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# A REMARK ON GENERALISED CUBIC PARTITIONS MODULO 5

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**Abstract**

Amdeberhan *et al.* ['Arithmetic properties for generalized cubic partitions and overpartitions modulo a prime', *Aequationes Math.* (2024), doi:10.1007/s00010-024-01116-7] defined the generalised cubic partition function $a_c(n)$ as the number of partitions of $n$ whose even parts may appear in $c \geq 1$ different colours and proved that $a_3(7n + 4) \equiv 0 \pmod 7$ and $a_5(11n + 10) \equiv 0 \pmod{11}$ for all $n \geq 0$ via modular forms. Recently, the author ['A note on congruences for generalized cubic partitions modulo primes', *Integers* **25** (2025), Article no. A20] gave elementary proofs of these congruences. We prove in this note two infinite families of congruences modulo 5 for $a_c(n)$ given by

$$a_{25c+4}(25n + 18) \equiv 0 \pmod 5,$$
$$a_{25c+9}(25n + 8) \equiv 0 \pmod 5$$

for all $c \geq 0$ and $n \geq 0$ using elementary $q$-series manipulations.

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## 1. Introduction

Throughout this paper, $f_m := \prod_{n\geq 1}(1 - q^{mn})$ for a positive integer $m$ and a complex number $q$ with $|q| < 1$. A (unrestricted) partition of a positive integer $n$ is a nonincreasing sequence of positive integers (called its parts) whose sum is $n$. The generating function due to Euler for the number $p(n)$ of partitions of $n$ with $p(0) := 1$ is

$$\sum_{n\geq 0} p(n)q^n = \frac{1}{f_1}.$$

Ramanujan [7, pages 210–213] showed that

$$\sum_{n\geq 0} p(5n + 4)q^n = 5\frac{f_5^5}{f_1^6}, \tag{1.1}$$

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which immediately yields the famous congruence

$$p(5n+4) \equiv 0 \pmod 5$$

for all $n \geq 0$.

Closely related to Ramanujan's cubic continued fraction, Chan [3] and Kim [10] introduced a cubic partition of $n$ as a partition of $n$ whose even parts may appear in two colours. The generating function for the number $a(n)$ of cubic partitions of $n$ with $a(0) := 1$ is

$$\sum_{n\geq 0} a(n)q^n = \frac{1}{f_1 f_2}.$$

Chan [3] established the identity

$$\sum_{n\geq 0} a(3n+2)q^n = 3\frac{f_3^3 f_6^3}{f_1^4 f_2^4},$$

analogous to (1.1), which implies that

$$a(3n+2) \equiv 0 \pmod 3$$

for all $n \geq 0$. Chan and Toh [4] and Xiong [11] independently obtained infinite families of congruences for $a(n)$ modulo powers of 5 using modular forms. Subsequently, Hirschhorn [9] gave elementary proofs of these congruences. For more results on congruences for $a(n)$, we refer the reader to [5] and the references therein.

Recently, Amdeberhan *et al.* [1] defined a generalised cubic partition of $n$ as a partition of $n$ whose even parts may appear in $c \geq 1$ different colours. If we let $a_c(n)$ be the number of such partitions of $n$ with $a_c(0) := 1$, then its generating function is

$$\sum_{n\geq 0} a_c(n)q^n = \frac{1}{f_1 f_2^{c-1}}.$$

Using modular forms, they proved the congruences

$$a_3(7n+4) \equiv 0 \pmod 7, \tag{1.2}$$

$$a_5(11n+10) \equiv 0 \pmod{11}. \tag{1.3}$$

Quite recently, the author [6] provided another proof of (1.2) and (1.3) using classical $q$-series manipulations and established infinite families of congruences for $a_c(n)$ modulo primes $p \not\equiv 1 \pmod 8$.

The goal of this note is to prove two new infinite families of congruences for $a_c(n)$ modulo 5.

THEOREM 1.1. *For all $c \geq 0$ and $n \geq 0$,*

$$a_{25c+4}(25n+18) \equiv 0 \pmod 5, \tag{1.4}$$

$$a_{25c+9}(25n+8) \equiv 0 \pmod 5. \tag{1.5}$$

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We prove Theorem 1.1 in Section 2 using elementary $q$-series techniques and 5-dissections.

