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COMPUTABILITY, ORDERS, AND SOLVABLE GROUPS

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Abstract. The main objective of this paper is the following two results. (1) There exists a computable bi-orderable group that does not have a computable bi-ordering; (2) there exists a bi-orderable, two-generated computably presented solvable group with undecidable word problem. Both of the groups can be found among two-generated solvable groups of derived length 3.

(1) [a]nswers a question posed by Downey and Kurtz; (2) answers a question posed by Bludov and Glass in Kourovka Notebook.

One of the technical tools used to obtain the main results is a computational extension of an embedding theorem of B. Neumann that was studied by the author earlier. In this paper we also compliment that result and derive new corollaries that might be of independent interest.

§1. Main results. A presentation $G = \langle X \mid R \rangle$ of a countable group is called *recursive presentation* if the sets X and $R \subseteq (X \cup X^{-1})^*$ are computably enumerated. It is said that $G = \langle X \rangle$ is a *computable group* with respect to the computably enumerated generating set X if the set $\{u \in (X \cup X^{-1})^* \mid u =_G 1\}$ is computable. The concept of computable groups was introduced by Rabin [11] and Mal'cev [8].

Note that in literature the terms "recursive" and "computable" are being used interchangeably. In this paper we will stick to the notion "computable" instead of "recursive", except for the term "recursive presentation", which is a well-established terminology in combinatorial group theory.

Finitely generated groups that are computable with respect to a finite generating set are called groups with decidable word problem. A well-known property of groups with decidable word problem is that decidability of the word problem does not depend on the choice of finite generating set, hence it is an intrinsic property of the group. This is in contrast with the general case of countable groups when the property of being computable depends on the choice of the generating set.

To formulate the first main theorem, we introduce the following definition, which is a weaker form of left- and bi-orderings on groups.

Definition 1 (p-order). For a given group G, we say that a binary relation $\preceq$ on G is a *p-order* if

- it is antisymmetric, that is, $g \preceq h$ and $h \preceq g$ imply $g = h$;
- $1 \preceq g$ implies $g^{-1} \preceq 1$;
- $1 \preceq g$ implies $1 \preceq g^n$ for any $n \in \mathbb{N}$.

If a group G possesses a p-order relation, then we say that G is *p-orderable*.

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Let us recall that G is a *left-orderable* group if there exists a linear order on G that is invariant under the left multiplication. If, in addition, the order on G is also invariant under the right multiplication, then the group G is called *bi-oderable*. Note that left- and bi-orders on groups are p-order relations.

DEFINITION 2. The linear order $\preceq$ on $G = \langle X \rangle$ is said to be computable with respect to the generating set $X = \{x_1, x_2, \ldots\}$ if the set $\{(u,v) \in (X \cup X^{-1})^* \times (X \cup X^{-1})^* \mid u \preceq v\}$ is computable.

Our first main result is the following.

THEOREM 1. *There exists a two-generated bi-orderable computable group that does not embed in any countable group with a computable p-order relation. Moreover, it can be chosen to be a solvable group of derived length* 3.

The question of Downey and Kurtz asked in [4] as Question 6.12 (ii) is as follows.

> Is every computable orderable group isomorphic to computably orderable group?

For the case of left-orderable groups a negative answer was recently obtained by Harrison-Trainor in [5]. However, as the author mentioned in the abstract of [5], the more general case of bi-orderable groups is left as open.

From Theorem 1 we immediately get the answer to the question of Downey and Kurtz for the general case of bi-orderable groups.

COROLLARY 1. *There exists a computable bi-orderable group G, which does not have a presentation with computable bi-order. G can be chosen to be two-generated solvable group of derived length* 3.

In [1], Bludov and Glass showed that every left-orderable computable group embeds into a finitely presentable left-orderable group with decidable word problem. The combination of these two results with Theorem 1 immediately leads to yet another strengthening of the result of Harrison-Trainor as follows.

COROLLARY 2. *There exists a finitely presentable left-orderable group with decidable word problem and without computable left-order.*

Addressing the question of Downey and Kurtz, Solomon showed in [13] that for the class of abelian groups (i.e., for solvable groups of derived length 1) the answer to the question is positive, that is, every bi-orderable computable abelian group possesses a presentation with computable bi-order. This result in combination with Corollary 1 leads to the following question.

