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RENEWAL THEOREMS FOR PROCESSES WITH DEPENDENT INTERARRIVAL TIMES

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Abstract

In this paper we develop renewal theorems for point processes with interarrival times $\xi(X_{n+1}X_n\cdots)$, where $(X_n)_{n\in\mathbb{Z}}$ is a stochastic process with finite state space Σ and $\xi\colon \Sigma_A \to \mathbb{R}$ is a Hölder continuous function on a subset $\Sigma_A \subset \Sigma^{\mathbb{N}}$. The theorems developed here unify and generalise the key renewal theorem for discrete measures and Lalley's renewal theorem for counting measures in symbolic dynamics. Moreover, they capture aspects of Markov renewal theory. The new renewal theorems allow for direct applications to problems in fractal and hyperbolic geometry, for instance to the problem of Minkowski measurability of self-conformal sets.

Keywords: Renewal theorem; dependent interarrival time; symbolic dynamics; key renewal theorem; Ruelle–Perron–Frobenius theory

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Secondary 28A80; 28A75

1. Introduction and statement of main results

We develop renewal theorems for renewal functions $N\colon \mathbb{R}\times\Sigma_A\to\mathbb{R}$ of the form

$$N(t,x)=\sum_{n=0}^{\infty}\sum_{\{y\in\Sigma_A\colon \sigma^n y=x\}}\chi(y)f_y(t-S_n\xi(y))\mathrm{e}^{S_n\eta(y)} \tag{1.1}$$

using the following notation and conditions. We denote by $\Sigma_A := \{x\in\Sigma^{\mathbb{N}} \mid A(x_k,x_{k+1})=1 \text{ for all } k\in\mathbb{N}\}$ the set of *one-sided infinite admissible words* over an *alphabet* $\Sigma := \{1,\dots,M\}$, $M\geq 2$, consistent with a primitive $M\times M$-incidence matrix $A=(A(i,j))_{i,j\in\Sigma}$ of 0s and 1s. By *primitive* we mean that there exists $n\in\mathbb{N}$ such that all entries of A^n are positive. Moreover, $\sigma\colon \Sigma_A\to\Sigma_A$ denotes the (left) shift map on Σ_A defined by $\sigma(\omega_1\omega_2\cdots) := \omega_2\omega_3\cdots$ for $\omega_1\omega_2\cdots\in\Sigma_A$. The functions $\chi,\xi,\eta\colon\Sigma_A\to\mathbb{R}$ are assumed to be α-Hölder continuous for some $\alpha\in(0,1)$ in the sense of Definition 2.1, with χ nonnegative. We write $S_0\xi := 0$ and $S_n\xi := \sum_{k=0}^{n-1}\xi\circ\sigma^k$ for the *nth Birkhoff sum* of ξ with $n\in\mathbb{N}$, and suppose that there exists $n\in\mathbb{N}$ for which $S_n\xi$ is strictly positive. Note that this condition is equivalent to ξ being cohomologous to a strictly positive function; see Remark 2.1. Some mild regularity conditions on the family $\{f_y\colon\mathbb{R}\to\mathbb{R}\mid y\in\Sigma_A\}$ are assumed (see Section 3) and $y\mapsto f_y$ is supposed to be α-Hölder continuous.

If, for instance, $\chi=\mathbf{1}_{\Sigma_A}$ and $f_y=\mathbf{1}_{[0,\infty)}$ for all $y\in\Sigma_A$, then $N(t,x)$ gives the expected number of renewals in the time interval $(0,t]$ provided that the history of the underlying stochastic process $(X_n)_{n\in\mathbb{Z}}$ coincides with x. In this model $\xi(X_{n+1}X_n\cdots)$ are the interarrival times and $\eta(ix)=\log\mathbb{P}(X_1=i\mid X_0X_{-1}\cdots=x)$ is the logarithmic transition probability.

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Note that the current setting extends and unifies the setting of established renewal theorems. In the context of classical renewal theory for finitely supported measures (in particular, of the key renewal theorem—see, e.g. [1] and [9]), η and ξ depend only on the first coordinate. When η and ξ depend only on the first two coordinates, we are in the setting of Markov renewal theory (see, e.g. [1] and [2]). If η is the constant zero function and $f_y = \mathbf{1}_{[0,\infty)}$, then we are precisely in the setting of [19], where renewal theorems for counting measures in symbolic dynamics were developed. In a forthcoming survey by the author, further details on these connections will be provided.

