109.08 Stirling's asymptotic approximations

For every positive real number s the Gamma function satisfies, and may be defined by, the relations

$$s\Gamma(s) = \Gamma(s+1) = \int_0^\infty x^s e^{-x} dx, \tag{1}$$

the left equality being a consequence of integration by parts.

Alternatively, the Bohr-Mollerup Theorem ([1], p. 14 Theorem 2.1) asserts that $\Gamma(s)$ is the unique *log convex function* (see [1]) which satisfies the left equality of (1) for all $s > 0$ and such that $\Gamma(1) = 1$. In particular, $\Gamma(n+1) = n!$ for every integer $n \geqslant 0$.

Because the Gamma and factorial functions have significant applications in mathematics and the mathematical sciences and both are fast increasing functions, historically [2] it has been important to find ways to estimate their values. Among the early achievements in this direction have been the Stirling asymptotic approximation formulas:

$$\Gamma(s+1) \sim \sqrt{2s\pi}\left(\frac{s}{e}\right)^s \quad \text{and} \quad n! \sim \sqrt{2n\pi}\left(\frac{n}{e}\right)^n,$$

the "$\sim$" denoting the equalities

$$1 = \lim_{s\to\infty} \frac{\Gamma(s+1)}{\sqrt{2s\pi}\left(\frac{s}{e}\right)^s} = \lim_{s\to\infty} \frac{n!}{\sqrt{2n\pi}\left(\frac{n}{e}\right)^n}. \tag{2}$$

The symbol "$\sim$" denotes an equivalence relation between numerator and denominator in the two cases. As calculations show, the given denominators of the fractions in (2) fall more and more short of the numerators as n and s grow. However, the difference between numerator and denominator as a percentage of the numerator in each case converges to zero as s, respectively n, gets large. This is equivalent to saying that the fractions converge to "1". Many functions are equivalent to Γ and $n!$ in the sense of "$\sim$", so Stirling's approximations only began the search for accurate asymptotic approximations to the gamma and factorial functions.

Multifarious proofs of (2) appear in the literature [3], [4], [5]; see the References for a sampling of recent proofs, including at least [6], [7] and [8] employing the integral expressions (3) and (4). The reader may regard our proof of (2) as detailing the construction of a generalised function or Dirac δ-function of u (see [9], p. 11).

Setting $x = s + \sqrt{s}z$ in (1), we obtain

$$\Gamma(s+1) = \int_{-\sqrt{s}}^\infty \left(s + \sqrt{s}z\right)^s e^{-\left(s+\sqrt{s}z\right)} \sqrt{s}\, dz, \tag{3}$$

which is equivalent to the ratio

$$\frac{\Gamma(s+1)}{s^{s+\frac{1}{2}}e^{-s}} = \int_{-\sqrt{s}}^\infty \left(1 + \frac{z}{\sqrt{s}}\right)^s e^{-\sqrt{s}z} dz. \tag{4}$$

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Thus our goal is to show that $\lim_{s\to\infty}$ for the integral of (4) is $\sqrt{2\pi}$. We set $u = \frac{z}{\sqrt{s}}$. Then:

$$\int_{-\sqrt{s}}^{\infty}\left(1+\frac{z}{\sqrt{s}}\right)e^{-\sqrt{s}z}dz = \sqrt{s}\int_{-1}^{\infty}(1+u)^s e^{-su}du = \sqrt{s}\int_{-1}^{\infty}\left(\frac{1+u}{e^u}\right)^s du. \qquad (5)$$

Write $\varphi(u) = (1+u)e^{-u} = \frac{1+u}{e^u}$ for $u > -1$.

Remarks:
(i) $\varphi(u) \geqslant 0$ for all $u \geqslant -1$; $\varphi(u) = 0$ only if $u = -1$.
(ii) Since the derivative $\varphi'(u) = -ue^{-u}$, $\varphi(u) = 0$ increases for negative u, decreases for positive u and the maximum value of $\varphi(u)$ is $\varphi(0) = 1$.
(iii) If $0 < y < x$ or $0 > y > x > -1$, then $1 > \varphi(y) > \varphi(x) > 0$.

Lemma 1: For any fixed real number $m > 1$,

$$\lim_{s\to\infty}\sqrt{s}\int_{-1}^{-1/m}\varphi(u)^s du = \lim_{s\to\infty}\sqrt{s}\int_{1/m}^{\infty}\varphi(u)^s du = 0. \qquad (6)$$

Proof: Let $m \geqslant 1$ be fixed and assume that s is a large positive real number. We will prove only that $\lim_{s\to\infty}\sqrt{s}\int_{1/m}^{\infty}\varphi(u)^s du = 0$, since the proof that $\lim_{s\to\infty}\sqrt{s}\int_{-1/m}^{1/m}\varphi(u)^s du = 0$ is similar. Since we know that $\varphi(u) < \varphi(1/m)$ for $0 < 1/m < u$, (Remarks (iii)) and the integral on the right in (7) is finite,

$$\sqrt{s}\int_{1/m}^{\infty}\varphi(u)^s du$$

$$= \sqrt{u}\int_{1/m}^{\infty}\varphi(u)^{s-1}\varphi(u)du < \sqrt{s}\,\varphi(1/m)^{s-1}\int_{1/m}^{\infty}\varphi(u)du \to 0, s\to\infty, \qquad (7)$$

the main point in (7) being that the product $\sqrt{s}\varphi(\pm 1/m)^{s-1}$ of a power function of s and an exponential function of s which is less than 1 will decrease to zero as s increases to ∞.