## 2. Proof of Theorem 1.1

To prove Theorem 1.1, we require the following 5-dissections for $f_1$, $1/f_1$ and $f_2^2/f_1$. For complex numbers $a$ and $q$ with $|q| < 1$, we define

$$(a;q)_\infty := \prod_{n\geq 0}(1-aq^n),$$
$$(a_1,a_2,\ldots,a_n;q)_\infty := (a_1;q)_\infty(a_2;q)_\infty\cdots(a_n;q)_\infty$$

and

$$R_m := R(q^m) \quad \text{where } R(q) := \frac{(q,q^4;q^5)_\infty}{(q^2,q^3;q^5)_\infty}.$$

LEMMA 2.1. *We have the identities*

$$f_1 = f_{25}\left(\frac{1}{R_5} - q - q^2R_5\right),$$
$$\frac{1}{f_1} = \frac{f_{25}^5}{f_5^6}\left(\frac{1}{R_5^4} + \frac{q}{R_5^3} + \frac{2q^2}{R_5^2} + \frac{3q^3}{R_5} + 5q^4 - 3q^5R_5 + 2q^6R_5^2 - q^7R_5^3 + q^8R_5^4\right),$$
$$\frac{f_2^2}{f_1} = (-q^{10},-q^{-15},q^{25};q^{25})_\infty + q(-q^5,-q^{-20},q^{25};q^{25})_\infty + q^3\frac{f_{50}^2}{f_{25}}.$$

PROOF. See [8, (8.1.4), (8.4.4)] for the first two identities. The third identity follows from [2, page 49, Corollary (ii)]. □

PROOF OF THEOREM 1.1. We first prove (1.4). Note that, by the binomial theorem, $f_k^5 \equiv f_{5k} \pmod 5$ for any $k \geq 1$. Thus, by Lemma 2.1, modulo 5,

$$\sum_{n\geq 0} a_{25c+4}(n)q^n = \frac{1}{f_1f_2^{25c+3}} \equiv \frac{1}{f_{10}f_{50}^c}\cdot\frac{f_2^2}{f_1}$$
$$\equiv \frac{1}{f_{10}f_{50}^c}\left((-q^{10},-q^{-15},q^{25};q^{25})_\infty + q(-q^5,-q^{-20},q^{25};q^{25})_\infty + q^3\frac{f_{50}^2}{f_{25}}\right).$$

Extracting the terms involving $q^{5n+3}$, dividing both sides by $q^3$ and then replacing $q^5$ with $q$ yields

$$\sum_{n\geq 0} a_{25c+4}(5n+3)q^n \equiv \frac{1}{f_2f_{10}^c}\cdot\frac{f_{10}^2}{f_5}$$
$$\equiv \frac{1}{f_5f_{10}^{c-2}}\cdot\frac{f_{50}^5}{f_{10}^6}\left(\frac{1}{R_{10}^4} + \frac{q^2}{R_{10}^3} + \frac{2q^4}{R_{10}^2} + \frac{3q^6}{R_{10}} + 5q^8 - 3q^{10}R_{10}\right.$$
$$\left. + 2q^{12}R_{10}^2 - q^{14}R_{10}^3 + q^{16}R_{10}^4\right) \pmod 5.$$

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Extracting again the terms involving $q^{5n+3}$, dividing both sides by $q^3$ and then replacing $q^5$ with $q$, we arrive at

$$\sum_{n\geq 0} a_{25c+4}(25n+18)q^n \equiv 5q\frac{f_{10}^5}{f_1 f_2^{c+4}} \equiv 0 \pmod 5,$$

which yields (1.4).

We next prove (1.5). Modulo 5,

$$\begin{aligned}\sum_{n\geq 0} a_{25c+9}(n)q^n &= \frac{1}{f_1 f_2^{25c+8}} \equiv \frac{1}{f_{10}^2 f_{50}^c}\cdot\frac{f_2^2}{f_1}\\ &\equiv \frac{1}{f_{10}^2 f_{50}^c}\Big((-q^{10},-q^{-15},q^{25};q^{25})_\infty + q(-q^5,-q^{-20},q^{25};q^{25})_\infty + q^3\frac{f_{50}^2}{f_{25}}\Big).\end{aligned}$$

We consider the terms involving $q^{5n+3}$, divide both sides by $q^3$ and then replace $q^5$ with $q$ giving

$$\begin{aligned}\sum_{n\geq 0} a_{25c+9}(5n+3)q^n &\equiv \frac{1}{f_2^2 f_{10}^c}\cdot\frac{f_{10}^2}{f_5} \equiv \frac{f_2^3}{f_5 f_{10}^{c-1}} \equiv \frac{f_{50}^3}{f_5 f_{10}^{c-1}}\Big(\frac{1}{R_{10}} - q^2 - q^4 R_{10}\Big)^3\\ &\equiv \frac{f_{50}^3}{f_5 f_{10}^{c-1}}\Big(\frac{1}{R_{10}^3} - \frac{3q^2}{R_{10}^2} + 5q^6 - 3q^{10}R_{10}^2 - q^{12}R_{10}^3\Big) \pmod 5.\end{aligned}$$

We finally consider the terms involving $q^{5n+1}$, divide both sides by $q$ and then replace $q^5$ with $q$, arriving at

$$\sum_{n\geq 0} a_{25c+9}(25n+8)q^n \equiv 5q\frac{f_{10}^3}{f_1 f_2^{c-1}} \equiv 0 \pmod 5,$$

which gives (1.5). □

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