*Math. Proc. Camb. Phil. Soc.* (1983), **94**, 149
*Printed in Great Britain*



# Almost periodic generalized functions

By H. BURKILL and B. C. RENNIE*
*University of Sheffield and James Cook University of North Queensland*

(*Received* 25 *November* 1981; *revised* 19 *July* 1982)

## 1. *Introduction*

In (4) a space $C$ of generalized functions was defined which is rather larger than the simple space used to such effect by Lighthill in (3). At the core of $C$ is the space $C_0 = T$ of test functions. These are entire (complex) functions $f$ such that all derivatives of $f$ and its Fourier transform $F$ have order of magnitude not exceeding $e^{-cx^2}$ as $x \to \pm\infty$, where $c$ is a positive number depending on the individual derivative concerned. If $f, g \in T$, the inner product $\langle f \mid g \rangle$ is defined to be

$$\int_{-\infty}^{\infty} f(x)\, g(x)\, dx.$$

A sequence $(f_n)$ of test functions is then said to converge weakly to the test function $f$ if $\langle f_n \mid g \rangle \to \langle f \mid g \rangle$ for every $g \in T$. The space $C_1$ is now defined to be the sequential completion of $C_0 = T$ in the sense of this weak topology, and the inner product $\langle f \mid g \rangle$ is obtained in the natural way for all $f \in C_1$ and $g \in T$. This process is continued so that $C_r$ and the inner product $\langle f \mid g \rangle$ $(f \in C_r, g \in T)$ are defined whenever $0 \leqslant r < \Omega$, where $\Omega$ is the first uncountable ordinal: when $r > 1$, $C_r$ is the sequential completion, with respect to the weak topology, of $\bigcup_{s<r} C_s$. Finally, $C$ is the space $\bigcup_{r<\Omega} C_r$ and $\langle f \mid g \rangle$ is defined for all $f \in C$ and $g \in T$. Two features of $C$ that may be picked out are that $C$ is sequentially complete and that it contains large 'ordinary' functions such as $e^{\lambda x}$.

Throughout this paper, when a generalized function is denoted by a lower case letter, the corresponding capital letter will stand for its Fourier transform. The 'Parseval identity' for generalized functions can then be written

$$\langle f \mid G \rangle = \langle F \mid g \rangle \quad (f \in C, g \in T).$$

The 'integration by parts' formula is

$$\langle f \mid g' \rangle = -\langle f' \mid g \rangle \quad (f \in C, g \in T).$$

One of the topics studied in (3) and (4) is that of Fourier series. Every periodic generalized function has a Fourier series associated with it which converges (weakly) to the generating function; and, conversely, if a trigonometric series

$$\sum_{n=-\infty}^{\infty} a_n e^{inx}$$

converges weakly, then the series is the Fourier series of its sum. The twin objects of this paper are, first, to define the property of almost periodicity for generalized func-

* This paper was written while the second author was a Visiting Professor at the University of Sheffield.

https://doi.org/10.1017/S0305004100060977 Published online by Cambridge University Press

tions and, secondly, to obtain results as far as possible analogous to those just described for the more general trigonometric series of the form

$$\sum_{n=-\infty}^{\infty} c_n e^{i\lambda_n x}.$$

In ((5), pp. 62–64) Schwartz briefly considers almost periodic distributions. We shall show that there is a natural correspondence between these and some of our almost periodic generalized functions. Another treatment of the subject of almost periodicity in the setting of generalized functions, also different from ours, appears in I. W. Doherty's thesis (2).

In the classical theory, an almost periodic function is one that almost repeats itself at fairly regular intervals; and the near repetition is interpreted in the sense of various norms. For generalized functions a norm appears to be inappropriate and instead we adopt a topological approach.

Our topology is based on the notion of *translate*. Let $k$ be any real number. For $f \in C$, the translate $f[k]$ of $f$ is defined inductively. First of all, if $f \in T$ $(= C_0)$, $f[k]$ is defined by

$$f[k](x) = f(x+k) \quad (x \in \mathbb{C}),$$

so that evidently $f[k] \in T$ and

$$\langle f[k] \mid g \rangle = \langle f \mid g[-k] \rangle \tag{1}$$

for any $g \in T$. Now suppose that $r \geqslant 1$, that $f[k]$ has been defined as a member of $C_s$ for $f$ in any space $C_s$ with $s < r$ and that (1) holds whenever $f \in C_s (s < r)$ and $g \in T$. If $f \in C_r$, then $f$ is the weak limit of a sequence $(f_n)$ in $\bigcup_{s<r} C_s$ and, as a result of (1), $(f_n[k])$ is a Cauchy sequence in $\bigcup_{s<r} C_s$. The limit of this sequence is a member of $C_r$ and is defined to be $f[k]$; moreover, clearly (1) holds for any $g \in T$. Since $C = \bigcup_{r<\Omega} C_r$, the translate $f[k]$ is therefore defined for every $f \in C$. Also (1) holds for all $f \in C$ and $g \in T$.

Next, let $g \in T$ and put

$$V_g = \{f \in C : |\langle f \mid g[y] \rangle| < 1 \text{ for all } y \in \mathbb{R}\}. \tag{2}$$

We wish to make $V_g$ a neighbourhood of the origin in $C$. To do so we define $\mathscr{S}$ to be the class of all translations

$$V_g + h \quad (g \in T, h \in C)$$

and take $\mathscr{T}$, the class of open sets in $C$, to be the class of all unions of finite intersections of members of $\mathscr{S}$ (so that $\mathscr{S}$ is an open subbase of $\mathscr{T}$).

Note that, if $f \in V_g$, then $f[y] \in V_g$ for every $y \in \mathbb{R}$.

In view of the definition of $\mathscr{T}$, the sequence $(f_n)$ converges to $f$ in $(C, \mathscr{T})$ if and only if, given $g \in T$, there exists an integer $n_0$ such that $f_n - f \in V_g$ for $n > n_0$, i.e.

$$|\langle f_n \mid g[y] \rangle - \langle f \mid g[y] \rangle| < 1 \quad \text{whenever} \quad y \in \mathbb{R} \quad \text{and} \quad n > n_0.$$

If $\epsilon > 0$, then $g/\epsilon \in T$ if and only if $g \in T$. Hence the above condition is equivalent to the existence of $n_1$ (depending on $g$ and $\epsilon$) such that

$$|\langle f_n \mid g[y] \rangle - \langle f \mid g[y] \rangle| < \epsilon \quad \text{whenever} \quad y \in \mathbb{R} \quad \text{and} \quad n > n_1.$$

Thus $f_n \to f$ in $(C, \mathscr{T})$ if and only if, for every $g \in T$,

$$\langle f_n \mid g[y] \rangle \to \langle f \mid g[y] \rangle \quad \text{as} \quad n \to \infty,$$

uniformly in $y$. We therefore call convergence in the topological space $(C, \mathscr{T})$ *weak uniform convergence*.

https://doi.org/10.1017/S0305004100060977 Published online by Cambridge University Press

We also need the concept of *weak continuity*. The generalized function $f$ is said to be weakly continuous if, for every $x_0 \in \mathbb{R}$,

$$f[x] \to f[x_0] \quad \text{weakly as} \quad x \to x_0,$$

i.e., for all $g \in T$,

$$\langle f[x] \mid g \rangle \to \langle f[x_0] \mid g \rangle \quad \text{as} \quad x \to x_0.$$

By (1), the weak continuity of $f$ is therefore equivalent to the property that, whenever $g \in T$, the function $\phi$ on $\mathbb{R}$ given by

$$\phi(y) = \langle f \mid g[y] \rangle \quad (y \in \mathbb{R})$$

is continuous.

