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On directional Whitney inequality

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Abstract. This paper studies a new Whitney type inequality on a compact domain $\Omega \subset \mathbb{R}^d$ that takes the form

$$\inf_{Q\in\Pi^d_{r-1}(\mathcal{E})} \|f-Q\|_p \le C(p,r,\Omega)\omega^r_{\mathcal{E}}(f,\operatorname{diam}(\Omega))_p, \quad r\in\mathbb{N}, \quad 0<p\le\infty,$$

where $\omega^r_{\mathcal{E}}(f,t)_p$ denotes the rth order directional modulus of smoothness of $f\in L^p(\Omega)$ along a finite set of directions $\mathcal{E}\subset\mathbb{S}^{d-1}$ such that $\operatorname{span}(\mathcal{E})=\mathbb{R}^d$, $\Pi^d_{r-1}(\mathcal{E}):=\{g\in C(\Omega): \omega^r_{\mathcal{E}}(g,\operatorname{diam}(\Omega))_p=0\}$. We prove that there does not exist a universal finite set of directions $\mathcal{E}$ for which this inequality holds on every convex body $\Omega\subset\mathbb{R}^d$, but for every connected C^2-domain $\Omega\subset\mathbb{R}^d$, one can choose $\mathcal{E}$ to be an arbitrary set of d independent directions. We also study the smallest number $\mathcal{N}_d(\Omega)\in\mathbb{N}$ for which there exists a set of $\mathcal{N}_d(\Omega)$ directions $\mathcal{E}$ such that $\operatorname{span}(\mathcal{E})=\mathbb{R}^d$ and the directional Whitney inequality holds on Ω for all $r\in\mathbb{N}$ and $p>0$. It is proved that $\mathcal{N}_d(\Omega)=d$ for every connected C^2-domain $\Omega\subset\mathbb{R}^d$, for $d=2$ and every planar convex body $\Omega\subset\mathbb{R}^2$, and for $d\ge 3$ and every almost smooth convex body $\Omega\subset\mathbb{R}^d$. For $d\ge 3$ and a more general convex body $\Omega\subset\mathbb{R}^d$, we connect $\mathcal{N}_d(\Omega)$ with a problem in convex geometry on the X-ray number of Ω, proving that if Ω is X-rayed by a finite set of directions $\mathcal{E}\subset\mathbb{S}^{d-1}$, then $\mathcal{E}$ admits the directional Whitney inequality on Ω for all $r\in\mathbb{N}$ and $0<p\le\infty$. Such a connection allows us to deduce certain quantitative estimate of $\mathcal{N}_d(\Omega)$ for $d\ge 3$.

A slight modification of the proof of the usual Whitney inequality in literature also yields a directional Whitney inequality on each convex body $\Omega\subset\mathbb{R}^d$, but with the set $\mathcal{E}$ containing more than $(cd)^{d-1}$ directions. In this paper, we develop a new and simpler method to prove the directional Whitney inequality on more general, possibly nonconvex domains requiring significantly fewer directions in the directional moduli.

1 Introduction

1.1 Definitions and notations

Let $\mathbb{S}^{d-1}:=\{x\in\mathbb{R}^d:\|x\|=1\}$ denote the unit sphere of $\mathbb{R}^d$. Here and throughout the paper, $\|\cdot\|$ denotes the Euclidean norm. Given $x\in\mathbb{R}^d$ and $r>0$, let $B_r(x):=\{y\in\mathbb{R}^d: \|x-y\|<r\}$ and $B_r[x]:=\overline{B_r(x)}$, where $\overline{\Omega}$ denotes the closure of $\Omega\subset\mathbb{R}^d$. A convex body in $\mathbb{R}^d$ is a compact convex subset of $\mathbb{R}^d$ with nonempty interior. Let Ω be a nonempty bounded measurable set in $\mathbb{R}^d$. We denote by $L^p(\Omega)$, $0<p<\infty$, the usual

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Lebesgue L^p-space on Ω. In the limiting case, we set $L^\infty(\Omega) = C(\overline{\Omega})$, the space of all continuous functions on $\overline{\Omega}$ equipped with the uniform norm.

The rth order directional modulus of smoothness of $f \in L^p(\Omega)$ in the direction of $\xi \in \mathbb{S}^{d-1}$ is defined by

$$\omega^r_\xi(f,t)_{L^p(\Omega)} := \sup_{|u|\le t} \|\triangle^r_{u\xi} f\|_{L^p(\Omega_{r,u\xi})}, \quad t>0, \quad 0<p\le\infty,$$

where $\Omega_{r,h} := \{x \in \Omega : \ x, x+h, \dots, x+rh \in \Omega\}$, and

$$\triangle^r_h f(x) := \sum_{j=0}^{r} (-1)^{r+j} \binom{r}{j} f(x+jh), \quad x \in \Omega_{r,h}, \quad h \in \mathbb{R}^d \backslash \{0\}.$$

Given a set of $\mathcal{E} \subset \mathbb{S}^{d-1}$ of directions, define

$$\omega^r_{\mathcal{E}}(f,t)_{L^p(\Omega)} := \sup_{\xi\in\mathcal{E}} \omega^r_\xi(f,t)_{L^p(\Omega)} \ \text{ and } \ \omega^r_{\mathcal{E}}(f;\Omega)_p := \omega^r_{\mathcal{E}}(f,\operatorname{diam}(\Omega))_{L^p(\Omega)},$$

where $\operatorname{diam}(\Omega) := \sup_{\xi,\eta\in\Omega} \|\xi-\eta\|$. In the case of $\mathcal{E} = \mathbb{S}^{d-1}$, we write $\omega^r(f,t)_p := \omega^r_{\mathbb{S}^{d-1}}(f,t)_p$.

Let Π^d_n denote the space of all real algebraic polynomials in d variables of total degree at most n. For $f \in L^p(\Omega)$ and $0<p\le\infty$, we define

$$E_r(f)_{L^p(\Omega)} := \inf\left\{\|f-q\|_p : \ q \in \Pi^d_{r-1}\right\}, \quad r=1,2,\dots$$

1.2 Whitney inequalities

The usual Whitney type inequality deals with approximation of a function f on a connected compact domain $\Omega \subset \mathbb{R}^d$ by polynomials of total degree $< r$, with the error $E_r(f)_{L^p(\Omega)}$ estimated in terms of the modulus of smoothness $\omega^r(f;\operatorname{diam}(\Omega))$. It takes the form

$$E_r(f)_{L^p(\Omega)} \le C\omega^r(f,\operatorname{diam}(\Omega))_p, \quad \forall f \in L^p(\Omega), \tag{1.1}$$

and allows one to obtain good approximation of f for each fixed r if $\operatorname{diam}(\Omega)$ is small. Note that then the Whitney inequality (1.1) is, in fact, an equivalence, because $\omega^r(Q;q)_p = 0$ for each $q \in \Pi^d_{r-1}$, and

$$\omega^r(f;\Omega)_p = \inf_{q\in\Pi^d_{r-1}} \omega^r(f-q;\Omega)_p \le C_{r,p} E_r(f)_{L^p(\Omega)}.$$

In many applications, it is very important to have certain quantitative estimates of the smallest constant C for which (1.1) holds. Such a constant is called the Whitney constant and is denoted by $w_r(\Omega)_p$. Thus,

$$w_r(\Omega)_p := \sup\left\{E_r(f)_{L^p(\Omega)} : \ f \in L^p(\Omega) \text{ and } \omega^r(f;\Omega)_p \le 1\right\}. \tag{1.2}$$

It can be easily seen that for each nonsingular affine transformation $T : \mathbb{R}^d \to \mathbb{R}^d$,

$$w_r(\Omega)_p = w_r(T(\Omega))_p. \tag{1.3}$$

The Whitney inequality had been studied extensively in various settings in literature. In the case of one variable, the classical Whitney theorem asserts that

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$w_r([0,1])_p \le C_{p,r} < \infty$ for all $r \in \mathbb{N}$ and $0 < p \le \infty$. This result was first proved by Whitney [30] for $p = \infty$, extended by Yu. Brudnyi [7] to $1 \le p < \infty$, and by Storozhenko [28] to $0 < p < 1$. For $p = \infty$, Yu. Brudnyi [7] also proved the estimate $w_r([0,1])_\infty \le Cr^{2r}$ for all $r \in \mathbb{N}$. This estimate was subsequently improved by a research team (K. Ivanov, Binev, and Takev) headed by Sendov [27] who finally showed that $w_r([0,1])_\infty \le 6$ (see [17, 21, 27] and the references therein). The best known result in this direction is due to Gilewicz, Kryakin, and Shevchuk [17], who proved that $w_r([0,1])_\infty \le 2 + e^{-2}$ for all $r \in \mathbb{N}$.

A multivariate extension of the Whitney inequality, proved by Yu. Brudnyi [6] for $p \ge 1$ and by Dekel and Leviatan [11] for the full range of $0 < p \le \infty$, asserts that $w_r(\Omega)_p \le C(p,d,r) < \infty$ for all $r \in \mathbb{N}$ and every convex body $\Omega \subset \mathbb{R}^d$. A similar estimate with the constant depending on certain geometric parameters of the domain Ω was obtained earlier by Yu. Brudnyi [6] for $p \ge 1$. In the case of $p = \infty$, Yu. Brudnyi and Kalton [8] obtained some sharp estimates of the Whitney constant $w_r(\Omega)_\infty$ for a convex body $\Omega \subset \mathbb{R}^d$ in terms of the dimension d. For more results on multivariate Whitney inequalities, we refer to [4, 5, 13, 16].

1.3 Directional Whitney inequalities

The main purpose in this paper is to establish a proper generalization of the Whitney inequality (1.1) for directional moduli of smoothness along a finite number of directions on a possibly nonconvex domain Ω. Such a generalization is motivated by a recent work of the current authors in an upcoming paper, where we establish both a direct Jackson inequality and its matching inverse for a new directional modulus of smoothness on a compact C^2-domain, and the usual Whitney inequality is not enough for our purposes.

The generalization requires approximation of functions from a space $\Pi^d_{r-1}(\mathcal{E})$ that is larger than Π^d_{r-1} and depends on the set $\mathcal{E} \subset \mathbb{S}^{d-1}$ of selected directions. To see this, assume that Ω is the closure of a connected open set. Then, a function $f \in L^p(\Omega)$ belongs to the space Π^d_{r-1} if and only if $\omega^r(f;\Omega)_p = 0$. However, for directional moduli, the equality $\omega^r_{\mathcal{E}}(f;\Omega)_p = 0$ may hold for functions $f \notin \Pi^d_{r-1}$, and therefore, the inequality

$$E_r(f)_{L^p(\Omega)} \le C\omega^r_{\mathcal{E}}(f;\Omega)_p$$

is, in general, not correct. For example, if $e_1 := (1,0,\dots,0), \dots, e_d = (0,\dots,0,1) \in \mathbb{R}^d$ and $\mathcal{E} = \{e_1,\dots,e_d\}$, then $\omega^r(f;\Omega)_p = 0$ if and only if f is a polynomial of degree $< r$ in each variable.