Renewal theorems deal with the asymptotic behaviour of renewal functions. It is well known that the limiting behaviour of a renewal function depends on the values that the interarrival times assume. Loosely speaking, if the possible values of the interarrival times form a spaced pattern, we are in the lattice situation, and otherwise in the nonlattice situation; see Definition 2.2. This lattice–nonlattice dichotomy also occurs in our main theorems; see Theorems 3.1 and 3.2. They are summarised in the following, where we call a function $f : \mathbb{R} \to \mathbb{R}$ *asymptotic* to a function $g : \mathbb{R} \to \mathbb{R}_{>0}$ as $t \to \infty$, written $f(t) \sim g(t)$ as $t \to \infty$, if $\lim_{t\to\infty} f(t)/g(t) = 1$.

Theorem 1.1. (Renewal theorem with dependent interarrival times.) *Assume that the family* $\{f_x(t) \mid x \in \Sigma_A\}$ *satisfies some regularity conditions as described in Section 3. There exists* $\delta > 0$ *given in Section 2.3 such that the following statements hold.*

(i) *If* ξ *is nonlattice then there exists an explicitly determined constant* $G(x)$ *such that*

$$N(t, x) \sim \mathrm{e}^{t\delta} G(x) \quad \text{as } t \to \infty,$$

uniformly for $x \in \Sigma_A$.

(ii) *If* ξ *is lattice then there exists an explicitly determined periodic function* $\tilde{G}_x$ *such that*

$$N(t, x) \sim \mathrm{e}^{t\delta} \tilde{G}_x(t) \quad \text{as } t \to \infty.$$

(iii) *We always have*

$$\lim_{t\to\infty} t^{-1} \int_0^t \mathrm{e}^{-T\delta} N(T, x)\, \mathrm{d}T = G(x).$$

We believe that the extension that we provide with this theorem is useful in a number of contexts. For example, in Section 4 we will show one application in geometry where the extension is essential; see Remark 4.1.

Renewal theory has attracted much attention in recent years and there is a variety of related results. In this paper we focus on the finite-alphabet case, and Hölder-continuous potential functions ξ and η. Recent results for infinite alphabets are given in [15], [16], and [25]. Our setting in particular implies existence of a finite invariant measure with finite mean, i.e. $\int \xi \, \mathrm{d}\mu_{\eta-\delta\xi} < \infty$, using notation from Section 2. Situations in which this fails are, for instance, considered in [23], [28], and the references therein.

The paper is organised as follows. In Section 2 we introduce the relevant notions from Perron–Frobenius theory, which appear in the statements of the main theorems. Section 3 is devoted to the presentation of the main results (Theorems 3.1 and 3.2). In Section 4 we present an application in fractal geometry, which motivated the development of the present renewal theorem. Finally, in Section 5 we provide the proofs of Theorems 3.1 and 3.2.

2. Ruelle–Perron–Frobenius theory

Here we assemble preliminaries and fix notation. References for the exposition below are [6] and [29].

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2.1. Subshifts of finite type

We call (Σ_A, σ) a *subshift of finite type*. If all entries of A are 1s then $\Sigma_A = \Sigma^{\mathbb{N}}$ and (Σ_A, σ) is called the *full shift* on M symbols. The set of *admissible words of length* $n \in \mathbb{N}$ is defined by

$$\Sigma_A^n := \{\omega \in \Sigma^n \mid A(\omega_k, \omega_{k+1}) = 1 \text{ for } k \le n-1\}.$$

If ω has infinite length or length $m \ge n$, we define $\omega|_n := \omega_1 \cdots \omega_n$ to be the subword of length n. Furthermore, $[\omega] := \{u_1 u_2 \cdots \in \Sigma_A \mid u_i = \omega_i \text{ for } i \le n\}$ is the *ω-cylinder set* for $\omega \in \Sigma_A^n$.

2.2. Continuous and Hölder-continuous functions

Equip $\Sigma^{\mathbb{N}}$ with the product topology of the discrete topologies on Σ and equip $\Sigma_A \subset \Sigma^{\mathbb{N}}$ with the subspace topology, i.e. the weakest topology with respect to which the canonical projections onto the coordinates are continuous. The spaces of continuous complex- and real-valued functions on Σ_A are respectively denoted by $\mathcal{C}(\Sigma_A)$ and $\mathcal{C}(\Sigma_A, \mathbb{R})$. Elements of $\mathcal{C}(\Sigma_A, \mathbb{R})$ are called *potential functions*.

Definition 2.1. For $\xi \in \mathcal{C}(\Sigma_A)$, $\alpha \in (0, 1)$, and $n \in \mathbb{N}_0$, define

$$\operatorname{var}_n(\xi) := \sup\{|\xi(\omega) - \xi(u)| \mid \omega, u \in \Sigma_A \text{ and } \omega_i = u_i \text{ for all } i \in \{1, \ldots, n\}\},$$

$$|\xi|_\alpha := \sup_{n \ge 0} \frac{\operatorname{var}_n(\xi)}{\alpha^n},$$

$$\mathcal{F}_\alpha(\Sigma_A) := \{\xi \in \mathcal{C}(\Sigma_A) \mid |\xi|_\alpha < \infty\},$$

$$\mathcal{F}_\alpha(\Sigma_A, \mathbb{R}) := \mathcal{F}_\alpha(\Sigma_A) \cap \mathcal{C}(\Sigma_A, \mathbb{R}).$$

Elements of $\mathcal{F}_\alpha(\Sigma_A)$ are called *α-Hölder continuous* functions on Σ_A.