QUESTION 1. *Is it true that every computable metabelian bi-orderable group possesses a presentation with computable bi-order*?

In Kourovka notebook [7, Question 17.28], Bludov and Glass asked the following question.

> Is there a solvable left-orderable group with undecidable word problem?

The next theorem answers this question.

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THEOREM 2. *There exists a two-generated computably presented bi-orderable solvable group of derived length* 3 *with undecidable word problem*.

In a personal communication, A.M.W. Glass mentioned to me the following strengthening of the above mentioned question.

QUESTION 2 (Bludov and Glass). *Does there exist a finitely presented bi-orderable solvable group with undecidable word problem*?

Note that answering an old question of Adian, the first example of finitely presented solvable group with undecidable word problem was constructed by Kharlampovich in [6]. Unfortunately, the group from [6] contains torsions, hence it is not bi-orderable.

Proofs of Theorems 1 and 2 are given in Sections 2 and 3, respectively.

1.1. Computability and group embeddings. The main technical tool of our paper is the following embedding theorem, which, in a weaker form and using a slightly different language, can be found in [2], and is based on a modified version of B. Neumann's construction from [9]. We will prove this theorem in Section 4.

THEOREM 3. *Let $H = \langle X \rangle$ be a group with countable generating set $X = \{x_1, x_2, ...\}$. Then there exists an embedding $\Phi_X : H \hookrightarrow G$ into a two-generated group $G = \langle f, s \rangle$ such that the following are true.*

(a) *There exists a computable map $\phi_X : \mathbb{N} \to \{f^{\pm 1}, s^{\pm 1}\}^*$ such that, for all $i \in \mathbb{N}$, $\phi_X(i)$ represents the element $\Phi_X(x_i)$ in G;*
(b) *If H is a solvable group of derived length l, then G is a solvable group of derived length $l + 2$;*
(c) *G has a recursive presentation if and only if H has a recursive presentation with respect to the generating set X;*
(d) *G has decidable word problem if and only if H is computable with respect to the generating set X;*
(e) *If H is a computable group with respect to the generating set X, then the membership problem for the subgroup $\Phi_X(H) \leq G$ is decidable, that is, there exists an algorithm that for any $g \in G$ decides whether or not $g \in \Phi_X(H)$;*
(f) *If H is left- or bi-orderable, then so is G and G extends the order on H. Moreover, if with respect to the generating set X there is a computable order on H, then G has a computable order as well that extends the order on H;*
(g) *There exist $N_1, N_2 \leq G$ such that $\Phi_X(H) \lhd N_1 \lhd N_2 \lhd G$, hence $\Phi_X(H)$ is a subnormal subgroup of G.*

REMARK 1. We would like to mention that the computational properties of the embedding Φ_X essentially depend on the choice of the generating set X. In fact, in applications in Theorems 1 and 2, Theorem 3 is applied on specific presentations of groups that are (group theoretically) isomorphic to $\bigoplus_{i=1}^{\infty} \mathbb{Z}_i$, where $\mathbb{Z}_i$ are copies of $\mathbb{Z}$.

Theorem 3 leads to the characterization of computable groups and groups with computable left- or bi-orders in terms of subgroups of finitely generated groups with decidable word problem and computable order, respectively. See the corollaries below.

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COROLLARY 3. *A countable group H is computable if and only if it is a (subnormal) subgroup of a finitely generated group with decidable word problem such that the membership problem for the subgroup H is decidable in the large group.*

PROOF. Let us assume that $G = \langle X \rangle$, $|X| < \infty$, is a finitely generated group with decidable word problem, and $H < G$ such that the membership problem for $H < G$ is decidable. Then, each element of H can be encoded by the lexicographically smallest word from $(X \cup X^{-1})^*$ representing that element. From the assumptions that the word problem for G is decidable and the membership problem for $H < G$ is decidable follows that the lexicographically smallest representatives from $(X \cup X^{-1})^*$ can be found computationally. These representatives give rise to set of generators of H with respect to which H is a computable group.