Definition 1.1 Given a set of directions $\mathcal{E} \subset {}^{d-1}$, we define $\Pi^d_{r-1}(\mathcal{E})$ to be the set of all real continuous functions Q on $\mathbb{R}^d$ such that for each fixed $x \in \mathbb{R}^d$ and $\xi \in \mathcal{E}$, the function $g(t) := Q(x + t\xi)$ is an algebraic polynomial of degree $< r$ in the variable $t \in \mathbb{R}$.

Clearly, $\Pi^d_{r-1}(\mathcal{E})$ is a linear subspace of $C(\Omega)$. Let us now state certain basic properties of this subspace in the following proposition, which will be proved in Section 2.

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Proposition 1.1 (i) *If* $\operatorname{span}(\mathcal{E}) \neq \mathbb{R}^d$, *then* $\dim \Pi^d_{r-1}(\mathcal{E}) = \infty$.

(ii) *If* $\operatorname{span}(\mathcal{E}) = \mathbb{R}^d$, *i.e.,* $\mathcal{E}$ *contains some* d *linearly independent directions, then* $\Pi^d_{r-1}(\mathcal{E})$ *is a finite-dimensional space of algebraic polynomials depending on* $\mathcal{E}$. *Moreover,* $\dim \Pi^d_{r-1}(\mathcal{E}) \le r^d$ *and* $\Pi^d_{r-1}(\mathcal{E}) \subset \Pi^d_{d(r-1)}$, *i.e., the monomials in any element from* $\Pi^d_{r-1}(\mathcal{E})$ *always have total degree not exceeding* $d(r-1)$.

In this paper, unless otherwise stated, we will always assume that $\operatorname{span}(\mathcal{E}) = \mathbb{R}^d$, so that $\dim \Pi^d_{r-1}(\mathcal{E}) < \infty$.

Remark 1.2 By the definition, if Ω is the closure of a connected open set, then a function $g \in L^p(\Omega)$ belongs to the space $\Pi^d_{r-1}(\mathcal{E})$ if and only if $\omega^r_{\mathcal{E}}(g;\Omega)_p = 0$.

Now, given a set of directions $\mathcal{E} \subset \mathbb{S}^{d-1}$, we define the best approximation of $f \in L^p(\Omega)$ by functions from the space $\Pi^d_{r-1}(\mathcal{E})$ in the L^p-metric by

$$E_r(f;\mathcal{E})_{L^p(\Omega)} := \inf\left\{\|f - Q\|_{L^p(\Omega)} : \ \ Q \in \Pi^d_{r-1}(\mathcal{E})\right\}, \ \ 0 < p \le \infty.$$

By Remark 1.2, we have

$$\omega^r_{\mathcal{E}}(f;\Omega)_p \le C_{p,r} E_r(f;\mathcal{E})_{L^p(\Omega)}.$$

Thus, an appropriate generalization of the Whitney inequality for directional moduli of smoothness takes the form

$$E_r(f;\mathcal{E})_{L^p(\Omega)} \le C\omega^r_{\mathcal{E}}(f,\Omega)_p, \tag{1.4}$$

where $C > 0$ is a constant independent of f.

For $r \in \mathbb{N}$, $0 < p \le \infty$, and a nonempty set $\mathcal{E} \subset \mathbb{S}^{d-1}$, we define the directional Whitney constant by

$$w_r(\Omega;\mathcal{E})_p := \sup\left\{E_r(f;\mathcal{E})_{L^p(\Omega)} : \ \ f \in L^p(\Omega), \ \ \omega^r_{\mathcal{E}}(f;\Omega)_p \le 1\right\}. \tag{1.5}$$

We say a set $\mathcal{E} \subset \mathbb{S}^{d-1}$ of directions admits a directional Whitney inequality on a domain $\Omega \subset \mathbb{R}^d$ if $w_r(\Omega;\mathcal{E})_p < \infty$ for all $p > 0$ and $r \in \mathbb{N}$.

Clearly, $w_r(\Omega)_p = w_r(\Omega;\mathbb{S}^{d-1})_p$, and

$$w_r(\Omega;\mathcal{E})_p = w_r(\Omega;\mathcal{E} \cup (-\mathcal{E}))_p. \tag{1.6}$$

Furthermore, it can be easily verified that for each nonsingular affine transformation $Tx = Ax + x_0$, $x, x_0 \in \mathbb{R}^d$, $A \in \mathbb{R}^{d\times d}$, we have

$$w_r(\Omega;\mathcal{E})_p = w_r(T(\Omega);\mathcal{E}_T)_p, \ \ \text{where } \mathcal{E}_T := \{A^{-1}\xi/\|A^{-1}\xi\| : \ \ \xi \in \mathcal{E}\}. \tag{1.7}$$

In this paper, we will develop a new method, which allows us to deduce a directional Whitney inequality (1.4) on a domain $\Omega \subset \mathbb{R}^d$ from a directional Whitney inequality on a geometrically simpler subdomain.

Remark 1.3 Ditzian and Ivanov [14] studied the equivalence of the moduli of smoothness $\omega^r(f,t)_{L^p(\Omega)}$ and $\omega^r_{\mathcal{E}}(f,t)_{L^p(\Omega)}$ for $p \ge 1$. They proved that under certain conditions on the set $\mathcal{E}$ of directions and the domain Ω,

$$C^{-1}\omega^r(f,t)_{L^p(\Omega)} \le \omega^r_{\mathcal{E}}(f,t)_{L^p(\Omega)} \le C\omega^r(f,t)_{L^p(\Omega)}, \ \ 0 < t < 1, \ \ p \ge 1, \tag{1.8}$$

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where $C > 0$ is a constant independent of t and f. This result, in particular, implies $\Pi^d_{r-1}(\mathcal{E}) = \Pi^d_{r-1}$. However, such an equivalence is not applicable to the case of $p < 1$, and often requires a larger number of directions. For example, for a cube in $\mathbb{R}^d$, the cone-type conditions in [14] may require $\geq c2^d$ different directions in $\mathcal{E}$. Our results in this paper show that the directional Whitney inequality (1.4) normally requires fewer directions.

1.4 Organization of the paper

This paper is organized as follows. Several preliminary results are proved in Section 2. A key lemma in this section asserts that if (K, J) is a pair of measurable subsets of $\mathbb{R}^d$ satisfying that $\bigcup_{j=1}^r (J - jh) \subset K$ for some $h \in \mathbb{R}^d \backslash \{0\}$ such that $h/\|h\| \in \mathcal{E} \subset \mathbb{S}^{d-1}$, then

$$w_r(K \cup J; \mathcal{E})_p^\theta \leq 1 + 2^r w_r(K; \mathcal{E})_p^\theta, \quad 0 < p \leq \infty, \quad \theta := \min\{p, 1\}. \tag{1.9}$$

This estimate can be applied iteratively to obtain a directional Whitney inequality on a general connected domain from a known Whitney inequality on a geometrically simpler subdomain (e.g., the rectangular box). The idea here plays a crucial role in latter sections.

In Section 3, to illustrate the idea of our method, we generalize the ordinary Whitney inequality to a class of more general, possibly nonconvex compact domains. Our proof is simpler than that in [11].

In Section 4, we prove that a set $\mathcal{E} \subset \mathbb{S}^{d-1}$ with $\operatorname{span}(\mathcal{E}) = \mathbb{R}^d$ is a universal set of directions that admits the directional Whitney inequality on every convex body in $\mathbb{R}^d$ if and only if $\overline{\mathcal{E} \cup (-\mathcal{E})} = \mathbb{S}^{d-1}$. This means that one has to take the domain into consideration when choosing the directions for the Whitney inequality. We also prove in Section 4 that if $\mathcal{E}_0 \subset \mathbb{S}^{d-1}$ is a set of directions that admits the directional Whitney inequality on a domain $\Omega \subset \mathbb{R}^d$, so is any larger set $\mathcal{E}$ of directions containing $\mathcal{E}_0$.

In Section 5, we prove that every set of d linearly independent directions is a universal set of directions that admits the directional Whitney inequality on all connected C^2-domains in $\mathbb{R}^d$.

Finally, in Section 6, we study the smallest number $\mathcal{N}_d(G) \in \mathbb{N}$ for each given convex body $G \subset \mathbb{R}^d$ such that there exists a set $\mathcal{E}$ of $\mathcal{N}_d(G)$ directions with $\operatorname{span}(\mathcal{E}) = \mathbb{R}^d$ for which $w_r(G; \mathcal{E})_p < \infty$ for all $p > 0$ and $r \in \mathbb{N}$. For $d = 2$, we prove that $\mathcal{N}_2(G) = 2$. For $d \geq 3$, we connect the number $\mathcal{N}_d(G)$ with a problem in convex geometry on the X-raying number $X(G)$ of $G \subset \mathbb{R}^d$, proving that if a convex body $G \subset \mathbb{R}^d$ is X-rayed by a finite set of directions $\mathcal{E} \subset \mathbb{S}^{d-1}$, then $w_r(\Omega; \mathcal{E})_p < \infty$ for all $r \in \mathbb{N}$ and $0 < p \leq \infty$. This, in particular, implies that $\mathcal{N}_d(\Omega) \leq X(\Omega)$. The connection also allows us to show that $\mathcal{N}_d(K) = d$ for a class of almost smooth convex bodies $K \subset \mathbb{R}^d$, and to obtain certain quantitative upper estimates of the number $\mathcal{N}_d(G)$ for more general convex bodies $G \subset \mathbb{R}^d$. The problem considered in this section may have potential applications in approximation of large data.

2 Preliminary results

We start with the proof of Proposition 1.1.

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Proof of Proposition 1.1 (i) If $\operatorname{span}(\mathcal{E}) \neq \mathbb{R}^d$, then there exists $\xi_0 \in {}^{d-1}$ such that $\xi_0 \cdot \xi = 0$ for all $\xi \in \mathcal{E}$. This implies that every ridge function of the form $f(x) := g(x \cdot \xi_0)$ with $g \in C(\mathbb{R})$ is contained in the space $\Pi^d_{r-1}(\mathcal{E})$, and thus $\dim \Pi^d_{r-1}(\mathcal{E}) = \infty$. (ii) Suppose $\operatorname{span}(\mathcal{E}) = \mathbb{R}^d$ and $\{\xi_1, \dots, \xi_d\} \subset \mathcal{E}$ is a set of d linearly independent directions. It is easy to see that the statement we need to prove is invariant under any affine change of variables (in particular, the space $\Pi^d_{d(r-1)}$ is), so we can assume that ξ_j is the jth basic unit vector. Then, for any $f \in \Pi^d_{r-1}(\mathcal{E})$, the directional modulus of smoothness $\omega^r_{\mathcal{E}}(f;\Omega)_\infty$ is zero for Ω being any compact parallelepiped with sides parallel to the coordinate axes, so by [13, Lemma 2.1] we obtain that f belongs to the space of polynomials of degree $< r$ in each of the d variables. This space has dimension r^d and is clearly a subspace of $\Pi^d_{d(r-1)}$. ∎

We will also need the lemma we just used in a somewhat more general form.

