
&= s_{\sigma\sigma'} \, p_{r(\beta\sigma')} \cdot p_{r(\alpha)} s_\mu^* \quad (\text{for } \sigma\sigma' = \mu) \\
&= s_{\sigma\sigma'} \cdot \sum_{\beta_i \beta\sigma_j \in \{0,1\}^{|\alpha|}} p_{r(\beta_i\beta\sigma_j)} \cdot p_{r(\alpha)} s_\mu^* \\
&= s_{\sigma\sigma'} p_{r(\alpha)} s_\mu^* \\
&= e \in \eta_\phi.
\end{aligned}$$

Since $f \geq fe$ and η_ϕ is a filter, we have $f \in \eta_\phi$, which proves that η_ϕ is an ultra-filter.

4. *Purely infinite labeled graph C^*-algebras*

In this section we find equivalent conditions for a labeled graph C^*-algebra $C^*(E, \mathcal{L}, \mathcal{E})$ to be purely infinite. We begin with the following Cuntz–Krieger uniqueness theorem for quotient labeled graph C^*-algebras. Although the theorem can be obtained by standard arguments as in the proof of [**5**, Theorem 5.5] for labeled graph C^*-algebras, we give a proof here only to make sure that our definition of a 'disagreeable' quotient labeled space (Definition 3.3) does not cause any degeneracy. We will use this uniqueness theorem later in Proposition 4.4.

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THEOREM 4.1. *Let $(E, \mathcal{L}, \mathcal{E}/R)$ be a disagreeable quotient labeled space and let $C^*(E, \mathcal{L}, \mathcal{E}/R) = C^*(s_a, p_{[A]})$. If $\{S_a, P_{[A]}\}$ is a representation of $(E, \mathcal{L}, \mathcal{E}/R)$ such that $P_{[A]} \neq 0$ for $[A] \neq [\emptyset]$ and $S_a \neq 0$ for $[r(a)] \neq [\emptyset]$, then the homomorphism*

$$\pi_{S,P} : C^*(E, \mathcal{L}, \mathcal{E}/R) \to C^*(S_a, P_{[A]})$$

with $\pi_{S,P}(s_a) = S_a$ and $\pi_{S,P}(p_{[A]}) = P_{[A]}$ is faithful.

Proof. As usual (for example, see [**5**, Theorem 5.5]), it is enough to show that:

(a) $\pi_{S,P}$ is faithful on $C^*(E, \mathcal{L}, \mathcal{E}/R)^\gamma$; and

(b) $\|\pi_{S,P}(\Phi(a))\| \leq \|\pi_{S,P}(a)\|$ for all $a \in C^*(E, \mathcal{L}, \mathcal{E}/R)$, where Φ is the conditional expectation onto the fixed point algebra $C^*(E, \mathcal{L}, \mathcal{E}/R)^\gamma$.

But (a) was shown in the proof of [**21**, Theorem 4.2], and thus we show (b). By (4), it suffices to prove (b) for an element a of the form

$$a = \sum_{(\alpha,[[w]_l],\beta)\in F} c_{\alpha,[[w]_l],\beta}\, s_\alpha p_{[[w]_l]} s_\beta^*,$$

where F is a finite set and $c_{\alpha,[[w]_l],\beta} \in \mathbb{C}$. Let $k = \max\{|\alpha|, |\beta| : (\alpha, [[w]_l], \beta) \in F\}$. We can choose $k \in \mathbb{N}$ by applying Definition 2.4(iv) and changing F if necessary such that $\min\{|\alpha|, |\beta|\} = k$ for every $(\alpha, [[w]_l], \beta) \in F$. Let $M = \max\{|\alpha|, |\beta| : (\alpha, [[w]_l], \beta) \in F\}$. We also may suppose that $l \geq M - k$. Since $|\alpha| = |\beta| = k$ whenever $|\alpha| = |\beta|$, we have

$$\Phi(a) = \sum_{\substack{(\alpha,[[w]_l],\beta)\in F\\ |\alpha|=|\beta|=k}} c_{\alpha,[[w]_l],\beta}\, s_\alpha p_{[[w]_l]} s_\beta^* \in \oplus_{[[w]_l]} \mathcal{F}^k([[w]_l]).$$

Thus the norm $\|\pi_{S,P}(\Phi(a))\|$ is attained on a direct summand, say $\mathcal{F}^k([[v]_l])$. For this $[[v]_l]$, let $F_{[v]_l} := \{(\alpha, [[v]_l], \beta) : (\alpha, [[v]_l], \beta) \in F\}$. Then we have

$$\|\pi_{S,P}(\Phi(a))\| = \left\| \sum_{\substack{(\alpha,[[v]_l],\beta)\in F_{[v]_l}\\ |\alpha|=|\beta|=k}} c_{\alpha,[[v]_l],\beta}\, S_\alpha P_{[[v]_l]} S_\beta^* \right\|.$$

