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Infinitely generated complements to partial tilting modules

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Abstract

We study the existence of complements to a partial tilting module over an arbitrary ring. As a consequence, we show that a finitely generated partial tilting module over an artin algebra has a (possibly infinitely generated) complement.

Tilting theory was introduced in the early 1980s in the context of finitely generated modules over artin algebras by Brenner and Butler [**7**] and by Happel and Ringel [**17**], and since then it has played a central role in the development of the representation theory of artin algebras. While the first papers were dealing with tilting modules of projective dimension at most 1, the theory was later extended to tilting modules of finite projective dimension by Miyashita [**19**].

Many important results on tilting modules remain true when passing to the more general setting of infinitely generated modules over arbitrary rings. This was shown by Colpi and Trlifaj [**10**] for tilting modules of projective dimension at most 1 (see also [**2**, **9**]). For tilting modules of higher projective dimension, we refer to [**1**], where we have extended Miyashita's notion of a tilting module of finite projective dimension to infinitely generated modules and have proven an analogue of Auslander–Reiten's result [**4**] which connects tilting theory to the notion of covariantly finiteness.

In this paper, we shall see that the general approach also yields some new information for the case of finitely generated modules over artin algebras. We will deal here with the question of the existence of complements to partial tilting modules. Recall that a module M over an arbitrary ring R is called a *tilting module* provided:

(T1) The projective dimension $\mathrm{pd}_R M$ of M is finite;
(T2) $\mathrm{Ext}^i_R(M, M^{(I)}) = 0$ for each $i > 0$ and all sets I;
(T3) there exists a long exact sequence $0 \longrightarrow R_R \longrightarrow M_0 \longrightarrow \cdots \longrightarrow M_r \longrightarrow 0$ with $M_i \in \mathrm{Add}M$ for each $0 \leqslant i \leqslant r$.

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A *partial tilting module* is a module M satisfying properties (T1) and (T2) above. A natural question here is to know whether a partial tilting module M *can be completed* to a tilting module, that is, whether there exists a module N such that $M \oplus N$ is tilting. Such an N, if it exists, will be called a *complement to* M.

If one restricts to finitely generated modules, it is not true in general that a partial tilting module admits a complement as shown by Rickard–Schofield in [**20**]. On the positive side, if M is a finitely generated partial tilting module over an artin algebra and has projective dimension at most one, then Bongartz [**6**] has shown that there exists a finitely generated module N of projective dimension at most one such that $M \oplus N$ is a tilting module. We refer to [**4**, **8**, **10**, **16**] for further discussion on this question.

Using our investigations in [**1**] and a recent result by [**11**], we prove the following theorem, which generalizes a corresponding result of [**10**] for the case $n \leqslant 1$. See below for the definitions.

THEOREM. *Let R be a ring and M_R be a partial tilting module of projective dimension n. Denote $\chi = M^{\perp}$. The following statements are equivalent.*

(1) *χ is closed under coproducts.*
(2) *The module M admits a complement N which is Ext-projective in χ.*

Moreover, the projective dimension of a module N as in condition (2) *is at most n.*

As a consequence, we will have the following result which in particular holds true for finitely generated partial tilting modules over artin algebras.

COROLLARY. *Let M_R be a finitely presented partial tilting module over a right coherent ring R. Then M has a* (*possibly infinitely generated*) *complement N which, in addition, is* Ext*-projective in $M^{\perp}$.*

1. *Preliminaries*

Let R be a ring. Denote the category of all R-modules of finite projective dimension by $\mathscr{P}^{<\infty}$, and for an integer $n \in \mathbb{N}$, denote by $\mathscr{P}^{\leqslant n}$ the category of all R-modules of finite projective dimension at most n. For a class of modules $\mathscr{C}$, we define

$$^{\perp}\mathscr{C} = \{Y \in \operatorname{Mod} R \colon \operatorname{Ext}^i_R(Y, C) = 0 \quad \text{for each } i > 0 \text{ and each } C \in \mathscr{C}\}$$

$$\mathscr{C}^{\perp} = \{Y \in \operatorname{Mod} R \colon \operatorname{Ext}^i_R(C, Y) = 0 \quad \text{for each } i > 0 \text{ and each } C \in \mathscr{C}\}.$$

If $\mathscr{C}$ consists of a unique module M, then we denote these subcategories by $^{\perp}M$ and $M^{\perp}$, respectively. Observe that $^{\perp}\mathscr{C}$ ($\mathscr{C}^{\perp}$, respectively) is a *resolving* (*coresolving*, respectively) subcategory, that is, it contains the projective (injective, respectively) modules, it is closed under extensions and under kernel of epimorphisms (under cokernel of monomorphisms, respectively).

