Proc. Camb. Phil. Soc. (1974), **76**, 423
PCPS 76–43
Printed in Great Britain

The Stokes phenomenon and generalized contiguous transformations of some generalized hypergeometric functions

By J. HEADING
University College of Wales, Aberystwyth

(*Received* 5 *November* 1973)

Abstract. Previous investigations by the author into the Stokes phenomenon pertaining to solutions of the differential equation $d^nu/dz^n = (-1)^n z^m u$ are extended in order to find when different equations have the same set of Stokes multipliers, with perhaps a series of zeros being additionally allowed. The reason for periodic cycles to exist (with n fixed and m varying), with the same Stokes multipliers regained after a complete cycle, is traced to certain transformation properties of the equations. Within the first cycle (with n fixed and m varying) further remarkable identities exist between the Stokes multipliers, and this also is traced to special transformations between the equations. Relations are found to exist when values of m are chosen so that the highest common factors of the two integers n and $n/(n+m)$ are identical. Finally, a transformation of the independent variable is deduced whereby the set of Stokes multipliers for an equation of order n is identical (apart from the additional zeros) to that for an equation of lower order. A hierarchy of equations is thrown up, whereby certain basic equations are transformed to yield more advanced equations of higher order and different m.

1. *Motivation and objectives.* The Stokes phenomenon is an important subject to investigate in its own right, though its application lies in the fact that approximate solutions of nth order ordinary differential equations can be found by a generalization of the so-called W.K.B.J. method.

Some years ago, the author in a series of paper (2, 3, 4) (designated respectively by I, II, III) investigated the Stokes phenomenon associated with solutions of the nth order differential equation with a transition point of order m (a rational fraction)

$$\frac{d^nu}{dz^n} = e^{n\pi i} z^m u \tag{1}$$

where $n+m > 0$. Recently, Dingle(1) has asserted that it was essentially these papers that demonstrated that the Stokes discontinuities follow a recognizable pattern. The present paper reports further discoveries in the structure of this pattern.

By means of the transformation derived in I,

$$w = z^{m+n}(m+n)^{-n}, \tag{2}$$

https://doi.org/10.1017/S0305004100049124 Published online by Cambridge University Press

equation (1) may be expressed in terms of the homogeneous operator $\vartheta = wd/dw$:

$$\left(\vartheta - \frac{n-1}{m+n}\right)\left(\vartheta - \frac{n-2}{m+n}\right) \dots \left(\vartheta - \frac{1}{m+n}\right)\left(\vartheta - \frac{0}{m+n}\right) u = e^{n\pi i} wu. \tag{3}$$

Solutions of (1) and (3) are designated by $u(n, m; z)$ and $u(n, m; w)$ respectively. In terms of the generalized hypergeometric function, particular solutions are

$$w^{p_r}\, {}_0F_{n-1}(p_r - p_1 + 1, p_r - p_2 + 1, \dots * \dots, p_r - p_n + 1; (-1)^n w),$$

where $p_r = (r-1)/(m+n)$, $r = 1, 2, \dots, n$.

The asymptotic expressions were found in I to be of the form

$$z^{-\frac{1}{2}m(n-1)/n} \exp\left[-ne^{2\pi iq/n} z^{(m+n)/n} (m+n)^{-1}\right], \tag{4}$$

where q is an integer. There is, of course, a descending asymptotic series attached to this asymptotic expression (4). Neither Slater(6) nor Luke(5) provides any pertinent information on the properties of the descending series, though it may be found formally by the prescription given by Luke, the recursive formula following ultimately by substitution into the differential equation. Whether explicit coefficients can be written down for the particular generalized hypergeometric functions considered here does not seem to be clear, but if they could then they would provide part of the solution to the problems presented in this paper.

Finally, in II, the Stokes phenomenon was considered for the expressions (4), which in themselves are valid only in specified restricted ranges of $\arg z$. The mode by which the constant attached to the expressions (4) changes with $\arg z$ is described by a set of n Stokes multipliers T_j ($T_1 \equiv 1$), given by

$$T_j = \frac{\sin\dfrac{\pi m}{m+n} \sin\dfrac{\pi(m+1)}{m+n} \dots \sin\dfrac{\pi(m+j-2)}{m+n}}{\sin\dfrac{\pi}{m+n} \sin\dfrac{2\pi}{m+n} \dots \sin\dfrac{\pi(j-1)}{m+n}} \exp\left[\pi i(j-1)/n\right], \tag{5}$$

with $2 \leqslant j \leqslant n$.

The use of these multipliers is explained in detail in II. Expressions (4) become dominant at particular values of $\arg z$ that make the square bracket real and positive. A particular solution of the differential equation (1) would be specified by the coefficients attached to n successive dominant terms. If $\arg z = \alpha$ is a particular value of $\arg z$ where dominancy occurs, and if $C_1, C_2, \dots, C_n$ are the coefficients of the previous n dominant terms occurring at

$$\arg z = \alpha - \frac{2\pi n}{m+n}, \quad \alpha - \frac{2\pi(n-1)}{m+n}, \quad \dots, \quad \alpha - \frac{2\pi}{m+n}$$

respectively, then the coefficient C_{n+1} of the dominant expression at $\arg z = \alpha$ is given by

$$C_{n+1} = T_1 C_1 + T_2 C_2 + \dots + T_n C_n \tag{6}$$

with $T_1 = 1$. An expression with the coefficient C_1 attached to it reappears after an interval $2\pi n/(m+n)$ of $\arg z$ with the new coefficient C_{n+1}. Note that the C's are labelled here in a more systematic manner than adopted in II. These Stokes multipliers are

https://doi.org/10.1017/S0305004100049124 Published online by Cambridge University Press

independent of α for the equations considered here, but for more complicated examples of the Stokes phenomenon they will also be functions of α.

In particular, if $T_2 = T_3 = \ldots = T_n \equiv 0$, the new coefficient C_{n+1} will equal C_1. There is then no Stokes phenomenon, and the solution must be expressible in terms of the elementary functions which have no asymptotic expansions. Such a situation will occur if the asymptotic series multiplying the expressions (4) terminate; for example, Bessel functions of order $n+\frac{1}{2}$ display this property. One objective of this paper is to find formal expressions for these terminating series in terms of the very elementary functions.

Moreover, if an equation with parameters n, m has the same Stokes multipliers as an equation with parameters n', m' ($n = n'$, or $n < n'$, not counting zeros that will occur in the latter case), we shall find transformation-relations between such equations and solutions. A hierarchy of equations and transformations will be built up, whereby equations with simple properties are transformed into equations with more complicated properties.

These results will generalize certain well-known properties of Bessel functions, given, for example, by Watson(7):

(i) $$\frac{d}{dz}[z^\nu J_\nu(z)] = z^\nu J_{\nu-1}(z),$$

relating to the case when $n = 2$ and varying m.

(ii) The following have been briefly discussed in II:

$$\frac{d^{2n}u}{dz^{2n}} = z^{-n}u$$

has solutions in terms of $z^{\frac{1}{2}n}J_n(2iz^{\frac{1}{2}})$, and hence possessing Stokes multipliers $1, 2i(-1)^n$;

$$\frac{d^{2n+1}u}{dz^{2n+1}} = z^{-n-\frac{1}{2}}u$$

has solutions in terms of $z^{\frac{1}{2}n+\frac{1}{4}}J_{n+\frac{1}{2}}(-2iz^{\frac{1}{2}})$, namely expressible in terms of elementary functions with Stokes multipliers $1, 0$.

(iii) The descending asymptotic series for Bessel functions of order ν contains the product

$$(4\nu^2-1^2)(4\nu^2-3^2)\ldots[4\nu^2-(2m-1)^2]$$

in the numerator of the coefficients of z^{-m}. Thus the series terminates with the term z^{-m} when $4\nu^2-(2m+1)^2 = 0$, namely $\nu = m+\frac{1}{2}$. In this case

$$J_{m+\frac{1}{2}}(z) = (-1)^m z^{m+\frac{1}{2}}\left(\frac{d}{z\,dz}\right)^n [z^{-\frac{1}{2}}J_{\frac{1}{2}}(z)],$$

with $J_{\frac{1}{2}}(z) = (2/\pi z)^{\frac{1}{2}}\sin z$.