It is easy to see that the $*$-algebra $\operatorname{span}\{S_\alpha P_{[[v]_l]} S_\beta^* : \alpha, \beta \in G\}$, where

$$G := \{\alpha \in \mathcal{L}_R(E^k) : \text{either } (\alpha, [[v]_l], \beta) \in F_{[v]_l} \text{ or } (\beta, [[v]_l], \alpha) \in F_{[v]_l} \text{ with } |\alpha| = |\beta|\},$$

is a finite-dimensional matrix algebra and contains the element

$$b_v := \sum_{\substack{(\alpha,[[v]_l],\beta)\in F_{[v]_l}\\ |\alpha|=|\beta|=k}} c_{\alpha,[[v]_l],\beta}\, S_\alpha P_{[[v]_l]} S_\beta^*.$$

Since $[[v]_l]$ is disagreeable (see Definition 3.3), we can choose a path $\lambda \in \mathcal{L}_R([[v]_l]E^{>M})$ which has no factorization $\lambda = \lambda'\lambda'' = \lambda''\gamma$ for $\lambda', \gamma \in \mathcal{L}_R^*(E)$ with $|\lambda'| = |\gamma| \leq M - k \leq l$ and $[r([v]_l, \lambda)] \neq [\emptyset]$. This gives a non-zero projection

$$Q = \sum_{\nu\in G} S_{\nu\lambda} P_{[r([v]_l,\lambda)]} S_{\nu\lambda}^*.$$

It is routine to check the following (for example, see [**5**, Theorem 5.5])

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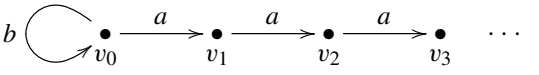
(i) $\|Q\,\pi_{S,P}(\Phi(a))\,Q\| = \|\pi_{S,P}(\Phi(a))\|$;
(ii) $Q\,S_\alpha P_{[[v]_l]} S_\beta^*\,Q = 0$ when $(\alpha, [[v]_l], \beta) \in F$ and $|\alpha| \neq |\beta|$.
Then we see that

$$\begin{aligned}
\|\pi_{S,P}(\Phi(a))\| &= \|Q\pi_{S,P}(\Phi(a))Q\| \\
&= \left\| Q\left(\sum_{\substack{(\alpha,[[v]_l],\beta)\in F_{[v]_l} \\ |\alpha|=|\beta|}} c_{\alpha,[[v]_l],\beta} S_\alpha P_{[[v]_l]} S_\beta^* \right) Q \right\| \\
&= \left\| Q\left(\sum_{(\alpha,[[v]_l],\beta)\in F_{[v]_l}} c_{\alpha,[[v]_l],\beta}\; S_\alpha P_{[[v]_l]} S_\beta^* \right) Q \right\| \\
&\leq \left\| \sum_{(\alpha,[[v]_l],\beta)\in F_{[v]_l}} c_{\alpha,[[v]_l],\beta} S_\alpha P_{[[v]_l]} S_\beta^* \right\| \\
&= \|\pi_{S,P}(a)\|.
\end{aligned}$$

This completes the proof. □

4.1. *Gauge-invariant ideals of C^*-algebras of strongly disagreeable labeled spaces.* Here we introduce a property, called 'strong disagreeableness', of a labeled space $(E, \mathcal{L}, \mathcal{E})$ which we believe is a necessary condition for the C^*-algebra $C^*(E, \mathcal{L}, \mathcal{E})$ to be purely infinite, at least in many practical examples including all the C^*-algebras of (locally finite) graphs.

As is seen from the following labeled graph, a C^*-algebra $C^*(E, \mathcal{L}, \mathcal{E})$ of a disagreeable labeled space $(E, \mathcal{L}, \mathcal{E})$ is not necessarily purely infinite if it has a non-disagreeable quotient labeled space:



Definition 4.2. We say that $(E, \mathcal{L}, \mathcal{E})$ is *strongly disagreeable* if the quotient labeled space $(E, \mathcal{L}, \mathcal{E}/H)$ is disagreeable for every hereditary saturated subset H of $\mathcal{E}$.

Remarks 4.3. We note the following facts related to condition (K).

(a) Let E be a locally finite graph with no sinks. E satisfies condition (K) if and only if the labeled space $(E, \mathcal{L}_{\mathrm{id}}, \mathcal{E})$ is strongly disagreeable: recall first that E satisfies condition (K) if and only if every ideal of $C^*(E)$ is gauge invariant (see [**3**, Lemma 2.2] for the fully general version). Thus Proposition 4.4 below implies that if $(E, \mathcal{L}_{\mathrm{id}}, \mathcal{E})$ is strongly disagreeable, then E has condition (K). The converse can be easily seen by considering a quotient graph because a hereditary saturated set H (respectively, $\mathcal{E}/H$) of $\mathcal{E}$ can be identified with a hereditary saturated set of E^0 (respectively, $E^0 \setminus H$).