Following [**13**], if $X \in \operatorname{Mod} R$ and $\mathscr{C}$ is a class of modules, we say that a morphism $f \colon X \to C$, with $C \in \mathscr{C}$ is a *$\mathscr{C}$-pre-envelope* of X if each morphism $g \colon X \to Y$ with $Y \in \mathscr{C}$ factors through f, or equivalently, if the abelian group homomorphism $\operatorname{Hom}_R(f, Y) \colon \operatorname{Hom}_R(C, Y) \to \operatorname{Hom}_R(X, Y)$ is an epimorphism for each $Y \in \mathscr{C}$. A $\mathscr{C}$-pre-envelope $f \colon X \to C$ is said to be *special* if f is a monomorphism and $\operatorname{Ext}^1_R(\operatorname{Coker} f, Y) = 0$ for all $Y \in \mathscr{C}$.

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Finally, a class $\mathscr{C}$ consisting of finitely presented modules is said to be *covariantly finite* if for every finitely presented module there is a $\mathscr{C}$-pre-envelope (cf. [**5**]). *Precovers* and *contravariantly finiteness* are defined dually. A module $B \in \mathscr{C}$ is said to be Ext-*projective* if $\mathrm{Ext}^1_R(B, X) = 0$ for all $X \in \mathscr{C}$. Ext-*injective* modules are defined dually.

We recall the following results from [**1**].

LEMMA 1·1 [**1**, (1·2)]. *Let $\mathscr{M} \subset \mathrm{Mod}\, R$ be a class of modules.*

(*a*) *Suppose $\mathscr{M}$ is closed under cokernels of monomorphisms and contains all injective modules. If C is a module with $\mathrm{Ext}^1_R(C, M) = 0$ for all $M \in \mathscr{M}$, then $C \in {}^{\perp}\mathscr{M}$.*

(*b*) *Suppose $\mathscr{M}$ is closed under kernels of epimorphisms and contains all projective modules. If C is a module with $\mathrm{Ext}^1_R(M, C) = 0$ for all $M \in \mathscr{M}$, then $C \in \mathscr{M}^{\perp}$.*

If $\mathscr{M} \subset \mathrm{Mod}\, R$, then $\check{\mathscr{M}}$ will denote the subcategory of all modules X_R such that there exists a long exact sequence

$$0 \longrightarrow X \longrightarrow M_0 \longrightarrow M_1 \longrightarrow \cdots \longrightarrow M_r \longrightarrow 0$$

with $r \geqslant 0$ and $M_i \in \mathscr{M}$ for $i = 0, \ldots, r$. Moreover, given a module M, we will write $\mathbf{Add}\, \boldsymbol{M}$ for the subcategory of all modules isomorphic to a direct summand of a coproduct of copies of M, and $\mathbf{Gen}\, \boldsymbol{M}$ for the subcategory of all M-generated modules, that is, modules X admitting an epimorphism $M' \to X$ with $M' \in \mathrm{Add}\, M$.

LEMMA 1·2. *Let M_R be a module of projective dimension n and denote $\chi = M^{\perp}$.*

(*a*) [**1**, (2·2)] *Then $\check{\chi} = \mathrm{Mod} R$ and all $Y \in {}^{\perp}\chi$ have $\mathrm{pd}_R Y \leqslant n$.*

(*b*) [**1**, (2·3, 2·4)] *If M is tilting, then for each $X \in \chi$, there exists a short exact sequence $0 \to K \to M' \to X \to 0$, with $M' \in \mathrm{Add}\, M$ and $K \in \chi$.*

THEOREM 1·3 [**1**, (4·4)]. *Let $M \in \mathrm{Mod}\, R$, and denote $\chi = M^{\perp}$. Then M is a tilting module if and only if M is a partial tilting module such that $\chi \subset \mathrm{Gen}\, M$.*

Observe that the term *partial tilting module* is sometimes used in a slightly different sense (e.g. in [**14**] or [**10**]).