Definition 2.2. Functions $\xi_1, \xi_2 \in \mathcal{C}(\Sigma_A)$ are called *cohomologous* if there exists $\psi \in \mathcal{C}(\Sigma_A)$ such that $\xi_1 - \xi_2 = \psi - \psi \circ \sigma$. A function $\xi \in \mathcal{C}(\Sigma_A, \mathbb{R})$ is said to be *lattice* if it is cohomologous to a function whose range is contained in a discrete subgroup of $\mathbb{R}$. Otherwise, we say that ξ is *nonlattice*.

Remark 2.1. The nth Birkhoff sum $S_n\xi$ being strictly positive for some $n \in \mathbb{N}$ is equivalent to ξ being cohomologous to a strictly positive function ζ. To see this, note that $\zeta, \xi \in \mathcal{C}(\Sigma_A, \mathbb{R})$ are cohomologous if and only if $S_m\zeta(x) = S_m\xi(x)$ for all $m \in \mathbb{N}$ and $x \in \Sigma_A$, with $\sigma^m x = x$. First, suppose that $S_n\xi$ is strictly positive. Let $\zeta := S_n\xi/n$. Then $S_m\xi(x) = S_m\zeta(x)$ for all $x \in \Sigma_A$, with $\sigma^m x = x$. Second, suppose that $\xi = \zeta + \psi - \psi \circ \sigma$ for some $\psi \in \mathcal{C}(\Sigma_A, \mathbb{R})$ and ζ satisfying $\zeta \ge \varepsilon > 0$ for all $x \in \Sigma_A$. Then $S_m\xi(x) = S_m\zeta(x) + \psi(x) - \psi \circ \sigma^{m+1}(x) \ge m\varepsilon + \operatorname{var}_0(\psi) > 0$ for sufficiently large $m \in \mathbb{N}$. The author is grateful to the anonymous referee for pointing out this connection.

2.3. Topological pressure function and Gibbs measures

The *topological pressure function* $P\colon \mathcal{C}(\Sigma_A, \mathbb{R}) \to \mathbb{R}$ is given by the well-defined limit

$$P(\xi) := \lim_{n \to \infty} n^{-1} \log \sum_{\omega \in \Sigma_A^n} \exp \sup_{u \in [\omega]} S_n\xi(u).$$

If $\xi, \eta \in \mathcal{C}(\Sigma_A, \mathbb{R})$ are such that $S_n\xi$ is strictly positive on Σ_A for some $n \in \mathbb{N}$ then the mapping $s \mapsto P(\eta + s\xi)$ is continuous, strictly monotonically increasing, and convex with $\lim_{s \to -\infty} P(\eta + s\xi) = -\infty$ and $\lim_{s \to \infty} P(\eta + s\xi) = \infty$. Hence, there is a unique $\delta \in \mathbb{R}$ for which $P(\eta - \delta\xi) = 0$.

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A finite Borel measure μ on Σ_A is said to be a *Gibbs measure* for $\xi \in \mathcal{C}(\Sigma_A, \mathbb{R})$ if there exists a constant $c > 0$ such that

$$c^{-1} \leq \frac{\mu([\omega|_n])}{\exp(S_n\xi(\omega) - nP(\xi))} \leq c \tag{2.1}$$

for every $\omega \in \Sigma_A$ and $n \in \mathbb{N}$.

2.4. Ruelle's Perron–Frobenius theorem

For $s \in \mathbb{C}$, define the *Ruelle–Perron–Frobenius operator* $\mathcal{L}_{\eta+s\xi} \colon \mathcal{C}(\Sigma_A) \to \mathcal{C}(\Sigma_A)$ to the complex potential function $\eta + s\xi$ by

$$\mathcal{L}_{\eta+s\xi}\chi(x) \coloneqq \sum_{\{y\in\Sigma_A \colon \sigma y = x\}} \chi(y)\mathrm{e}^{\eta(y)+s\xi(y)}.$$