The inverse statement of the corollary follows from Theorem 3, parts (d) and (e). ⊣

COROLLARY 4. *A countable group H has a computable left- or bi-ordering if and only if it is a (subnormal) subgroup of a finitely generated group with computable left- or bi-order, respectively, such that the membership problem for the subgroup H is decidable in the large group. Moreover, for any fixed computable order on H we can assume that the large group extends the order on H.*

PROOF. Let us assume that $G = \langle X \rangle$, $|X| < \infty$, is a finitely generated group with decidable word problem and a computable left- or bi-order '$\prec$', and $H < G$ such that the membership problem for $H < G$ is decidable. Then, as it is mentioned in the proof of Corollary 4, the lexicographically smallest representatives from $(X \cup X^{-1})^*$ induce a set of generators from H with respect to which H is computable. Moreover, the induced from '$\prec$' order on H also will be computable left- (resp. bi-) order on H with respect to that set of generators.

The inverse statement of the corollary follows from Theorem 3, parts (e) and (f). ⊣

REMARK 2. For a group H both the properties of being computable and having computable left- or bi-orders depends on the presentation of the group. One of the noteworthy aspects of Corollaries 3 and 4 is that they reformulate these computability properties for H in terms that do not refer to the presentation of the group.

§2. Proof of Theorem 1.

LEMMA 2.1. *Let G be a finitely generated group with decidable word problem. Then G has a presentation with computable (p-, left-, or bi-) order if and only if that order is computable with respect to any finite generating set.*

PROOF. Let us assume that $G = \langle X \rangle$, $X = \{x_1^{\pm 1}, x_2^{\pm 1}, \ldots\}$, such that for a given (p-, left-, or bi-) order $\preceq_G$ on G, the set $\{(u,v) \subseteq X^* \times X^* \mid u \preceq_G v\}$ is computable. Then, by Corollary 4, there exists a two-generated group $H = \langle a,b \rangle$ such that $G \leq H$, H has order $\preceq_H$ that extends the order $\preceq_G$ and is computable, and the membership problem for $G \leq H$ is decidable.

Now let us assume that G is also generated by a finite set $S = \{s_1^{\pm 1}, \ldots, s_n^{\pm 1}\}$. Let $u_1, u_2, \ldots, u_n \in \{a^{\pm 1}, b^{\pm 1}\}^*$ be such that they represent the elements $s_1, \ldots, s_n$ in G. Let

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$\phi : S^* \to X^*$ be the map induces by $s_1 \mapsto u_1, ..., s_n \mapsto u_n$. Then, for any $w_1, w_2 \in S^*$, $w_1 \preceq_G w_2$ is equivalent to $\phi(w_1) \preceq_H \phi(w_2)$. Therefore, since $\preceq_H$ is a computable order, we get that $\{(u,v) \subseteq S^* \times S^* \mid u \preceq_G v\}$ is computable as well. Since S is an arbitrarily chosen generating set, one direction of the lemma is proved. The other direction is trivial. ⊣

LEMMA 2.2. *If a finitely generated group has a computable p-order, then the induced p-order on any finitely generated subgroup is also computable*

PROOF. Indeed, let $G = \langle S \rangle$ be a finitely generated group with a computable p-order relation $\preceq_G$, that is, the set $\{u \in (S \cup S^{-1})^* \mid 1 \preceq_G u\}$ is computable. Let $H = \langle X \rangle \leq G$, $|X| < \infty$, be a finitely generated subgroup of G with the induced p-order relation $\preceq_H$ (i.e., for $h_1, h_2 \in H$, $h_1 \preceq_H h_2$ if and only if $h_1 \preceq_G h_2$.) Then $\preceq_H$ is a computable relation with respect to the generating set X, because every word $w \in (X \cup X^{-1})^*$ can be computably rewritten as a word $v \in (S \cup S^{-1})^*$ such that w and v represent the same element of H. The last observation means that computability of $\{u \in (S \cup S^{-1})^* \mid 1 \preceq_G u\}$ implies computability of $\{u \in (X \cup X^{-1})^* \mid 1 \preceq_H u\}$. ⊣

Let us fix a computably enumerable and computably inseparable pair $(\mathcal{M}, \mathcal{N})$ of subsets of $\mathbb{N}$. That is $\mathcal{M}$ and $\mathcal{N}$ are computably enumerable and there is no computable subset of $\mathbb{N}$ that contains $\mathcal{M}$ and is disjoint from $\mathcal{N}$. (For the existence of such a pair see [12, 13].) Let $\mathcal{M} = \{m_1, m_2, ...\}$ and $\mathcal{N} = \{n_1, n_2, ...\}$, such that their enumerations are computable.