(b) A *strongly aperiodic* higher-rank graph was introduced in [**23**], and it was shown [**30**, Corollary 3.9] that a k-graph Λ is strongly aperiodic if and only if every ideal of the C^*-algebra $C^*(\Lambda)$ associated to Λ is gauge invariant. Thus strong aperiodicity can be viewed as the higher-rank graph analogue of condition (K).

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PROPOSITION 4.4. *If $(E, \mathcal{L}, \mathcal{E})$ is strongly disagreeable, every ideal of $C^*(E, \mathcal{L}, \mathcal{E})$ is gauge invariant.*

Proof. Let I be an ideal of $C^*(E, \mathcal{L}, \mathcal{E})$. Then $H_I = \{A \in \mathcal{E} : p_A \in I\}$ is a saturated hereditary subset of $\mathcal{E}$, and the ideal I_{H_I} generated by the projections $\{p_A : A \in H_I\}$ is gauge invariant [**21**, Lemma 3.9]. Since $I_{H_I} \subseteq I$, the quotient map

$$q : C^*(E, \mathcal{L}, \mathcal{E})/I_{H_I} \to C^*(E, \mathcal{L}, \mathcal{E})/I$$

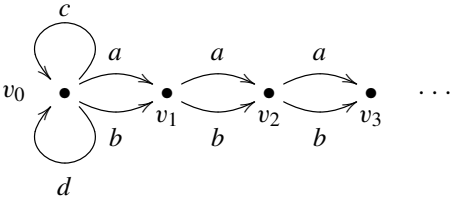
given by $q(s + I_{H_I}) := s + I$ for $s \in C^*(E, \mathcal{L}, \mathcal{E})$, is well defined. From [**21**, Theorem 5.2], we have an isomorphism $\pi : C^*(E, \mathcal{L}, \mathcal{E}/H_I) \to C^*(E, \mathcal{L}, \mathcal{E})/I_{H_I}$ which maps the canonical generators to the canonical generators. Then the composition map $q \circ \pi : C^*(E, \mathcal{L}, \mathcal{E}/H_I) \to C^*(E, \mathcal{L}, \mathcal{E})/I$ satisfies

$$q \circ \pi(p_{[A]}) = q(p_A + I_{H_I}) = p_A + I,$$
$$q \circ \pi(s_a) = q(s_a + I_{H_I}) = s_a + I$$

for $[A] \in \mathcal{E}/H_I$ and $a \in \mathcal{A}_{H_I}$. If $p_{[A]} \neq 0$, then $[A] \neq [\emptyset]$ in $\mathcal{E}/H_I$, hence $A \notin H_I$. Thus $p_A + I \in C^*(E, \mathcal{L}, \mathcal{E})/I$ is a non-zero projection. Also for $a \in \mathcal{A}_{H_I}$ (namely $[r(a)] \neq [\emptyset]$ in H_I), we have $q \circ \pi(s_a) = q(s_a + I_{H_I}) = s_a + I \neq 0$ because otherwise $p_{r(a)} = s_a^* s_a \in I$, contradicting $[r(a)] \neq [\emptyset]$ in H_I. Since the quotient labeled space $(E, \mathcal{L}, \mathcal{E}/H_I)$ is disagreeable by our assumption, we see that the map $q \circ \pi$ is injective by Theorem 4.1. Thus q is be injective, so that I must coincide with the gauge-invariant ideal I_{H_I}.

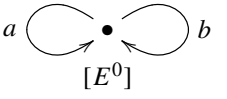
Note that if I is an ideal such that $H_I = \{\emptyset\}$, then $I_{H_I} = \{0\}$, and $q \circ \pi$ is the quotient map $q : C^*(E, \mathcal{L}, \mathcal{E}) \to C^*(E, \mathcal{L}, \mathcal{E})/I$. Since the family $\{p_A + I,\ s_a + I\}$ is a representation of the labeled space $(E, \mathcal{L}, \mathcal{E})$ in the C^*-algebra $C^*(E, \mathcal{L}, \mathcal{E})/I$ such that $p_A + I \neq 0$ and $s_a + I \neq 0$ for each $A \in \mathcal{E}$ and $a \in \mathcal{A}$, the disagreeability of $(E, \mathcal{L}, \mathcal{E})$ asserts by the Cuntz–Krieger unique theorem that q is injective. Thus we have $I = \{0\}$. □

Example 4.5. For the labeled graph

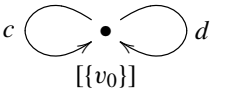


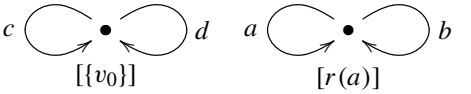
it is not hard to see that $\mathcal{E} = \{A \subset E^0 : A \text{ is finite or cofinite}\}$. The labeled space $(E, \mathcal{L}, \mathcal{E})$ is disagreeable, and there are only three non-trivial hereditary saturated subsets of $\mathcal{E}$: $H_1 = \{A \subset E^0 : |A| < \infty\}$, $H_2 = \{A \subset E^0 \setminus \{v_0\} : |A| < \infty \text{ or } |A^c| < \infty\}$, and $H_3 = \{A \subset E^0 \setminus \{v_0\} : |A| < \infty\}$. Note that $H_3 \subset H_2 \cap H_1$. To see what each quotient algebra $C^*(E, \mathcal{L}, \mathcal{E}/H_i)$ would be like, it is helpful to view each quotient labeled space $(E, \mathcal{L}, \mathcal{E}/H_i)$ as if it is a labeled space:

$(E, \mathcal{L}, \mathcal{E}/H_1)$:



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$(E, \mathcal{L}, \mathcal{E}/H_2)$: 

$(E, \mathcal{L}, \mathcal{E}/H_3)$: 

Then it is rather obvious that $(E, \mathcal{L}, \mathcal{E}/H_i)$ is disagreeable for all $i = 1, 2, 3$, and thus by Proposition 4.4 every ideal of $C^*(E, \mathcal{L}, \mathcal{E})$ is gauge invariant. Consequently, the C^*-algebra $C^*(E, \mathcal{L}, \mathcal{E})$ has only three non-trivial ideals I_{H_i}, $i = 1, 2, 3$, by Remark 2.5(a).

4.2. *Purely infinite C^*-algebras of strongly disagreeable labeled spaces.* If a graph C^*-algebra $C^*(E)$ is purely infinite, then by Theorems 1.1 and 1.2 we know that every loop in E must have an exit. For labeled graph C^*-algebras, the following lemma shows that if $C^*(E, \mathcal{L}, \mathcal{E})$ is purely infinite, then every loop (α, A) has an exit provided A is a minimal set.

LEMMA 4.6. *Let $(E, \mathcal{L}, \mathcal{E})$ have a loop (α, A) with no exits. If $A \in \mathcal{E}_{\min}$, then $C^*(E, \mathcal{L}, \mathcal{E})$ contains a hereditary subalgebra which is isomorphic to $M_n(C(\mathbb{T}))$ for some $n \geq 1$.*

Proof. We may assume that $\alpha = \alpha_1 \cdots \alpha_n$, $n = |\alpha|$, has the smallest length among all the loops at A. Since (α, A) has no exits, we have $A = r(A, \alpha)$ and $\mathcal{L}(AE^{|\alpha|}) = \{\alpha\}$. Thus

$$\mathcal{L}(AE^{\geq 1}) = \{\alpha^k \alpha' : k \geq 0, \alpha' \text{ is an initial path of } \alpha\}, \tag{12}$$

and $p_A = s_{\alpha_1} p_{r(A,\alpha_1)} s_{\alpha_1}^* = s_{\alpha_{[1,2]}} p_{r(A,\alpha_{[1,2]})} s_{\alpha_{[1,2]}}^* = \cdots = s_\alpha p_{r(A,\alpha)} s_\alpha^*$. Moreover, the relative ranges $r(A, \alpha_{[1,i]})$, $1 \leq i \leq |\alpha|$, are all minimal and mutually disjoint. In fact, if $r(A, \alpha_{[1,i]})$ is the union of two disjoint sets $B_1, B_2 \in \mathcal{E}$, then A itself must be a union of two disjoint sets since

$$A = r(A, \alpha) = r(r(A, \alpha_{[1,i]}), \alpha_{[i+1,|\alpha|]}) = r(B_1, \alpha_{[i+1,|\alpha|]}) \cup r(B_2, \alpha_{[i+1,|\alpha|]}).$$

From minimality of A, one of the two sets should be empty. But then $B_1 = \emptyset$ or $B_2 = \emptyset$ because both sets B_1 and B_2 emit the path $\alpha_{[i+1,|\alpha|]}$ otherwise. Hence, $r(A, \alpha_{[1,i]})$ is minimal for $1 \leq i \leq |\alpha|$. Also it follows easily from minimality of the sets $r(A, \alpha_{[1,i]})$ and our assumption for α having the smallest length that

$$r(A, \alpha_{[1,i]}) \cap r(A, \alpha_{[1,j]}) = \emptyset, \quad 1 \leq i < j \leq |\alpha|.$$

Let $A_i := r(A, \alpha_{[1,i]})$, $1 \leq i \leq n$, and

$$p := p_{A_1} + \cdots + p_{A_n}.$$

Now we claim that $pC^*(E, \mathcal{L}, \mathcal{E})p \cong C(\mathbb{T}) \otimes M_n$. For $s_\mu p_C s_\nu^* \in C^*(E, \mathcal{L}, \mathcal{E})$, if

$$p(s_\mu p_C s_\nu^*)p = \sum_{i,j} s_\mu p_{r(A_i,\mu)\cap C\cap r(A_j,\nu)} s_\nu^* \neq 0,$$

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then $s_\mu p_{r(A_i,\mu)\cap C\cap r(A_j,\nu)} s_\nu^* \neq 0$ for some i, j. Hence

$$r(A_i, \mu) \cap r(A_j, \nu) = r(A, \alpha_{[1,i]}\mu) \cap r(A, \alpha_{[1,j]}\nu) \neq \emptyset.$$