By [6, Theorem 1.7] and [30, Theorem 2.16, Corollary 2.17], for each $\xi \in \mathcal{F}_\alpha(\Sigma_A, \mathbb{R})$ and some $\alpha \in (0, 1)$, there exists a unique Borel probability measure ν_ξ on Σ_A satisfying $\mathcal{L}^*_\xi \nu_\xi = \gamma_\xi \nu_\xi$ for some $\gamma_\xi > 0$, where $\mathcal{L}^*_\xi$ denotes the dual operator acting on the space of Borel-probability measures supported on Σ_A. This equation uniquely determines γ_ξ, which satisfies $\gamma_\xi = \exp(P(\xi))$ and which coincides with the spectral radius of $\mathcal{L}_\xi$. Furthermore, there exists a unique strictly positive eigenfunction $h_\xi \in \mathcal{C}(\Sigma_A, \mathbb{R})$ satisfying $\mathcal{L}_\xi h_\xi = \gamma_\xi h_\xi$ and $\int h_\xi \, \mathrm{d}\nu_\xi = 1$. Define μ_ξ by $\mathrm{d}\mu_\xi / \mathrm{d}\nu_\xi = h_\xi$. This is the unique σ-invariant Gibbs measure for the potential function ξ. Additionally, under some normalisation assumptions, we have convergence of the iterates of the Ruelle–Perron–Frobenius operator to the projection onto the one-dimensional subspace generated by its eigenfunction h_ξ, namely,

$$\lim_{m\to\infty} \|\gamma_\xi^{-m}\mathcal{L}^m_\xi \psi - \textstyle\int \psi \, \mathrm{d}\nu_\xi \cdot h_\xi\|_\infty = 0 \quad \text{for all } \psi \in \mathcal{C}(\Sigma_A, \mathbb{R}).$$

Section 2.3 and the relation $\gamma_\xi = \exp(P(\xi))$ imply the following.

Proposition 2.1. *Let $\xi, \eta \in \mathcal{C}(\Sigma_A, \mathbb{R})$ be such that, for some $n \in \mathbb{N}$, the nth Birkhoff sum $S_n\xi$ of ξ is strictly positive on Σ_A. Then $s \mapsto \gamma_{\eta+s\xi}$ is continuous, strictly monotonically increasing, log-convex in $s \in \mathbb{R}$ with $\lim_{s\to-\infty} \gamma_{\eta+s\xi} = 0$, and satisfies $\lim_{s\to\infty} \gamma_{\eta+s\xi} = \infty$. The unique $\delta \in \mathbb{R}$ from Section 2.3 is the unique $\delta \in \mathbb{R}$ for which $\gamma_{\eta-\delta\xi} = 1$.*

Remark 2.2. In the proof of our renewal theorems we will work with an analytic continuation of the pressure function to the complex domain and with complex Perron–Frobenius theory. These complex quantities and their properties will be presented in Section 5.1.

3. Renewal theorems with dependent interarrival times

Here we make our regularity conditions on the functions f_y from (1.1) as well as the statement of the results more precise. We fix $x \in \Sigma_A$ and take $\alpha \in (0, 1)$. Recall that $\xi, \eta, \chi \in \mathcal{F}_\alpha(\Sigma_A, \mathbb{R})$ are such that χ is nonnegative but not identically 0 and such that $S_n\xi$ is strictly positive on Σ_A for some $n \in \mathbb{N}$; see also Remark 2.1. Henceforth, we let $\delta > 0$ denote the unique real for which $\gamma_{\eta-\delta\xi} = 1$ (see Proposition 2.1). One of the regularity conditions on $f_y \colon \mathbb{R} \to \mathbb{R}$ for $y \in \Sigma_A$ requires equi directly Riemann integrability, a condition which is motivated by an assumption of the classical key renewal theorem.

Definition 3.1. For a function $f \colon \mathbb{R} \to \mathbb{R}$, $h > 0$, and $k \in \mathbb{Z}$, set

$$\underline{m}_k(f, h) \coloneqq \inf\{f(t) \mid (k-1)h \leq t < kh\},$$
$$\overline{m}_k(f, h) \coloneqq \sup\{f(t) \mid (k-1)h \leq t < kh\}.$$

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The function f is called *directly Riemann integrable (dRi)* if, for some sufficiently small $h > 0$,

$$\underline{R}(f,h) := \sum_{k\in\mathbb{Z}} h\underline{m}_k(f,h) \quad \text{and} \quad \overline{R}(f,h) := \sum_{k\in\mathbb{Z}} h\overline{m}_k(f,h)$$

are finite and tend to the same limit as $h \to 0$. We call $\{f_x : \mathbb{R} \to \mathbb{R} \mid x \in I\}$ with some index set I *equi directly Riemann integrable (equi dRi)* if f_x is dRi for all $x \in I$ and if

$$\sum_{k\in\mathbb{Z}} h \sup_{x\in I}(\underline{m}_k(f_x,h) - \overline{m}_k(f_x,h))$$

tends to 0 as $h \to 0$.

Direct Riemann integrability excludes wild oscillations of the function at ∞ and is stronger than Riemann integrability. For further insights into this notion, we refer the reader to [2, Chapter B.V] and [9, Chapter XI].