Let $\mathcal{P} = \{p_1, p_2, ...\}$ be the set of primes listed in its natural order.

Now, let us consider the abelian group $A \simeq \bigoplus_{i=1}^{\infty} \mathbb{Z}_i$ given by the presentation

$$A = \langle a_i, i = 1, 2, ... \mid a_i a_j a_i^{-1} a_j^{-1} = 1, a_{2n_i} = a_{2n_i-1}^{p_i}, a_{2m_i} = a_{2m_i-1}^{-p_i}, i, j \in \mathbb{N} \rangle. \quad (1)$$

LEMMA 2.3. *The group A is computable with respect to the enumerated generating set $\{a_1, a_2, ...\}$.*

PROOF. Indeed, any element of the abelian group A can be presented as a finite length word in the alphabet $\{a_1^{\pm 1}, a_2^{\pm 1}, ...\}$. On its own turn, any such word can be computationally transformed into another, reduced, word that represents the same element and is of the form

$$a_{i_1}^{k_1} a_{i_2}^{k_2} ... a_{i_n}^{k_n}, \quad (2)$$

where $k_1, ..., k_n \in \mathbb{Z}$ and $i_1 > i_2 > \cdots > i_n$. If this word is not empty and it represents the trivial element, then it must contain a subword of one of the following forms: $a_{2n_i} a_{2n_i-1}^{-p_i}$, $a_{2n_i}^{-1} a_{2n_i-1}^{p_i}$, $a_{2m_i} a_{2m_i-1}^{p_i}$ or $a_{2m_i}^{-1} a_{2m_i-1}^{-p_i}$. On the other hand, since the set of primes is computable and $\mathcal{M}, \mathcal{N}$ are computably enumerated, we get that the set of words

$$\mathcal{W} = \{a_{2n_i} a_{2n_i-1}^{-p_i},\ a_{2n_i}^{-1} a_{2n_i-1}^{p_i},\ a_{2m_i} a_{2m_i-1}^{p_i},\ a_{2m_i}^{-1} a_{2m_i-1}^{-p_i} \mid i \in \mathbb{N}\} \subset \{a_1^{\pm 1}, a_2^{\pm 1}, ...\}^*$$

is computable. Therefore, for a word w of the form (2), one can algorithmically find all subwords of w that are from $\mathcal{W}$.

For example, given a word w of the form (2), in order to detect a subword in w of the form $a_{2n_i} a_{2n_i-1}^{-p_i}$, $i \in \mathbb{N}$, one can first find all subwords of the form $a_{2n} a_{2n-1}^{-p}$ for p being a prime number, then assuming that p is the ith prime number, one can

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check whether or not $n = n_i$. Since the enumeration $\{n_1, n_2, \ldots\}$ is computable, the described procedure is computable as well.

As a conclusion, in order to check whether or not a word w of the form (2) represents the trivial element of the group A, we can iteratively find subwords from $\mathcal{W}$ in w and remove them until this procedure cannot be continued further. If the procedure ends up with the empty word, then w represents the trivial element of A, otherwise, it does not. Since the described procedure is computable, we get that the group A is computable with respect to the enumerated generating set $\{a_1, a_2, \ldots\}$. $\dashv$

Now, let G be the group obtained after embedding the group A into a two-generated group according to the embedding from Theorem 3 (we consider the embedding with respect to the presentation (1)). Let the embedding be $\Phi : A \hookrightarrow G$.

By the properties of the embedding of Theorem 3, $G = \langle x^{\pm 1}, y^{\pm 1} \rangle$ will have decidable word problem (part (d)), will be bi-orderable (part (f)), and will be solvable of derived length 3 (part (b)).