There is the expected connection between weak continuity and weak uniform convergence.

THEOREM 1·1. *If the sequence $(f_n)$ of weakly continuous generalized functions converges uniformly weakly to the generalized function $f$, then $f$ is weakly continuous.*

*Proof.* Let $g$ be any test function and define the functions $\phi$, $\phi_n (n = 1, 2, \ldots)$, by

$$\phi(y) = \langle f \mid g[y] \rangle, \quad \phi_n(y) = \langle f_n \mid g[y] \rangle \quad (y \in \mathbb{R}).$$

Then each function $\phi_n$ is continuous and $\phi_n \to \phi$ uniformly. Therefore $\phi$ is continuous and it follows that $f$ is weakly continuous.

## 2. *Definition of almost periodicity*

We first recall the basic definitions of the classical theory.

A subset $E$ of $\mathbb{R}$ is said to be *relatively dense* if there exists a positive number $p$ (called an *inclusion number*) such that every interval of length $p$ contains a member of $E$.

Next, let $f$ be a complex-valued function on $\mathbb{R}$.

If $\epsilon > 0$, the real number $\tau$ is called an *$\epsilon$-translation number* of $f$ if

$$\sup_{-\infty < x < \infty} |f(x+\tau) - f(x)| < \epsilon.$$

The set of all $\epsilon$-translation numbers of $f$ is denoted by $E(f, \epsilon)$.

Finally, the function $f$ is *uniformly almost periodic* (*u.a.p.*) if $f$ is continuous and, for every $\epsilon > 0$, the set $E(f, \epsilon)$ is relatively dense.

In the space $C$ of generalized functions we base the notion of a translation number on the topology $\mathscr{T}$ introduced in the previous section.

DEFINITION 2·1. *Let $f$ be a generalized function. If $V$ is a neighbourhood of the origin in $C$, the real number $\tau$ is called a* $V$-translation number *of $f$ if*

$$f[\tau] - f \in V.$$

*The set of all $V$-translation numbers of $f$ is denoted by $E(f, V)$.*

The definition of almost periodicity in $C$ now takes its natural form.

DEFINITION 2·2. *The generalized function $f$ is said to be* weakly almost periodic (w.a.p.) *if $f$ is weakly continuous and, given any neighbourhood $V$ of the origin, the set $E(f, V)$ is relatively dense.*

It is easy to see that, if the weakly continuous function $f$ is periodic, i.e. if there

https://doi.org/10.1017/S0305004100060977 Published online by Cambridge University Press

exists a non-zero real number $l$ such that $f[l] = f$, then $f$ is w.a.p. For if $a$ is any real number, then $f[a+l] = f[a]$, since

$$\langle f[a+l] \,|\, g\rangle = \langle f[l] \,|\, g[-a]\rangle = \langle f \,|\, g[-a]\rangle = \langle f[a] \,|\, g\rangle$$

when $g$ is any test function; and so, if $V$ is an arbitrary neighbourhood of the origin, the set $E(f, V)$ contains the relatively dense set $\{0, \pm l, \pm 2l, \ldots\}$.

In (4) a measurable complex-valued function $f^*$ on $\mathbb{R}$ is called *ordinary* if it is absolutely integrable over every finite interval and if

$$\int_{-\infty}^{\infty} |f^*(x)|\, e^{-cx^2}\, dx < \infty$$

for all $c > 0$. Such a function may be regarded as an element of $C$, since to it there corresponds a generalized function $f$ such that

$$\langle f \,|\, g\rangle = \int_{-\infty}^{\infty} f^* g$$

for all test functions $g$. ((4), Theorem 6.)

THEOREM 2·3. *Let $f^*$ be a u.a.p. function. Then $f^*$ is an ordinary function and $f$ is w.a.p.,*

*Proof.* Since $f^*$ is continuous and bounded, it is certainly ordinary. In fact, $f^*$ is uniformly continuous and this entails the weak continuity of $f$.

Now take any neighbourhood $V$ of the origin. It contains a set

$$\bigcap_{k=1}^{n} V_{g_k}.$$

(see (2)), where $g_1, \ldots, g_n \in T$; let

$$K = \max_{1 \leqslant k \leqslant n} \int_{-\infty}^{\infty} |g_k|.$$

Then every $1/K$-translation number of $f^*$ is a $V$-translation number of $f$, and so the set $E(f, V)$ is relatively dense.

### 3. *Intersections of relatively dense sets*

If the functions $f_1, f_2$ are u.a.p. and $\epsilon > 0$, then

$$E(f_1, \epsilon) \cap E(f_2, \epsilon) \tag{4}$$

is relatively dense. This is proved, for instance, in ((1) pp. 4–5). However, the standard proof does not immediately extend to the case of an arbitrary intersection

$$\bigcap_{k=1}^{n} E(f_k, \epsilon) \tag{5}$$

with $n \geqslant 2$; and the relative density of any set (5) is just what is needed to help establish the most useful characterization of weak almost periodicity.

The reason why the set (4) is relatively dense when the intersection of two arbitrary relatively dense sets certainly need not be relatively dense (and can, of course, even be empty) is that sets of translation numbers of u.a.p. functions have significant properties besides relative density.

https://doi.org/10.1017/S0305004100060977 Published online by Cambridge University Press

DEFINITION 3·1. *A subset $E$ of $\mathbb{R}$ is called* uniform *if* (i) *it is relatively dense*, (ii) *it is symmetrical about the origin*, (iii) *it contains an interval $(-\delta, \delta)$, where $\delta > 0$.*

*A uniform set $E$ is called* additive *if it contains a sequence of uniform sets $E_1 \supseteq E_2 \supseteq \ldots$ such that, whenever $x, y \in E_n$, then $x+y \in E_{n-1} (n = 1, 2, \ldots; E_0 = E)$.*

We shall see that the combined property of uniformity and additivity is transmitted to the intersection of two such sets, so that therefore the intersection of any finite collection of uniform and additive sets is again uniform and additive (and, in particular, relatively dense).

It is easy to see that, if $f$ is u.a.p. and $\epsilon > 0$, then $E(f, \epsilon)$ is uniform and additive. Property (i) of definition 3·1 is inherent in the definition of uniform almost periodicity; (ii) follows from the fact that

$$\sup_{-\infty<x<\infty} |f(x+\tau)-f(x)| = \sup_{-\infty<x<\infty} |f(x)-f(x-\tau)|;$$

and (iii) is a consequence of the uniform continuity of $f$. Finally, the sets $E_n$ in the definition of additivity may clearly be taken to be the sets $E(f, \epsilon/2^n) (n = 1, 2, \ldots)$. The relative density of the sets (5) now follows at once.

The proofs of the lemma below and the theorem following it are based on the argument in ((1), pp. 4–5). We shall consider two uniform and additive sets $E, F$ and we shall take $(E_n)$, $(F_n)$ to be sequences of the type occurring in definition 3·1 and associated with the sets $E, F$ respectively. Also, given $x \in \mathbb{R}$ and $A \subseteq \mathbb{R}$, we denote by $d(x, A)$ the distance of $x$ from $A$, i.e. $\inf\{|x-y| : y \in A\}$.