Then by (12) we see that the paths μ, ν are of the form

$$\mu = \alpha_{[i+1,,n]}\alpha^l\alpha_{[1,k]}, \quad \nu = \alpha_{[j+1,n]}\alpha^m\alpha_{[1,k']}$$

for some $l, m \geq 0$ and $0 \leq k, k' < n$. Then $\emptyset \neq r(A, \alpha_{[1,i]}\mu) \cap r(A, \alpha_{[1,j]}\nu) = r(A, \alpha^{l+1}\alpha_{[1,k]}) \cap r(A, \alpha^{m+1}\alpha_{[1,k']}) = r(A, \alpha_{[1,k]}) \cap r(A, \alpha_{[1,k']}) = A_k \cap A_{k'}$. Thus, $k = k'$ (which implies $r(A_i, \mu) = r(A_j, \nu)$), and

$$\begin{aligned} s_\mu p_{r(A_i,\mu)\cap C\cap r(A_j,\nu)} s_\nu^* &= s_{\alpha_{[i+1,n]}\alpha^l\alpha_{[1,k]}} p_{r(A,\alpha_{[i,k]})} s^*_{\alpha_{[j+1,n]}\alpha^m\alpha_{[1,k]}} \\ &= s_{\alpha_{[i+1,n]}\alpha^l\alpha_{[1,k]}} p_{A_k} s^*_{\alpha_{[j+1,n]}\alpha^m\alpha_{[1,k]}}. \end{aligned}$$

This means that the hereditary subalgebra $pC^*(E, \mathcal{L}, \mathcal{E})p$ is generated by the elements $p_{A_{i-1}} s_{\alpha_i} (= s_{\alpha_i} p_{r(A_{i-1},\alpha_i)} = s_{\alpha_i} p_{A_i})$. Let γ be the restriction of the gauge action on $C^*(E, \mathcal{L}, \mathcal{E})$ to the hereditary algebra $pC^*(E, \mathcal{L}, \mathcal{E})p$ which is obviously gauge invariant, and let β be the gauge action of the universal (graph) C^*-algebra $C(\mathbb{T}) \otimes M_n$ generated by the partial isometries $t_1, \ldots, t_n$ satisfying the relations

$$t_i^* t_i = t_{i+1} t_{i+1}^*, \quad t_n^* t_n = t_1 t_1^* \quad \text{and} \quad \sum_{j=1}^{n} t_j^* t_j = 1$$

for $1 \leq i \leq n-1$. But the partial elements $s_{\alpha_i} p_{A_i}$, $1 \leq i \leq n$, satisfy the above relations with p in place of 1, hence there exists a homomorphism

$$\pi : C(\mathbb{T}) \otimes M_n \to pC^*(E, \mathcal{L}, \mathcal{E})p$$

such that $\pi(t_i) = s_{\alpha_i} p_{A_i}$ for $1 \leq i \leq n$. It is then immediate that $\pi(\beta_z(t_i)) = \gamma_z(\pi(t_i))$ for all i and the gauge-invariant uniqueness theorem [**4**, Theorem 5.3] proves that π is an isomorphism. □

As we reviewed in Theorem 1.2, several equivalent conditions for a graph C^*-algebra $C^*(E)$ to be purely infinite are known from [**17**]. To state one of them for a row-finite graph E: E satisfies condition (K) and every vertex in a maximal tail connects to a loop in the maximal tail. Thus if α is a loop in E, there is another loop β which is neither an initial path of α nor an extension of α whenever $C^*(E)$ is purely infinite. In the same vein, we have the following proposition.

PROPOSITION 4.7. *Let $(E, \mathcal{L}, \mathcal{E})$ be a labeled space with a loop at a minimal set $A \in \mathcal{E}_{\min}$. If $C^*(E, \mathcal{L}, \mathcal{E})$ is purely infinite, then there exist at least two loops (α, A) and (β, A) such that $\beta^m \neq \alpha^k$ for all $m, k \geq 1$.*

Proof. Let (α, A) have the smallest length among the loops at A. To prove the proposition, we show that if every loop at A is of the form α^k, then there is a hereditary saturated subset H of $\mathcal{E}$ such that the quotient $C^*(E, \mathcal{L}, \mathcal{E})/I_H$ contains a hereditary subalgebra which is isomorphic to $M_n(C(\mathbb{T}))$ for some $n \geq 1$.

First observe that for the minimal set $A \in \mathcal{E}_{\min}$,

$$(\beta, A) \text{ is a loop if and only if } \beta \in \mathcal{L}(AE^{\geq 1}A),$$

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which is immediate from $A \subset r(A, \beta)$ if and only if $r(A, \beta) \cap A \neq \emptyset$. Suppose that

$$\text{if } \beta \in \mathcal{L}(AE^{\geq 1}A) \text{ then } \beta^m \in \{\alpha^k : k \geq 1\}, \quad \text{for some } m \geq 1, \tag{13}$$

and let

$$H := \{B \in \mathcal{E} : r(B, \beta) \cap A = \emptyset \text{ for all } \beta \in \mathcal{L}^*(E)\}.$$

Then $H \neq \mathcal{E}$ since $A \notin H$. We will show that H is a non-empty hereditary saturated subset of $\mathcal{E}$.