However, below we show that G does not possess any computable p-order with respect to any presentation. First, note that, by Lemma 2.1, if $\preceq_G$ is a computable preorder on G with respect to some presentation, then it is a computable p-order with respect to the generating set $\{x^{\pm 1}, y^{\pm 1}\}$, that is, the set

$$\mathcal{O} = \{(u, v) \in \{x^{\pm 1}, y^{\pm 1}\}^* \times \{x^{\pm 1}, y^{\pm 1}\}^* \mid u \preceq_G v\}$$

is computable. It follows from part (a) of Theorem 3 that there exist a computable map $\phi : \mathbb{N} \to \{x^{\pm 1}, y^{\pm 1}\}^*$ such that $\phi(i) = u_i \in \{x^{\pm 1}, y^{\pm 1}\}^*$ is a word representing the element $\Phi(a_i) \in G$. Therefore, the set

$$\mathcal{L} = \{n \in \mathbb{N} \mid [u_{2n} \preceq_G 1 \ \& \ u_{2n-1} \preceq_G 1] \mathrm{OR} [1 \preceq_G u_{2n} \ \& \ 1 \preceq_G u_{2n-1}]\}$$

is computable as well. However, since $\preceq_G$ is a p-order relation, we have that $\mathcal{N} \subseteq \mathcal{L}$ and $\mathcal{M} \cap \mathcal{L} = \emptyset$. We get to a contradiction with the assumption that the pair $(\mathcal{M}, \mathcal{N})$ is computably inseparable. Therefore, G does not possess a computable p-order with respect to any presentation.

Now, let us show that the group G cannot be embedded into a countable group with a presentation possessing a computable p-order. Indeed, towards a contradiction, let us assume that H is a countable group with a computable preorder $\preceq_H$ in which G is embedded. It follows from Theorem 3 and Lemma 2.1 that without loss of generality we can assume that H is finitely generated and the computability of $\preceq_H$ is with respect to (any) finite generating set. However, since G does not have any computable p-order, by Lemma 2.2 we immediately reach a contradiction.

REMARK 3. One of the key ingredients in the presented proof of Theorem 1 is the utilization of the computably inseparable and computably enumerable pair of sets in a group theoretical structure. We would like to mention that, in a similar manner, computably inseparable and computably enumerable pairs where applied in [3] to produce groups with decidable word problem that are not subgroups of groups with decidable conjugacy problem. One of the earliest applications of this concept can be found in [12].

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§3. Proof of Theorem 2. Let us fix a computably enumerated noncomputable set $\mathcal{N} = \{n_1, n_2, \ldots\}$. Let us consider the following presentation of the group $\bigoplus_{i=1}^{\infty} \mathbb{Z}_i$:

$$A = \langle a_i, i = 1,2, \ldots \mid a_i a_j a_i^{-1} a_j^{-1} = 1, a_{2n_i} = a_{2n_i - 1}, i, j \in \mathbb{N} \rangle. \tag{3}$$

Since $\mathcal{N}$ is computably enumerable, the presentation (3) is recursive. However, since $a_{2n} = a_{2n-1}$ is equivalent to $n \in \mathcal{N}$ and $\mathcal{N}$ is not computable, we get that A is not computable with respect to the enumerated generating set $\{a_1, a_2, \ldots\}$. Now, let A be embedded into a two-generated group G according to Theorem 3. Then, G is solvable of derived length 3 and is bi-orderable. By part (c) of Theorem 3, G is computably presentable, by part (d), the word problem in G is undecidable.

§4. Proof of Theorem 3. The proof of Theorem 3 uses the embedding construction described in [2]. Moreover, the proof is in a way a commentary on the proof of the main theorem from [2]. There are also differences in notations compared with the exposition from [2]. For example, the embedding Φ_X described below is the embedding ϕ from [2], the map f defined below is c in [2], f_i is $b^{(i)}$, etc.

Let us recall the definition of wreath product. Given two groups A and B, the base subgroup A^B is the set of functions from B to A with pointwise multiplication and the group B acts on A^B from the left by automorphisms, such that for $f \in A^B$ and $b \in B$, the resulting function bf is given by

$$(bf)(x) = f(xb), \forall x \in B.$$

For the convenience we will denote bf by f^b.