LEMMA 3·2. *If $E, F$ are uniform and additive sets, and if $c$ is any positive number, then the set*

$$U = \{x \in E : d(x, F) < c\}$$

*is uniform.*

*Proof.* Let $k$ be a positive integer sufficiently large for $kc$ to be an inclusion number of $E_1$ and of $F_1$. In each interval $((n-1)kc, nkc) (n = 0, \pm 1, \pm 2, \ldots)$ choose a point $x_n \in E_1$ and a point $y_n \in F_1$.

Denote by $\lambda_i$ the interval $[(i-1)c, ic) (i = 0, \pm 1, \pm 2, \ldots)$. For every $n$, $x_n - y_n$ lies in an interval $\lambda_i$ such that $-k+1 \leqslant i \leqslant k$. So there is only a finite number of intervals $\lambda_i$ which contain points $x_n - y_n$. Hence there exists an integer $N$ with the property that, to every $n$ there corresponds an integer $\nu(n)$, with $-N \leqslant \nu(n) \leqslant N$, for which $x_{\nu(n)} - y_{\nu(n)}$ belongs to the same interval $\lambda_i$ as $x_n - y_n$. Thus

$$x_n - y_n = x_{\nu(n)} - y_{\nu(n)} + \theta c,$$

where $|\theta| < 1$, or

$$x_n - x_{\nu(n)} = y_n - y_{\nu(n)} + \theta c. \tag{6}$$

But $-x_{\nu(n)} \in E_1$ and $-y_{\nu(n)} \in F_1$, so that $\xi_n = x_n - x_{\nu(n)} \in E$ and $y_n - y_{\nu(n)} \in F$. Therefore $d(\xi_n, F) < c$ for all $n$. Moreover

$$(n-N-1)kc < \xi_n < (n+N+1)kc$$

for every $n$, and so the set $\{\xi_n : n = 0, \pm 1, \pm 2, \ldots\}$ is relatively dense.

By (6), $U \supseteq \{\xi_n : n = 0, \pm 1, \pm 2, \ldots\}$. Hence $U$ is relatively dense. Also $U$ is clearly symmetrical about the origin and, since both $E$ and $F$ contain 0 as an interior point, so does $U$. Therefore $U$ is uniform.

https://doi.org/10.1017/S0305004100060977 Published online by Cambridge University Press

THEOREM 3·3. *If $E, F$ are uniform and additive sets, then the set $E \cap F$ is uniform and additive.*

*Proof.* We begin by noting that all the sets $E_1, E_2, \ldots, F_1, F_2, \ldots$ are uniform and additive.

Suppose that $F_1$ contains the interval $(-\delta_1, \delta_1)$. By the preceding lemma, the set

$$U_1 = \{x \in E : d(x, F_1) < \delta_1\}$$

is uniform. But, by additivity,

$$\{x \in \mathbb{R} : d(x, F_1) < \delta_1\} \subseteq F.$$

Hence $U_1 \subseteq F$ and so $U_1 \subseteq E \cap F$. Therefore $E \cap F$ is relatively dense. Also $E \cap F$ contains an interval $(-\delta, \delta)$; and, since $E$ and $F$ are symmetrical about the origin, $E \cap F$ has the same property. Hence $E \cap F$ is uniform.

We have

$$E \cap F \supseteq E_1 \cap F_1 \supseteq E_2 \cap F_2 \supseteq \ldots$$

and all the sets $E_n \cap F_n$ are uniform. Now let $x, y \in E_n \cap F_n$ $(n \geqslant 1)$. Then $x, y \in E_n$ and so $x+y \in E_{n-1}$. Similarly, $x+y \in F_{n-1}$ and so $x+y \in E_{n-1} \cap F_{n-1}$. Therefore $E \cap F$ is additive, and the theorem is proved.

It is now easy to establish the characterization of weak almost periodicity referred to at the beginning of the section.

THEOREM 3·4. *A necessary and sufficient condition for the generalized function $f$ to be w.a.p. is that, whenever $g \in T$, the function $\phi$ on $\mathbb{R}$ given by*

$$\phi(y) = \langle f \mid g[y] \rangle \quad (y \in \mathbb{R}) \tag{7}$$

*is u.a.p.*

*Proof. Necessity.* Suppose that $f \in C$ is w.a.p. Let $g \in T$ and let the function $\phi$ be defined by (7). We have previously noted that $\phi$ is continuous. Next, take any $\epsilon > 0$. Then $V_{g/\epsilon}$ is a neighbourhood of the origin and the set $E(f, V_{g/\epsilon})$ is relatively dense. If $\tau$ belongs to this set, then

$$|\langle f[\tau] - f \mid g[y]/\epsilon \rangle| < 1 \quad \text{for all} \quad y \in \mathbb{R},$$

i.e.

$$|\phi(y-\tau) - \phi(y)| < \epsilon \quad \text{for all} \quad y \in \mathbb{R}.$$

Thus $-\tau$ is an $\epsilon$-translation number of $\phi$, and so is $\tau$. Hence $E(\phi, \epsilon)$ is relatively dense and $\phi$ is u.a.p.

*Sufficiency.* Suppose that, for every $g \in T$, the function $\phi$ given by (7) is u.a.p. Since every $\phi$ is continuous, $f$ is weakly continuous. Now let $V$ be a neighbourhood of the origin, so that $V$ contains a set

$$\bigcap_{k=1}^{n} V_{g_k},$$

where $g_1, \ldots, g_n \in T$. Each of the functions $\phi_k (k = 1, \ldots, n)$ given by

$$\phi_k(y) = \langle f \mid g_k[y] \rangle \quad (y \in \mathbb{R})$$

is u.a.p. and therefore each set $E(\phi_k, 1)$ is uniform and additive. Hence the set

$$D = \bigcap_{k=1}^{n} E(\phi_k, 1)$$

https://doi.org/10.1017/S0305004100060977 Published online by Cambridge University Press

is also uniform and additive, and is consequently relatively dense. Let $\tau \in D$, so that

$$|\langle f[\tau]-f \mid g_k[y]\rangle| < 1 \quad \text{for all} \quad y \in \mathbb{R} \quad (k = 1, \ldots, n).$$

Then

$$f[\tau]-f \in \bigcap_{k=1}^{n} V_{g_k} \subseteq V,$$

i.e.

$$\tau \in E(f, V).$$

Thus $E(f, V)$ contains $D$ and is therefore relatively dense. Hence $f$ is w.a.p.

When $f_1, f_2 \in L^2(\mathbb{R})$, the *convolution* $f_1 * f_2$ is the function on $\mathbb{R}$ given by

$$(f_1 * f_2)(y) = \int_{-\infty}^{\infty} f_1(x) f_2(y-x)\, dx \quad (y \in \mathbb{R}).$$

Now let $f \in C$, $g \in T$. It is natural to denote by $f * g$ the function defined by

$$(f * g)(y) = \langle f \mid g_-[-y]\rangle = \langle f[y] \mid g_-\rangle \quad (y \in \mathbb{R}),$$

where

$$g_-(x) = g(-x) \quad (x \in \mathbb{R}).$$

With this notation theorem 3·4 takes the following form: The generalized function $f$ is w.a.p. if and only if, for every $g \in T$, the convolution $f * g$ is u.a.p.

### 4. *Operations on w.a.p. functions*

By use of theorem 3·4 it is easy to prove that several operations on w.a.p. functions yield w.a.p. functions.