Since $C^*(E, \mathcal{L}, \mathcal{E})$ is purely infinite, by Lemma 4.6, α must have an exit. If (α, A) has an exit of type (i), that is, if there is a path $\gamma \in \mathcal{L}(AE^{|\alpha|})$ with $\gamma \neq \alpha$, then $r(A, \gamma) \in H$. Since $r(r(A, \gamma), \beta) \cap A = r(A, \gamma\beta) \cap A \neq \emptyset$ for some β implies that $\gamma\beta \in \mathcal{L}(AE^{\geq 1}A)$, $(\gamma\beta, A)$ is a loop, and by (13), $(\gamma\beta)^m = \alpha^k$ for some $m, k \geq 1$, which gives $\gamma = \alpha$, a contradiction. Hence (α, A) must have an exit of type (ii), namely $A \subsetneq r(A, \alpha)$. Then $H \neq \emptyset$ with $r(A, \alpha) \setminus A \in H$. In fact, if $B := r(A, \alpha) \setminus A \notin H$, then there is a $\beta \in \mathcal{L}^*(E)$ such that $r(B, \beta) \cap A \neq \emptyset$. Again by minimality of A, we have

$$A \subset r(B, \beta) = r(r(A, \alpha) \setminus A, \beta) \subset r(A, \alpha\beta), \tag{14}$$

which shows that $(\alpha\beta, A)$ is a loop, and

$$A \subset r(A, \alpha) \subset r(r(B, \beta), \alpha) = r(B, \beta\alpha) = r(r(A, \alpha) \setminus A, \beta\alpha). \tag{15}$$

By (13) together with (14), there exist $m, k \geq 1$ such that $(\alpha\beta)^m = \alpha^k$, but then $\alpha(\beta\alpha)^{m-1}\beta = (\alpha\beta)^m = \alpha^k = \alpha\alpha^{k-1}$ and thus $(\beta\alpha)^{m-1}\beta = \alpha^{k-1}$. Multiplying both sides by α from the right, we have $(\beta\alpha)^m = \alpha^k = (\alpha\beta)^m$, or $\beta\alpha = \alpha\beta$. Thus we have from (14) and (15) a contradiction to our standing assumption that $(E, \mathcal{L}, \mathcal{E})$ is weakly left-resolving, and see that H is non-empty. To show that H is hereditary, let $B \in H$ and $\gamma \in \mathcal{L}^*(E)$. If $r(B, \gamma) \notin H$, then $r(r(B, \gamma), \beta) \cap A \neq \emptyset$ for some $\beta \in \mathcal{L}^*(E)$, but then $r(B, \gamma\beta) \cap A \neq \emptyset$, in contradiction to $B \in H$. Thus H is closed under relative ranges. It is rather obvious that H is closed under taking finite unions and subsets. In order to see that H is saturated, let $B \in \mathcal{E}$ satisfy $r(B, \gamma) \in H$ for all $\gamma \in \mathcal{L}^*(E)$. If $B \notin H$, then $r(B, \beta) \cap A \neq \emptyset$ for some β. Thus again from minimality of A, we have $A \subset r(B, \beta)$ and then

$$A \subset r(A, \alpha) \subset r(r(B, \beta), \alpha)$$

which means $r(B, \beta) \notin H$, a contradiction.

Now consider the quotient labeled space $(E, \mathcal{L}, \mathcal{E}/H)$. For $B \in \mathcal{E}$, note that $[B] = [\emptyset]$ occurs exactly when $B \in H$, from which $[A] \neq [\emptyset]$ in $\mathcal{E}/H$ follows. Also $r(A, \alpha) \setminus A \in H$, as seen above, implies $[A] = [r(A, \alpha)] (= r([A], \alpha))$, and thus we obtain a loop $(\alpha, [A])$ in the quotient labeled space $(E, \mathcal{L}, \mathcal{E}/H)$ which has no exits of type (ii). If it has an exit of type (i) with a path $\mu \in \mathcal{L}_H([A]E^{|\alpha|})$, $\mu \neq \alpha$ and $r([A], \mu) \neq [\emptyset]$, then $[r(A, \mu)] \neq [\emptyset]$, that is, $r(A, \mu) \notin H$. Thus there exists a path $\beta \in \mathcal{L}^*(E)$ such that $r(r(A, \mu), \beta) \cap A \neq \emptyset$, or equivalently $A \subset r(A, \mu\beta)$. But then by (13), $(\mu\beta)^m = \alpha^n$ for some $m, n \geq 1$, which gives $\mu = \alpha$, a contradiction. Hence the loop $(\alpha, [A])$ of $(E, \mathcal{L}, \mathcal{E}/H)$ does not have any exits. On the other hand, minimality of the set $[A] \in \mathcal{E}/H$ easily follows from minimality of A in $\mathcal{E}$ and thus the same arguments as in the proof of Lemma 4.6 can be applied to conclude that the C^*-algebra $C^*(E, \mathcal{L}, \mathcal{E}/H)$ contains a hereditary subalgebra isomorphic to $M_n(C(\mathbb{T}))$ for some $n \geq 1$. □