Definition 4.1. The wreath product $A \wr B$ is defined as a semidirect product $A^B \rtimes B$, with the multiplication $(f_1 b_1)(f_2 b_2) = f_1 f_2^{b_1} b_1 b_2$ for all $f_1, f_2 \in A^B$, $b_1, b_2 \in B$.

Let $H = \langle X \rangle$, where $X = \{x_1, x_2, \ldots\}$, be the initial group, which we want to embed into a two-generated group G as in Theorem 3. In order to construct the group from G of Theorem 3, first, we will describe an embedding of the group H into a subgroup L of the full wreath product $H \wr \langle z \rangle$, where $\langle z \rangle$ is an infinite cyclic group generated by z, as follows. Define the functions $f_i : \langle z \rangle \to H$, $i = 1,2, \ldots$, as

$$f_i(z^n) = \begin{cases} x_i & \text{if } n > 0, \\ 1 & \text{otherwise.} \end{cases}$$

Note that

$$[z, f_i](z^n) = (z f_i z^{-1} f_i^{-1})(z^n) = \begin{cases} x_i & \text{if } n = 0, \\ 1 & \text{otherwise.} \end{cases} \tag{4}$$

Denote $L = \langle z, f_1, f_2, \ldots \rangle \leq H \wr \langle z \rangle$. Since $x_i \mapsto f_i$, $i \in \mathbb{N}$, embeds H into L, we can conclude with the following.

Observation 4.1. *The group L is computable with respect to the generating set $\{z, f_1, f_2, \ldots\} \leq H \wr \langle z \rangle$ if and only if H is computable with respect to the generating set X.*

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PROOF. Indeed, every word from $\{z^{\pm 1}, f_1^{\pm 1}, f_2^{\pm 1}, \ldots\}^*$ can be computationally transformed into a word of the form

$$w = f_{i_1}^{z^{k_1}} f_{i_2}^{z^{k_2}} \ldots f_{i_l}^{z^{k_l}} z^n,$$

where, in fact, $f_{i_j}^{z^{k_j}}$ is the left k_j-shift of f_{i_j}. w represents the trivial element of L if and only if $n = 0$ and $f_{i_1}^{z^{k_1}} f_{i_2}^{z^{k_2}} \ldots f_{i_l}^{z^{k_l}}$ is the identity map. Therefore, since for each $1 \leq j \leq l$, $f_{i_j}^{z^{k_j}}$ is constantly equal to z_{i_j} at the points z^n for $n > -k_j$, and is equal to 1 otherwise, the computability of L with respect to the generating set $\{z, f_1, f_2, \ldots\}$ reduces to the computability of H with respect to the generating set $X = \{x_1, x_2, \ldots\}$. ⊣

If we regard the maps $f_i : \langle z \rangle \to H$ as elements of the wreath product $H \wr \langle z \rangle$, then the next observation follows from (4).

OBSERVATION 4.2. *The map $x_i \mapsto [z, f_i]$, $i \in \mathbb{N}$, induces an isomorphism between H and $\langle [z, f_i] \mid i \in \mathbb{N} \rangle < L$.*

The next step is to embed the group L into a two generated subgroup $G = \langle f, s \rangle$ of the full wreath product $L \wr \langle s \rangle < H \wr \langle z \rangle \wr \langle s \rangle$, where $\langle s \rangle$ is another infinite cyclic group generated by s, as follows. Define the function $f : \langle s \rangle \to L$ as

$$f(s^n) = \begin{cases} z & \text{if } n = 1, \\ f_i & \text{if } n = 2^i, i \in \mathbb{N}, \\ 1 & \text{otherwise.} \end{cases}$$

Then

$$[f, f^{s^{2^i - 1}}](s^n) = (f f^{s^{2^i - 1}} f^{-1} f^{-s^{2^i - 1}})(s^n) = \begin{cases} [z, f_i] & \text{if } n = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Therefore, by Observation 4.2, the map $x_i \mapsto [f, f^{s^{2^i - 1}}]$ induces an embedding of the group H into $G = \langle f, s \rangle \leq L \wr \langle s \rangle$. Now, we are ready to define the embedding $\Phi_X : H \hookrightarrow G$ as the one induced by the map $x_i \mapsto [f, f^{s^{2^i - 1}}]$, $i \in \mathbb{N}$. Since this map is computable, it implies part (a) of Theorem 3: take $\phi_X(i) = [f, f^{s^{2^i - 1}}]$. Also, part (b) of Theorem 3 follows from the observation that G is a subgroup of $H \wr \langle z \rangle \wr \langle s \rangle$.