THEOREM 4·1. (*a*) *If the generalized function $f$ is w.a.p., then so is $f'$.* (*b*) *If $f_1, f_2$ are w.a.p., then $f_1+f_2$ is w.a.p.*

*Proof.* Let $g \in T$. (*a*) Since $g' \in T$ and $f$ is w.a.p., the function $\phi$ on $\mathbb{R}$ given by

$$\phi(y) = \langle f' \mid g[y]\rangle = -\langle f \mid g'[y]\rangle \quad (y \in \mathbb{R})$$

is u.a.p., and it follows that $f'$ is w.a.p. (*b*) It is sufficient to remark that

$$\langle f_1+f_2 \mid g[y]\rangle = \langle f_1 \mid g[y]\rangle + \langle f_2 \mid g[y]\rangle$$

and that the sum of two u.a.p. functions is u.a.p.

THEOREM 4·2. *The set of w.a.p. functions is sequentially closed in $(C, \mathscr{T})$.*

*Proof.* We have to show that, if $(f_n)$ is a sequence of w.a.p. functions and $f_n \to f$ uniformly weakly, then $f$ is w.a.p.

Let $g \in T$ and define the functions $\phi_n (n = 1, 2, \ldots)$ and $\phi$ on $\mathbb{R}$ by

$$\phi_n(y) = \langle f_n \mid g[y]\rangle, \quad \phi(y) = \langle f \mid g[y]\rangle \quad (y \in \mathbb{R}).$$

Each $\phi_n$ is u.a.p. and we noted in section 1 that $\phi_n \to \phi$ uniformly. Hence $\phi$ is u.a.p. and so $f$ is w.a.p.

Theorems 4·1 (*a*) and 4·2 could have been proved directly from the definition of weak almost periodicity (and with the additional help of theorem 1·1 in the case of the second result).

If we wished to establish theorem 4·1 (*b*) with minimum reliance on the theory of u.a.p. functions, we should need to know that, for any neighbourhood $V$ of the origin

$$E(f_1, V) \cap E(f_2, V)$$

https://doi.org/10.1017/S0305004100060977 Published online by Cambridge University Press

is relatively dense or, equivalently, that, whenever $g \in T$,

$$E(f_1, V_g) \cap E(f_2, V_g)$$

is relatively dense. This follows from theorem 3·3 provided that, for all w.a.p. functions $f$, the set $E(f, V_g)$ can be relied on to be uniform and additive.

To establish the uniformity of $E(f, V_g)$ we first note that the set is, of course, relatively dense and that it is symmetrical about 0 for if $f[\tau] - f \in V_g$, then $f - f[-\tau] \in V_g$. We also require $E(f, V_g)$ to contain an interval $(-\delta, \delta)$. This follows from the first part of theorem 3·4 and the uniform continuity of all u.a.p. functions.† Also

$$E_k = E(f, V_{2^k g}) \quad (k = 1, 2, \ldots)$$

is a sequence in $E(f, V_g)$ of the type required in the definition of additivity.

The product of two generalized functions is, as a rule, undefined. Nevertheless, as has been pointed out in (4), a meaning can be attached to the product $fv_\lambda$ ($\lambda$ real) when $f$ is a generalized function and $v_\lambda$ is the function given by

$$v_\lambda(x) = e^{-i\lambda x} \, (x \in \mathbb{C}). \tag{8}$$

This product will subsequently be of some importance. If $f \in C_0 (= T)$, then $fv_\lambda$ has its usual meaning and the product belongs to $T$ ($\subseteq C$) ((4), theorem 1). Also

$$\langle fv_\lambda \mid g \rangle = \langle f \mid v_\lambda g \rangle \tag{9}$$

for any $g \in T$, since both sides are equal to $\int_{-\infty}^{\infty} fv_\lambda g$. The definition can be completed inductively for all $f \in C = \bigcup_{r<\Omega} C_r$; moreover (9) holds whenever $f \in C$ and $g \in T$. For the details see ((4), p. 148).

THEOREM 4·3. *If the generalized function $f$ is w.a.p., then so is $fv_\lambda$, where $v_\lambda$ is defined by* (8).

*Proof.* Take any $g \in T$.

In the argument below it is best to sacrifice rigorous notation to clarity. We shall, for instance, write $\langle f(x) \mid g(x) \rangle$ for $\langle f \mid g \rangle$ or $\langle f(x) \mid g(x+y) \rangle$ for $\langle f \mid g[y] \rangle$.

As $g(x)\, e^{-i\lambda x} \in T$,

$$\langle f(x) \mid g(x+y)\, e^{-i\lambda(x+y)} \rangle$$

is a u.a.p. function of $y$. It follows that

$$\langle f(x)\, e^{-i\lambda x} \mid g(x+y) \rangle = \langle f(x) \mid g(x+y)\, e^{-i\lambda(x+y)} \rangle\, e^{i\lambda y}$$

is u.a.p. (in $y$). Since this is true for all $g \in T$, $f(x)\, e^{-i\lambda x}$ is w.a.p.

The last identity also shows that, if the generalized function $f$ is weakly continuous (without necessarily being w.a.p.), then $fv_\lambda$ is weakly continuous as well.

† All the classical a.p. functions are uniformly continuous in some sense. Weak almost periodicity shares this feature, since the uniform continuity, for all $g \in T$, of the functions $\phi$ defined by

$$\phi(y) = \langle f \mid g[y] \rangle \quad (y \in \mathbb{R})$$

is equivalent to the property that

$$\langle f[x_1] - f[x_2] \mid g \rangle \to 0 \quad \text{as} \quad x_1 - x_2 \to 0, \quad \text{uniformly for} \quad x_1, x_2 \in \mathbb{R},$$

which may naturally be regarded as uniform weak continuity.

https://doi.org/10.1017/S0305004100060977 Published online by Cambridge University Press

### 5. *Means*

Let $f$ be a complex-valued function on $\mathbb{R}$. If

$$\lim_{X\to\infty}\frac{1}{2X}\int_{-X}^{X} f$$

exists, then this limit, the mean of $f$, is denoted by $M(f)$ or by $M_x(f(x))$. When $f$ is u.a.p., then

$$c(\lambda) = M_x(f(x)\,e^{-i\lambda x})$$

exists for every real $\lambda$ and is non-zero only when $\lambda$ belongs to a certain countable subset $\Lambda$ of $\mathbb{R}$, called the spectrum of $f$. If $\Lambda = \{\lambda_1, \lambda_2, \ldots\}$ and $c_n = c(\lambda_n)$ $(n = 1, 2, \ldots)$, then the series

$$\sum_n c_n e^{i\lambda_n x}$$

is said to be the Fourier series of $f$.

In the case of w.a.p. functions, the first step towards the concept of a Fourier series must be a definition of the mean of a generalized function.

DEFINITION 5·1. *Let $g$ be a test function with $\int_{-\infty}^{\infty} g \neq 0$. If $f$ is a generalized function and $M_y(\langle f \mid g[y]\rangle)$ exists, then*

$$\frac{M_y(\langle f \mid g[y]\rangle)}{\int_{-\infty}^{\infty} g}$$

*is called the* $g$-mean *of $f$ and is denoted by $M^g(f)$. (Note that $g[y]$ can be replaced by $g[-y]$, since $M_y(\phi(y)) = M_y(\phi(-y))$.)*

When $f$ is w.a.p., $M^g(f)$ certainly exists, since the function $\phi$ given by $\phi(y) = \langle f \mid g[y]\rangle$ $(y \in \mathbb{R})$ is u.a.p. The next result confirms that the $g$-mean is a suitable object for use in the context of weak almost periodicity.