2. *The periodic transformation cycle.* In this section, we shall discuss and prove the following

https://doi.org/10.1017/S0305004100049124 Published online by Cambridge University Press

THEOREM I. *Differential operators exist such that solutions of equation* (1) *for given* m *may be transformed into solutions of a similar equation of order* n *but with* m *replaced by*

$$m_N = \frac{m - Nn(m+n)}{1 + N(m+n)},$$

N being an integer.

The general Stokes multiplier $T_j(n,m)$ given by (5) may be rewritten in the form

$$T_j(n,m) = \frac{\sin\frac{\pi n}{m+n}\sin\frac{\pi(n-1)}{m+n}\dots\sin\frac{\pi(n-j+2)}{m+n}}{\sin\frac{\pi}{m+n}\sin\frac{2\pi}{m+n}\dots\sin\frac{\pi(j-1)}{m+n}} e^{\pi i(j-1)/n} \tag{7}$$

with $2 \leqslant j \leqslant n$ and $T_1 \equiv 1$.

If for a given n we now introduce a second value of m, m_N say, defined by

$$\frac{1}{m_N+n} = \frac{1}{m+n} + N, \tag{8}$$

where N is an integer, then the corresponding values of the Stokes multipliers become

$$\begin{aligned} T_j(n,m_N) &= \frac{(-1)^{nN+(n-1)N+\dots+(n-j+2)N}}{(-1)^{N+2N+\dots+(j-1)N}} T_j(n,m) \\ &= (-1)^{N(n+1)(j-1)} T_j(n,m). \end{aligned} \tag{9}$$

If n is odd, $(n+1)$ is even, so for all N and j

$$T_j(n,m_N) = T_j(n,m).$$

If N is even, the same result is valid. But if n is even and N odd, we have

$$T_j(n,m_N) = (-1)^{j-1} T_j(n,m). \tag{10}$$

In particular, this holds when n is even and $N = 1$.

Apart from these changes of sign under these circumstances, it appears that the Stokes phenomenon for the equation with parameters n and m_N is governed by identical Stokes multipliers as for the equation with parameters n and m. This suggests that the solutions of these two distinct equations are in some sense related. For example, we may quote the two equations when $n = 7$, $m = 5$, $N = 2$:

$$\frac{d^7u}{dz^7} = -z^5u; \quad \frac{d^7u}{dz^7} = -z^{-163/25}u.$$

Solving equation (8) for m_N, we obtain,

$$m_N = \frac{m - Nn(m+n)}{1 + N(m+n)}. \tag{11}$$

The differential equations satisfied by $u(n,m;z)$ are

$$\frac{d^nu(n,m;z)}{dz^n} = e^{n\pi i} z^m u(n,m;z),$$

$$\left(\vartheta - \frac{n-1}{m+n}\right)\left(\vartheta - \frac{n-2}{m+n}\right)\dots\left(\vartheta - \frac{1}{m+n}\right)\vartheta u(n,m;w) = e^{n\pi i} w u(n,m;w), \tag{12}$$

https://doi.org/10.1017/S0305004100049124 Published online by Cambridge University Press

where $$w = \frac{z^{m+n}}{(m+n)^n}, \quad \vartheta \equiv w\frac{d}{dw}.$$

Similarly, the equations governing $u(n, m_N; z)$ are

$$\frac{d^n u(n, m_N; z)}{dz^n} = e^{n\pi i} z^{m_N} u(n, m_N; z),$$

$$\left(\vartheta - \frac{(n-1)[1+N(m+n)]}{m+n}\right)\left(\vartheta - \frac{(n-2)[1+N(m+n)]}{m+n}\right)\dots$$

$$\left(\vartheta - \frac{1+N(m+n)}{m+n}\right)\vartheta u(n, m_N; z) = e^{n\pi i} w u(n, m_N; z), \qquad (13)$$

where $$w = \frac{z^{m_N+n}}{(m_N+n)^n}, \quad \vartheta \equiv w\frac{d}{dw}.$$

We propose to demonstrate how equation (12) transforms into equation (13). Evidently it is only necessary to consider the case when $N = 1$; all other cases follow systematically. Writing for convenience $a \equiv 1/(m+n)$, we have the two equations

$$[\vartheta - (n-1)a][\vartheta - (n-2)a]\dots(\vartheta - a)\vartheta u(n, m; w) = e^{n\pi i} w u(n, m; w),$$

$$[\vartheta - (n-1)(a+1)][\vartheta - (n-2)(a+1)]\dots[\vartheta - (a+1)]\vartheta u(n, m_1; w) = e^{n\pi i} w u(n, m_1; w).$$

To avoid a multiplicity of symbols in the following transformation processes, we shall write

$$(\vartheta - ra) \equiv [ra],$$

the number-component of the operator being all-important. When such an operator acts on a power of w, w^p say, this factor w^p may be carried to the left of the operator by replacing ϑ by $\vartheta + p$. In other words,

$$(\vartheta - ra) w^p f \equiv w^p(\vartheta + p - ra) f,$$

so
$$[ra] w^p \to w^p[ra - p],$$

and in particular,
$$[ra] w \to w[ra - 1].$$

We require to transform, if possible, the equation

$$[(n-1)a][(n-2)a]\dots[a][0] u(n, m; w) = e^{n\pi i} w u(n, m; w) \qquad (14)$$

into

$$[(n-1)(a+1)][(n-2)(a+1)]\dots[a+1][0] u(n, m_1; w) = e^{n\pi i} w u(n, m_1; w). \qquad (15)$$

We operate on equation (14) by a series of operators designed, if possible, to produce equation (15). In the present case, it is always possible to produce a suitable transformation operator, but in further cases to be considered later in this paper it will not always be possible except under restricted conditions.

The single operator $[ra + s]$ yields

$$[ra+s][(n-1)a][(n-2)a]\dots[a][0] u(n, m; w) = e^{n\pi i}[ra+s] w u(n, m; w)$$
$$= e^{n\pi i} w[ra + s - 1] u(n, m; w).$$

https://doi.org/10.1017/S0305004100049124 Published online by Cambridge University Press

The required operator is suggested by examining the structure in simple cases. The general operator is

$$\prod_{r=1}^{n-1}\prod_{s=1}^{r}[ra+s] \equiv \prod_{r=1}^{n-1}\prod_{s=1}^{r}[\vartheta-(ra+s)].$$

This operator may be separated into two parts, one part containing the operators involved in (15):

$$\prod_{r=1}^{n-1}[ra+r]\times\prod_{r=2}^{n-1}\prod_{s=1}^{r-1}[ra+s]. \tag{16}$$

Written more explicitly, this operator contains the individual terms

$$\begin{array}{ccccc} [a+1] & [2a+1] & [3a+1] & \dots & [(n-1)a+1] \\ & [2a+2] & [3a+2] & \dots & [(n-1)a+2] \\ & & [3a+3] & \dots & [(n-1)a+3] \\ & & & \dots & \dots \\ & & & & [(n-1)a+n-1], \end{array}$$

the last entry in each column being separated to form the first part of the operator (16). This operator (16) now operates on equation (14), yielding

$$\begin{aligned} \prod_{r=1}^{n-1}[ra+r]\times\prod_{r=2}^{n-1}\prod_{s=1}^{r-1}[ra+s]\times\prod_{r=1}^{n-1}[ra][0]u(n,m;w) &= e^{n\pi i}\prod_{r=1}^{n-1}\prod_{s=1}^{r}[ra+s]\,wu(n,m;w) \\ &= e^{n\pi i}w\prod_{r=1}^{n-1}\prod_{s=1}^{r}[ra+s-1]\,u(n,m;w), \end{aligned}$$

or, by rearrangement,

$$\prod_{r=0}^{n-1}[ra+r]\times\prod_{r=1}^{n-1}\prod_{s=0}^{r-1}[ra+s]\,u(n,m;w) = e^{n\pi i}\prod_{r=1}^{n-1}\prod_{s=0}^{r-1}[ra+s]\,u(n,m;w). \tag{17}$$