We finish this section with the following example suggested by C. M. Ringel.

Example 1·4. Let R be a finite dimensional tame hereditary k-algebra, where k is a field. For each simple regular S, there exists an indecomposable Prüfer module denoted S^{ω} (see [**21**]). Consider T the sum of a copy of each indecomposable Prüfer module S^{ω} and the (unique) indecomposable torsion-free divisible module Q. Clearly, $\mathrm{pd}_R T \leqslant 1$ and since T is a regular divisible module, it follows from [**21**, (4·6, 4·7)] that $\mathrm{Ext}^1_R(T, T^{(\mathrm{I})}) = 0$ for all sets I. Further, from [**21**, (4·6 and 5·5)] we infer that there exists a short exact sequence $0 \to R \to q^{(\mathrm{I})} \to T' \to 0$, where I is some set and T' is a direct sum of Prüfer modules. So, we conclude that T is a tilting module which is an infinite direct sum of infinite-dimensional indecomposable modules.

2. *The results*

In this section, we shall prove our main result and look at some consequences. Note that our theorem generalizes a corresponding result of [**10**, theorem 1·9] for the case $n \leqslant 1$.

THEOREM 2·1. *Let R be a ring and M_R be a partial tilting module of projective dimension n. Denote $\chi = M^{\perp}$. The following statements are equivalent.*

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(1) χ *is closed under coproducts.*

(2) *The module* M *admits a complement* N *which is* Ext*-projective in* χ.

Moreover, the projective dimension of a module N *as in condition (2) is at most* n.

Proof. (1) $\Rightarrow$ (2): by a result of [**11**], every module has a special χ-pre-envelope. Let $f\colon R \to N$ be a special χ-pre-envelope of the regular module R. So, there exists a short exact sequence

$$0 \longrightarrow R \xrightarrow{f} N \xrightarrow{g} C \longrightarrow 0,$$

where $\operatorname{Ext}^1_R(C, X) = 0$ for all $X \in \chi$. Using Lemma 1·1, we conclude in fact that $C \in {}^{\perp}\chi$. Since also $R \in {}^{\perp}\chi$, it follows that $N \in {}^{\perp}\chi$, and in particular we infer from 1·2 that N has projective dimension at most n. Further, N generates all modules from χ by [**2**, (1·1)].

We now show that N is a complement for M. Indeed, it follows from our assumption on χ that $T = M \oplus N$ is a partial tilting module with $T^{\perp} = \chi \subset \operatorname{Gen} N = \operatorname{Gen} T$, and so T is even a tilting module by Theorem 1·3. Clearly, N is Ext-projective in χ.

(2) $\Rightarrow$ (1): set $T = M \oplus N$. Since N is Ext-projective, it follows from Lemma 1·1 that $N \in {}^{\perp}\chi$, hence $\chi \subset N^{\perp}$ and $\chi = N^{\perp} \cap \chi = T^{\perp}$. Let now $X_i \in \chi$, $i \in I$, and set $X = \coprod_{i\in I} X_i$. By Lemma 1·2, there is a short exact sequence

$$0 \longrightarrow \coprod K_i \longrightarrow \coprod T_i \longrightarrow X \longrightarrow 0,$$

where $T_i \in \operatorname{Add} T$ and $K_i \in \chi$. By iterating this procedure, we obtain a long exact sequence

$$0 \longrightarrow A \longrightarrow B_n \longrightarrow \cdots \longrightarrow B_1 \longrightarrow X \longrightarrow 0$$

with $B_1, \ldots, B_n \in \operatorname{Add} T$. Since $\operatorname{Add} T \subset M^{\perp}$, we conclude by dimension shifting that $\operatorname{Ext}^i_R(M, X) \cong \operatorname{Ext}^{i+n}_R(M, A) = 0$, hence $X \in \chi$ and the result is proven.

Observe that a partial tilting module M with $M^{\perp}$ not closed under coproducts can still have a complement. This happens for instance when we consider the direct summand $M = \mathbb{Q}$ of the tilting module $T = \mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}$ over the ring $R = \mathbb{Z}$, for which, as pointed out in [**10**, (1·5)], the category $M^{\perp}$ is not closed under coproducts.

We now apply Theorem 2·1 to the situation usually considered in representation theory.