Lemma 2.1 [13, Lemma 2.1] *Let S be a compact parallelepiped in $\mathbb{R}^d$, and let $\mathcal{E}_S \subset \mathbb{S}^{d-1}$ denote the set of all edge directions of S. Then,*

$$w_r(S;\mathcal{E}_S)_p \le C(p,d,r) < \infty, \quad r \in \mathbb{N}, \quad 0 < p \le \infty.$$

Proof Lemma 2.1 was proved in [13, Lemma 2.1] in the case when S is a rectangular box. The more general case follows from (1.7). ∎

Our second lemma will be used repeatedly in this paper and is the heart of the matter.

Lemma 2.2 *Let $r \in \mathbb{N}$ and $\mathcal{E} \subset \mathbb{S}^{d-1}$. Assume that (K, J) is a pair of measurable subsets of $\mathbb{R}^d$ satisfying that $J \subset \bigcap_{j=1}^r (K + jh)$ for some $h \in \mathbb{R}^d \backslash \{0\}$ such that $h/\|h\| \in \mathcal{E}$. Then, for any $0 < p \le \infty$,*

$$w_r(K \cup J;\mathcal{E})^\theta_p \le 1 + 2^r w_r(K;\mathcal{E})^\theta_p, \quad \theta := \min\{p,1\}. \tag{2.1}$$

Proof Given a function $F : K \cup J \to \mathbb{R}$, we have

$$(-1)^r \triangle^r_h F(\xi - rh) = \triangle^r_{-h} F(\xi) = \sum_{j=0}^r (-1)^j \binom{r}{j} F(\xi - jh), \quad \xi \in J,$$

and thus,

$$|F(\xi)| \le |\triangle^r_h F(\xi - rh)| + \sum_{j=1}^r \binom{r}{j} |F(\xi - jh)|, \quad \forall \xi \in J.$$

Because $J \subset \bigcap_{j=1}^r (K + jh)$, it follows that

$$\|F\|^\theta_{L^p(J)} \le \|\triangle^r_h F\|^\theta_{L^p(J-rh)} + (2^r - 1)\|F\|^\theta_{L^p(K)},$$

which, in turn, implies

$$\|F\|^\theta_{L^p(K\cup J)} \le \|\triangle^r_h F\|^\theta_{L^p(J-rh)} + 2^r \|F\|^\theta_{L^p(K)}. \tag{2.2}$$

Because $\triangle^r_h Q = 0$ for each $Q \in \Pi^d_{r-1}(\mathcal{E})$, using (2.2) with $f - Q$ in place of F, we deduce

$$\|f - Q\|^\theta_{L^p(K\cup J)} \le \|\triangle^r_h f\|^\theta_{L^p(J-rh)} + 2^r \|f - Q\|^\theta_{L^p(K)}, \quad \forall Q \in \Pi^d_{r-1}(\mathcal{E}).$$

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It follows that

$$\begin{aligned} E_r(f;\mathcal{E})^\theta_{L^p(K\cup J)} &\le \omega^r_{h/\|h\|}(f;K\cup J)^\theta_p + 2^r E_r(f,\mathcal{E})^\theta_{L^p(K)} \\ &\le \omega^r_{h/\|h\|}(f;K\cup J)^\theta_p + 2^r w_r(K;\mathcal{E})^\theta_p \omega^r_{\mathcal{E}}(f;K)^\theta_p \\ &\le \omega^r_{\mathcal{E}}(f;K\cup J)^\theta_p\big(1+2^r w_r(K;\mathcal{E})^\theta_p\big), \end{aligned}$$

which proves (2.1). ∎

To establish a directional Whitney type inequality on a domain $\Omega \subset \mathbb{R}^d$ for a set of directions $\mathcal{E} \subset \mathbb{S}^{d-1}$, we often need to apply the estimate (2.1) iteratively. Indeed, we have the following useful lemma.

Lemma 2.3 *Let $\mathcal{E} \subset \mathbb{S}^{d-1}$ be a set of directions. Assume that $\Omega \subset \mathbb{R}^d$ is a finite union $\Omega = \bigcup_{k=0}^m E_k$ of measurable subsets $E_k \subset \mathbb{R}^d$ satisfying that for each $1 \le k \le m$, there exists a vector $h_k \in \mathbb{R}^d\backslash\{0\}$ whose direction lies in $\mathcal{E}$ such that*

$$E_k \subset \bigcap_{j=1}^r (\Omega_{k-1} + jh_k), \tag{2.3}$$

where $\Omega_k := \bigcup_{j=0}^k E_j$, $k = 0, 1, \dots, m$. Then,

$$w_r(\Omega;\mathcal{E})^\theta_p \le 2^{mr} w_r(\Omega_0;\mathcal{E})^\theta_p + \frac{2^{mr}-1}{2^r-1}. \tag{2.4}$$

Proof With the decomposition $\Omega = \bigcup_{k=0}^m E_k$, we may apply (2.1) recursively to the pairs of sets $(K, J) = (\Omega_{k-1}, E_k)$, $k = 1, 2, \dots, m$, to obtain

$$w_r(\Omega_k;\mathcal{E})^\theta_p \le 1 + 2^r w_r(\Omega_{k-1};\mathcal{E})^\theta_p, \quad k = 1, 2, \dots, m.$$

Inequality (2.4) then follows. ∎

Remark 2.4 To apply Lemma 2.3 to establish a directional Whitney inequality in a domain, the crucial step is to decompose the domain Ω as a finite union $\Omega = \bigcup_{k=0}^m E_k$, so that the condition (2.3) is satisfied. With this approach, we often choose an initial set E_0 to be geometrically simpler, so that its directional Whitney constant $w_r(E_0;\mathcal{E})_p$ is known to be finite. For instance, according to Lemma 2.1, we may choose E_0 to be any compact parallelepiped in $\mathbb{R}^d$. We need to assume that the direction of h_j for each j lies in the set $\mathcal{E}$. If too many directions are involved, we may end up with $\Pi^d_{r-1}(\mathcal{E}) = \Pi^d_{r-1}$.

For latter applications, we introduce the following definition.

Definition 2.1 Given $\xi \in \mathbb{S}^{d-1}$, we say $G \subset \mathbb{R}^d$ is a regular ξ-directional domain with parameter $L \ge 1$ if there exists a rotation $\boldsymbol{\rho} \in SO(d)$ such that

(1) $\boldsymbol{\rho}(0, \dots, 0, 1) = \xi$, and G takes the form

$$G := \boldsymbol{\rho}\big(\{(x, y) : \ x \in D,\ g_1(x) \le y \le g_2(x)\}\big), \tag{2.5}$$

where $D \subset \mathbb{R}^{d-1}$ is compact and $g_i : D \to \mathbb{R}$ are measurable;

(2) there exist an affine function $H : \mathbb{R}^{d-1} \to \mathbb{R}$ and a constant $\delta > 0$ such that $S \subset G \subset S_L$, where

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$$\boldsymbol{\rho}^{-1}(S) := \{(x,y):\ \ x\in D,\ \ H(x)-\delta\le y\le H(x)+\delta\}, \tag{2.6}$$

$$\boldsymbol{\rho}^{-1}(S_L) := \{(x,y):\ \ x\in D,\ \ H(x)-L\delta\le y\le H(x)+L\delta\}. \tag{2.7}$$

In this case, we say S is the base of G.

Lemma 2.5 *Let $G\subset\mathbb{R}^d$ be a regular ξ-directional domain with parameter $L\ge 1$ and base S as given in Definition 2.1 for some $\xi\in\mathbb{S}^{d-1}$. Let $\mathcal{E}\subset\mathbb{S}^{d-1}$ be a set of directions containing ξ. Assume that K is a measurable subset of $\mathbb{R}^d$ such that $S\subset K\cap G$ and $w_r(K;\mathcal{E})_p<\infty$ for some $r\in\mathbb{N}$, $0<p\le\infty$. Then,*

$$w_r(G\cup K;\mathcal{E})_p\le C_{p,r}L^{r-1+2/p}(1+w_r(K;\mathcal{E})_p), \tag{2.8}$$

where the constant $C_{p,r}$ depends only on p and r.

Proof Without loss of generality, we may assume $\xi=e_d:=(0,\dots,0,1)$ and $\boldsymbol{\rho}=I$ in Definition 2.1. Let $A_1:=w_r(K;\mathcal{E})_p$, let $f\in L^p(K\cup G)$, and let $Q\in\Pi^d_{r-1}(\mathcal{E})$ be such that

$$\|f-Q\|_{L^p(K)}\le A_1\omega^r_{\mathcal{E}}(f;K)_p. \tag{2.9}$$

It is enough to show that

$$\|f-Q\|_{L^p(G)}\le C_{p,r}L^{r-1+2/p}(1+A_1)\omega^r_{\mathcal{E}}(f;K\cup G)_p. \tag{2.10}$$

For simplicity, we may assume that $p<\infty$. (The case $p=\infty$ can be treated similarly.) By the Whitney inequality in one variable (see [12, pp. 183, 374]), for each fixed $x\in D$, there exists $\varphi_x\in\Pi^1_{r-1}$ such that

$$\begin{aligned}\int_{g_1(x)}^{g_2(x)}|f(x,y)-\varphi_x(y)|^p\,dy&\le C_{p,r}\sup_{0<hr<g_2(x)-g_1(x)}\int_{g_1(x)}^{g_2(x)-rh}|\triangle^r_{he_d}f(x,y)|^p dy\\&\le\frac{C_{p,r}}{g_2(x)-g_1(x)}\int_0^{(g_2(x)-g_1(x))/r}\Big[\int_{I_{x,hr}}|\triangle^r_{he_d}f(x,y)|^p dy\Big]dh,\end{aligned}$$

where $I_{x,rh}:=\big\{y\in\mathbb{R}:\ \ y,y+rh\in[g_1(x),g_2(x)]\big\}$, and the second step uses (5.17) of [12, p. 373]. Because $S\subset G\subset S_L$, we have $\delta\le g_2(x)-g_1(x)\le 2L\delta$ for any $x\in D$. It follows that

$$\begin{aligned}\int_D\int_{g_1(x)}^{g_2(x)}|f(x,y)-\varphi_x(y)|^p\,dydx&\le\frac{C_{p,r}}{\delta}\int_0^{2L\delta}\int_D\int_{I_{x,rh}}|\triangle^r_{he_d}f(x,y)|^p dydxdh\\&\le C_{p,r}L\cdot\omega^r_{e_d}(f;G)^p_p.\end{aligned} \tag{2.11}$$

Thus,

$$\begin{aligned}\|f-Q\|^p_{L^p(G)}&=\int_D\int_{g_1(x)}^{g_2(x)}|f(x,y)-Q(x,y)|^p\,dydx\\&\le C_{p,r}L\cdot\omega^r_{e_d}(f;G)^p_p+2^p\int_D\int_{g_1(x)}^{g_2(x)}|\varphi_x(y)-Q(x,y)|^p\,dydx.\end{aligned}$$

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Because $e_d \in \mathcal{E}$, $Q(x,y) \in \Pi^d_{r-1}(\mathcal{E})$ is an algebraic polynomial of degree $< r$ in the variable $y \in \mathbb{R}$. It follows by the Remez inequality in one variable that

$$\int_D \int_{g_1(x)}^{g_2(x)} |\varphi_x(y) - Q(x,y)|^p \, dy dx \\ \le C_{p,r} L^{(r-1)p+1} \int_D \int_{H(x)-\delta}^{H(x)+\delta} |\varphi_x(y) - Q(x,y)|^p \, dy dx,$$

which, using (2.9) and (2.11), is estimated above by

$$\le C_{p,r} L^{(r-1)p+2} \cdot \omega^r_{e_d}(f;G)^p_p + C_{p,r} L^{(r-1)p+1} \|f - Q\|^p_{L^p(S)} \\ \le C_{p,r} L^{(r-1)p+2} \cdot \omega^r_{e_d}(f;G)^p_p + C_{p,r} L^{(r-1)p+1} A_1^p \omega^r_{\mathcal{E}}(f;K)^p_p \\ \le C_{p,r} L^{(r-1)p+2} (1 + A_1^p) \omega_{\mathcal{E}}(f; K \cup G)^p_p.$$

This proves (2.10). ∎

Remark 2.6 If, in addition, we assume the domain D in Definition 2.1 is a compact parallelepiped in $\mathbb{R}^{d-1}$, then we may choose $K = S$ and use Lemma 2.1 to obtain

$$w_r(G;\mathcal{E})_p \le C(p,r) L^{r-1+2/p} < \infty, \tag{2.12}$$

where $\mathcal{E} \subset \mathbb{S}^{d-1}$ is the set of the edge directions of S.