Note that the final group G depends on the initial choice and enumeration of the generating set of H. This property is of essential importance for our applications, as the initial group H that we consider in applications is isomorphic to $\bigoplus_i \mathbb{Z}_i$, and only for its particular presentation it leads to the desirable computational properties for the end group G.

Note that every word w from $\{f^{\pm 1}, s^{\pm 1}\}^*$ can be computationally rewritten in the form

$$w' = (f^{s^{\gamma_1}})^{\beta_1} (f^{s^{\gamma_2}})^{\beta_2} \ldots (f^{s^{\gamma_n}})^{\beta_n} s^{\delta}, \tag{5}$$

where $\delta, \gamma_i, \beta_i, i = 1, \ldots n$, are some integers, such that w' represents the same element of G as w.

The proof of the next lemma can be found in [2].

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LEMMA 4.1 (Lemma 4, [2]). *The word w' from (5) represents the trivial element in G if and only if the following conditions take place.*

- $\delta = 0$,
- $(f^{s^{\gamma_1}})^{\beta_1}(f^{s^{\gamma_2}})^{\beta_2}\dots(f^{s^{\gamma_n}})^{\beta_n}$, *regarded as a map* $\langle s\rangle \to L$, *takes trivial values on each point* s^μ, *where* μ *ranges from* $-3\max\{\|\gamma_1\|,\dots,\|\gamma_n\|\}$ *to* $3\max\{\|\gamma_1\|,\dots,\|\gamma_n\|\}$,
- $\sum_{j\in B_i}\beta_j = 0$ *for* $i = 1,\dots,n$, *where* $B_i = \{k\in\{1,\dots,n\} \mid \gamma_j = \gamma_i\}$.

Note that for each $\gamma,\beta\in\mathbb{Z}$, $(f^{s^\gamma}) : \langle s\rangle \to L$ is computable, because for every $\mu\in\mathbb{Z}$ we have

$$(f^{s^\gamma})^\beta(s^\mu) = (f(s^{\gamma+\mu})(s^\mu))^\beta = \begin{cases} f_i^\beta & \text{if } \gamma+\mu = 2^i, i\in\mathbb{N},\\ z^\beta & \text{if } \gamma+\mu = 1,\\ 1 & \text{otherwise.}\end{cases}$$

Therefore, Lemma 4.1 implies that the word problem in G can be computationally reduced to the computability property of the group L with respect to the generating set $\{z, f_1, f_2, \dots\}$. This observation combined with Observation 4.1 implies part (d) of Theorem 3. Moreover, it implies that if G has a recursive presentation, then so does L with respect to $L = \langle z, f_1, f_2, \dots\rangle$ hence, by Observation 4.1, also H with respect to X. Combined with the fact that Φ_X is a computable map, it also implies the inverse. Thus part (c) of Theorem 3 is confirmed as well.

Proof of part (e) of Theorem 3 is based on the following two lemmas from [2]. We refer to [2] for more details.

LEMMA 4.2 (Lemma 6, [2]). *The word w' from (5) represents an element from $\Phi_X(H)$ if and only if*

- $\delta = 0$,
- $(f^{s^{\gamma_1}})^{\beta_1}(f^{s^{\gamma_2}})^{\beta_2}\dots(f^{s^{\gamma_n}})^{\beta_n}$, *regarded as a map* $\langle s\rangle \to L$, *takes trivial values on each point* s^μ, *where* $\mu\neq 1$ *and ranges from* $-3\max\{\|\gamma_1\|,\dots,\|\gamma_n\|\}$ *to* $3\max\{\|\gamma_1\|,\dots,\|\gamma_n\|\}$,
- $\sum_{j\in B_i}\beta_j = 0$ *for* $i = 1,\dots,n$, *where* $B_i = \{k\in\{1,\dots,n\} \mid \gamma_j = \gamma_i\}$, *and*
- $(f^{s^{\gamma_1}})^{\beta_1}(f^{s^{\gamma_2}})^{\beta_2}\dots(f^{s^{\gamma_n}})^{\beta_n}(s)\in\langle [z,f_i] \mid i\in\mathbb{N}\rangle$.