COROLLARY 2·2. *Let* M_R *be a finitely presented partial tilting module over a right coherent ring* R. *Then* M *has a* (*possibly infinitely generated*) *complement* N *which is* Ext*-projective in* $M^{\perp}$.

In particular, this holds true for finitely generated partial tilting modules over artin algebras.

Proof. If M_R is a finitely presented module over a right coherent ring R, then it is well known that the functors $\operatorname{Ext}^i_R(M, -), i \geqslant 0$, commute with coproducts, and so the assumption of the Theorem is satisfied.

For the reader's convenience, we repeat here the argument. For $i = 0$, the claim follows from the fact that M is finitely generated. For $i \geqslant 1$, we consider a projective presentation

$$\cdots P_n \xrightarrow{f_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \xrightarrow{f_0} M \longrightarrow 0$$

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Observe that we can choose all P_n finitely generated, hence coherent, and so all the kernels $K_n = \mathrm{Ker} f_n$ are finitely presented. Then, since $\mathrm{Ext}^1_R(P_0, -)$ vanishes and $\mathrm{Hom}_R(K_1, -)$ and $\mathrm{Hom}_R(P_0, -)$ commute with coproducts, so does $\mathrm{Ext}^1_R(M, -)$. For $i > 1$, we continue inductively using dimension shift.

An important feature in the proof of Theorem 2·1 was the result of Eklof–Trlifaj which establishes the existence of special $M^\perp$-pre-envelopes for every module in Mod R [**11**]. It is not known whether the dual statement is true, that is, if every R-module has a special ${}^\perp M$-precover. However, Eklof–Trlifaj have shown the existence of special ${}^\perp M$-(pre)covers under the additional assumption that M is pure-injective [**12**]. Using dual arguments, we can then obtain a statement dual to Theorem 2·1 for partial co-tilting modules which are pure-injective. Here a module is said to be *partial co-tilting* if it satisfies (C1) $\mathrm{id}_R M < \infty$; and (C2) $\mathrm{Ext}^i_R(M^I, M) = 0$ for each $i > 0$ and all sets I. Moreover, a *complement* to a partial co-tilting module M is a module N such that $M \oplus N$ is a *co-tilting module*, that is, a module satisfying (C1), (C2) and (C3): there exists an injective cogenerator Q and a long exact sequence $0 \to M_n \to \cdots \to M_0 \to Q \to 0$, with $M_i \in \mathrm{Prod}\, M$ for each $0 \leqslant i \leqslant r$. We then have that a partial co-tilting module M which is pure-injective admits a complement which is Ext-injective in ${}^\perp M$ if and only if ${}^\perp M$ is closed for products.

We shall see in the sequel how the existence of special ${}^\perp M$-(pre)covers is also related to the existence of an Ext-injective complement to a partial tilting module.

If Λ is an artin algebra, then it was proven by Happel and Unger in [**15**, (2·1)] that every resolving subcategory $\mathscr{C}$ of mod Λ contained in $\mathscr{P}^{\leqslant n}$ for some $n \in \mathbb{N}$ is contravariantly finite provided there is a tilting module which is Ext-injective in $\mathscr{C}$. The following proposition extends their result to the general situation and, moreover, shows that their result even holds when $\mathscr{C}$ is a resolving sub-category of mod Λ which is just contained in $\mathscr{P}^{<\infty}$.

Proposition 2·3. *Let R be a ring and $\mathscr{C}$ be a resolving subcategory of* Mod R *contained in $\mathscr{P}^{<\infty}$, and assume there is a tilting module T such that all modules in* Add T *are* Ext*-injective in $\mathscr{C}$. Then every module has a special $\mathscr{C}$-precover.*

Proof. Set $\chi = T^\perp$ and $\omega = \mathscr{C}$. Since T is a tilting module, we know from Lemma 1·2 that, $\check{\chi} = \mathrm{Mod}\, R$, and that for each $X \in \chi$, there exists a short exact sequence $0 \to K \to M' \to X \to 0$ with $K \in \chi$ and $M' \in \mathrm{Add}\, M$. By assumption we further have $\mathrm{Add}\, T \subset \omega$ and $\check{\omega} = \mathscr{C}$.