3 The ordinary multivariate whitney inequality

It was shown in [11] that for all $r \in \mathbb{N}$ and $p > 0$, $\sup_K w_r(K)_p \le C_{p,d} < \infty$, where the supremum is taken over all convex bodies in $\mathbb{R}^d$. To illustrate the main idea of our method, we extend this result to a class of more general, and possibly nonconvex domains.

Definition 3.1 A domain $\Omega \subset \mathbb{R}^d$ is star-shaped with respect to a ball B in $\mathbb{R}^d$ if for each $x \in \Omega$, the closed convex hull of $\{x\} \cup B$ is contained in Ω. Given a constant $R > 1$, we say a domain $\Omega \subset \mathbb{R}^d$ belongs to the class $\mathcal{S}^d(R)$ if there exists an affine transformation $T : \mathbb{R}^d \to \mathbb{R}^d$ such that $T(\Omega)$ is a star-shaped domain with respect to the unit ball $B_1[0]$ in $\mathbb{R}^d$ and is contained in the ball $B_R[0]$.

Remark 3.1 There are many nonconvex sets in the class $\mathcal{S}^d(R)$. A simple example is as follows. Let $A = \{x_1, x_2\} \subset R^{d-1}$ be a set of two points distance δ apart, where $0 < \delta < 2\sqrt{R^2 - 1}$. Take Ω to be the union over $x \in A$ of the convex hulls of $\{x\} \cup B_1[0]$. Then, Ω is not convex and belongs to $\mathcal{S}^d(R)$. Various other choices of A are possible.

According to John's theorem ([11, Proposition 2.5], [22]), every convex body in $\mathbb{R}^d$ belongs to the class $\mathcal{S}^d(d)$.

Theorem 3.2 *Given $r \in \mathbb{N}$, $0 < p \le \infty$, and $R > 1$, there exists a constant $C(p,d,r,R)$ depending only on p, d, r, and R when $R \to \infty$ such that*

$$\sup_{\Omega \in \mathcal{S}^d(R)} w_r(\Omega)_p \le C(p,d,r,R) < \infty. \tag{3.1}$$

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Proof By (1.7), without loss of generality, we may assume that Ω is a star-shaped domain with respect to the unit ball $B_1[0]$ and $\Omega \subset B_R[0]$. Following [15], we may decompose Ω as a finite union $\Omega = \bigcup_{j=0}^{m} G_j$ of $(m+1) \le (CRd)^{d-1}$ regular directional domains G_j with parameters $\le C_d R$ such that $G_0 := [-\frac{1}{\sqrt{d}}, \frac{1}{\sqrt{d}}]^d$, and the base of each G_j, $j \ge 1$, is contained in G_0. Indeed, the decomposition can be constructed as follows. First, cover the sphere $\{\xi \in \mathbb{R}^d : \ \|\xi\| = R\}$ with $m \le (CRd)^{d-1}$ open balls $B_1, \dots, B_m$ of radius $\frac{1}{2d}$ in $\mathbb{R}^d$. Denote by ξ_j the unit vector pointing from the origin to the center of the ball B_j, and let I_j denote a $(d-1)$-dimensional cube with side length $\frac{1}{d}$ centered at the origin and perpendicular to ξ_j. For each point $x \in \Omega \backslash B_{1/(2d)}[0]$, the perpendicular from x onto I_j belongs to the closed convex hull of x and $B_{1/(2d)}[0]$, where j is such that B_j contains $x\frac{R}{\|x\|}$. Indeed, this readily follows from the fact that the two parallel lines having the direction ξ_j and passing through the origin and through x are at most $\frac{1}{2d}$ far apart. Define $G_0 := [-\frac{1}{\sqrt{d}}, \frac{1}{\sqrt{d}}]^d$ and

$$G_j := \left\{\eta + t\xi_j \in \Omega : \ \ \eta \in I_j, \ \ t \ge 0\right\}, \ \ j = 1, \dots, m.$$

Because Ω is star-shaped with respect to the unit ball $B_1[0]$ and $B_1[0] \subset \Omega \subset B_R[0]$, it is easily seen that $\Omega = \bigcup_{j=0}^{m} G_j$ and each G_j, $j \ge 1$ is a regular ξ_j-directional domain with parameter $\le C_d R$ and base

$$S_j := \{\eta + t\xi_j : \ \ \eta \in I_j, \ \ 0 \le t \le \sqrt{d-1}/(2d)\} \subset G_0.$$

Now, let $\Omega_j := \bigcup_{k=0}^{j} G_k$ for $j = 0, 1, \dots, m$. Applying Lemma 2.5 iteratively to the pairs $(G, K) = (G_j, \Omega_{j-1})$ for $j = 1, \dots, m$, we obtain

$$w_r(\Omega_j)_p = w_r(\Omega_{j-1} \cup G_j)_p \le C(p, r, d, R)\big(1 + w_r(\Omega_{j-1})_p\big), \ \ j = 1, 2, \dots, m.$$

By Lemma 2.1, this implies that

$$w_r(\Omega)_p = w_r(\Omega_m)_p \le C(p, r, d, R)\big(1 + w_r(\Omega_0)_p\big) \le C(p, r, d, R) < \infty. \qquad \blacksquare$$

Remark 3.3 The above proof also yields a directional Whitney inequality with the number of directions required in the directional modulus $\ge (cdR)^{d-1}$. Indeed, from the above proof, if $\Omega \subset B_R[0]$ is star-shaped with respect to the unit ball $B_1[0]$, and $\mathcal{E} \subset \mathbb{S}^{d-1}$ is a finite set of directions satisfying that $\mathbb{S}^{d-1} \subset \bigcup_{\xi \in \mathcal{E}} B_{1/2Rd}(\xi_j)$, then $w_r(\Omega; \mathcal{E})_p \le C(p, r, d, R) < \infty$ for all $r \in \mathbb{N}$ and $0 < p \le \infty$.

4 On choices of directions

As pointed out in Remark 3.3, the proof of Theorem 3.2 yields a directional Whitney inequality on each convex body, but with the set of directions depending on the domain. It is therefore natural to ask whether there exists a universal set of directions $\mathcal{E} \ne \mathbb{S}^{d-1}$ for which $\operatorname{span}(\mathcal{E}) = \mathbb{R}^d$ and $w_r(\Omega; \mathcal{E})_p < \infty$ holds for all $r \in \mathbb{N}$, $p > 0$, and every convex body $\Omega \subset \mathbb{R}^d$. We will prove two results related to this question in this section.

Theorem 4.1 *Let $d \ge 2$ and let $\mathcal{E} \subset \mathbb{S}^{d-1}$ be a set of directions such that $\operatorname{span}(\mathcal{E}) = \mathbb{R}^d$ and $\overline{\mathcal{E} \cup (-\mathcal{E})} \ne \mathbb{S}^{d-1}$. Then, there exists a convex body $K \subset \mathbb{R}^d$ such that $w_r(K; \mathcal{E})_\infty = \infty$ for all $r \in \mathbb{N}$.*

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By Theorem 4.1, given any $r \in \mathbb{N}$, there does not exist a universal finite set of directions $\mathcal{E}$ such that $\operatorname{span}(\mathcal{E}) = \mathbb{R}^d$ and $w_r(\Omega;\mathcal{E})_\infty < \infty$ for every convex body $\Omega \subset \mathbb{R}^d$. Thus, one has to take the domain into consideration when choosing the directions for the Whitney inequality.

Using Theorem 4.1, Remark 3.3, and John's theorem [11, Proposition 2.5] for convex bodies, we immediately derive the following characterization of universal sets of directions $\mathcal{E}$ for which $w_r(\Omega;\mathcal{E})_p < \infty$ for all $r \in \mathbb{N}$, $p > 0$, and every convex body $\Omega \subset \mathbb{R}^d$.

Corollary 4.2 *Let $d \geq 2$ and let $\mathcal{E} \subset \mathbb{S}^{d-1}$ be a set of directions such that $\operatorname{span}(\mathcal{E}) = \mathbb{R}^d$. Then, in order that $w_r(\Omega;\mathcal{E})_p < \infty$ for every $r \in \mathbb{N}$, $p > 0$ and every convex body $\Omega \subset \mathbb{R}^d$, it is necessary and sufficient that $\overline{\mathcal{E} \cup (-\mathcal{E})} = \mathbb{S}^{d-1}$.*

Our second result shows that the directional Whitney inequality remains valid if the set of directions is enlarged, which, in particular, implies that if $\mathcal{E}_0 \subset \mathbb{S}^{d-1}$ is a universal set of directions for the directional Whitney inequality on a class of domains, so is any larger set of directions $\mathcal{E} \supset \mathcal{E}_0$.

Theorem 4.3 *Let G be the closure of a bounded connected nonempty open set in $\mathbb{R}^d$. Assume that $\mathcal{E}_0 \subset {}^{d-1}$ is a set of directions such that $\operatorname{span}(\mathcal{E}) = \mathbb{R}^d$ and $w_r(G;\mathcal{E}_0)_p < \infty$ for some $r \in \mathbb{N}$ and $0 < p \leq \infty$. Then, for any $\mathcal{E}_0 \subset \mathcal{E} \subset {}^{d-1}$,*

$$w_r(G;\mathcal{E})_p \leq C(w_r(G;\mathcal{E}_0)_p + 1), \tag{4.1}$$

where the constant $C > 0$ depends only on p, r, and the set G.