THEOREM 5·2. *Let the function $f^*$ on $\mathbb{R}$ be u.a.p. (and therefore ordinary), and let $f$ be the generalized (w.a.p.) function corresponding to $f^*$. Also let $g$ be any test function such that $\int_{-\infty}^{\infty} g \neq 0$. Then $M^g(f)$ exists and is equal to $M(f^*)$.*

*Proof.* We first take

$$f^*(x) = e^{i\lambda x}\,(x \in \mathbb{R}),$$

where $\lambda$ is real, so that

$$M(f^*) = \begin{cases} 1 & (\lambda = 0), \\ 0 & (\lambda \neq 0). \end{cases} \tag{10}$$

Then

$$\langle f \mid g[y]\rangle = \int_{-\infty}^{\infty} e^{i\lambda x} g(x+y)\,dx = e^{-i\lambda y} G(-\lambda/2\pi)$$

and so, by (10), $M^g(f) = G(0)\Big/\int_{-\infty}^{\infty} g = 1$ when $\lambda = 0$ and $M^g(f) = 0$ when $\lambda \neq 0$. Thus the required result holds when $f^*(x) = e^{i\lambda x}$. It follows at once that the result also holds when $f^*$ is a trigonometric polynomial.

https://doi.org/10.1017/S0305004100060977 Published online by Cambridge University Press

If $f^*$ is an arbitrary u.a.p. function, there is a sequence $(f_n^*)$ of trigonometric polynomials which converges uniformly to $f$ ((1), p. 29). Evidently

$$|M(f_n^*) - M(f^*)| \leqslant \sup_{-\infty<x<\infty} |f_n^*(x) - f^*(x)|$$

and so

$$M(f_n^*) \to M(f^*) \quad \text{as} \quad n \to \infty. \tag{11}$$

Also, denoting the generalized function corresponding to $f_n^*$ by $f_n$, we have

$$\begin{aligned} |M_y(\langle f_n \,|\, g[y]\rangle) - M_y(\langle f \,|\, g[y]\rangle)| &= \left| M_y\left( \int_{-\infty}^{\infty} (f_n^*(x) - f^*(x))\, g(x+y)\, dx \right) \right| \\ &\leqslant \sup_{-\infty<y<\infty} \int_{-\infty}^{\infty} |f_n^*(x) - f^*(x)|\, |g(x+y)|\, dx \\ &\leqslant \sup_{-\infty<x<\infty} |f_n^*(x) - f^*(x)| \int_{-\infty}^{\infty} |g(t)|\, dt \end{aligned}$$

and the last term tends to 0 as $n \to \infty$. Hence, by (11),

$$M_y(\langle f \,|\, g[y]\rangle) = \lim_{n\to\infty} M_y(\langle f_n \,|\, g[y]\rangle) = \lim_{n\to\infty} M(f_n^*) \int_{-\infty}^{\infty} g = M(f^*) \int_{-\infty}^{\infty} g.$$

This proves the theorem.

*Note.* It can be shown that $M^g(f)$ exists and is equal to $M(f^*)$ whenever $f^*$ is an ordinary function such that not only $M(f^*)$ exists, but also

$$\frac{1}{X} \int_0^X f^* \to M(f^*) \quad \text{as} \quad X \to \pm\infty$$

(which is a property possessed by u.a.p. functions). However, the existence of $M(f^*)$ is not, by itself, sufficient to ensure the existence of $M^g(f)$ or the equality of the two means if $M^g(f)$ does exist. For instance, take

$$g(x) = e^{-\pi(x-1)^2},$$

so that

$$\int_{-\infty}^{\infty} g(x)\, dx = 1.$$

If

$$f_1^*(x) = x, \quad f_2^*(x) = x^3,$$

then

$$M(f_1^*) = M(f_2^*) = 0.$$

On the other hand,

$$\langle f_1 \,|\, g[y]\rangle = \int_{-\infty}^{\infty} x\, e^{-\pi(x+y-1)^2}\, dx = \int_{-\infty}^{\infty} (t+1-y)\, e^{-\pi t^2}\, dt = 1 - y$$

and therefore

$$M^g(f_1) = 1,$$

while a similar argument shows that $M^g(f_2)$ does not exist.

### 6. *Weak differentiability*

The mean of definition 5·1 becomes much easier to handle when we restrict ourselves to reasonably smooth w.a.p. functions. The requisite additional condition is discussed in this section.

https://doi.org/10.1017/S0305004100060977 Published online by Cambridge University Press

DEFINITION 6·1. *The generalized function $f$ is said to be* weakly differentiable *if*

$$\frac{f[\delta]-f}{\delta} \to f' \quad \textit{weakly as} \quad \delta \to 0,$$

*i.e. if, for every $g \in T$,*

$$\left\langle \frac{f[\delta]-f}{\delta} \middle| g \right\rangle \to \langle f' \mid g \rangle \quad \textit{as} \quad \delta \to 0,$$

*or, equivalently,*

$$\left\langle f \middle| \frac{g[\delta]-g}{\delta} \right\rangle \to \langle f \mid g' \rangle \quad \textit{as} \quad \delta \to 0.$$

THEOREM 6·2. *Every generalized function corresponding to an ordinary function is weakly differentiable.*

*Proof.* Let $f$ be the generalized function corresponding to the ordinary function $f^*$ and let $g$ be any test function. Then

$$\begin{aligned}\langle f \mid g[\delta]-g \rangle/\delta &= \int_{-\infty}^{\infty} f^*(x)\,(g(x+\delta)-g(x))/\delta\,dx \\ &= \int_{-\infty}^{\infty} f^*(x)\,g'(x+\theta\delta)\,dx,\end{aligned}$$

where $\theta = \theta(x,\delta) \in (0,1)$. The integrand tends to $f^*(x)\,g'(x)$ and there are constants $K, c > 0$ such that

$$|g'(x+\theta\delta)| < K e^{-cx^2}$$

for all $x \in \mathbb{R}$ and $\delta \in (-1,1)$, say. Since

$$\int_{-\infty}^{\infty} |f^*(x)|\, e^{-cx^2}\,dx < \infty,$$

the theorem on dominated convergence ensures that

$$\langle f \mid g[\delta]-g \rangle/\delta \to \int_{-\infty}^{\infty} f^* g' = \langle f \mid g' \rangle.$$

Having shown that there is a plentiful supply of weakly differentiable generalized functions, we now turn to the general theory.

THEOREM 6·3. *Let the generalized function $f$ be weakly differentiable. Then* (i) *all derivatives of $f$ are weakly differentiable; and* (ii) *if $g \in T$ and the function $\phi$ on $\mathbb{R}$ is given by*

$$\phi(y) = \langle f \mid g[y] \rangle \quad (y \in \mathbb{R}),$$

*then $\phi$ has derivatives of all orders and, for $n = 1, 2, \ldots$,*

$$\phi^{(n)}(y) = \langle f \mid g^{(n)}[y] \rangle. \tag{12}$$

*Proof.* (i) It is only necessary to show that $f'$ is weakly differentiable.
Let $g \in T$. Then $g' \in T$ and

$$\langle f'[\delta]-f' \mid g \rangle/\delta = -\langle f[\delta]-f \mid g' \rangle/\delta \to -\langle f' \mid g' \rangle = \langle f'' \mid g \rangle$$

as $\delta \to 0$. Thus $f'$ is weakly differentiable. (ii) If $g \in T$, then, for every $y$, $g[y] \in T$ and so

$$(\phi(y+\delta)-\phi(y))/\delta = \langle f \mid g[y+\delta]-g[y] \rangle/\delta \to \langle f \mid g'[y] \rangle.$$

https://doi.org/10.1017/S0305004100060977 Published online by Cambridge University Press

Thus $\phi$ is differentiable at $y$ and

$$\phi'(y) = \langle f \mid g'[y] \rangle.$$

The general result is obtained by induction.