Explicitly, the left-hand side may be written in the form

$$\begin{array}{ccccccc} [0] & [a] & [2a] & [3a] & \dots & [(n-1)a] & u(n,m;w) \\ & [a+1] & [2a+1] & [3a+1] & \dots & [(n-1)a+1] & \\ & & [2a+2] & [3a+2] & \dots & [(n-1)a+2] & \\ & & & [3a+3] & \dots & [(n-1)a+3] & \\ & & & & \dots & \dots & \\ & & & & & [(n-1)a+(n-1)] & \end{array}$$

and the right-hand side in the form

$$\begin{array}{cccccc} e^{n\pi i}w & [a] & [2a] & & [(n-1)a] & u(n,m;w) \\ & & [2a+1] & \dots & [(n-1)a+1] & \\ & & & \dots & \dots & \\ & & & & [(n-1)a+(n-2)]. & \end{array}$$

https://doi.org/10.1017/S0305004100049124 Published online by Cambridge University Press

Evidently, the operators that appear on the right-hand side also appear on the left; thus if we define

$$\prod_{r=1}^{n-1}\prod_{s=0}^{r-1}[ra+s]\,u(n,m;w) \equiv u(n,m_1;w),$$

we see that equation (17) reduces to equation (15).

N of these cycles may be carried out in succession. To pass from the case $N=1$ to $N=2$, all that must be done to equations (14) and (15) is to replace a by $a+1$, giving

$$\prod_{r=1}^{n-1}\prod_{s=0}^{r-1}[r(a+1)+s]\,u(n,m_1;w) \equiv u(n,m_2;w).$$

The elimination of $w(n,m_1;w)$ yields

$$u(n,m_2;w) \equiv \prod_{r=1}^{n-1}\prod_{s=0}^{2r-1}[ra+s]\,u(n,m;w).$$

Generally, for N cycles, we conclude that

$$u(n,m_N;w) \equiv \prod_{r=1}^{n-1}\prod_{s=0}^{Nr-1}[ra+s]\,u(n,m;w). \tag{18}$$

To illustrate this, consider the case when $n=4$, $m=3$; the basic equation is then

$$\frac{d^4u(4,3;z)}{dz^4} = z^3u(4,3;z).$$

When $w = z^7/7^4$, the equation for $u(4,3;w)$ is

$$(\vartheta-\tfrac{3}{7})(\vartheta-\tfrac{2}{7})(\vartheta-\tfrac{1}{7})\,\vartheta u(4,3;w) = wu(4,3;w).$$

Taking $N=1$, 2, in turn we have $m_1 = -25/8$, $m_2 = -53/15$. Then a solution $u(4,-\frac{25}{8};z)$ is obtained by taking a solution $u(4,3;z)$, transforming to $u(4,3;w)$ by using $w = z^7/7^4$, carrying out the operation

$$\prod_{r=1}^{3}\prod_{s=0}^{r-1}[\vartheta-(ra+s)]\,u(4,3;w) \quad (a=\tfrac{1}{7})$$
$$\equiv (\vartheta-\tfrac{1}{7})(\vartheta-\tfrac{2}{7})(\vartheta-\tfrac{3}{7})(\vartheta-\tfrac{9}{7})(\vartheta-\tfrac{10}{7})(\vartheta-\tfrac{17}{7})\,u(4,3;w),$$

and replacing w by $z^{(m_1+4)}/(m_1+4)^4 \equiv z^{-7/8}/(7/8)^4$. This resulting function will satisfy the equation

$$\frac{d^4u}{dz^4} = z^{-25/8}u.$$

Similarly, when $N=2$,

$$u(4,-\tfrac{53}{15};w) \equiv \prod_{r=1}^{3}\prod_{s=0}^{2r-1}[\vartheta-(\tfrac{1}{7}r+s)]\,u(4,3;w),$$

with $w = z^{(m_2+4)}/(m_2+4)^4 \equiv z^{-7/15}/(7/15)^4$ is a solution of

$$\frac{d^4u}{dz^4} = z^{-53/15}u.$$

https://doi.org/10.1017/S0305004100049124 Published online by Cambridge University Press

We return to the general solution $u(n, m; z)$. The transformation from $u(n, m; w)$ to $u(n, m_N; w)$ involves a maximum of $\frac{1}{2}(n-1)\,nN$ derivatives in the operator product. The asymptotic expressions (4) involved in $u(n, m; z)$ for large $|z|$ are

$$z^{-\frac{1}{2}m(n-1)/n} \exp\left[-n e^{2\pi i q/n} z^{(m+n)/n}/(m+n)\right] \propto w^{-\frac{1}{2}m(n-1)/n(m+n)} \exp\left(-n e^{2\pi i q/n} w^{1/n}\right),$$

since $w = z^{(m+n)}/(m+n)^n$. The asymptotic expression of the solution (18) of the transformed equation will be given by

$$\vartheta^{\frac{1}{2}(n-1)\,nN} w^{-\frac{1}{2}m(n-1)/n(m+n)} \exp\left(-n e^{2\pi i q/n} w^{1/n}\right),$$

the leading term being given by causing the operator d/dw to jump over all the powers of w and to operate solely on the exponential function. We obtain an expression proportional to

$$w^{\frac{1}{2}(n-1)\,nN} . w^{-\frac{1}{2}m(n-1)/n(m+n)} . e^{\pi i q(n-1)\,N} . w^{\frac{1}{2}(n^{-1}-1)(n-1)\,nN} \exp\left(-n e^{2\pi i q/n} w^{1/n}\right)$$
$$= e^{\pi i q(n-1)\,N} . w^{\frac{1}{2}(n-1)[Nn(m+n)-m]/n(m+n)} \exp\left(-n e^{2\pi i q/n} w^{1/n}\right).$$

For the transformed equation, we use

$$w = z^{(m_N+n)}/(m_N+n)^n, \quad m_N = [m - Nn(m+n)]/[1 + N(m+n)].$$

The new asymptotic expression now reduces to

$$\exp\left[\pi i q(n-1)\,N\right] . z^{-\frac{1}{2}m_N(n-1)/n} \exp\left[-n e^{2\pi i q/n} z^{(m_N+n)/n}/(m_N+n)\right]$$

after considerable simplification. These are the expected forms, of course, relating to the transformed equation, with the exception of the factors $\exp[\pi i q(n-1)\,N]$. When n is odd, this additional factor is unity. When N is even, it is also unity. But when N is odd and n even, the factor has the form $(-1)^q$. This is sufficient to explain the alternation in sign of the Stokes multipliers (10), for they refer to transformed asymptotic expressions that also alternate in sign.

The alternation in sign of the Stokes multipliers (10) when N is odd and n even is therefore a trivial phenomenon, and need not be considered further. In no way does it preclude the underlying solutions from being derived as transformations of the original functions.

However, this general transformation theory does not exhibit the really interesting cases that arise. These specialized cases are considered in the following sections.

3. *The vanishing of the Stokes multipliers within the first cycle.* We now examine the formula for the T's when T_2 vanishes. Since T_3 is formed by multiplying T_2 by a further factor, T_3 will also vanish, unless the denominator of the additional factor vanishes. Since the vanishing factors are only sines, a limiting value of the ratio would then occur, yielding a non-vanishing value for T_3. Such limiting ratios may occur several times in the evaluation of the n multipliers T_j. It was the systematic distribution of these zeros that led to the investigation of the subsequent transformation theory.