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Since pure infiniteness of $C^*(E)$ implies that every quotient graph of E satisfies condition (L), one might expect that a labeled space $(E, \mathcal{L}, \mathcal{E})$ should be strongly disagreeable whenever its C^*-algebra $C^*(E, \mathcal{L}, \mathcal{E})$ is purely infinite. In fact, this is true in some cases including all (locally finite directed) graphs.

PROPOSITION 4.8. *Let $(E, \mathcal{L}, \mathcal{E})$ be a labeled space such that every quotient labeled space has sufficiently many minimal sets. If $C^*(E, \mathcal{L}, \mathcal{E})$ is purely infinite, then $(E, \mathcal{L}, \mathcal{E})$ is strongly disagreeable.*

Proof. We only prove that $(E, \mathcal{L}, \mathcal{E})$ is disagreeable since the same argument also shows that $(E, \mathcal{L}, \mathcal{E}/H)$ is disagreeable for any hereditary saturated subset H.

Suppose $(E, \mathcal{L}, \mathcal{E})$ is not disagreeable. By Proposition 3.2 (or Remark 3.4), there exist $A \in \mathcal{E}$ and $\delta \in \mathcal{L}^*(E)$ satisfying $\mathcal{L}(AE^{|\delta|n}) = \{\delta^n\}$ for all $n \geq 1$. By choosing δ with the smallest length, this is equivalent to saying that

$$\mathcal{L}(AE^{\geq 1}) = \{\delta^n \delta' : n \geq 0,\ \delta' \text{ is an initial path of } \delta\}. \tag{16}$$

Since A contains a minimal subset and (16) holds true for any subset of A in $\mathcal{E}$, we may assume that A is minimal. Moreover, any loop at A is of the form δ^n. In fact, if $\alpha = \delta^n \delta'$ is a loop for some $n \geq 1$ and δ' with $\delta = \delta'\delta''$, then $\alpha^k = (\delta^n \delta')^k = \delta^l$ for some $k, l \geq 1$, by (16), which implies $\delta'\delta'' = \delta''\delta'$. Since δ has the smallest length among the paths satisfying (16), we must have $\delta = \delta'$ (see the proof of Lemma 3.1). Then by Proposition 4.7, we see that there are no loops at the minimal set A, that is,

$$r(A, \alpha) \cap A = \emptyset \quad \text{for all } \alpha \in \mathcal{L}(AE^{\geq 1}),$$

since $C^*(E, \mathcal{L}, \mathcal{E})$ is purely infinite. Now consider the sets

$$\{B_n : n \geq 1\} := \{B \in \mathcal{E}_{\min} : B \subset r(A, \alpha) \text{ for some } \alpha \in \mathcal{L}(AE^{\geq 1})\}$$

as in the proof of Theorem 3.14. Then the previous arguments for A can be applied to B_n to see that

$$r(B_n, \alpha) \cap B_n = \emptyset \tag{17}$$

for all $n \geq 1$ and $\alpha \in \mathcal{L}(B_n E^{\geq 1})$. If $n \neq m$ and $B_n \cap r(B_m, \lambda) \neq \emptyset$ for some $\gamma \in \mathcal{L}^*(E)$ (hence, if $B_n \subseteq r(B_m, \lambda)$), we write $B_m > B_n$. Note that once $B_m > B_n$ holds, $B_n > B_m$ is not possible because of (17). Then the hereditary subset H of $\mathcal{E}$ consisting of all finite unions of B_n gives rise to an ideal I_H, and this ideal can be shown to be an AF algebra in the same way as before, in contradiction to Remark 2.7(c). □

In view of the fact that if $C^*(E, \mathcal{L}, \mathcal{E})$ is purely infinite, then every quotient C^*-algebra (in particular, every C^*-algebra of a quotient labeled space) is purely infinite, Proposition 4.8 supports our stance of discussing the pure infiniteness of labeled graph C^*-algebras only for strongly disagreeable labeled spaces. The following theorem below is a non-simple version of Corollary 3.9. Note that the equivalent conditions in the theorem are also equivalent to the following: every non-zero hereditary subalgebra of $C^*(E, \mathcal{L}, \mathcal{E}/H)$ contains an infinite projection for every hereditary saturated subset H of $\mathcal{E}$.