Corollary 1: If one limit of (8) exists then so does the other and they are equal.

$$\lim_{s\to\infty}\sqrt{s}\int_{-1}^{\infty}\varphi(u)^s du = \lim_{s\to\infty}\sqrt{s}\int_{-1/m}^{1/m}\varphi(u)^s du, \qquad (m \geqslant 1). \qquad (8)$$

The right side of (8) is independent of $m \geqslant 1$.

Proof: On the left we have for $m \geqslant 1$ the following decomposition of the left side of (8):

$$\lim_{s\to\infty}\sqrt{s}\int_{-1}^{1/m}\varphi(u)^s du + \lim_{s\to\infty}\sqrt{s}\int_{-1/m}^{1/m}\varphi(u)^s du + \lim_{s\to\infty}\sqrt{s}\int_{1/m}^{\infty}\varphi(u)^s du$$

$$= \lim_{s\to\infty}\sqrt{s}\int_{-1/m}^{1/m}\varphi(u)^s du. \qquad (9)$$

Equation (6) implies (9), which proves Corollary 1.

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To prove (2) it is now enough to prove:

Proposition: For every $m > 1$

$$\lim_{s\to\infty} \sqrt{s} \int_{-1/m}^{1/m} \varphi(u)^s \, du = \int_{-\infty}^{\infty} e^{-z^2} dz. \tag{10}$$

We know ([10], xvi or [11], p. 164) that $\int_{-\infty}^{\infty} e^{-z^2/2} dz = \sqrt{2\pi}$. In fact, for any $a > 0$, $\int_{-\infty}^{\infty} e^{-z^2/a} dz = \sqrt{a\pi}$. Actually, in view of Corollary 1, to prove the Proposition it is enough to prove that there is at least one $m \geqslant 1$ satisfying (10).

We change variables to write

$$\sqrt{s} \int_{-1/m}^{1/m} \varphi(u)^s \, du = \sqrt{s} \int_0^{1/m} \left(\varphi(u)^s + \varphi(-u)^s\right) du. \tag{11}$$

Using the power series for $\log(1+u)$ we obtain for $\log \varphi(u)^s = s\left(\log(1+u) - u\right)$:

$$s\left[\left(u - \frac{1}{2}u^2 + \frac{1}{3}u^3 - \frac{1}{4}u^4 + \dots\right) - u\right] = -\frac{su^2}{2} + s\sum_{j=3}^{\infty} (-1)^{j-1} \frac{u^j}{j} \tag{12}$$

and, using the power series of $\log(1-u)$ we obtain for $\log \varphi(-u)^s = s\left(\log(1-u) + u\right)$:

$$s\left[\left(-u - \frac{u^2}{2} - \frac{u^3}{3} - \frac{u^4}{4} - \dots\right) + u\right] = -s\frac{u^2}{2} - s\sum_{j=3}^{\infty} \frac{u^j}{j}. \tag{13}$$

The power series in (12) and (13) are both absolutely and uniformly convergent for $|u| \leqslant \frac{1}{m} < 1$; thus, these series define bounded continuous functions on $\left[0, \frac{1}{m}\right]$. Putting (12) and (13) together we get

$$\sum_{j=3}^{\infty} (-1)^{j-1} \frac{u^j}{j} = \frac{u^3}{3} - \sum_{j=2}^{\infty} \left(\frac{u^{2j}}{2j} - \frac{u^{2j+1}}{2j+1}\right) < \frac{u^3}{3} < \sum_{j=3}^{\infty} \frac{u^j}{j} \qquad (0 < u < 1). \tag{14}$$

Thus, $-\sum_{j=3}^{\infty} \frac{u^j}{j} < -\frac{u^3}{3} < 0$ and $0 < \sum_{j=3}^{\infty} (-1)^{j-1} \frac{u^j}{j} < \frac{u^3}{3}$. (14) implies

$$\exp\left(-\sum_{j=3}^{\infty} \frac{u^j}{j}\right) < \exp\left(-\frac{u^3}{3}\right) \text{ and } \exp \sum_{j=3}^{\infty} (-1)^{j-1} \frac{u^j}{j} < \exp \frac{u^3}{3}. \tag{15}$$