The previous theory shows that the set of T's for given n repeats itself for successive values of m designated by m, m_1, m_2, ... (except for irrelevant changes of sign), the values of m_N being related to m by

$$\frac{1}{m_N+n} = \frac{1}{m+n} + N.$$

https://doi.org/10.1017/S0305004100049124 Published online by Cambridge University Press

We are therefore interested only in the intermediate values of m_I lying in the range $m \leqslant m_I < m_1$, the previous cycling rules being applicable to higher values of m_I.

The first sine factor in the numerator of T_j vanishes when

$$\sin \frac{\pi m}{m+n} = 0,$$

namely,

$$\frac{m}{m+n} = -I \quad \text{or} \quad m = -\frac{nI}{I+1}.$$

Whatever the value of the integer I (except -1), it is obvious that $m \leqslant 0$, the equality holding when $I = 0$. We shall concern ourselves with the n cases:

$$\begin{array}{lccccc} I: & 0 & 1 & 2 & \dots & n-1 \\ m_I: & 0 & -\frac{1}{2}n & -\frac{2}{3}n & \dots & -(n-1). \end{array}$$

The following value of I, given by $I = n$, yields $m = -n^2/(n+1)$. If we refer to (11) with $m = 0$, the general cycle will occur when

$$m_N = -\frac{Nn^2}{1+Nn},$$

namely, $-n^2/(1+n)$ when $N = 1$. Consequently, the solution in the case when $I = n$ is obtained from a transformation of the solution when $I = 0$, so is irrelevant in our discussion of the transformation properties within the first cycle.

We shall designate the n dependent variables within this first cycle by $u_0, u_1, \dots, u_{n-1}$ for the n values of I. Since $1/(m+n) = (I+1)/n$, the equation for u_I in terms of

$$w = z^{n/(I+1)} \left(\frac{I+1}{n}\right)^n$$

is

$$\left[\vartheta - \frac{(n-1)(I+1)}{n}\right]\left[\vartheta - \frac{(n-2)(I+1)}{n}\right] \dots \left[\vartheta - \frac{I+1}{n}\right] \vartheta u_I = e^{n\pi i} w u_I. \tag{19}$$

The n parameters in the equation are in arithmetic progression with common difference $(I+1)/n$. In particular, u_0 satisfies

$$[\vartheta - (1-n^{-1})][\vartheta - (1-2n^{-1})] \dots (\vartheta - n^{-1})\vartheta u_0 = e^{n\pi i} w u_0. \tag{20}$$

The fundamental question is, what transformation relations exist between the solutions u_I within the first cycle $0 \leqslant I \leqslant n-1$? Moreover, can any of the solution u_I (solutions of an nth order equation) be expressed in terms of solutions of an equation of lower order? The relevance of this last question may be seen by noting that solutions u_0 of the equation

$$\frac{d^n u(n, 0; z)}{dz^n} = e^{n\pi i} u(n, 0; z)$$

are $\exp(-e^{2\pi i q/n} z)$, immediately related to the basic solution e^z of the first order equation $du/dz = u$.

The evaluation of the Stokes multipliers T_j for the n specified values of I may be carried out either analytically or numerically. The author has used a small pro-

https://doi.org/10.1017/S0305004100049124 Published online by Cambridge University Press

grammable desk electronic calculator with ten stores, the Sumlock Compucorp 322G Scientist, possessing the almost-instantaneous facility for the calculation of the special functions involved in the ratio. The following tables present the results for $2 \leqslant n \leqslant 10$; some remarkable features are demonstrated, which therefore need explanation. The complex factor $e^{\pi i(j-1)/n}$ attached to T_j is omitted for convenience.

n	I	m	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}
2	0	0	1	0	—	—	—	—	—	—	—	—
2	1	−1	1	−2	—	—	—	—	—	—	—	—
3	0	0	1	0	0	—	—	—	—	—	—	—
3	1	−3/2	1	0	0	—	—	—	—	—	—	—
3	2	−2	1	3	3	—	—	—	—	—	—	—
4	0	0	1	0	0	0	—	—	—	—	—	—
4	1	−2	1	0	2	0	—	—	—	—	—	—
4	2	−8/3	1	0	0	0	—	—	—	—	—	—
4	3	−3	1	−4	6	−4	—	—	—	—	—	—
5	0	0	1	0	0	0	0	—	—	—	—	—
5	1	−5/2	1	0	0	0	0	—	—	—	—	—
5	2	−10/3	1	0	0	0	0	—	—	—	—	—
5	3	−15/4	1	0	0	0	0	—	—	—	—	—
5	4	−4	1	5	10	10	5	—	—	—	—	—
6	0	0	1	0	0	0	0	0	—	—	—	—
6	1	−3	1	0	0	−2	0	0	—	—	—	—
6	2	−4	1	0	3	0	3	0	—	—	—	—
6	3	−9/2	1	0	0	2	0	0	—	—	—	—
6	4	−24/5	1	0	0	0	0	0	—	—	—	—
6	5	−5	1	−6	15	−20	15	−6	—	—	—	—
7	0	0	1	0	0	0	0	0	0	—	—	—
7	1	−7/2	1	0	0	0	0	0	0	—	—	—
7	2	−14/3	1	0	0	0	0	0	0	—	—	—
7	3	−21/4	1	0	0	0	0	0	0	—	—	—
7	4	−28/4	1	0	0	0	0	0	0	—	—	—
7	5	−35/6	1	0	0	0	0	0	0	—	—	—
7	6	−6	1	7	21	35	35	21	7	—	—	—
8	0	0	1	0	0	0	0	0	0	0	—	—
8	1	−4	1	0	0	0	2	0	0	0	—	—
8	2	−16/3	1	0	0	0	0	0	0	0	—	—
8	3	−6	1	0	4	0	6	0	4	0	—	—
8	4	−32/5	1	0	0	0	0	0	0	0	—	—
8	5	−20/3	1	0	0	0	2	0	0	0	—	—
8	6	−48/7	1	0	0	0	0	0	0	0	—	—
8	7	−7	1	−8	28	−56	70	−56	28	−8	—	—
9	0	0	1	0	0	0	0	0	0	0	0	—
9	1	−9/2	1	0	0	0	0	0	0	0	0	—
9	2	−6	1	0	0	3	0	0	3	0	0	—
9	3	−27/4	1	0	0	0	0	0	0	0	0	—
9	4	−36/5	1	0	0	0	0	0	0	0	0	—
9	5	−15/2	1	0	0	3	0	0	3	0	0	—
9	6	−54/7	1	0	0	0	0	0	0	0	0	—
9	7	−63/8	1	0	0	0	0	0	0	0	0	—
9	8	−8	1	9	36	84	126	126	84	36	9	—
10	0	0	1	0	0	0	0	0	0	0	0	0
10	1	−5	1	0	0	0	0	−2	0	0	0	0

https://doi.org/10.1017/S0305004100049124 Published online by Cambridge University Press

n	I	m	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}
10	2	$-20/3$	1	0	0	0	0	0	0	0	0	0
10	3	$-15/2$	1	0	0	0	0	2	0	0	0	0
10	4	-8	1	0	5	0	10	0	10	0	5	0
10	5	$-25/3$	1	0	0	0	0	-2	0	0	0	0
10	6	$-60/7$	1	0	0	0	0	0	0	0	0	0
10	7	$-35/4$	1	0	0	0	0	2	0	0	0	0
10	8	$-80/9$	1	0	0	0	0	0	0	0	0	0
10	9	-9	1	-10	45	-120	210	-252	210	-120	45	-10

The negative signs occurring in some of these entries are irrelevant. The particular case $I = 1$ for all n corresponds to the special equations noted in II, when solutions can be expressed in terms of Bessel functions, but this particular feature will be shown to be but a simple example of a much more general property.

An inspection of these tabulated values reveals the following features. Either all the T's vanish (except $T_1 \equiv 1$), or the binomial coefficients are very much in evidence, appearing oftentimes with zeros in many rows. At first sight, it is not immediately obvious when particular binomial coefficients occur, but this may be ascertained by rearranging the vertical arrangements of the rows.