Next, we show (cf. [**15**, p. 451]), that $\omega \subset {}^\perp\chi$. By Lemma 1·1, it suffices to verify that $\mathrm{Ext}^1_R(C, X) = 0$ for every $C \in \omega = \mathscr{C}$ and $X \in \chi$. By iterating the above procedure, we construct a long exact sequence

$$0 \longrightarrow K_r \longrightarrow M_{r-1} \longrightarrow \cdots \longrightarrow M_0 \longrightarrow X \longrightarrow 0$$

of length $r = \mathrm{pd}_R C$ where $M_0, \ldots, M_{r-1} \in \mathrm{Add}\, T$ and $K_r \in \chi$. We know by assumption that $M_0, \ldots, M_{r-1}$ are Ext-injective in $\mathscr{C}$, hence in $\mathscr{C}^\perp$ by Lemma 1·1. Clearly, by dimension shifting, $\mathrm{Ext}^1_R(C, X) \cong \mathrm{Ext}^{r+1}_R(C, K_r) = 0$.

Now we can proceed as in the Theorem of Auslander and Buchweitz (dual version of [**3**, (1·1, 2·2, 2·3, 2·5)], see also [**15**, p. 450]), and we obtain that for all $X \in \mathrm{Mod}\, R$

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there exist short exact sequences

$$0 \longrightarrow K_X \longrightarrow M_X \xrightarrow{g_X} X \longrightarrow 0 \quad \text{with } K_X \in \chi, M_X \in \breve{\omega} \text{ and}$$

$$0 \longrightarrow X \xrightarrow{f_X} B_X \longrightarrow C_X \longrightarrow 0 \quad \text{with } B_X \in \chi, C_X \in \breve{\omega} \text{ and}$$

where g_X is a special $\mathscr{C}$-precover (and f_X is a special χ-pre-envelope).

We now give an application of Proposition 2·3. Following [**18**], we will say that a module M is *product-complete* provided $\operatorname{Add} M$ is closed under products.

PROPOSITION 2·4. *Let R be a ring and M_R be a partial tilting module which is product-complete. Assume that $\mathscr{C}_M = {}^{\perp}M \bigcap \mathscr{P}^{<\infty}$ is contained in $\mathscr{P}^{\leqslant n}$ for some $n \in \mathbb{N}$. Then the following statements are equivalent.*

(1) *M has a complement C which is* Ext*-injective in $\mathscr{C}_M$ and product-complete.*
(2) *There is a module C which is product-complete, Ext-injective in $\mathscr{C}_M$ and cogenerates all modules from $\mathscr{C}_M$.*
(3) *Every module has a special $\mathscr{C}_M$-precover, and for any injective module X the special $\mathscr{C}_M$-precover $0 \to K \to C \to X \to 0$ can be chosen with C product-complete.*
(4) *There is a minimal injective cogenerator Q with a special $\mathscr{C}_M$-precover $0 \to K \to C \to Q \to 0$ where C is product-complete.*

Proof. (1) $\Rightarrow$ (3): we know that C is Ext-injective in $\mathscr{C}_M$, and since $\mathscr{C}_M \subset {}^{\perp}M$, so is M. Further, by [**18**] the module $T = M \oplus C$ is product-complete and therefore satisfies $\operatorname{Add} T = \operatorname{Prod} T$. So, T is a tilting module such that all modules in $\operatorname{Add} T$ are Ext-injective in $\mathscr{C}_M$, and as a consequence of Proposition 2·3, every module has a special $\mathscr{C}_M$-precover. Moreover, if $X \in T^{\perp}$, we can choose a special $\mathscr{C}_M$-precover as follows. Let $I = \operatorname{Hom}_R(T, X)$ and $g\colon T^{(I)} \to X$ be the coproduct map induced by all elements of I. Then g is an $\operatorname{Add} T$-precover, and it is surjective since $T^{\perp} \subset \operatorname{Gen} T$ by Theorem 1·3. Further, $K = \operatorname{Ker} g$ is contained in $T^{\perp}$, and from the proof of Proposition 2·3 we know that $\operatorname{Ext}^1_R(C, K) = 0$ for all $C \in \mathscr{C}_M$. So, we conclude that the exact sequence $0 \to K \to T^{(I)} \xrightarrow{g} X \to 0$ is a special $\mathscr{C}_M$-precover where the middle-term $T^{(I)}$ is product-complete by [**18**].

(3) $\Rightarrow$ (4): is trivial.