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THEOREM 4.9. *Let $(E, \mathcal{L}, \mathcal{E})$ be a strongly disagreeable labeled space. Then the following are equivalent,*

(a) *Every non-zero hereditary subalgebra in every quotient of $C^*(E, \mathcal{L}, \mathcal{E})$ contains an infinite projection.*
(b) *$C^*(E, \mathcal{L}, \mathcal{E})$ is purely infinite.*
(c) *Every non-zero projection p_A is properly infinite for $A \in \mathcal{E}$.*

Proof. We only need to show that (c) implies (a). If $C^*(E, \mathcal{L}, \mathcal{E})$ is simple, the assertion follows from Corollary 3.9. If $C^*(E, \mathcal{L}, \mathcal{E}) = C^*(s_a, p_A)$ is not simple, and I is a non-zero ideal of $C^*(E, \mathcal{L}, \mathcal{E})$, then by Proposition 4.4, I is gauge invariant, that is, $I = I_H$ for a non-empty hereditary saturated subset H (in fact, $H = \{A \in \mathcal{E} : p_A \in I\}$). Moreover, with $C^*(E, \mathcal{L}, \mathcal{E}/H) = C^*(t_a, p_{[A]})$, there is an isomorphism

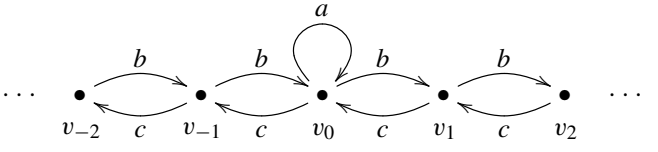
$$\psi : C^*(E, \mathcal{L}, \mathcal{E}/H) \to C^*(E, \mathcal{L}, \mathcal{E})/I$$

such that $\psi(p_{[A]}) = p_A + I$ and $\psi(t_a) = s_a + I$ for $[A] \in \mathcal{E}/H$ and $a \in \mathcal{A}_H$ (Remark 2.5). Now we show that a non-zero hereditary subalgebra B of $C^*(E, \mathcal{L}, \mathcal{E}/H)$ has an infinite projection. Since $(E, \mathcal{L}, \mathcal{E}/H)$ is disagreeable, applying Proposition 3.8 (see also Remark 3.16(a)), we have a non-zero projection $p \in B$ such that $s_\mu p_{[A]} s_\mu^* \preceq p$ for some $\mu \in \mathcal{L}_H^*(E)$ and $[A] \in \mathcal{E}/H$, $[A] \neq [\emptyset]$. On the other hand, the projection $p_A + I \neq 0$ is properly infinite in $C^*(E, \mathcal{L}, \mathcal{E})/I$ because p_A is properly infinite by (c). But the property of being properly infinite is preserved under a $*$-homomorphism, hence the projection $p_{[A]} = \psi^{-1}(p_A + I)$ is properly infinite. Thus p is infinite in B. □

COROLLARY 4.10. *Let $(E, \mathcal{L}, \mathcal{E})$ be a strongly disagreeable labeled space. If for every hereditary saturated subset H of $\mathcal{E}$, every vertex connects to a loop in the quotient labeled space $(E, \mathcal{L}, \mathcal{E}/H)$, then $C^*(E, \mathcal{L}, \mathcal{E})$ is purely infinite.*

Proof. This follows from Theorem 4.9 and Remark 3.16 . □

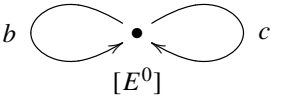
Example 4.11. For the labeled graph $(E, \mathcal{L})$ given by



the accommodating set $\mathcal{E}$ consists of the finite or cofinite vertex subsets, and

$$H := \{A \subset E^0 : A \text{ is finite}\}$$

is the only non-trivial hereditary and saturated subset of $\mathcal{E}$. Each single vertex set $\{v_i\} = [v_i]_{|i|} \in \mathcal{E}$, $i \in \mathbb{Z}$, admits a loop, hence the projections $p_{\{v_i\}}$ are all infinite because the labeled space $(E, \mathcal{L}, \mathcal{E})$ is disagreeable. Since $\mathcal{A}_H = \{b, c\}$, $\mathcal{E}/H = \{[\emptyset], [E^0]\}$, and $[r(b)] = [r(c)] = [E^0]$, the quotient labeled space $(E, \mathcal{L}, \mathcal{E}/H)$ can be visualized as follows:



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Clearly $(E, \mathcal{L}, \mathcal{E}/H)$ is disagreeable, hence the labeled space $(E, \mathcal{L}, \mathcal{E})$ is strongly disagreeable. For each $A \in \mathcal{E}$, the infinite projection $p_A \in C^*(E, \mathcal{L}, \mathcal{E})$ is either zero or infinite in the quotient algebra $C^*(E, \mathcal{L}, \mathcal{E})/I_H \cong C^*(E, \mathcal{L}, \mathcal{E}/H)$, and thus p_A is properly infinite [**27**, Proposition 3.14]. Therefore the C^*-algebra $C^*(E, \mathcal{L}, \mathcal{E})$ is purely infinite by Theorem 4.9.

Acknowledgements. The first and second authors' research is partially supported by NRF-2015R1C1A2A01052516. The third author's research is partially supported by Hanshin University.

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