An immediate consequence of the second part of the theorem is that weak differentiability implies weak continuity.

The next result may be compared with theorem 1·1.

THEOREM 6·4. *If the sequence $(f_n)$ of weakly differentiable generalized functions converges uniformly weakly to the generalized function $f$, then $f$ is weakly differentiable.*

*Proof.* Let $g \in T$ and define the functions $\phi_n (n = 1, 2, \ldots)$, $\phi$ by

$$\phi_n(y) = \langle f_n \mid g[y] \rangle, \quad \phi(y) = \langle f \mid g[y] \rangle \quad (y \in \mathbb{R}).$$

Then, as $n \to \infty$,

$$\phi_n(y) \to \phi(y)$$

(in fact uniformly); and, by (12),

$$\phi_n'(y) = \langle f_n \mid g'[y] \rangle \to \langle f \mid g'[y] \rangle$$

uniformly in $y$. Therefore $\phi$ is differentiable and

$$\phi'(y) = \langle f \mid g'[y] \rangle$$

for all $y$. The particular case $y = 0$ shows that $f$ is weakly differentiable.

It is obvious that the sum of two weakly differentiable generalized functions is weakly differentiable. There is also the corresponding result for certain products.

THEOREM 6·5. *If the generalized function $f$ is weakly differentiable, then so is $fv_\lambda$, where $v_\lambda$ is the function defined by* (8).

*Proof.* Let $g$ be an arbitrary test function. We wish to show that

$$\langle fv_\lambda \mid g[\delta] - g \rangle / \delta \to \langle fv_\lambda \mid g' \rangle \quad \text{as} \quad \delta \to 0. \tag{13}$$

Since $v_\lambda[\delta] = e^{-i\lambda\delta} v_\lambda$, the left hand side of (13) is

$$\begin{aligned} \langle f \mid v_\lambda g[\delta] - v_\lambda g \rangle / \delta &= \langle f \mid v_\lambda[\delta] g[\delta] e^{i\lambda\delta} - v_\lambda g \rangle / \delta \\ &= e^{i\lambda\delta} \langle f \mid (v_\lambda g)[\delta] - v_\lambda g \rangle / \delta + \langle f \mid v_\lambda g \rangle (e^{i\lambda\delta} - 1)/\delta. \end{aligned}$$

But $v_\lambda g$ is a test function and therefore, as $\delta \to 0$, the last line tends to

$$\langle f \mid (v_\lambda g)' \rangle + i\lambda \langle f \mid v_\lambda g \rangle = \langle f \mid v_\lambda g' \rangle = \langle fv_\lambda \mid g' \rangle.$$

DEFINITION 6·6. *A generalized function which is weakly differentiable and weakly almost periodic is called a* weakly differentiable almost periodic (w.d.a.p.) *function.*

*Notes.* (i) Every generalized function corresponding to a u.a.p. function is both w.a.p. (theorem 2·3) and weakly differentiable (theorem 6·2) and is therefore w.d.a.p. (ii) In view of theorems 4·2 and 6·4, the set of w.d.a.p. functions is sequentially closed in $(C, \mathscr{T})$. The results of section 4 and of this section also prove that the set of w.d.a.p. functions is closed under the operations of addition, multiplication by $v_\lambda$, and differentiation.

The next theorem shows that a w.d.a.p. function has a property which might be called *uniform* weak differentiability.

https://doi.org/10.1017/S0305004100060977 Published online by Cambridge University Press

THEOREM 6·7. *If the generalized function $f$ is w.d.a.p., then*

$$\frac{f[\delta]-f}{\delta}\to f'$$

*uniformly weakly.*

*Proof.* Let $g\in T$ and define the u.a.p. function $\phi$ by

$$\phi(y)=\langle f\mid g[y]\rangle \quad (y\in\mathbb{R}).$$

By theorem 6·3 (ii), $\phi$ is differentiable and so, given the real numbers $y$, $\delta$, there exists $\theta\in(0,1)$ such that

$$(\phi(y+\delta)-\phi(y))/\delta=\phi'(y+\theta\delta).$$

Now $\phi'=\langle f\mid g'[y]\rangle$ (see (12)) and, since $g'\in T$, this means that $\phi'$ is u.a.p. and therefore uniformly continuous. It follows that, as $\delta\to 0$,

$$(\phi(y+\delta)-\phi(y))/\delta\to\phi'(y)$$

uniformly for $y\in\mathbb{R}$. Thus

$$\langle f[\delta]-f\mid g[y]\rangle/\delta\to-\phi'(y)=-\langle f\mid g'[y]\rangle=\langle f'\mid g[y]\rangle$$

uniformly in $y$, i.e.

$$(f[\delta]-f)/\delta\to f'$$

uniformly weakly.

We have seen (theorem 5·2) that the $g$-mean of a w.a.p. function which corresponds to an (ordinary) u.a.p. function has the same value for all $g\in T$ (with $\int_{-\infty}^{\infty}g\neq 0$). The reason for the introduction of w.d.a.p. functions is that for these also the mean is independent of $g$.

THEOREM 6·8. *If the generalized function $f$ is w.d.a.p., then $M^g(f)$ has the same value for all test functions $g$ such that $\int_{-\infty}^{\infty}g\neq 0$.*

*Proof.* Let $g,h$ be test functions such that $\int_{-\infty}^{\infty}g\neq 0$ and $\int_{-\infty}^{\infty}h\neq 0$. We may clearly suppose that

$$\int_{-\infty}^{\infty}g=\int_{-\infty}^{\infty}h=1.$$

Then the function $q$ on $\mathbb{C}$ defined by

$$q(x+iy)=\int_{-\infty+iy}^{x+iy}(g-h)\quad (x,y\in\mathbb{R})$$

belongs to $T$ and is such that $q'=g-h$.

The function $\phi$ on $\mathbb{R}$ given by

$$\phi(y)=\langle f\mid q[y]\rangle$$

is u.a.p. and, by theorem 6·3 (ii), it is differentiable with

$$\phi'(y)=\langle f\mid q'[y]\rangle.$$

Since $\phi$ is bounded,

$$M_y(\langle f\mid q'[y]\rangle)=M(\phi')=0.$$

Thus

$$M_y(\langle f\mid g[y]\rangle)=M_y(\langle f\mid h[y]\rangle),$$

i.e.

$$M^g(f)=M^h(f).$$

The theorem allows us to refer unambiguously to *the* mean $M(f)$ of a w.d.a.p. function $f$.



https://doi.org/10.1017/S0305004100060977 Published online by Cambridge University Press

### 7. *Fourier series of w.d.a.p. functions*

In this section the function $v_\lambda$ defined by

$$v_\lambda(x) = e^{-i\lambda x} \quad (x \in \mathbb{C})$$

will repeatedly occur.