The *regularity conditions* are as follows.

(A) *Lebesgue integrability.* For any $x \in \Sigma_A$, the Lebesgue integral

$$\int_{-\infty}^{\infty} \mathrm{e}^{-t\delta}|f_x(t)|\,\mathrm{d}t$$

exists.

(B) *Boundedness of N.* There exists $\mathfrak{C} > 0$ such that $\mathrm{e}^{-t\delta}N^{\mathrm{abs}}(t,x) \le \mathfrak{C}$ for all $x \in \Sigma_A$ and $t \in \mathbb{R}$, where

$$N^{\mathrm{abs}}(t,x) := \sum_{n=0}^{\infty} \sum_{\{y:\,\sigma^n y = x\}} \chi(y)|f_y(t - S_n\xi(y))|\mathrm{e}^{S_n\eta(y)}.$$

(C) *Exponential decay of N on the negative half-axis.* There exist $\tilde{\mathfrak{C}} > 0$, $s > 0$, and $t^* \in \mathbb{R}$ such that $\mathrm{e}^{-t\delta}N^{\mathrm{abs}}(t,x) \le \tilde{\mathfrak{C}}\mathrm{e}^{st}$ for all $t \le t^*$.

In the nonlattice situation we additionally need the following condition.

(D) *Nonoscillatory.* The renewal function N does not oscillate wildly at ∞, in particular at least one of the following assertions are satisfied.

(i) The function $t \mapsto f_x(t)$ is monotonic for any $x \in \Sigma_A$.

(ii) The family $\{t \mapsto \mathrm{e}^{-t\delta}|f_x(t)| \mid x \in \Sigma_A\}$ is equi dRi.

(iii) The oscillation of N is small in the sense that

$$\begin{aligned}&\liminf_{\varepsilon\to 0} \limsup_{r\to\infty} \sup_{\tilde{\varepsilon}\in[0,\varepsilon]} \mathrm{e}^{-(r-\tilde{\varepsilon})\delta}N(r-\tilde{\varepsilon},x)\\ &\quad = \limsup_{\varepsilon\to 0} \liminf_{r\to\infty} \inf_{\tilde{\varepsilon}\in[0,\varepsilon]} \mathrm{e}^{-(r-\tilde{\varepsilon})\delta}N(r-\tilde{\varepsilon},x).\end{aligned}$$

Note that conditions (A)–(D) are a weakening of imposed assumptions of known renewal theorems. In Theorem 3.2 it is, for instance, shown that equi dRi of $\{t \mapsto \mathrm{e}^{-t\delta}|f_x(t)| \mid x \in \Sigma_A\}$ and exponential decay of $t \mapsto \mathrm{e}^{-t\delta}f_x(t)$ on the negative half-axis imply (A)–(D). Conditions (A)–(C) and (D)(i) are motivated by [19].

For $t \in \mathbb{R}$, we write $\lfloor t \rfloor$ for the largest integer $k \in \mathbb{Z}$ satisfying $k \le t$. Moreover, we set $\{t\} := t - \lfloor t \rfloor \in [0,1)$. Note that, for $t \in \mathbb{R}$ positive, $\lfloor t \rfloor$ is the integer part and $\{t\}$ is the fractional part of t.

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Theorem 3.1. (Renewal theorem.) *Assume that $x \mapsto f_x(t)$ is α-Hölder continuous for any $t \in \mathbb{R}$ and that conditions (A)–(C) hold.*

(i) *If ξ is nonlattice and (D) is satisfied, then*

$$N(t,x) \sim \mathrm{e}^{t\delta} \underbrace{\frac{h_{\eta-\delta\xi}(x)}{\int \xi \,\mathrm{d}\mu_{\eta-\delta\xi}} \int_{\Sigma_A} \chi(y) \int_{-\infty}^{\infty} \mathrm{e}^{-T\delta} f_y(T)\,\mathrm{d}T\,\mathrm{d}\nu_{\eta-\delta\xi}(y)}_{=:G(x)}$$

as $t \to \infty$, uniformly for $x \in \Sigma_A$.