4.1 Proof of Theorem 4.1

Proof By (1.6), we may assume, without loss of generality, that $\mathcal{E} = -\mathcal{E}$ and $\overline{\mathcal{E}} \neq \mathbb{S}^{d-1}$. Our aim is to find a convex body $K \subset \mathbb{R}^d$ such that

$$w_r(K;\mathcal{E})_\infty = \infty. \tag{4.2}$$

Fix a direction, $\xi \in \mathbb{S}^{d-1} \backslash \overline{\mathcal{E}}$. Clearly, there exists a constant $\delta \in (0,1)$ such that

$$\eta \cdot \xi \leq 1 - \delta, \ \forall \eta \in \mathcal{E}. \tag{4.3}$$

Let $0 < \varepsilon < \delta$ be a constant, and define

$$K := \big\{x \in \mathbb{R}^d : \ \|x\|(1-\varepsilon) \leq x \cdot \xi \leq 1\big\}.$$

Then, K is a convex body in $\mathbb{R}^d$. Consider a sequence of continuous functions on K given by

$$f_n(x) := g_n(x \cdot \xi), \ \ x \in K, \ \ n = 1, 2, \ldots,$$

where $g_n(0) = -n$ and $g_n(t) := \max\{-n, \ln t\}$ for $t > 0$. Clearly, to show (4.2), it is sufficient to prove that

$$\sup_n \omega_{\mathcal{E}}^r(f_n;K)_\infty \leq C_r < \infty, \tag{4.4}$$

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and

$$\lim_{n\to\infty} E_r(f_n;\mathcal{E})_{L^\infty(K)} = \infty. \tag{4.5}$$

To this end, we first claim that if $x, y \in K$ and $(y-x)/\|y-x\| \in \mathcal{E}$, then

$$c^{-1}(x\cdot\xi) \le y\cdot\xi \le c(x\cdot\xi), \tag{4.6}$$

where $c = \frac{2-\delta}{\delta-\varepsilon}\frac{1}{1-\varepsilon}$. Indeed, because $\frac{y-x}{\|y-x\|} \in \mathcal{E}$, we obtain from (4.3) that

$$(y-x)\cdot\xi = \|y-x\|\frac{y-x}{\|y-x\|}\cdot\xi \le (1-\delta)\|y-x\| \le (1-\delta)(\|x\|+\|y\|).$$

On the other hand, by the definition of the set K,

$$(y-x)\cdot\xi = y\cdot\xi - x\cdot\xi \ge (1-\varepsilon)\|y\| - \|x\|.$$

Thus,

$$(1-\varepsilon)\|y\| - \|x\| \le (1-\delta)(\|x\|+\|y\|),$$

which implies

$$\|y\| \le \frac{2-\delta}{\delta-\varepsilon}\|x\|.$$

Inequality (4.6) then follows by symmetry and the inequality

$$(1-\varepsilon)\|z\| \le z\cdot\xi \le \|z\|, \quad \forall z \in K.$$

Next, we prove (4.4). Because

$$\omega^r_{\mathcal{E}}(f_n;K)_\infty \le C_r\omega_{\mathcal{E}}(f_n;K)_\infty,$$

it is enough to prove (4.4) for $r = 1$. Assume that $x, x+s\eta \in K$ for some $s > 0$ and $\eta \in \mathcal{E}$. If both $x\cdot\xi$ and $(x+s\eta)\cdot\xi$ lie in the interval $[0, e^{-n}]$, then $\triangle_{s\eta}f(x) = 0$. If both $x\cdot\xi$ and $(x+s\eta)\cdot\xi$ lie in the interval $[e^{-n}, 1]$, then using (4.6) and the mean value theorem, we have

$$|\triangle_{s\eta}f(x)| = \left|g_n(x\cdot\xi) - g_n\big((x+s\eta)\cdot\xi\big)\right| \le C\frac{(x+s\eta)\cdot\xi + x\cdot\xi}{x\cdot\xi} \le C < \infty.$$

If only one of the numbers $x\cdot\xi$ and $(x+s\eta)\cdot\xi$ lies in the interval $[e^{-n}, 1]$, say, $(x+s\eta)\cdot\xi \ge e^{-n}$ and $x\cdot\xi < e^{-n}$, then there exists a number $0 \le s_0 \le s$ such that $\xi\cdot(x+s_0\eta) = e^{-n}$, and hence,

$$|\triangle_{s\eta}f(x)| = |f(x+s\eta) - f(x+s_0\eta)| = |\triangle_{(s-s_0)\eta}f(x+s_0\eta)| \le C.$$

Thus, in all the cases, we have shown that

$$\sup_{\eta\in\mathcal{E}}\max_{x, x+s\eta\in K}|f(x+s\eta) - f(x)| \le C.$$

Inequality (4.4) for all $r \in \mathbb{N}$ then follows.

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Finally, we prove (4.5). Let $Q \in \Pi^d_{r-1}(\mathcal{E})$ be such that $E_r(f_n)_{L^\infty(K)} = \|f_n - Q\|_{L^\infty(K)}$. Let $h = \frac{1}{dr}\xi$. Because $\operatorname{span}(\mathcal{E}) = \mathbb{R}^d$, we have $\Pi^d_{r-1}(\mathcal{E}) \subset \Pi^d_{d(r-1)}$. Thus, for $n > \ln(dr)$,

$$\begin{aligned} C_r\|f_n - Q\|_{L^\infty(K)} &\geq |\triangle_h^{dr}(f_n - Q)(0)| = |\triangle_h^{dr} f_n(0)| \\ &= \left|\sum_{j=0}^{dr}(-1)^{j+1}\binom{dr}{j} f_n(jh)\right| \geq n - 2^{dr}\ln(dr) \\ &\to \infty, \quad \text{as } n \to \infty. \end{aligned}$$

This shows (4.5). ∎

4.2 Proof of Theorem 4.3

Proof Let $A_0 := w_r(G;\mathcal{E}_0)_p$. Because $\operatorname{span}(\mathcal{E}_0) = \mathbb{R}^d$, it follows by Proposition 1.1 that $\Pi^d_{r-1}(\mathcal{E}_0)$ is a finite-dimensional vector space, and $\Pi^d_{r-1}(\mathcal{E}) \subset \Pi^d_{r-1}(\mathcal{E}_0)$. We claim that

$$E_r(f;\mathcal{E})_{L^p(G)} \leq C\omega^r_{\mathcal{E}}(f;G)_p \quad \text{for any} \quad f \in \Pi^d_{r-1}(\mathcal{E}_0), \tag{4.7}$$

where $C > 0$ depends only on p, r, and G. Indeed, from the definition of $\Pi^d_{r-1}(\mathcal{E})$ and Remark 1.2, for any $g \in C(G)$,

$$\omega^r_{\mathcal{E}}(g;G)_p = 0 \iff g \in \Pi^d_{r-1}(\mathcal{E}) \iff E_r(g;\mathcal{E})_{L^p(G)} = 0.$$

This implies that both the mappings

$$g \mapsto \omega^r_{\mathcal{E}}(g;G)_p^{\min\{1,p\}} \quad \text{and} \quad g \mapsto E_r(g;\mathcal{E})_{L^p(G)}^{\min\{1,p\}}$$

are quasi-norms on the quotient space $\Pi^d_{r-1}(\mathcal{E}_0)/\Pi^d_{r-1}(\mathcal{E})$. Because this space is finite-dimensional, the norms are equivalent and (4.7) follows.

Now, we can show (4.1). Fix $f \in L^p(G)$. Let $Q \in \Pi^d_{r-1}(\mathcal{E}_0)$ be such that

$$\|f - Q\|_{L^p(G)} \leq A_0\omega^r_{\mathcal{E}_0}(f;G)_p.$$

Next, by (4.7), there exists $R \in \Pi^d_{r-1}(\mathcal{E})$ such that

$$E_r(Q;\mathcal{E})_p = \|Q - R\|_{L^p(G)} \leq C\omega^r_{\mathcal{E}}(Q;G)_p.$$

Thus,

$$\begin{aligned} E_r(f;\mathcal{E})_{L^p(G)} &\leq \|f - R\|_{L^p(G)} \leq \|f - Q\|_{L^p(G)} + \|Q - R\|_{L^p(G)} \\ &\leq A_0\omega^r_{\mathcal{E}_0}(f;G)_p + C\omega^r_{\mathcal{E}}(Q;G)_p \\ &\leq A_0\omega^r_{\mathcal{E}_0}(f;G)_p + C\omega^r_{\mathcal{E}}(f;G)_p + C\|f - Q\|_{L^p(G)} \\ &\leq CA_0\omega^r_{\mathcal{E}_0}(f;G)_p + C\omega^r_{\mathcal{E}}(f;G)_p \leq C(A_0+1)\omega^r_{\mathcal{E}}(f;G)_p. \end{aligned}$$

∎

5 Universal sets of directions for smooth domains

By Theorem 4.1, one cannot find a finite set of directions $\mathcal{E}$ satisfying $\operatorname{span}(\mathcal{E}) = \mathbb{R}^d$ that admits the directional Whitney inequality on every convex body in $\mathbb{R}^d$. It is therefore natural to ask that for which domains $G \subset \mathbb{R}^d$ a given set of d linearly

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independent directions $\mathcal{E} \subset \mathbb{S}^{d-1}$ admits the directional Whitney inequality. In this section, we will give an affirmative answer to this question, proving that if Lip-2 condition is imposed on the domain, then the choice of the directions can be an arbitrary set of d linearly independent directions.

Definition 5.1 A compact set $G \subset \mathbb{R}^d$ is said to be a Lip-2 domain with parameter $L > 1$ if there exists a constant $\delta > 0$ such that $\operatorname{diam}(G) \le L\delta$, and each point $x \in G$ is contained in a closed ball $B_x \subset G$ of radius δ.

As is well known, every connected C^2-domain is a Lip-2 domain.

Theorem 5.1 *Let $\mathcal{E} \subset {}^{d-1}$ be a set of d linearly independent directions with*

$$\min_{x \in {}^{d-1}} \max_{\xi \in \mathcal{E}} |\xi \cdot x| \ge \varepsilon_0 > 0. \tag{5.1}$$

If $G \subset \mathbb{R}^d$ is a connected and compact Lip-2 domain with parameter $L > 1$, then for any $0 < p \le \infty$, $r \in \mathbb{N}$, and $f \in L^p(G)$,

$$w_r(G; \mathcal{E})_p \le C(p, d, r, L, \varepsilon_0) < \infty. \tag{5.2}$$

Note that according to Theorem 4.1, Theorem 5.1 is not true without the Lip-2 assumption, at least for $p = \infty$.

The proof of Theorem 5.1 relies on two technical lemmas, which will be stated in the following subsection.

5.1 Two technical lemmas

Given $a > 0$ and a ball $B = B_\delta(x_0)$, we denote by aB the dilation $B_{a\delta}(x_0)$.