(4) $\Rightarrow$ (2): in the special $\mathscr{C}_M$-precover $0 \to K \to C \to Q \to 0$ we have $\operatorname{Ext}^1_R(X, K) = 0 = \operatorname{Ext}^1_R(X, Q)$ for all $X \in \mathscr{C}_M$, hence C is Ext-injective in $\mathscr{C}_M$. Moreover, C cogenerates all modules from $\mathscr{C}_M$ by [**2**, 1·1].

(2) $\Rightarrow$ (1): as in [**8**, 2·2], we prove that the module C is a complement for M. Indeed, $T = M \oplus C$ is a partial tilting module. Moreover, since T is product-complete, we know from [**18**, 3·1] that for all $A \in \mathscr{C}_M$ there is an $\operatorname{Add} T$-pre-envelope $f\colon A \to B$ which is a monomorphism since A is cogenerated by C. Then we obtain an exact sequence $0 \to A \to B \to Z \to 0$ where Z lies in ${}^{\perp}T \bigcap \mathscr{P}^{<\infty}$ and in particular in $\mathscr{C}_M$. Setting $A = R$ and iterating this argument, we obtain a long exact sequence $0 \to R \xrightarrow{f_0} B_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} B_n \to Z_n \to 0$ where $B_i \in \operatorname{Add} T$ and $Z_i = \operatorname{Coker} f_i \in \mathscr{C}_M$ for all $0 \leqslant i \leqslant n$. But then $B_0, \ldots, B_{n-1} \in Z_n^{\perp}$, so we infer from Lemma [**1**, 1·4] and from the assumption $\mathscr{C}_M \subset \mathscr{P}^{\leqslant n}$ that $\operatorname{Ext}^1_R(Z_n, Z_{n-1}) \cong \operatorname{Ext}^{n+1}_R(Z_n, R) = 0$. This shows that also the last term Z_n lies in $\operatorname{Add} T$, and T is a tilting module.

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Observe that over a representation-finite ring, any module is endofinite [**22**, theorem 6] and in particular product-complete. Moreover, every product-complete module M is pure-injective by [**18**], and therefore all modules have a ${}^{\perp}M$-(pre)cover by the result of [**12**] mentioned above. So, in particular, we obtain the following immediate consequence from the above Proposition.

Corollary 2·5. *Over a ring of finite representation type and finite global dimension, every partial tilting module admits a complement.*

In the artin algebra case, we can use the above arguments to refine a criterion for the existence of complements given by the second named author, Happel and Unger [**8**, 2·2].

Proposition 2·6. *Let Λ be an artin algebra and M_Λ a finitely generated partial tilting module. Denote $\mathscr{C}_M = {}^{\perp}M \bigcap \mathscr{P}^{<\infty}$. Then the following statements are equivalent.*

(1) *M has a complement C which is* Ext*-injective in $\mathscr{C}_M \bigcap \operatorname{mod}\Lambda$.*
(2) *There is an integer $n \in \mathbb{N}$ and a module $C \in \mathscr{C}_M \bigcap \operatorname{mod}\Lambda$ such that all modules in $\mathscr{C}_M \bigcap \operatorname{mod}\Lambda$ are cogenerated by C and have projective dimension at most n.*
(3) *$\mathscr{C}_M \bigcap \operatorname{mod}\Lambda$ is contravariantly finite.*

Proof. We only show how the statement is related to Proposition 2·4. Assume that $\mathscr{C} \subset \operatorname{mod}\Lambda$ is a contravariantly finite resolving subcategory which is contained in $\mathscr{P}^{<\infty}$. Then by Wakamatsu's Lemma (see [**4**, 1·3]) every finitely generated module has a special $\mathscr{C}$-precover, and moreover, $\mathscr{C}$ is contained in $\mathscr{P}^{\leqslant n}$ for some $n \in IN$ by [**4**, 3·9].

Assume now that $\mathscr{C} \subset \operatorname{mod}\Lambda$ is a subcategory which is closed under extensions and has a module $C \in \mathscr{C}$ which cogenerates all modules from $\mathscr{C}$. Of course, since C is a finite direct sum of indecomposable modules, we can choose C minimal with respect to this property, that is, such that no proper direct summand of C cogenerates all modules from $\mathscr{C}$. Then we can deduce from [**5**, 2·4] that C is Ext-injective in $\mathscr{C}$.

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