(ii) *Assume that ξ is lattice, and let $\zeta, \psi \in \mathcal{C}(\Sigma_A, \mathbb{R})$ satisfy the relation*

$$\xi - \zeta = \psi - \psi \circ \sigma,$$

where $\zeta(\Sigma_A) \subseteq a\mathbb{Z}$ for some $a > 0$. Suppose that ξ is not cohomologous to any function with values in a proper subgroup of $a\mathbb{Z}$. Then

$$N(t,x) \sim \mathrm{e}^{t\delta} \tilde{G}_x(t)$$

as $t \to \infty$, uniformly for $x \in \Sigma_A$. Here $\tilde{G}_x$ is the following periodic function with period a

$$\tilde{G}_x(t) = \frac{a\mathrm{e}^{\delta\psi(x)-a\{(t+\psi(x))/a\}\delta}}{\int \zeta \,\mathrm{d}\mu_{\eta-\delta\zeta}} h_{\eta-\delta\zeta}(x) \times \int_{\Sigma_A} \chi(y) \sum_{\ell=-\infty}^{\infty} \mathrm{e}^{-a\ell\delta} f_y\left(a\ell + a\left\{\frac{t+\psi(x)}{a}\right\} - \psi(y)\right) \mathrm{d}\nu_{\eta-\delta\zeta}(y).$$

(iii) *We always have*

$$\lim_{t\to\infty} t^{-1} \int_0^t \mathrm{e}^{-T\delta} N(T,x)\,\mathrm{d}T = G(x).$$

Moreover, in the lattice situation, the relation $\int_0^a \tilde{G}_x(T)\,\mathrm{d}T = aG(x)$ holds.

Theorem 3.2. *Assume that $x \mapsto f_x(t)$ is α-Hölder continuous for $t \in \mathbb{R}$ and that the family $\{t \mapsto \mathrm{e}^{-t\delta}|f_x(t)| \mid x \in \Sigma_A\}$ is equi dRi. If there exist $\mathfrak{C}', s > 0$ such that $\mathrm{e}^{-t\delta}|f_x(t)| \le \mathfrak{C}'\mathrm{e}^{st}$ for $t < 0$ and $x \in \Sigma_A$, then (A)–(D) are satisfied and, thus, the statements of Theorem 3.1 hold.*

4. An application—Minkowski content of self-conformal sets

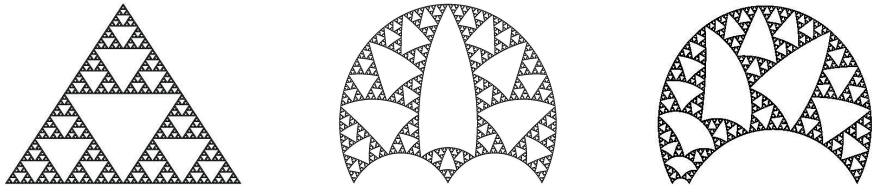
A direct application of the renewal theorems with dependent interarrival times in geometry, namely existence of the Minkowski content of self-conformal sets, is the focus of the current section. It is this application which led to developing the renewal theorems.

Let $B \subset \mathbb{R}^d$ be a bounded subset of d-dimensional Euclidean space $(\mathbb{R}^d, \|\cdot\|_2)$. We wish to understand the asymptotic behaviour of the (e^{-t})-*parallel volume* $\lambda_d(B_{\mathrm{e}^{-t}})$ of B as $t \to \infty$. Here λ_d denotes the d-dimensional Lebesgue measure and $B_{\mathrm{e}^{-t}} := \{x \in \mathbb{R}^d \mid \inf_{b\in B} \|b - x\|_2 \le \mathrm{e}^{-t}\}$ for $t \in \mathbb{R}$. More precisely, we study the existence of the *Minkowski content*

$$\mathcal{M}(B) := \lim_{t\to\infty} \mathrm{e}^{t(d-D)} \lambda_d(B_{\mathrm{e}^{-t}}) \tag{4.1}$$

of B and determine its value when the limit exists. Equation (4.1) implicitly assumes existence of the *Minkowski dimension* $D := d + \lim_{t\to\infty} t^{-1} \log \lambda_d(B_{\mathrm{e}^{-t}})$, which coincides with the box-counting dimension; see [8, Proposition 3.2]. If $D \in \mathbb{N}$ and $\lambda_D(B) > 0$, $\mathcal{M}(B) = \lambda_D(B)$. Otherwise, $\lambda_d(B) = 0$ and the Minkowski content, when it exists, can be interpreted as

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(a) (b) (c)

FIGURE 1: The self-conformal sets from Example 4.1; (a) shows the Sierpiński gasket (self-similar), and (b) and (c) show conformal images of the Sierpiński gasket (self-conformal but not self-similar).

the D-dimensional volume of B, giving a substitute of the notion of volume for noninteger dimensions. Besides this geometric relevance, the Minkowski content plays a major role in the Weyl–Berry conjecture, which is concerned with spectral asymptotics of Laplace operators on domains with irregular boundaries; see [4], [5], and [20]. These are two of the reasons why the Minkowski content has attracted much attention in recent years; see, e.g. [18] for a more in depth introduction.