Lemma 5.2 *Let G be a Lip-2 domain in $\mathbb{R}^d$ with parameter $L \ge 1$ and constant $\delta > 0$ as given in Definition 5.1. Let $\varepsilon \in (0,1)$ be an arbitrarily given constant. Then, the set G can be represented as a finite union $G = \bigcup_{j=1}^m E_j$ of possibly repeated subsets $E_j \subset G$ such that*

(1) *$m \le C(d, L/\varepsilon) < \infty$;*
(2) *for each $1 \le j \le m$, there exists an open ball B_j of radius δ such that $B_j \subset E_j \subset (1+\varepsilon)B_j$;*
(3) *$E_j \cap E_{j+1}$ contains an open ball of radius $\varepsilon\delta/4$ for each $1 \le j < m$.*

Proof First, we cover the domain G with $m_0 \le C_d(L/\varepsilon)^d$ open balls $B_{\varepsilon\delta/4}(y_j)$, $j = 1, \dots, m_0$ with centers $y_1, \dots, y_{m_0} \in G$. For each j, we can find an open ball $B_j = B_\delta(x_j)$ of radius δ such that $x_j \in \overline{B_j} \subset G$, implying that $B_{\varepsilon\delta/4}(y_j) \subset (1 + \frac{\varepsilon}{4})B_j$. Thus,

$$G = \bigcup_{j=1}^{m_0} G_j, \quad \text{where } G_j := \big((1 + \frac{\varepsilon}{4})B_j\big) \cap G.$$

Because G is connected and each G_j is open relative to the topology of G, we can form a sequence of sets $\{\widehat{E}_j\}_{j=1}^m$ from possibly repeated copies of the sets G_j, $1 \le j \le m_0$, such that $m \le 2m_0^2$,

$$G = \bigcup_{j=1}^{m} \widehat{E}_j \text{ and } \widehat{E}_j \cap \widehat{E}_{j+1} \ne \varnothing \quad j = 1, 2, \dots, m-1.$$

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Next, for each $1 \le j \le m$, we define

$$G_j^* := \big((1+\varepsilon)B_j\big) \cap G.$$

Then,

$$B_j \subset G_j^* \subset (1+\varepsilon)B_j.$$

If $G_i \cap G_j \neq \varnothing$, then

$$\Big(\big(1+\frac{\varepsilon}{4}\big)B_i\Big) \cap \Big(\big(1+\frac{\varepsilon}{4}\big)B_j\Big) \neq \varnothing,$$

and hence $\|x_i - x_j\| \le (2+\frac{\varepsilon}{2})\delta$. Because

$$G_i^* \cap G_j^* = (1+\varepsilon)B_i \cap G_j^* \supset (1+\varepsilon)B_i \cap B_j,$$

this implies that $G_i^* \cap G_j^*$ contains a ball of radius $\varepsilon\delta/4$ if $G_i \cap G_j \neq \varnothing$.

Finally, to complete the proof, we define $E_j = G_i^*$ if $\widehat{E}_j = G_i$. ∎

Lemma 5.3 *Let $\mathcal{E} \subset {}^{d-1}$ be a set of d linearly independent directions such that $\min_{x\in{}^{d-1}} \max_{\xi\in\mathcal{E}} |x\cdot\xi| \ge \varepsilon_0 > 0$. Let $r \in \mathbb{N}$ and $\sigma := 1 + \frac{\varepsilon_0^2}{4r}$. Let (S_0, E) be a pair of bounded measurable subsets of $\mathbb{R}^d$. Let $L \ge 1$ be a given parameter. Assume that there exists an open ball B of radius $L\delta$ for some constant $\delta > 0$ such that $B \subset E \subset \sigma B$ and $S_0 \cap B$ contains an open ball B_0 of radius δ. Then, for any $0 < p \le \infty$,*

$$w_r(S_0 \cup E; \mathcal{E})_p \le C(L, d, p, r, \varepsilon_0)\big(1 + w_r(S_0; \mathcal{E})_p\big). \tag{5.3}$$

The proof of Lemma 5.3 is quite technical, so we postpone it until Section 5.3. For the moment, we take it for granted and proceed with the proof of Theorem 5.1 in the next subsection.

5.2 Proof of Theorem 5.1

Proof First, we prove that $w_r(B; \mathcal{E})_p \le C(p, d, r, \varepsilon_0) < \infty$ for every open ball $B \subset \mathbb{R}^d$. Because dilations and translations do not change the directions in the set $\mathcal{E}$, we may assume that $B = B_1(0)$.

Let $\mathcal{E} = \{\xi_1, \dots, \xi_d\}$. Let A be the $d \times d$ matrix whose jth column vector is ξ_j. Then, by (5.1),

$$\|A^t x\| = \left(\sum_{j=1}^d |x\cdot\xi_j|^2\right)^{\frac{1}{2}} \ge \varepsilon_0\|x\|, \quad \forall x \in \mathbb{R}^d,$$

so from the fact that a matrix and its transpose have the same singular values, we get

$$\varepsilon_0\|x\| \le \left\|\sum_{j=1}^d x_j\xi_j\right\| \le \sqrt{d}\|x\|, \quad \forall x = (x_1, \dots, x_d) \in \mathbb{R}^d.$$

In particular, this implies that the parallelepiped

$$H := \left\{\sum_{j=1}^d x_j\xi_j : \ x = (x_1, \dots, x_d) \in \left[-\frac{1}{d}, \frac{1}{d}\right]^d\right\},$$

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whose edge directions lie in the set $\mathcal{E}$, satisfies $B_{\varepsilon_0/d}[0] \subset H \subset B_1[0]$. Thus, applying Lemma 5.3 to the pair of sets $(S_0, E) = (H, B)$ and the ball $B = B_1(0)$, we obtain

$$w_r(B;\mathcal{E})_p = w_r(H \cup B;\mathcal{E})_p \le C(p,r,d,\varepsilon_0) w_r(H;\mathcal{E})_p.$$

By Lemma 2.1, this implies that $w_r(B;\mathcal{E})_p \le C(p,d,r,\varepsilon_0) < \infty$.

Next, we set $\varepsilon := \frac{\varepsilon_0^2}{4r}$, and apply Lemma 5.2 to write the domain G as a finite union $G = \bigcup_{j=1}^m E_j$ such that $m \le C(d,r,\varepsilon_0) < \infty$, $B_j \subset E_j \subset (1+\varepsilon)B_j$ for an open ball B_j of radius $\delta > 0$, and $E_j \cap E_{j+1}$ contains an open ball of radius $\varepsilon\delta/4$. Setting $E_0 := B_1$, and applying Lemma 5.3 iteratively to the pairs of sets

$$(S_0, E) := \left(\bigcup_{i=0}^{j} E_i, E_{j+1} \right) \quad \text{for} \quad j = 0, 1, \dots, m-1,$$

we obtain

$$w_r\left(\bigcup_{i=0}^{j+1} E_i;\mathcal{E} \right)_p \le C(d,p,r,\varepsilon_0)\left(1 + w_r\left(\bigcup_{i=0}^{j} E_i;\mathcal{E} \right)_p\right), \quad j = 0, 1, \dots, m-1.$$

This implies that

$$\begin{aligned} w_r(G;\mathcal{E})_p = w_r\left(\bigcup_{i=0}^{m} E_i;\mathcal{E} \right)_p &\le C(d,p,r,\varepsilon_0,L)\left(1 + w_r(B_1;\mathcal{E})_p\right) \\ &\le C(d,p,r,\varepsilon_0,L) < \infty. \end{aligned}$$

■

5.3 Proof of Lemma 5.3

Proof Without loss of generality, we assume that $w_r(S_0;\mathcal{E})_p < \infty$ (otherwise, there is nothing to prove). Let $\mathcal{E} = \{\xi_1, \dots, \xi_d\}$, and let $\xi_{d+j} = -\xi_j$ for $1 \le j \le d$. Then,

$$\mathbb{S}^{d-1} := \bigcup_{j=1}^{2d} \{\xi \in \mathbb{S}^{d-1} : \ \xi \cdot \xi_j \ge \varepsilon_0\}. \tag{5.4}$$

We break the proof into several steps.

In the first step, we prove that for each pair (S_0, E) of measurable sets satisfying $B_0 \subset E \subset \sigma B_0$ and $B_0 \subset S_0$,

$$w_r(S_0 \cup E;\mathcal{E})_p \le C(p,r,\varepsilon_0,d)\big(w_r(S_0;\mathcal{E})_p + 1\big), \tag{5.5}$$

which particularly implies (5.3) for the special case of $B = B_0$.

Without loss of generality, we may assume that $B_0 = B_1(0)$. Using (5.4), we may decompose the ball σB_0 as $\sigma B_0 = \bigcup_{j=1}^{2d} A_j$, where

$$A_j := \big\{x \in \mathbb{R}^d : \|x\| < \sigma, \ \ x \cdot \xi_j \ge \varepsilon_0 \|x\|\big\}.$$

We claim that for each $1 \le j \le 2d$ and $h_j := \frac{\varepsilon_0}{2r}\xi_j$,

$$\bigcup_{x \in A_j} [x - r h_j, x - h_j] \subset B_0. \tag{5.6}$$

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Indeed, each $y \in [x - rh_j, x - h_j]$ can be written in the form $y = x - u\xi_j$ with $\frac{\varepsilon_0}{2r} \le u \le \frac{\varepsilon_0}{2}$. Because $x \in A_j$, we have

$$\|y\|^2 \le \|x\|^2 + u^2 - 2\|x\|u\varepsilon_0 \le (\sigma - u\varepsilon_0)^2 + u^2 < 1 - \frac{u\varepsilon_0}{2} + u^2 \le 1,$$

proving the claim (5.6).

Because $E \subset \sigma B_0 = \bigcup_{j=1}^{2d} A_j$, we may decompose $E \cup S_0$ as

$$E \cup S_0 = \bigcup_{j=0}^{2d} E_j, \quad \text{where } E_0 := S_0, \quad E_j := E \cap A_j \text{ for } j \ge 1.$$

Let $\Omega_k := \bigcup_{j=0}^{k} E_j$ for $0 \le k \le 2d$. Using the claim (5.6), we obtain that for any $1 \le k \le m$,

$$\bigcup_{x \in E_k} [x - rh_k, x - h_k] \subset B_0 \subset S_0 \subset \Omega_{k-1},$$

which, in particular, implies that the condition (2.3) is satisfied. Thus, using Lemma 2.3 and (2.4), we deduce the desired estimate (5.5).

In the second step, we prove that for any ball $B_0 \subset S_0$ and constant $a \ge 1$,

$$w_r(S_0 \cup aB_0; \mathcal{E})_p \le C(p, r, d, \varepsilon_0, a)\big(w_r(S_0; \mathcal{E})_p + 1\big). \tag{5.7}$$

To this end, let ℓ be a nonnegative integer such that $\sigma^\ell \le a < \sigma^{\ell+1}$. Let

$$S_{j,0} := S_0 \cup \sigma^{j-1}B_0, \quad E_j := \sigma^j B_0, \quad j = 1, 2, \dots, \ell,$$

and let

$$S_{\ell+1,0} := S_0 \cup \sigma^\ell B_0, \; E_{\ell+1} := aB_0.$$

Note that for $1 \le j \le \ell + 1$,

$$\sigma^{j-1}B_0 \subset E_j \subset \sigma^j B_0 \text{ and } \sigma^{j-1}B_0 \subset S_{j,0}.$$

Thus, applying (5.5) iteratively to the ball $\sigma^{j-1}B_0$ and the pair of sets $(S_{j,0}, E_j)$ for $j = 1, 2, \dots, \ell + 1$, we obtain

$$w_r(S_{j,0} \cup E_j; \mathcal{E})_p \le C(p, r, \varepsilon_0, d)\big(w_r(S_{j,0}; \mathcal{E})_p + 1\big), \quad j = 1, 2, \dots, \ell + 1.$$

Because $S_{1,0} = S_0$, the desired estimate (5.7) then follows.