Binomial coefficients	n	I
$(1+x)^1$	2	0
	3	0, 1
	4	0, 2
	5	0, 1, 2, 3
	6	0, 4
	7	0, 1, 2, 3, 4, 5
	8	0, 2, 4, 6
	9	0, 1, 3, 4, 6, 7
	10	0, 2, 6, 8
$(1+x)^2$	2	1
	4	1
	6	1, 3
	8	1, 5
	10	1, 3, 5, 7
$(1+x)^3$	3	2
	6	2
	9	2, 5
$(1+x)^4$	4	3
	8	3
$(1+x)^5$	5	4
	10	4
$(1+x)^6$	6	5
$(1+x)^7$	7	6
$(1+x)^8$	8	7
$(1+x)^9$	9	8
$(1+x)^{10}$	10	9

Let (a, b) denote the highest common factor of the integers a and b. Then the non-zero T's for the equation defined by the parameters n and I equal the binomial coefficients appearing in the expansion of $(1+x)^{(n, I+1)}$, the minus signs and the factor

https://doi.org/10.1017/S0305004100049124 Published online by Cambridge University Press

$e^{\pi i(j-1)/n}$ being disregarded. An inspection of the above scheme shows this immediately, and it is a general property for arbitrary n and I, $0 \leqslant I \leqslant n-1$.

This equality of the Stokes multipliers (but not of course of the corresponding asymptotic expressions) leads us to formulate the two questions more formally.

(i) For given n, of the set of integers I defined by $\{I, 0 \leqslant I \leqslant n-1\}$ consider the subset $\{I_h\}$ such that the corresponding set of highest common factors $\{(n, I_h+1)\}$ has a common value h. Thus when $n = 8$, we have

$\{I_1\} = \{0, 2, 4, 6\}$ with $(8, 1) = (8, 3) = (8, 5) = (8, 7) = 1$ (namely $I+1$ is prime to n for all elements of this set);

$\{I_2\} = \{1, 5\}$ with $(8,2) = (8, 6) = 2$;

$\{I_4\} = \{3\}$ with $(8, 4) = 4$;

$\{I_8\} = \{7\}$ with $(8, 8) = 8$.

For a particular set $\{I_h\}$, let I_L denote the smallest integer in the set. Is it possible to relate solutions of the set of differential equations defined by the particular parameters n and the set $\{I_h$, omit $I_L\}$ to solutions of the particular equation defined by the parameters n and I_L, the relation being expressed in terms of products of differential operators operating on the solution u_{I_L}? If this is so, solutions of all equations arising from the set $\{I_1$, omit $0\}$ can be expressed in terms of solutions of the equation with $I = m = 0$, namely in terms of the elementary functions. The Stokes phenomenon disappears for such elementary functions. The answer to this question will be in the affirmative.

We may add that solutions of all equations arising from the set $\{I_2$, omit $1\}$ can be expressed in terms of Bessel functions. It may also be possible *sometimes* to express solutions u_{I_h} corresponding to the set of parameters $\{I_h$, omit $I_L\}$ in terms of a solution u_{I_M}, where I_M lies within the set and $I_h > I_M > I_L$. But this feature is not relevant to the subject of this present paper.

(ii) Consider now the set of all integers n and I, with $0 \leqslant I \leqslant n-1$, such that the highest common factors $(n, I+1)$ has a specified value h. Thus, when $h = 3$, we have the pairs of parameters

$$n = 3, \quad I = 2; \qquad n = 6, \quad I = 2; \qquad n = 9, \quad I = 2, \quad 5; \quad \ldots.$$

Is it possible to express solutions of all equations defined by such pairs of parameters in terms of the lowest order differential equation specified by $n = h$, $I = h-1$? The answer to this question is in the affirmative. This means that the zeros in the tabulated rows of Stokes multipliers may be discarded in deciding the type of differential equation yielding the basic solution. Thus when $h = 3$, the solution of the equation defined by $n = 3$, $I = 2$, $n = -2$, namely

$$\frac{d^3u}{dz^3} = -z^{-2}u,$$

may be used to form solutions of the equations

$$\frac{d^6u}{dz^6} = z^{-4}u \quad (n = 6, I = 2, m = -4);$$

$$\frac{d^9u}{dz^9} = -z^{-6}u \quad (n = 9, I = 2, m = -6);$$

https://doi.org/10.1017/S0305004100049124 Published online by Cambridge University Press

$$\frac{d^9u}{dz^9} = -z^{-15/2}u \quad (n = 9, I = 5, m = -15/2)$$

and so on.

4. *The second theorem: transformation within the first cycle.* In these two sections, we shall discuss and prove the following

THEOREM II. *Let I_L denote the lowest integer (including zero) for which $(n, I_L+1) = h$, and I any other integer such that $(n, I+1) = h$, where $I_L < I \leqslant n-1$. Then differential operators exist such that solutions of equation (1) with $m = -nI_L/(I_L+1)$ may be transformed into solutions of the similar equation of order n but with m replaced by $-nI/(I+1)$.*

COROLLARY. *If $I_L = 0$, $h = 1$, $m = 0$, then solutions of the equation specified by the further values of m are expressible in terms of the elementary functions.*

In the notation introduced in paragraph 2, the equation defined by the parameters n and I, $0 \leqslant I \leqslant n-1$, is

$$[0][(I+1)n^{-1}][2(I+1)n^{-1}]\dots[(n-1)(I+1)n^{-1}]u(n, m_I; w) = e^{n\pi i}wu(n, m_I; w). \quad (21)$$

For fixed n, we seek a transformation that will transform this equation from a lower value of I to a higher value of I within the first cycle. For given highest common factor $(n, I+1) = h$, it is possible to transform the equation defined by the particular parameter I_L to all successive equations defined by the set of parameters $\{I_h$, omit $I_L\}$.

An operator $\Pi[a_r]$ must be discovered that operates on equation (21) with $I = I_L$, yielding an identical equation with $I = I_h$. The equation becomes

$$\prod_r [a_r] \times \prod_{r=0}^{n-1} [r(I_L+1)n^{-1}]u(n, m_{I_L}; w) = e^{n\pi i}w \prod_r [a_r - 1]u(n, m_{I_L}; w);$$

so if we replace

$$u(n, m_{I_h}; w) \equiv \prod_r [a_r - 1]u(n, m_{I_L}; w)$$

and

$$\prod_r [a_r] \times \prod_{r=0}^{n-1} [r(I_L+1)n^{-1}] \equiv \prod_{r=0}^{n-1} [r(I_h+1)n^{-1}] \times \prod_r [a_r - 1],$$

the theorem is proved.

To commence with, we consider the case when $(n, I_h+1) = 1$, namely those values of I such that $I+1$ is prime to n. The value $I = 0$ is therefore included, so $I_L = 0$, and the basic equation becomes

$$[0][n^{-1}][2n^{-1}]\dots[(n-1)n^{-1}]u(n, 0; w) = e^{n\pi i}wu(m, 0; w). \quad (22)$$

The operator to be produced on the left-hand side is

$$[0][(I_h+1)n^{-1}][2(I_h+1)n^{-1}]\dots[(n-1)(I_h+1)n^{-1}], \quad (23)$$

where $h = 1$ in the present investigation. To produce this, we insert operators on the left-hand side of (22), ensuring that, when the operators (23) are extracted, there remain operators identical to those produced on the right-hand side of (22). In fact, we use a series of contiguous operators, namely those appearing in (22) but with unity

https://doi.org/10.1017/S0305004100049124 Published online by Cambridge University Press

successively added until the appropriate form (23) is attained. That it is attained is simple to prove.