For defining self-conformal sets, let $X \subset \mathbb{R}^d$ be a nonempty compact set with $X = \overline{\operatorname{int} X}$. Define $\Phi := \{\phi_1, \dots, \phi_M : X \to X\}$ to be an iterated function system (IFS) consisting of contractive conformal $\mathcal{C}^{1+\alpha}$-diffeomorphisms ϕ_i with $M \geq 2$ and $\alpha \in (0, 1)$. Let F be the unique compact nonempty set satisfying $F = \bigcup_{i=1}^{M} \phi_i F$. It exists due to [11] and is called the *self-conformal set* associated with Φ. We call F *self-similar* if all ϕ_i are *similitudes*, i.e. if $\|\phi_i(x) - \phi_i(y)\|_2 = c_i \|x - y\|_2$ for all $x, y \in X$ and $i \in \{1, \dots, M\} =: \Sigma$ with $c_i \in (0, 1)$. Self-conformal sets provide prominent examples of fractal sets; see Example 4.1 and Figure 1.

Next, we explain how to obtain the asymptotic behaviour of $\lambda_d(F_{\mathrm{e}^{-t}})$ as $t \to \infty$. Assume that Φ satisfies the *open set condition*, i.e. there exists a *feasible open set* $O \subset X$, which is nonempty and which satisfies $\phi_i O \subset O$ and $\phi_i O \cap \phi_j O = \varnothing$ for $i \neq j \in \Sigma$. Assume without loss of generality that O is bounded. For $\omega = \omega_1 \cdots \omega_n \in \Sigma^n$, write $\phi_\omega := \phi_{\omega_1} \circ \cdots \circ \phi_{\omega_n}$. Set $\Gamma := O \setminus \bigcup_{i=1}^{M} \phi_i O$ and $\Phi\Gamma := \bigcup_{i=1}^{M} \phi_i \Gamma$, and note that

$$O = \bigcup_{n=0}^{\infty} \bigcup_{u \in \Sigma^n} \phi_u \Gamma \cup \bigcap_{n=0}^{\infty} \Phi^n O, \tag{4.2}$$

where the unions are disjoint. For the sets of Figure 1, we depict in Figure 2 the associated sets O, ΦO, and Γ. We have $\Phi(\bigcap_{n=0}^{\infty} \Phi^n O) = \bigcap_{n=0}^{\infty} \Phi^n O$. Thus, $\bigcap_{n=0}^{\infty} \Phi^n O$ is either empty or coincides with F by uniqueness of the self-conformal set. Therefore, $\lambda_d(\bigcap_{n=0}^{\infty} \Phi^n O) \leq \lambda_d(F)$. Let D denote the Minkowski dimension of F. If $D < d$ then $\lambda_d(F) = 0$ and whence $\lambda_d(\bigcap_{n=0}^{\infty} \Phi^n O) = 0$. In the following we assume that O can be chosen so that $\lambda_d(F_{\mathrm{e}^{-t}} \cap \Gamma) = \mathfrak{o}(\mathrm{e}^{t(D-d)})$ as $t \to \infty$ with the little Landau symbol $\mathfrak{o}$. (For functions $f, g : \mathbb{R} \to \mathbb{R}$, we write $f = \mathfrak{o}(g)$ as $t \to \infty$ if $\lim_{t \to \infty} f(t)/g(t) = 0$.) This is a mild condition, which is always satisfied for self-similar systems with any feasible open set O (see [31]). Equation (4.2) thus implies for $D < d$ that

$$\lambda_d(F_{\mathrm{e}^{-t}} \cap O) = \sum_{n=0}^{\infty} \sum_{u \in \Sigma^n} \lambda_d(F_{\mathrm{e}^{-t}} \cap \phi_u \Gamma) = \sum_{\omega \in \Sigma^m} \sum_{n=0}^{\infty} \sum_{u \in \Sigma^n} \lambda_d(F_{\mathrm{e}^{-t}} \cap \phi_u \phi_\omega \Gamma) + \mathfrak{o}(\mathrm{e}^{t(D-d)})$$

for any $m \in \mathbb{N}$. Furthermore, suppose that $\lambda_d(F_{\mathrm{e}^{-t}} \cap \phi_u \phi_\omega \Gamma) = \lambda_d((\phi_u F)_{\mathrm{e}^{-t}} \cap \phi_u \phi_\omega \Gamma)$. By [31], for self-similar systems, this assumption always holds for some feasible open set O.