Finally, in the last step, we prove (5.3) in the general case, where $B = B_{L\delta}(x_0)$ and the center x_0 may not be the same as that of B_0. Without loss of generality, we assume that $B_0 = B_\delta(0)$.

Because

$$B \subset E \subset \sigma B, \; B \subset S_0 \cup B, \text{ and } S_0 \cup E = (S_0 \cup B) \cup E,$$

it follows by the already proven estimate (5.5) that

$$w_r(E \cup S_0; \mathcal{E})_p \le C(p, r, \varepsilon_0, d)\big(w_r(S_0 \cup B; \mathcal{E})_p + 1\big).$$

Thus, it is enough to show that

$$w_r(S_0 \cup B; \mathcal{E})_p \le C(p, r, \varepsilon_0, d, L)\big(w_r(S_0; \mathcal{E})_p + 1\big). \tag{5.8}$$

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We consider the following two cases:

Case 1. $\|x_0\| \le \frac{\delta}{2}$.

In this case,

$$\widetilde{B} := B_{\delta/2}(x_0) \subset B_\delta(0) = B_0 \subset S_0.$$

Because $B = (2L)\widetilde{B}$, (5.8) follows from (5.7).

Case 2. $\|x_0\| > \frac{\delta}{2}$.

In this case, we will construct a set K such that $S_0 \cup \widehat{B} \subset K \subset S_0 \cup B$ with $\widehat{B} := B_{\delta/\sigma}(x_0)$, and

$$w_r(K;\mathcal{E})_p \le C(p,r,\varepsilon_0,d,L)\big(w_r(S_0;\mathcal{E})_p + 1\big). \tag{5.9}$$

For the moment, we assume such a set K is constructed and proceed with the proof of (5.8). Because $(L\sigma)\widehat{B} = B$ and $K \cup B = S_0 \cup B$, we obtain from (5.7) that

$$w_r(S_0 \cup B;\mathcal{E})_p = w_r(K \cup (L\sigma)\widehat{B};\mathcal{E})_p \le C(p,r,\varepsilon_0,d,L)\big(w_r(K;\mathcal{E})_p + 1\big),$$

which combined with (5.9) implies (5.8).

It remains to construct the set K. Because $B_\delta(0) \subset B$, we have $\frac{\delta}{2} < \|x_0\| \le (L-1)\delta$. Let n_0 be a positive integer $< L\sigma/(\sigma-1)$ such that

$$n_0\frac{(\sigma-1)\delta}{\sigma} < \|x_0\| \le (n_0+1)\frac{(\sigma-1)\delta}{\sigma}.$$

Let

$$y_j = j\frac{(\sigma-1)\delta}{\sigma}\frac{x_0}{\|x_0\|}, \quad j = 0,1,\dots,n_0,$$

be the equally spaced points on the line segment $[0,x_0]$. Define $\widehat{B}_j := B_{\delta/\sigma}(y_j)$ for $j = 0,1,\dots,n_0$. A straightforward calculation shows that

$$B_{\delta/\sigma}(x_0) \subset \sigma\widehat{B}_{n_0} \subset B \ \text{ and } \ \widehat{B}_j \subset \sigma\widehat{B}_{j-1} \subset B \ \text{ for } j = 1,\dots,n_0. \tag{5.10}$$

Define $K_0 = S_0$ and $K_j := K_{j-1} \cup \sigma\widehat{B}_j$ for $j = 1,\dots,n_0$. Because $\sigma\widehat{B}_0 = B_0 \subset K_0$ and $\widehat{B}_j \subset \sigma\widehat{B}_{j-1} \subset K_{j-1}$ for $1 \le j \le n_0$, it follows by (5.8) that

$$w_r(K_j;\mathcal{E})_p = w_r(K_{j-1} \cup \sigma\widehat{B}_j;\mathcal{E})_p \le C(p,r,d,\varepsilon_0)\big(w_r(K_{j-1};\mathcal{E})_p + 1\big),$$

where $j = 1,2,\dots,n_0$. This implies that

$$w_r(K_{n_0};\mathcal{E})_p \le C(p,r,d,\varepsilon_0,L)\big(w_r(S_0;\mathcal{E})_p + 1\big),$$

and (5.9) is satisfied with $K := K_{n_0}$. Furthermore, using (5.10), we have

$$S_0 \subset K_{n_0} = \bigcup_{j=0}^{n_0}(S_0 \cup \sigma\widehat{B}_j) \subset S_0 \cup B \ \text{ and } \ B_{\delta/\sigma}(x_0) \subset \sigma\widehat{B}_{n_0} \subset K_{n_0}.$$

Thus, the set $K = K_{n_0}$ has all the desired properties. ∎

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6 Connection with X-ray numbers of convex bodies

Let $d \geq 2$. For each convex body $G \subset \mathbb{R}^d$, we define $\mathcal{N}_d(G)$ to be the smallest number $n \in \mathbb{N}$ for which there exists a set of n directions $\mathcal{E}$ such that $\operatorname{span}(\mathcal{E}) = \mathbb{R}^d$, and $w_r(G;\mathcal{E})_p < \infty$ for all $r \in \mathbb{N}$ and $p > 0$. Clearly, $\mathcal{N}_d(G) \geq d$ for every convex body $G \subset \mathbb{R}^d$. Moreover, by Theorem 5.1, if the convex body $G \subset \mathbb{R}^d$ is Lip-2, then $\mathcal{N}_d(G) = d$.

Next, we define $\mathcal{N}_d$ to be the smallest number $n \in \mathbb{N}$ such that for every convex body $G \subset \mathbb{R}^d$, there exists a set of n directions $\mathcal{E}$ for which $\operatorname{span}(\mathcal{E}) = \mathbb{R}^d$ and $w_r(G;\mathcal{E})_p < \infty$ for all $0 < p \leq \infty$ and $r \in \mathbb{N}$. According to Theorem 4.3, we have

$$\mathcal{N}_d = \sup_G \mathcal{N}_d(G), \tag{6.1}$$

where the supremum is taken over all convex bodies $G \subset \mathbb{R}^d$. Moreover, by Remark 3.3 and John's theorem for convex bodies, we have

$$d \leq \mathcal{N}_d \leq (cd)^{2(d-1)}. \tag{6.2}$$

The upper estimate in (6.2) is far from being optimal, especially as $d \to \infty$. In this section, we will show that such an estimate can be significantly improved, proving that $\mathcal{N}_2 = \mathcal{N}_2(G) = 2$ for every planar convex body $G \subset \mathbb{R}^2$, and $\mathcal{N}_d(G) = d$ for $d \geq 3$ and every "almost smooth" convex body $G \subset \mathbb{R}^d$. A crucial ingredient in our approach for $d \geq 3$ is to establish a connection between the number $\mathcal{N}_d$ with the X-raying numbers of convex bodies from convex geometry.

We start with the following result for $d = 2$, which implies that $\mathcal{N}_2 = \mathcal{N}_2(G) = 2$ for every planar convex body $G \subset \mathbb{R}^2$.

Theorem 6.1 *If G is a convex body in $\mathbb{R}^2$, then there exist two linearly independent vectors $\xi_1, \xi_2 \in {}^1$ such that for all $0 < p \leq \infty$ and $r \in \mathbb{N}$,*

$$w_r(G;\{\xi_1,\xi_2\})_p \leq C(p,r) < \infty, \tag{6.3}$$

where $C(p,r) > 0$ is a constant depending only on p and r.

We will present two short proofs.

Proof A A geometric result of Besicovitch [1] asserts that it is possible to inscribe an affine image of a regular hexagon into any planar convex body. Because (6.3) does not change under affine transform, we may assume the hexagon with the vertices $(\pm 1, \pm 1)$ and $(0, \pm 2)$ (which are the vertices of a regular hexagon after a proper dilation along one of the coordinate axes) is inscribed into G, i.e., each vertex of the hexagon is on the boundary of G. Convexity of G then implies that G contains the hexagon and is contained in a nonconvex 12-gon (resembling the outline of the star of David) obtained by extending the sides of the hexagon, i.e., having additional nodes $(\pm 2, 0)$ and $(\pm 1, \pm 3)$. Choosing $\xi_1 = (1,0)$ and $\xi_2 = (0,1)$, the proof is now completed by two applications of Lemma 2.5 for $S = [-1,1]^2$: first in the direction of ξ_1 with $L = 2$, and second in the direction of ξ_2 with $L = 3$. ∎

Proof B By John's theorem, without loss of generality, we may assume that $B_1[0] \subset G \subset B_2[0]$. Let $a, b \in G$ be such that $\|a - b\| = \operatorname{diam}(G) = L \leq 4$. Let ξ_1 be the unit vector in the direction of $b - a$ and ξ_2 the unit vector that is perpendicular to ξ_1.

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Without loss of generality, we may assume that $\xi_1 = (0,1)$ and $\xi_2 = (1,0)$, because otherwise we may consider a rotation of G. Then, $a = (0,-L_1)$ and $b = (0,L_2)$ for some $L_1, L_2 \geq 1$ such that $L_1 + L_2 = L$, and $S := [-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]^2 \subset B_1[0] \subset G$. Let $G_1 := ([-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}] \times [-L_1, L_2]) \cap G$. Because G is convex, the set G_1 satisfies the conditions of Corollary 2.5 in the direction of e_2. Thus, using Theorem 2.5 and Remark 2.6, we get

$$E_{r-1}(f)_{L^p(G_1)} \leq C(p,r)\omega^r(f, G_1; \{e_1, e_2\})_p.$$

On the other hand, because $\operatorname{diam}(G) = \|a - b\| = L_1 + L_2$, it follows that for each $(x,y) \in G$, $-L_1 \leq y \leq L_2$, which implies that $(0,y) \in G_1$, and $[(0,y),(x,y)] \subset G$. Thus, applying Remark 2.6 in the direction of e_1 with $K = G_1$, we obtain the desired inequality (6.3). ∎

Remark 6.2 Proof A is a bit shorter and provides explicit small values of the parameter L from Lemma 2.5, which would result in a better value of the constant in (6.3). We are unaware of any algorithms for construction of an inscribed affine-regular hexagon into given planar convex body, although the variational proof from [20, Theorem 2.3] may be a basis for such an algorithm. Proof B uses John's theorem, i.e., the largest area ellipse inscribed into the convex body. This ellipse can be constructed algorithmically; see, e.g., [9, 10, 29] and the references therein.

The problem of finding the exact value of the number $\mathcal{N}_d$ for $d \geq 3$ appears to be more challenging. The proofs of Theorem 6.1 above cannot be immediately extended to the higher-dimensional convex bodies.

For the remainder of this section, we assume that $d \geq 3$. Our aim is to relate the directional Whitney inequality to the concept of X-raying from convex geometry, which allows us to deduce improved upper estimates of $\mathcal{N}_d$ for $d \geq 3$ from certain known results on X-ray numbers of convex bodies.