Arranging the new operators in columns, we obtain

$$\left.\begin{array}{llll} [0] & [n^{-1}] & [2n^{-1}] & [3n^{-1}] \ldots u(n,0;w) = e^{n\pi i} w \quad [n^{-1}] \quad [2n^{-1}] \ldots u(n,0;w) \\ & [n^{-1}+1] & [2n^{-1}+1] & [3n^{-1}+1] \ldots \qquad\qquad [n^{-1}+1]\,[2n^{-1}+1] \ldots \\ & [n^{-1}+2] & [2n^{-1}+2] & [3n^{-1}+2] \ldots \qquad\qquad \ldots\ldots\ldots\ldots \\ & \ldots\ldots\ldots\ldots & & \end{array}\right\}. \tag{24}$$

The operators on the right-hand side are identical to those on the left, except for the last entries in each of the n columns. We assert that it is possible to extend these n columns so that the last entries reproduce (in some order) the operators required in (23).

Alternatively, if unity is successively subtracted from these n numbers appearing in (23), then the numbers $0, n^{-1}, 2n^{-1}, \ldots$ are attained without repetition, such numbers being less than unity. For generally, let

$$r(I_h+1)\,n^{-1} = M + pn^{-1},$$
$$s(I_h+1)\,n^{-1} = N + qn^{-1},$$

where $0 \leqslant r, s, p, q \leqslant n-1$, M, N integers. If p and q are never identical, the theorem is proved. If they are identical, subtraction yields

$$(r-s)\,(I_h+1)\,n^{-1} = M - N.$$

But n cannot cancel with I_h+1 since they are relatively prime; moreover, $r-s < n-1$, so n cannot cancel with the integer $r-s$ to yield the integer $M-N$. It is therefore impossible for p and q to be identical when r and s are distinct. Thus the theorem is proved, in that the final operators in the $n-1$ columns on the left-hand side of (24) reproduce (in some order) the operator (23). Consequently the function

$$\begin{array}{rllll} u(n, m_{I_h}; w) \equiv & [n^{-1}] & [2n^{-1}] & \ldots & [(n-1)\,n^{-1}] \quad u(n,0;w) \\ & [n^{-1}+1] & [2n^{-1}+1] & \ldots & [(n-1)\,n^{-1}+1] \\ & [n^{-1}+2] & [2n^{-1}+2] & \ldots & [(n-1)\,n^{-1}+2] \\ & & \ldots\ldots\ldots\ldots\ldots\ldots & & \\ & [(I_h+1)\,n^{-1}-1] & \ldots & [(n-1)\,(I_h+1)\,n^{-1}-1] & \end{array}$$

(in some order)

satisfies equation (21) with $I = I_h$ and $I+1$ prime to n.

Every such solution defined by the parameters n and I_h is thereby expressed in terms of derivatives of the elementary function

$$\exp(-e^{2\pi i q/n} z) \equiv u(n,0;z). \tag{25}$$

The operators will operate on function (25) with the prior substitution $w = (z/n)^n$. There will be produced the exponential function together with a factor consisting of a truncated descending series. The corresponding function $u(n, m_{I_h}; z)$ is obtained by substituting (after the operation) the change of variable

$$w = z^{(m_{I_h}+n)}/(m_{I_h}+n)^n.$$

https://doi.org/10.1017/S0305004100049124 Published online by Cambridge University Press

The number of individual operations involved is clearly seen. It equals

$$\sum_{r=1}^{n-1}[r(I_h+1)n^{-1}]-\sum_{r=1}^{n-1}(rn^{-1}),$$

since unity is being added successively to each bracket on the left-hand side of (24). This equals

$$\sum_{r=1}^{n-1}(rI_hn^{-1})=\tfrac{1}{2}(n-1)I_h.$$

This is, of course, an integer. If n is odd, the 2 cancels with the $(n-1)$; if n is even, then I_h must be even (the 2 thereby cancelling) in order that I_h+1 should be prime to n.

The actual asymptotic expression for $u(n, m_{I_h}; w)$ (namely, without the truncated descending series) involves the operator

$$w^{\frac{1}{2}(n-1)I_h}\left(\frac{d}{dw}\right)^{\frac{1}{2}(n-1)I_h}$$

operating only on exponential functions and not on powers of w multiplying these exponential functions.

As an example, consider the case when $n=6$, $I=4$, $h=1$. Then

$$[0]\quad[\tfrac{1}{6}]\quad[\tfrac{2}{6}]\quad[\tfrac{3}{6}]\quad[\tfrac{4}{6}]\quad[\tfrac{5}{6}]$$

is transformed to

$$[0]\quad[\tfrac{5}{6}]\quad[\tfrac{10}{6}]\quad[\tfrac{15}{6}]\quad[\tfrac{20}{6}]\quad[\tfrac{25}{6}]$$

by the scheme

$$\begin{array}{ccccc}[0] & [\frac{1}{6}] & [\frac{2}{6}] & [\frac{3}{6}] & [\frac{4}{6}] \quad [\frac{5}{6}],\\ & [\frac{7}{6}] & [\frac{8}{6}] & [\frac{9}{6}] & [\frac{10}{6}] \\ & [\frac{13}{6}] & [\frac{14}{6}] & [\frac{15}{6}] & \\ & [\frac{19}{6}] & [\frac{20}{6}] & & \\ & [\frac{25}{6}] & & & \end{array}$$

giving

$$u(6,m_4;w)\equiv\begin{array}{ccccc}[\frac{1}{6}] & [\frac{2}{6}] & [\frac{3}{6}] & [\frac{4}{6}] & u(6,0;w),\\ [\frac{7}{6}] & [\frac{8}{6}] & [\frac{9}{6}] & & \\ [\frac{13}{6}] & [\frac{14}{6}] & & & \\ [\frac{19}{6}] & & & & \end{array}$$

the number of additional operators involved being

$$\tfrac{1}{2}(n-1)I_h=\tfrac{1}{2}\times 5\times 4=10.$$

5. *The second theorem when* $(I_h+1, n)=h\neq 1$.

The proof given in the previous section when $h=1$ must now be modified. We return to equation (21). Write $n/h=p$, $(I_h+1)/h=q$; hence $(I_L+1)/h=1$, where

$$(I_L+1, n)=(I_h+1, n)=h\neq 1.$$

https://doi.org/10.1017/S0305004100049124 Published online by Cambridge University Press

Then the n operators (21)

$$[0] \quad [(I_L+1)n^{-1}] \quad [2(I_L+1)n^{-1}] \quad \dots \quad [(n-1)(I_L+1)n^{-1}]$$

must be transformed to

$$[0] \quad [(I_h+1)n^{-1}] \quad [2(I_h+1)n^{-1}] \quad \dots \quad [(n-1)(I_h+1)n^{-1}].$$

Divide both of these sets of operators into h equal groups, each consisting of p operators; the first two groups will be

$$\begin{array}{l} [0] \quad [(I_L+1)n^{-1}] \dots [(p-1)(I_L+1)n^{-1}] \,\vdots\, [p(I_L+1)n^{-1}] \dots [(2p-1)(I_L+1)n^{-1}] \\ [0] \quad [(I_h+1)n^{-1}] \dots [(p-1)(I_h+1)n^{-1}] \,\vdots\, [p(I_h+1)n^{-1}] \dots [(2p-1)(I_h+1)n^{-1}] \end{array}.$$

The first group in each row becomes

$$\left.\begin{array}{llll} [0] & [p^{-1}] & [2p^{-1}] & \dots \quad [(p-1)p^{-1}], \\ [0] & [qp^{-1}] & [2qp^{-1}] & \dots \quad [(p-1)qp^{-1}]. \end{array}\right\} \tag{26}$$

The theorem proved in the last section is obviously appropriate to this first group.

The second group is obtained from the first by adding 1 to all operators in the first row of the first group (26), and q to all operators in the second row of the first group (26). Because q is an integer, and since the theorem holds for the first group, then it will also hold for the second group. Similarly for all subsequent groups.