For s large and positive we get from (12), (13), (14) and (15)

$$\varphi(u)^s + \varphi(-u)^s = \exp\left(-\frac{su^2}{2}\right)\left(\exp s\sum_{j=3}^{\infty} (-1)^{j-1} \frac{u^j}{j} + \exp\left(-s\sum_{j=3}^{\infty} \frac{u^j}{j}\right)\right)$$

$$< 2\exp\left(-\frac{su^2}{2}\right)\cosh\left(\frac{su^3}{3}\right). \tag{16}$$

Lemma 2: For any fixed $M > 1$

$$\lim_{s\to\infty} \sqrt{s} \int_{-M/\sqrt{s}}^{M/\sqrt{s}} \varphi(u)^s \, du = \int_{-M}^{M} \exp\left(-\frac{z^2}{2}\right) dz. \tag{17}$$

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Proof: We make the change of variables $z = \sqrt{s}\,u$. Then (12), (13) and (14) imply that for $\sqrt{s} > M$

$$\sqrt{s}\int_0^{M/\sqrt{s}} \left(\varphi(u)^s + \varphi(-u)^s\right)$$

$$= \int_0^M \exp\left(-\frac{z^2}{2}\right)\left(\exp\frac{1}{\sqrt{s}}\sum_{j=3}^{\infty}\frac{(-1)^{j-1}z^j}{js^{(j-3)/2}} + \exp-\frac{1}{\sqrt{s}}\sum_{j=3}^{\infty}\frac{z^j}{js^{(j-3)/2}}\right)dz$$

$$< 2\int_0^M \exp\left(-\frac{z^2}{2}\right)\cosh\left(\frac{M^3}{3\sqrt{s}}\right)dz. \tag{18}$$

The Lemma follows by letting $\sqrt{s}$ go to ∞ and making use of the symmetries.

Now we proceed toward evaluating (4). Since the improper integral $\int_{-\infty}^{\infty} e^{-z^2/2}dz$ converges, for any $\varepsilon > 0$ there exists $M(\varepsilon)$ such that for any $M \geqslant M(\varepsilon) > 1$

$$\int_{-\infty}^{\infty} e^{-z^2/2}dz > \int_{-M}^{M} e^{-z^2/2}dz > \int_{-\infty}^{\infty} e^{-z^2/2}dz - \varepsilon. \tag{19}$$

Since $\int_{-M}^{M} \exp\left(-z^2/2\right)du < 3$ and, for all $x > 0$,

$$3(\cosh x - 1) < \cosh 3x - 1 < \sinh 3x,$$

with $\sqrt{s} = M^n$ we get

$$\int_{-M}^{M} \exp\left(-\frac{z^2}{2}\right)\left(\cosh\frac{M^{3-n}}{3} - 1\right)dz < 3\left(\cosh\frac{M^{3-n}}{3} - 1\right)$$

$$< \sinh M^{3-n}. \tag{20}$$

Using (17), (18), (19) and (20) we see that for $\sqrt{s} = M^n$ with n sufficiently large Lemma 2 implies

$$\int_{-\infty}^{\infty} \exp\left(-\frac{z^2}{2}\right)dz > \sqrt{s}\int_{-M/\sqrt{s}}^{M/\sqrt{s}} \varphi(u)^s\,dz > \int_{-\infty}^{\infty} \exp\left(-\frac{z^2}{2}\right)dz - \varepsilon. \tag{21}$$

If $mM = \sqrt{s}$, then $1/m = M\sqrt{s}$ and, therefore, for all $t \geqslant 1$

$$\sqrt{st}\int_{-1/m\sqrt{t}}^{1/m\sqrt{t}} \varphi(u)^{st}\,du = \sqrt{st}\int_{-M/\sqrt{st}}^{M/\sqrt{st}} \varphi(u)^{st}\,du, \tag{22}$$

since $mM = \sqrt{s}$ implies $\left(m\sqrt{t}\right)M = \sqrt{st}$. Thus, from equations (19), (20), (21) and (22) we get:

Corollary 2: For $1/m = M/\sqrt{s}$ with $\sqrt{s} \geqslant M^n$ with $n > 3$ sufficiently large

$$\int_{-\infty}^{\infty} \exp\left(-\frac{z^2}{2}\right)dz > \sqrt{s}\int_{-1/m}^{1/m} \varphi(u)^s\,dz > \int_{-\infty}^{\infty} \exp\left(-\frac{z^2}{2}\right)dz - \varepsilon. \tag{23}$$

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Both $\varepsilon > 0$ and $M \geqslant M(\varepsilon)$ are arbitrary, so

$$\lim_{s \to \infty} \sqrt{s} \int_{-1/m}^{1/m} \varphi(u)^s \, du = \int_{-\infty}^{\infty} \exp\left(-\frac{z^2}{2}\right) dz, \tag{24}$$

which proves (10) and (2).

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ALLAN SILBERGER
1573 Kew Rd, Cleveland Hts,
OH 44118 USA
e-mail: *allanjsilberger@gmail.com*
RASUL KHAN
Cleveland State University,
e-mail: *r.khan@csuohio.edu*

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