We need to introduce some definitions from convex geometry, which can be found in [2, 3]. Let G be a convex body in $\mathbb{R}^d$. We say a point x on the boundary of G is illuminated in the direction $e \in {}^{d-1}$ if the ray $\{x + te : t \geq 0\}$ intersects the interior of G. A set of directions $\mathcal{E} \subset {}^{d-1}$ is said to illuminate G if every boundary point of G is illuminated in a direction from the set $\mathcal{E}$. One can consider illumination of $G \subset \mathbb{R}^d$ by parallel beams of light. The illumination number $I_d(G)$ of G is the smallest $n \in \mathbb{N}$ for which G can be illuminated by a set of n directions. Next, we say G is X-rayed by a set of directions $\mathcal{E} \subset {}^{d-1}$ if G is illuminated by the set $\mathcal{E} \cup (-\mathcal{E})$, and define the X-ray number $X_d(G)$ of G to be the smallest number $n \in \mathbb{N}$ for which G can be X-rayed by a set of n directions. By the definition, we have

$$X_d(G) \leq I_d(G) \leq 2X_d(G). \tag{6.4}$$

Finally, for $d \geq 3$, we say $G \subset \mathbb{R}^d$ is almost smooth if for each boundary point $x \in \partial G$, and any two outer unit normal vectors u_1, u_2 of the supporting hyperplane(s) of G at x, the inequality $u_1 \cdot u_2 \geq (d-2)/(d-1)$ holds.

In this section, we will prove the following result.

Theorem 6.3 *If $G \subset \mathbb{R}^d$ is a convex body in $\mathbb{R}^d$ X-rayed by a finite set of directions $\mathcal{E} \subset {}^{d-1}$, then for any $r \in \mathbb{N}$ and $0 < p \leq \infty$,*

$$w_r(G; \mathcal{E})_p \leq C_{p,r,G} < \infty. \tag{6.5}$$

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In particular, this implies that for every convex body $G \subset \mathbb{R}^d$,

$$\mathcal{N}_d(G) \le X_d(G). \tag{6.6}$$

We postpone the proof of Theorem 6.3 until Section 6.1. For the moment, we take it for granted and prove a few useful corollaries for $d \ge 3$. First, it was shown in [3, Theorem 2.3] that $X_d(G) = d$ for every almost smooth convex body $G \subset \mathbb{R}^d$. This combined with Theorem 6.3 implies the following corollary.

Corollary 6.4 *If $d \ge 3$ and G is an almost smooth convex body in $\mathbb{R}^d$, then $\mathcal{N}_d(G) = d$.*

Next, we define

$$X_d := \sup_G X_d(G),$$

where the supremum is taken over all convex bodies G in $\mathbb{R}^d$. For the maximal X-ray number X_d, it is conjectured and proved only for $d = 2$ that $X_d \le 3 \cdot 2^{d-2}$ (see [3] and the references therein). A better studied concept is the illumination number, which is related to the X-ray number via (6.4). The well-known illumination conjecture in convex geometry states [2] that

$$I_d := \sup_G I(G) \le 2^d,$$

where the supremum is taken over all convex bodies $G \subset \mathbb{R}^d$. Again, this conjecture was proved for $d = 2$ only (see [2]). Let us provide a summary of known upper bounds for I_d:

- By [23], $I_3 \le 16$.
- The estimates $I_4 \le 96$, $I_5 \le 1091$, and $I_6 \le 15373$ were recently obtained in [24].
- For $d \ge 7$, the following explicit bound follows from the results of Rogers and Shepard [25, 26] (see also [1, Section 2.2]):

$$I_d \le \binom{2d}{d} d(\ln d + \ln\ln d + 5),$$

  where 5 can be replaced with 4 for sufficiently large d.
- The following remarkable asymptotic estimate of I_d with an implicit constant $c_0 > 0$ was obtained very recently in [19]:

$$I_d \le \binom{2d}{d} e^{-c_0\sqrt{d}}.$$

Using (6.4), (6.6), and these known bounds for I_d, we obtain the following corollary.

Corollary 6.5 *For $d \ge 3$, we have*

$$\mathcal{N}_d \le \begin{cases} 16, & \text{if } d = 3, \\ 96, & \text{if } d = 4, \\ 1091, & \text{if } d = 5, \\ 15373, & \text{if } d = 6, \\ \binom{2d}{d} \min\{d(\ln d + \ln\ln d + 5), e^{-c_0\sqrt{d}}\}, & \text{if } d \ge 7. \end{cases}$$

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6.1 Proof of Theorem 6.3

Proof Without loss of generality, we may assume that $\mathcal{E} = -\mathcal{E}$ and $0 \in G^\circ$, where G° denotes the interior of G. For a direction $e \in {}^{d-1}$, we define

$$R(S,e) := \{x - te : x \in S,\ t \geq 0\} \text{ for any } S \subset G,$$
$$\varphi(x,e) := \max\{t \geq 0 :\ \ x - te \in G\} \text{ for any } x \in G.$$

Clearly, $0 \leq \varphi(x,e) \leq \operatorname{diam}(G)$, and

$$R(S,e) \cap G = \{x - te :\ \ x \in S,\ \ 0 \leq t \leq \varphi(x,e)\} \text{ for any } S \subset G.$$

We shall use the following known result from convex geometry (see [18]): there exists an increasing sequence $\{S_n\}_{n=1}^\infty$ of open, convex, and C^2-subsets of G such that $G^\circ = \bigcup_{n=1}^\infty S_n$,

$$\left(1 - \frac{1}{n+1}\right)G \subset S_n \subset \left(1 - \frac{1}{n+2}\right)G, \quad n = 1, 2, \ldots \tag{6.7}$$

(Indeed, one can choose each S_n to be an algebraic domain.) We break the rest of the proof into several steps.

First, we claim that there exists a positive integer n_0 such that

$$G \subset \bigcup_{e \in \mathcal{E}} R(S_{n_0}, e). \tag{6.8}$$

Assume otherwise that for each n,

$$T_n := G \backslash \left(\bigcup_{e \in \mathcal{E}} R(S_n, e) \right) \neq \varnothing.$$

Because $\{T_n\}_{n \geq 1}$ is a decreasing sequence of closed subsets of G, by Cantor's intersection theorem, there exists a point $x \in \bigcap_{n=1}^\infty T_n$. Because $S_n \subset R(S_n, e)$ for any $e \in \mathbb{S}^{d-1}$ and $\bigcup_{n=1}^\infty S_n = G^\circ = G \backslash \partial G$, we have $x \in \partial G$. Thus, x must be illuminated in a direction $e \in \mathcal{E}$, which means that there exists $t \geq 0$ and $e \in \mathcal{E}$ such that $x + te \in G^\circ = \bigcup_{n=1}^\infty S_n$. This implies that $x \in R(S_n, e)$ for some n and direction $e \in \mathcal{E}$, which is impossible as

$$x \in T_n = G \backslash \left(\bigcup_{e \in \mathcal{E}} R(S_n, e) \right), \quad \forall n \in \mathbb{N}.$$

This proves the claim (6.8).

Next, we set $n_1 = n_0 + 2$ and

$$G_e := R(S_{n_0}, e) \cap G, \quad e \in \mathcal{S}.$$

We prove that for each direction $e \in \mathcal{E}$ and measurable set $S_{n_1} \subset S \subset G$,

$$w_r(G_e \cup S; \mathcal{E})_p \leq C(p, r, G)\big(1 + w_r(S; \mathcal{E})_p\big). \tag{6.9}$$

Indeed, by (6.7),

$$\overline{S_{n_0}} \subset \left(1 - \frac{1}{n_0 + 2}\right)G \subset \left(1 - \frac{1}{n_0 + 3}\right)G^\circ \subset S_{n_1}.$$

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Thus, there exists $\delta > 0$ such that

$$\overline{S_{n_0}} + B_{r\delta}[0] \subset S_{n_1}. \tag{6.10}$$

Now, we define

$$\begin{aligned} K_{e,0} &:= \{x - te : \ x \in S_{n_0}, \ 0 \le t \le r\delta\}, \\ K_{e,j} &:= \big\{x - te : \ x \in S_{n_0}, \ (r+j-1)\delta \le t \le \min\{(r+j)\delta, \varphi(x,e)\}\big\}, \\ &\qquad\qquad j = 1, 2, \dots, \ell_e, \end{aligned}$$

where $\ell_e \le \frac{\operatorname{diam} G}{\delta}$ is the largest integer such that $(r + \ell_e - 1)\delta \le \varphi(x, e)$ for some $x \in S_{n_0}$. By (6.10), $K_{e,0} \subset S_{n_1} \subset G$, and thus, for $S_{n_1} \subset S \subset G$,

$$G_e \cup S = S \cup \left(\bigcup_{j=1}^{\ell_e} K_{e,j} \right).$$

Furthermore, for $h = \delta e$ and each $1 \le j \le \ell_e$, we have

$$\bigcup_{i=1}^{r} (K_{e,j} - ih) \subset S \cup \left(\bigcup_{v=1}^{j-1} K_{e,v} \right).$$

The estimate (6.9) then follows by Lemma 2.3.

Third, we show that

$$w_r(G; \mathcal{E})_p \le C\big(1 + w_r(\overline{S_{n_1}}; \mathcal{E})_p\big). \tag{6.11}$$

Let $\mathcal{E} = \{\xi_j\}_{j=1}^m$. By (6.8), we have

$$G = \bigcup_{j=1}^{m} \big(R(S_{n_0}, \xi_j) \cap G\big) = \bigcup_{j=0}^{m} G_j,$$

where

$$G_0 = \overline{S_{n_1}} \ \text{ and } \ G_j = G_{\xi_j} \cup S_{n_1} \ \text{ for } 1 \le j \le m.$$

Using (6.9) with $S = \bigcup_{j=0}^{k-1} G_j$ iteratively for $k = 1, 2, \dots, m$, we obtain

$$w_r\left(\bigcup_{j=0}^{k} G_j; \mathcal{E} \right)_p \le C\left(1 + w_r\left(\bigcup_{j=0}^{k-1} G_j; \mathcal{E} \right)_p\right), \ \ k = 1, 2, \dots, m.$$

Thus,

$$w_r(G; \mathcal{E})_p = w_r\left(\bigcup_{j=0}^{m} G_j; \mathcal{E} \right)_p \le C\left(1 + w_r(\overline{S_{n_1}}; \mathcal{E})_p\right).$$

Finally, we show that

$$w_r(\overline{S_{n_1}}; \mathcal{E})_p \le C_{p,r,G} < \infty,$$

which combined with (6.11) will imply the desired estimate $w_r(G; \mathcal{E})_p < \infty$. Because $\overline{S_{n_1}}$ is a convex C^2-domain, by Theorems 4.3 and 5.1, it suffices to show that $\operatorname{span}(\mathcal{E}) = \mathbb{R}^d$.

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Assume otherwise. Then, there exists a direction $\eta \in {}^{d-1}$ orthogonal to any vector in $\mathcal{E}$. Let H be one of the two supporting hyperplanes to G orthogonal to η. Then, any point of $H \cap G$ cannot be illuminated by any direction of $\mathcal{E}$, obtaining a contradiction.

This completes the proof of Theorem 6.3. ∎

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