THEOREM 7·1. *Let the generalized function $f$ be w.d.a.p. and let $k$ be a positive real number. If $u_k$ is the test function defined by*

$$u_k(x) = e^{-\pi k^2 x^2} \quad (x \in \mathbb{C})$$

*and $\psi_k$ is the u.a.p. function given by*

$$\psi_k(y) = \langle f \mid u_k[-y] \rangle \quad (y \in \mathbb{R}),$$

*then, for every real number $\lambda$,*

$$M(fv_\lambda) = k e^{\lambda^2/4\pi k^2} M(\psi_k v_\lambda).$$

*Proof.* Define the test function $u_{k,\lambda}$ by

$$u_{k,\lambda}(x) = e^{-\pi k^2 x^2 + i\lambda x} \quad (x \in \mathbb{C}).$$

We then have

$$\begin{aligned} M(fv_\lambda) &= M^{u_{k,\lambda}}(fv_\lambda) \\ &= M_y(\langle f(x)\, e^{-i\lambda x} \mid e^{-\pi k^2(x-y)^2 + i\lambda(x-y)} \rangle) \Big/ \int_{-\infty}^{\infty} e^{-\pi k^2 x^2 + i\lambda x}\, dx \\ &= k e^{\lambda^2/4\pi k^2} M_y(e^{-i\lambda y} \langle f(x) \mid e^{-\pi k^2(x-y)^2} \rangle) \\ &= k e^{\lambda^2/4\pi k^2} M(\psi_k v_\lambda). \end{aligned}$$

COROLLARY. *If $f$ is w.d.a.p., then $M(fv_\lambda) = 0$ except when $\lambda$ belongs to a countable set.*

*Proof.* In the theorem take a particular value of $k$, say 1, and put $u_1 = u$, $\psi_1 = \psi$, so that

$$u(x) = e^{-\pi x^2} \quad (x \in \mathbb{C}),$$

$$\psi(y) = \langle f \mid u[-y] \rangle \quad (y \in \mathbb{R}), \tag{14}$$

and

$$M(fv_\lambda) = e^{\lambda^2/4\pi} M(\psi v_\lambda). \tag{15}$$

Thus $M(fv_\lambda) = 0$ if and only if $M(\psi v_\lambda) = 0$ and, since $\psi$ is u.a.p., $M(\psi v_\lambda) = 0$ except when $\lambda$ belongs to a countable set.

DEFINITION 7·2. *If the generalized function $f$ is w.d.a.p., denote by $\Lambda = \{\lambda_1, \lambda_2, \ldots\}$ the countable set of real numbers for which $M(fv_\lambda) \neq 0$ and put*

$$c_n = M(fv_{\lambda_n}) \quad (n = 1, 2, \ldots).$$

*Then $\Lambda$ is called the* spectrum *of $f$, the $\lambda_n$ are called the* Fourier exponents *and the $c_n$ the* Fourier coefficients *of $f$; also*

$$\sum_n c_n e^{i\lambda_n x}$$

*is called the* Fourier series *of $f$.*

*Notes.* (i) When $f^*$ is u.a.p. and $f$ is the corresponding w.d.a.p. function, then $M(f^* v_\lambda) = M(fv_\lambda)$, by theorem 5·2. Hence $f^*$ and $f$ have the same Fourier series. (ii) Whenever $f$ is w.a.p., the function $\psi$ defined by (14) is u.a.p. and therefore has a Fourier series

$$\sum_n \gamma_n e^{i\lambda_n x}.$$

https://doi.org/10.1017/S0305004100060977 Published online by Cambridge University Press

The identity (15), which presupposes the weak differentiability of $f$, nevertheless suggests that the Fourier series of an arbitrary w.a.p. function $f$ might be defined as

$$\sum_n c_n e^{i\lambda_n x},$$

where

$$c_n = e^{\lambda_n^2/4\pi}\gamma_n \quad (n = 1, 2, \ldots).$$

However, it appears to be difficult to derive any worth-while consequences from so formal a definition.

Theorem 7·1 not only enables us to define the Fourier series of a w.d.a.p. function, but also shows that the order of magnitude of the Fourier coefficients is restricted.

THEOREM 7·3. *If the w.d.a.p. function $f$ has Fourier series*

$$\sum_n c_n e^{i\lambda_n x},$$

*then, given $\epsilon > 0$, there exists a constant $K$ (depending on $\epsilon$) such that*

$$|c_n| < K e^{\epsilon\lambda_n^2}$$

*for all $n$.*

*Proof.* Take $k = 1/\sqrt{(4\pi\epsilon)}$ in theorem 7·1 and note that the means $M(\psi_k v_{\lambda_n})$, i.e. the Fourier coefficients of the u.a.p. function $\psi_k$, are bounded (since either the spectrum of $\psi_k$ is finite or else $M(\psi_k v_{\lambda_n}) \to 0$ as $n \to \infty$).

UNIQUENESS THEOREM 7·4. *Two w.d.a.p. functions with the same Fourier series are identical.*

*Proof.* It is sufficient to show that, if $f$ is w.d.a.p. and $M(fv_\lambda) = 0$ for all real $\lambda$, then $f$ is the zero element of $C$, i.e.

$$\langle f \mid g\rangle = 0 \quad \text{for all} \quad g \in T.$$

Take any test function $g$ and any real number $\lambda$. Put $g_\lambda = gv_{-\lambda}$, so that

$$g_\lambda(x) = g(x)\, e^{i\lambda x} \quad (x \in \mathbb{C})$$

and $g_\lambda \in T$.

If $\int_{-\infty}^{\infty} g_\lambda = 0$, then the function $q$ on $\mathbb{C}$ given by

$$q(x+iy) = \int_{-\infty+iy}^{x+iy} g_\lambda \quad (x, y \in \mathbb{R})$$

belongs to $T$ and $q' = g_\lambda$. Since $fv_\lambda$ is w.d.a.p., part of the argument used in proving theorem 6·8 shows that

$$M_y(\langle fv_\lambda \mid g_\lambda[-y]\rangle) = M_y(\langle fv_\lambda \mid q'[-y]\rangle) = 0.$$

If $\int_{-\infty}^{\infty} g_\lambda \neq 0$, then

$$M^{g_\lambda}(fv_\lambda) = M(fv_\lambda) = 0,$$

and so

$$M_y(\langle fv_\lambda \mid g_\lambda[-y]\rangle) = 0.$$

Hence, whatever the value of $\int_{-\infty}^{\infty} g_\lambda$,

$$M_y(\langle f(x)\, e^{-i\lambda x} \mid g(x-y)\, e^{i\lambda(x-y)}\rangle) = 0,$$

i.e.

$$M_y(e^{-i\lambda y}\langle f(x) \mid g(x-y)\rangle) = 0.$$



https://doi.org/10.1017/S0305004100060977 Published online by Cambridge University Press

This means that the u.a.p. function $\psi$ given by

$$\psi(y) = \langle f \mid g[-y] \rangle \quad (y \in \mathbb{R})$$

is such that $M_y(\psi(y)\, e^{-i\lambda y}) = 0$ for all real $\lambda$. Therefore, by the uniqueness theorem for u.a.p. functions ((1), p. 27), $\psi(y) = 0$ and, in particular,

$$\langle f \mid g \rangle = \psi(0) = 0.$$

THEOREM 7·5. *If the generalized function $f$ is w.d.a.p., then, for every real number $\lambda$,*

$$M(f' v_\lambda) = i\lambda M(f v_\lambda),$$

*so that the Fourier series of $f'$ is obtained from the Fourier series of $f$ by formal differentiation.*

*Proof.* If $\lambda \in \mathbb{R}$ and $g$ is a test function such that $\int_{-\infty}^{\infty} g \neq 0$,

$$\begin{aligned}\left(\int_{-\infty}^{\infty} g\right) M(f' v_\lambda) &= M_y(\langle f' v_\lambda \mid g[y] \rangle) = M_y(\langle f' \mid v_\lambda g[y] \rangle) \\ &= -M_y(\langle f \mid v_\lambda g'[y] - i\lambda v_\lambda g[y] \rangle) \\ &= -M_y(\langle f v_\lambda \mid g'[y] \rangle) + i\lambda M_y(\langle f v_\lambda \mid g[y] \rangle) \\ &= i\lambda \left(\int_{-\infty}^{\infty} g\right) M(f v_\lambda);\end{aligned}$$

for, as in the proof of theorem 6·8, $M_y(\langle f v_\lambda \mid g'[y] \rangle) = 0$.