The number of additional operators required in the first group to transform the first row into the second is $\frac{1}{2}(p-1)(q-1)$ as before. The second group contains this number of terms plus an additional rectangular block (by its mode of construction) with p terms across and $q-1$ vertically, namely $p(q-1)$ operators. The third group will contain this rectangular block twice, and so on up to the hth group. The total is then

$$\tfrac{1}{2}(p-1)(q-1)h + p(q-1)\sum_{r=1}^{h-1} r = \tfrac{1}{2}h(q-1)(ph-1).$$

To illustrate this theorem, consider the case when $n = 16$, $I_L = 3$, $I_h = 11$, $h = 4$. Then $p = 4$, $q = 3$. The operator, to be divided into $h = 4$ groups,

$$[0] \quad [\tfrac{4}{16}] \quad [\tfrac{8}{16}] \quad [\tfrac{12}{16}] \,\vdots\, [\tfrac{16}{16}] \quad [\tfrac{20}{16}] \quad [\tfrac{24}{16}] \quad [\tfrac{28}{16}] \,\vdots\, [\tfrac{32}{16}] \quad [\tfrac{36}{16}] \quad [\tfrac{40}{16}] \quad [\tfrac{44}{16}] \,\vdots\, [\tfrac{48}{16}] \quad [\tfrac{52}{16}] \quad [\tfrac{56}{16}] \quad [\tfrac{60}{16}]$$

is to be transformed into the operator

$$[0] \quad [\tfrac{12}{16}] \quad [\tfrac{24}{16}] \quad [\tfrac{36}{16}] \,\vdots\, [\tfrac{48}{16}] \quad [\tfrac{60}{16}] \quad [\tfrac{72}{16}] \quad [\tfrac{84}{16}] \,\vdots\, [\tfrac{96}{16}] \quad [\tfrac{108}{16}] \quad [\tfrac{120}{16}] \quad [\tfrac{132}{16}] \,\vdots\, [\tfrac{144}{16}] \quad [\tfrac{156}{16}] \quad [\tfrac{168}{16}] \quad [\tfrac{180}{16}].$$

The required blocks and additional operators will be

$$\begin{array}{cccc:cccc:cccc:cccc}
[0] & [\tfrac{4}{16}] & [\tfrac{8}{16}] & [\tfrac{12}{16}] & [\tfrac{16}{16}] & [\tfrac{20}{16}] & [\tfrac{24}{16}] & [\tfrac{28}{16}] & [\tfrac{32}{16}] & [\tfrac{36}{16}] & [\tfrac{40}{16}] & [\tfrac{44}{16}] & [\tfrac{48}{16}] & [\tfrac{52}{16}] & [\tfrac{56}{16}] & [\tfrac{60}{16}] \\
\multicolumn{2}{c}{[\tfrac{20}{16}] \quad [\tfrac{24}{16}]} & & & [\tfrac{32}{16}] & [\tfrac{36}{16}] & [\tfrac{40}{16}] & [\tfrac{44}{16}] & [\tfrac{48}{16}] & [\tfrac{52}{16}] & [\tfrac{56}{16}] & [\tfrac{60}{16}] & [\tfrac{64}{16}] & [\tfrac{68}{16}] & [\tfrac{72}{16}] & [\tfrac{76}{16}] \\
[\tfrac{36}{16}] & & & & [\tfrac{48}{16}] & [\tfrac{52}{16}] & [\tfrac{56}{16}] & [\tfrac{60}{16}] & [\tfrac{64}{16}] & [\tfrac{68}{16}] & [\tfrac{72}{16}] & [\tfrac{76}{16}] & [\tfrac{80}{16}] & [\tfrac{84}{16}] & [\tfrac{88}{16}] & [\tfrac{92}{16}] \\
\multicolumn{4}{c:}{} & [\tfrac{68}{16}] & [\tfrac{72}{16}] & & & [\tfrac{80}{16}] & [\tfrac{84}{16}] & [\tfrac{88}{16}] & [\tfrac{92}{16}] & [\tfrac{96}{16}] & [\tfrac{100}{16}] & [\tfrac{104}{16}] & [\tfrac{108}{16}] \\
\multicolumn{4}{c:}{} & [\tfrac{84}{16}] & & & & [\tfrac{96}{16}] & [\tfrac{100}{16}] & [\tfrac{104}{16}] & [\tfrac{108}{16}] & [\tfrac{112}{16}] & [\tfrac{116}{16}] & [\tfrac{120}{16}] & [\tfrac{124}{16}] \\
\multicolumn{8}{c:}{} & [\tfrac{116}{16}] & [\tfrac{120}{16}] & [\tfrac{128}{16}] & [\tfrac{132}{16}] & [\tfrac{136}{16}] & [\tfrac{140}{16}] \\
\multicolumn{8}{c:}{} & [\tfrac{132}{16}] & & [\tfrac{144}{16}] & [\tfrac{148}{16}] & [\tfrac{152}{16}] & [\tfrac{156}{16}] \\
\multicolumn{12}{c:}{} & [\tfrac{164}{16}] & [\tfrac{168}{16}] \\
\multicolumn{12}{c:}{} & [\tfrac{180}{16}] &
\end{array}$$

https://doi.org/10.1017/S0305004100049124 Published online by Cambridge University Press

The additional number of operators is given by

$$\tfrac{1}{2}h(q-1)(ph-1) = 60,$$

as may be checked by counting the number of operators lying under the first row.

The solution $u(16, m_3; w)$ is transformed into $u(16, m_{11}; w)$ by using all the operators in this scheme except those 16 lying at the bottom of the 16 columns. Using in both cases the appropriate relations between w and z, this shows how solutions of the equation

$$\frac{d^{16}u}{dz^{16}} = z^{-12}u$$

are related to solutions of the equation

$$\frac{d^{16}u}{dz^{16}} = z^{-44/3}u.$$

6. *The third theorem: transformation of the order of the differential equations.*

We finally consider the transformation properties between solutions of equations of different orders but with a common value h, by discussing and proving the following

THEOREM III. *There exist differential operators that transform solutions of the differential equation of order h*

$$\frac{d^h u}{dz^h} = e^{h\pi i} z^{-(h-1)}u$$

into solutions of equation (1), *where*

$$n = Rh, \quad m = \frac{Rh(I_h + MRh)}{1 + I_h + MRh},$$

R and M being integers, and $(Rh, I_h + 1) = h$.

The basic equation lying within the first cycle is taken to be that yielding Stokes multipliers equivalent to the coefficients of the expansion of $(1+x)^h$, with no interspersed zeros. The equation is governed (see the first table in section 3 and the last row for each value of n) by the parameters $n = h$, $I = h-1$, $m = -(h-1)$, namely

$$\frac{d^h u[h, -(h-1); z]}{dz^h} = (-1)^h z^{-(h-1)} u[h, -(h-1); z],$$

or, when $w = z$, $\vartheta \equiv wd/dw$,

$$\vartheta(\vartheta-1)(\vartheta-2)\ldots[\vartheta-(h-1)]\,u[h, -(h-1); w] = e^{h\pi i} wu[h, -(h-1); w],$$

or, expressed in our abbreviated notation,

$$[0]\ [1]\ [2]\ \ldots\ [h-1]u[h, -(h-1); w] = e^{h\pi i} wu[h, -(h-1); w]. \tag{27}$$

Section 2, dealing with the general transformation cycle, has shown how further equations of the same order may be produced, but with the same Stokes multipliers. Equation (8) becomes

$$\frac{1}{m_N + h} = 1 + N, \tag{28}$$

https://doi.org/10.1017/S0305004100049124 Published online by Cambridge University Press

giving the equation $$\frac{d^h u(h, m_N; z)}{dz^h} = (-1)^h z^{m_N} u(h, m_N; z),$$

or when $$w = z^{m_N+h}(m_N+h)^h = (1+N)^h z^{1/(1+N)},$$

$$[0][1+N][2(1+N)]\dots[(h-1)(1+N)]\, u(h, m_N; w) = e^{h\pi i} w u(h, m_N; w). \qquad (29)$$

(27) may be transformed to (29) by using result (18).