LEMMA 4.3 (Lemma 7, [2]). $\bar f := (f_1^{z^{\eta_1}})^{\xi_1}(f_2^{z^{\eta_2}})^{\xi_2}\dots(f_m^{z^{\eta_m}})^{\xi_m} z^\eta \in L$ *belongs to* $\langle [z,f_i] \mid i\in\mathbb{N}\rangle \le L$ *if and only if* $\eta = 0$ *and* $\bar f(z^{\pm 1}) = \bar f(z^{\pm 2}) = \dots = \bar f(z^{\pm\eta_0}) = 1$, *where* $\eta_0 = \max\{|\eta_1|,\dots,|\eta_m|\}$.

Part (f) of Theorem 3 is a direct consequence of the following lemma.

LEMMA 4.4. *Let K be a finitely generated subgroup of a wreath product $A\wr B$ of two left- or bi- orderable groups such that for each $f\in A^B$ and $b\in B$, $fb\in K$ implies that $supp(f)$ is well-orderable with respect to an order on B. Then the group K is correspondingly left- or bi- orderable.*

Moreover, if K has decidable word problem, A and B possess computable orders $\preceq_A$ and $\preceq_B$, respectively, and the set $\{(f_1b, f_2b)\in K\times K \mid f_1(x_0)\prec_A f_2(x_0), x_0 = \min\{x\in supp(f_1f_2^{-1})\}\}$ is computable, then K has a computable order.

PROOF. Define $f_1b_1 \prec_K f_2b_2$ if either $b_1\prec_B b_2$ or $b_1 = b_2$ and $f_1\neq f_2$ and $f_1(x_0)\prec_A f_2(x_0)$, where $x_0 = \min\{x\in supp(f_1f_2^{-1})\}$. It is straightforward to show

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that the order $\prec_K$ is a bi-order in case $\prec_A$ and $\prec_B$ are bi-orders on A and B, respectively. See [9] for details. The case for left-orders can be shown identically.

Proof of the second statement: As for the first part, define $f_1b_1 \prec_K f_2b_2$ if either $b_1 \prec_B b_2$ or $b_1 = b_2$ and $f_1 \neq f_2$ and $f_1(x_0) \prec_A f_2(x_0)$, where $x_0 = \min\{x \in supp(f_1 f_2^{-1})\}$. Therefore, since K and $\preceq_B$ are computable and, by our assumption, $\{(f_1b, f_2b) \in K \times K \mid f_1(x_0) \prec_A f_2(x_0), x_0 = \min\{x \in supp(f_1 f_2^{-1})\}\}$ is computable, we get that $\preceq_K$ is a computable order. ⊣

Now, part (f) of Theorem 3 follows from Lemma 4.4 and the observation that the groups $L \leq H \wr \langle z \rangle$ and correspondingly $G = \langle f, s \rangle \leq L \wr \langle s \rangle$ satisfy the assumptions from Lemma 4.4.

Finally, for part (e) of Theorem 3, note that we have

$$\Phi_X(H) = \langle [f, f^{s^{2^i-1}}] \mid i \in \mathbb{N} \rangle \lhd \langle [f, f^j f^{s^{2^i-1}} f^{-j}] \mid i, j \in \mathbb{N} \rangle \lhd \langle f^{s^i} \mid i \in \mathbb{N} \rangle \lhd \langle f, s \rangle = G.$$

Therefore, one can take $N_1 = \langle [f, f^j f^{s^{2^i-1}} f^{-j}] \mid i, j \in \mathbb{N} \rangle$ and $N_2 = \langle f^{s^i} \mid i \in \mathbb{N} \rangle$.

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