THEOREM 7·6. *If $(f_n)$ is a sequence of w.d.a.p. functions converging uniformly weakly to the generalized function $f$ (which is therefore also w.d.a.p.), then, for each real $\lambda$,*

$$M(f_n v_\lambda) \to M(f v_\lambda) \quad \text{as} \quad n \to \infty,$$

*so that the Fourier series of the $f_n$ may be said to tend to the Fourier series of $f$.*

*Proof.* Let $\psi_n$ $(n = 1, 2, \ldots)$, $\psi$ be the u.a.p. functions defined by

$$\psi_n(y) = \langle f_n \mid u[-y] \rangle, \quad \psi(y) = \langle f \mid u[-y] \rangle \quad (y \in \mathbb{R}),$$

where

$$u(x) = e^{-\pi x^2} \quad (x \in \mathbb{C}).$$

Since $f_n \to f$ uniformly weakly, $\psi_n \to \psi$ uniformly on $\mathbb{R}$.

For each $\lambda$, we now have

$$\begin{aligned}|M(f_n v_\lambda) - M(f v_\lambda)| &= e^{\lambda^2/4\pi} |M(\psi_n v_\lambda) - M(\psi v_\lambda)| \\ &\leqslant e^{\lambda^2/4\pi} \sup_{-\infty < y < \infty} |\psi_n(y) - \psi(y)|\end{aligned}$$

and the last term tends to 0 as $n \to \infty$.

In the corollary below and the following theorem it is convenient to use the notation $e^{i\lambda x}$ not only for the ordinary function which this expression defines, but also for the corresponding generalized function.

COROLLARY. *If the series*

$$\sum_{n=1}^{\infty} c_n e^{i\lambda_n x}$$

*converges uniformly weakly to the generalized function $s$, then $s$ is w.d.a.p. and the series is the Fourier series of $s$.*

https://doi.org/10.1017/S0305004100060977 Published online by Cambridge University Press

*Proof.* Each partial sum $\sum_{r=1}^{n} c_r e^{i\lambda_r x}$ corresponds to a u.a.p. function and is therefore w.d.a.p. Since the set of w.d.a.p. functions is sequentially closed, it follows that $s$ is w.d.a.p. (See the notes after definition 6·6.)

By note (i) after definition 7·2, $\sum_{r=1}^{n} c_r e^{i\lambda_r x}$ is its own Fourier series. Hence, by the theorem, $\sum_{r=1}^{\infty} c_r e^{i\lambda_r x}$ is the Fourier series of $s$.

THEOREM 7·7. *If the w.d.a.p. function $f$ has Fourier series*

$$\sum_{n=1}^{\infty} c_n e^{i\lambda_n x}, \tag{16}$$

*where*

$$\lambda_n^2/\log n \to \infty \quad as \quad n \to \infty,$$

*then the series converges uniformly weakly to $f$.*

*Proof.* Let $g$ be any test function. By taking Fourier transforms we see that

$$\begin{aligned} |\langle c_n e^{i\lambda_n x} \,|\, g(x+y)\rangle| &= |\langle c_n \delta(x-\lambda_n/2\pi) \,|\, e^{-2\pi i x y} G(-x)\rangle| \\ &= |c_n e^{-i\lambda_n y} G(-\lambda_n/2\pi)| \\ &= |c_n G(-\lambda_n/2\pi)|. \end{aligned} \tag{17}$$

Now there is a real number $a > 0$ such that

$$|G(x)| = O(e^{-8a\pi^2 x^2}) \quad \text{as} \quad x \to \pm\infty$$

and, by theorem 7·3,

$$|c_n| = O(e^{a\lambda_n^2}) \quad \text{as} \quad n \to \infty.$$

Hence

$$|c_n G(-\lambda_n/2\pi)| = O(e^{-a\lambda_n^2}) \quad \text{as} \quad n \to \infty.$$

Also, for sufficiently large $n$, $\lambda_n^2 > (2/a)\log n$, so that

$$|c_n G(-\lambda_n/2\pi)| = O(1/n^2) \quad \text{as} \quad n \to \infty.$$

It follows that the sequence

$$\left\langle \sum_{r=1}^{n} c_r e^{i\lambda_r x} \,|\, g(x+y) \right\rangle$$

converges uniformly for $y \in \mathbb{R}$ and this means that the sequence of generalized functions

$$\sum_{r=1}^{n} c_r e^{i\lambda_r x}$$

converges uniformly weakly.

Now, by the corollary to theorem 7·6, the sum $s$ of the series (16) is w.d.a.p. and (16) is the Fourier series of $s$. Thus the two w.d.a.p. functions $s, f$ have the same Fourier series and so, by theorem 7·4, $s = f$.

Finally we touch on the connection between w.d.a.p. functions and Schwartz's almost periodic distributions. Every such distribution is the sum of derivatives (of some order) of finitely many distributions corresponding to ordinary u.a.p. functions. According to the comments after definition 6·6, analogously defined generalized functions are w.d.a.p. On the other hand, there are w.d.a.p. functions which are not of this special form. For instance the series

$$\sum_{n=1}^{\infty} 2^n e^{in^k x},$$

https://doi.org/10.1017/S0305004100060977 Published online by Cambridge University Press

where $k > 0$, converges uniformly weakly to a w.d.a.p. function $s$ of which it is the Fourier series, since (17) shows that, whenever $g \in T$, there exist $\alpha, \beta$ such that $\alpha \geqslant \beta > 0$ and

$$|\langle 2^n e^{in^k x} | g(x+y)\rangle| = |2^n G(-n^k/2\pi)| = O(2^n e^{-\alpha n^{2k}}) = O(e^{-\beta n^{2k}}).$$

However, $s$ is not a special w.d.a.p. function. To prove this it is only necessary to verify that a subseries

$$\sum_{j=1}^{\infty} 2^{n_j} e^{in_j^k x} \tag{18}$$

cannot converge uniformly weakly to a derivative of a w.d.a.p. function $f$ which corresponds to an ordinary u.a.p. function $f^*$. If the sum of (18) were such a derivative, then, by theorem 7·5, the Fourier series of $f$ and so also of $f^*$ would be

$$\sum_{j=1}^{\infty} 2^{n_j} (in_j^k)^{-r} e^{in_j^k x},$$

where $r$ is a non-negative integer. But no series of this kind can be the Fourier series of a u.a.p. function since its coefficients do not tend to 0.

## REFERENCES

(1) BESICOVITCH, A. S. *Almost periodic functions* (Cambridge University Press, 1932).
(2) DOHERTY, I. W. *Almost periodic generalized functions* (Ph.D. thesis, University of Melbourne, 1973).
(3) LIGHTHILL, M. J. *Fourier analysis and generalised functions* (Cambridge University Press, 1958).
(4) RENNIE, B. C. On generalized functions. *J. App. Prob.* **19***A* (1982), 139–156.
(5) SCHWARTZ, L. *Théorie des distributions*, vol. 2 (Hermann, Paris, 1951).

https://doi.org/10.1017/S0305004100060977 Published online by Cambridge University Press