Consider now an equation of higher order n, but possessing the same Stokes multipliers with zeros interspersed. Within its first cycle, values of I_h are allowed such that the highest common factor $(n, I_h+1) = h$. The corresponding value of m_{I_h} is given by $-nI_h/(I_h+1)$. Further equations may be formed possessing the same Stokes multipliers by employing the transformations of the periodic cycles, namely by using values m_M satisfying

$$\frac{1}{m_M+n} = \frac{1}{m_I+n} + M,$$

where M is an integer, or

$$m_M = \frac{m_I - Mn(m_I+n)}{1+M(m_I+n)} = -\frac{n(I_h+Mn)}{1+I_h+Mn}.$$

If $n = Rh$, we choose in particular in (28) the special value of N given by $R-1$. Then we may form the following hierarchy of equations:

	Order	I	m	Notes
(i)	h	$h-1$	$-(h-1)$	At end of first cycle
(ii)	h	—	$\frac{1}{R}-h \equiv m_{R-1}$	Obtained after $N = R-1$ periodic transformations from (i)
(iii)	$n = Rh$	$I_L = h-1$	$-R(h-1)$	With $(n, I_L+1) = h$
(iv)	$n = Rh$	I_h	$-\frac{RhI_h}{I_h+1}$	With $(n, I_h+1) = h$ within the first cycle
(v)	$n = Rh$	—	$-\frac{Rh(I_h+MRh)}{1+I_h+MRh}$	Obtained after M periodic transformations from (iv)

These equations are related amongst themselves:

(i) is transformed to (ii) by using section 2;

(ii) and (iii) are not, as yet, related;

(iii) is transformed to (iv) by using sections 4 or 5;

(iv) is transformed to (v) by using section 2.

Thus the basic equation (i) is transformed into the most general equation (v), all solutions possessing the same Stokes multipliers given by the coefficients appearing in $(1+x)^h$, provided (iii) can be obtained from (ii). That this is always possible we now proceed to show.

The equation corresponding to (iii) is

$$[0]\quad \left[\frac{I_L+1}{n}\right]\dots\left[\frac{(n-1)(I_L+1)}{n}\right] u(n, m_{I_L}; w) = e^{n\pi i} w u(n, m_{I_L}; w),$$

https://doi.org/10.1017/S0305004100049124 Published online by Cambridge University Press

namely, when $n/(I_L+1) = n/h = R$,

$$[0] \quad \left[\frac{1}{R}\right]\left[\frac{2}{R}\right]\dots\left[\frac{n-1}{R}\right] u(n, m_{I_L}; w) = e^{n\pi i} w u(n, m_{I_L}; w). \tag{30}$$

We divide the operators on the left into h groups, each containing R operators:

$$\vdots\ [0] \quad \left[\frac{1}{R}\right]\dots\left[\frac{R-1}{R}\right] \vdots \left[\frac{R}{R}\right]\dots\left[\frac{2R-1}{R}\right] \vdots\dots\vdots \left[\frac{(h-1)R}{R}\right]\dots\left[\frac{n-1}{R}\right] \vdots. \tag{31}$$

Consider now equation (29) with $N = R-1$:

$$[0][R][2R]\dots[(h-1)R]u(h, m_{R-1}; w) = e^{h\pi i} w u(h, m_{R-1}; w). \tag{32}$$

All these operators on its left occur, when divided by R, in the larger set (31). We now apply the operator $[1][R+1]\dots[(h-1)R+1]$, yielding on the right-hand side of (32),

$$\begin{aligned} e^{h\pi i}[1][R+1]\dots[(h-1)R+1] w u(h, m_{R-1}; w) &= e^{h\pi i} w [0][R]\dots[(h-1)R] u(h, m_{R-1}; w) \\ &= e^{2h\pi i} w^2 u(h, m_{R-1}; w). \end{aligned}$$

Similarly, the operator $[2][R+2]\dots[(h-1)R+2]$ will yield $e^{3h\pi i} w^3 u$ on the right-hand side. This process is continued, the final operator $[R-1][2R-1]\dots[n-1]$ yielding $e^{Rh\pi i} w^R u$. The resulting equation is

$$[0]\ [1]\ [2]\dots[n-1]u(h, m_{R-1}; w) = e^{Rh\pi i} w^R u(h, m_{R-1}; w),$$

similar to equation (30).

We finally introduce the change of independent variable

$$\begin{aligned} w^R &= e^{n\pi i} e^{-Rh\pi i} R^n t \\ &= R^n t, \end{aligned}$$

with

$$w\frac{d}{dw} = Rt\frac{d}{dt},$$

giving

$$[0]\left[\frac{1}{R}\right]\dots\left[\frac{n-1}{R}\right] u(h, m_{R-1}; Rt^{1/R}) = e^{n\pi i} t u(h, m_{R-1}; Rt^{1/R}).$$

Hence we conclude that the transformation from (ii) to (iii) is given by

$$u(n, m_{I_L}; w) = u(h, m_{R-1}; Rw^{1/R}),$$

that is,

$$u(Rh, -R(h-1); w) = u(h, R^{-1}-h; Rw^{1/R}). \tag{33}$$

The set of transformations is therefore complete.

As a concluding illustration, let $h = 3$, the basic equation being

$$\frac{d^3 u(3, -2; z)}{dz^3} = -z^{-2} u(3, -2; z),$$

with Stokes multipliers 1, 3, 3 apart from the exponential factors. In operator notation, the equation is

$$[0]\ [1]\ [2]\ u(3, -2; w) = -wu(3, -2; w).$$

https://doi.org/10.1017/S0305004100049124 Published online by Cambridge University Press

Taking $R = 4$, after 3 periodic cycles we obtain the further equation

$$[0]\ [3]\ [6]\ u(3, -\tfrac{11}{4}; w) = -wu(3, -\tfrac{11}{4}; w). \tag{34}$$

Consider the equation of order 12, with $h = 3$, the lowest value of I_h within the first cycle being $I_L = 2$. In terms of z the equation is

$$\frac{d^{12}u(12, -8; z)}{dz^{12}} = z^{-8}u(12, -8; z).$$

The Stokes multipliers will be 1, 0, 0, 0, 3, 0, 0, 0, 3, 0, 0, 0, apart from the exponential factors. The equation may be upgraded within the first cycle to higher values of I_h, and then it may be upgraded using periodic cycles, still maintaining the same set of Stokes multipliers. In terms of w, the equation using I_L may be written

$$[0]\ [\tfrac{1}{4}]\ [\tfrac{2}{4}]\ \dots\ [\tfrac{11}{4}]u(12, -8; w) = wu(12, -8; w). \tag{35}$$

Operating on equation (34) four times in succession, we have

$$[1]\ [4]\ [7]\ [0]\ [3]\ [6]\ u(3, -\tfrac{11}{4}; w) = w^2u(3, -\tfrac{11}{4}; w),$$

and

$$[0]\ [1]\ [2]\ \dots\ [11]\ u(3, -\tfrac{11}{4}; w) = w^4u(3, -\tfrac{11}{4}; w).$$

Finally, the change of independent variable $w = 4t^{\frac{1}{4}}$ yields equation (35), where

$$u(12, -8; w) = u(3, -\tfrac{11}{4}; 4w^{\frac{1}{4}}).$$

REFERENCES

(1) DINGLE, R. B. *Asymptotic expansions: their derivation and interpretation* (Academic Press; London and New York, 1973).
(2) HEADING, J. *Proc. Cambridge Philos. Soc.* **53** (1957), 399.
(3) HEADING, J. *Proc. Cambridge Philos. Soc.* **53** (1957), 419.
(4) HEADING, J. *Proc. Cambridge Philos. Soc.* **56** (1960), 329.
(5) LUKE, Y. L. *The special functions and their approximations*, vol. 1 (Academic Press; London and New York, 1969).
(6) SLATER, L. J. *Generalized hypergeometric functions* (Cambridge, 1966).
(7) WATSON, G. N. *A treatise on the theory of Bessel functions*, 2nd ed. (Cambridge, 1944).

https://doi.org/10.1017/S0305004100049124 Published online by Cambridge University Press