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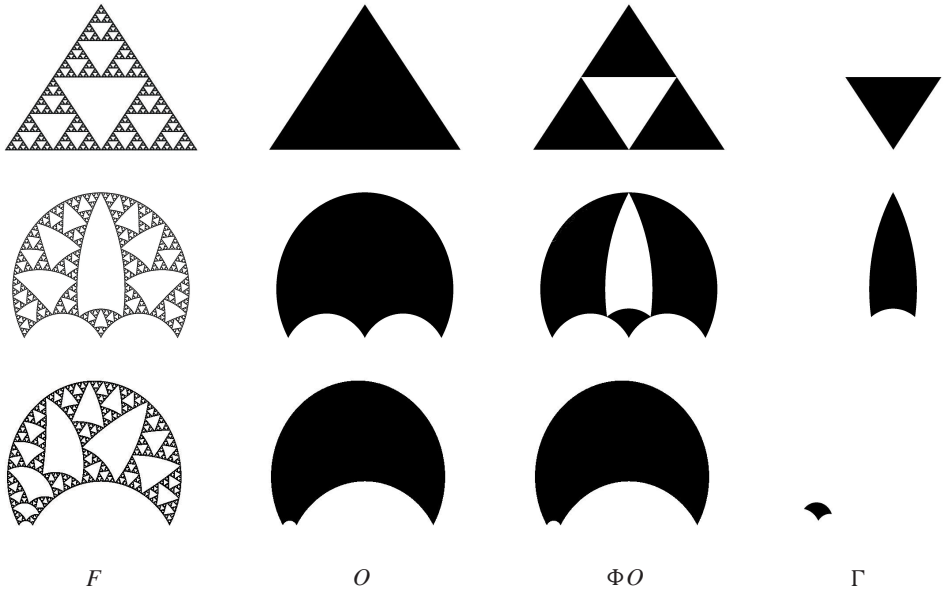


FIGURE 2: The self-conformal sets F from Example 4.1 (see Figure 1), together with examples of associated feasible open sets O, ΦO, and Γ.

(The open sets O of Figure 2 satisfy the above conditions; see Example 4.1.) If m is large, $\phi_\omega\Gamma$ is small, as each ϕ_i is strictly contracting. Since conformal maps locally behave like similarities and λ_d is homogeneous of degree d, we can approximate $\lambda_d((\phi_u F)_{\mathrm{e}^{-t}} \cap \phi_u\phi_\omega\Gamma)$ by

$$|\phi_u'(\pi\sigma\omega x)|^d \lambda_d(F_{\mathrm{e}^{-t}/|\phi_u'(\pi\sigma\omega x)|} \cap \phi_\omega\Gamma) \tag{4.3}$$

with an arbitrary $x \in \Sigma^{\mathbb{N}}$. Here $\pi\colon \Sigma^{\mathbb{N}} \to F$ is the *code map* defined by

$$\{\pi(\omega)\} \coloneqq \bigcap_{n=0}^{\infty} \phi_{\omega|_n}(X).$$

In order to bring (4.3) into a form to apply the renewal theorem, we define $\xi\colon \Sigma^{\mathbb{N}} \to \mathbb{R}$ by $\xi(\omega) \coloneqq -\log|\phi_{\omega_1}'(\pi\sigma\omega)|$. The function ξ is called the *geometric potential function* associated with the IFS Φ. It carries important geometric information of Φ and F. By definition, $\exp(-S_n\xi(u\omega x)) = |\phi_u'(\pi\sigma\omega x)|$. Thus, $\lambda_d(F_{\mathrm{e}^{-t}} \cap O)$ can be approximated by

$$\sum_{\omega\in\Sigma^m}\sum_{n=0}^{\infty}\sum_{u\in\Sigma^n} \mathrm{e}^{-dS_n\xi(u\omega x)}\lambda_d(F_{\mathrm{e}^{-t+S_n\xi(u\omega x)}} \cap \phi_\omega\Gamma) + \mathfrak{o}(\mathrm{e}^{t(D-d)}).$$

Setting $f_y(t) \coloneqq \lambda_d(F_{\mathrm{e}^{-t}} \cap \phi_\omega\Gamma)$ for $y \in \Sigma^{\mathbb{N}}$, $\chi \coloneqq \mathbf{1}_{\Sigma^{\mathbb{N}}}$ and $\eta \coloneqq -d\xi$, and assuming the regularity conditions of Section 3, we can apply Theorem 3.1 and, if ξ is nonlattice, obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{u\in\Sigma^n} \mathrm{e}^{-dS_n\xi(u\omega x)}\lambda_d(F_{\mathrm{e}^{-t+S_n\xi(u\omega x)}} \cap \phi_\omega\Gamma)\\
&\quad = N(t,\omega x)\\
&\quad \sim \mathrm{e}^{t\delta}\frac{h_{-(d+\delta)\xi}(\omega x)}{\int \xi\,\mathrm{d}\mu_{-(d+\delta)\xi}}\int_{-\infty}^{\infty}\mathrm{e}^{-T\delta}\lambda_d(F_{\mathrm{e}^{-T}} \cap \phi_\omega\Gamma)\,\mathrm{d}T,
\end{aligned}$$

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where $\delta > 0$ is the unique value for which $P(-(d+\delta)\xi) = 0$; see Section 2.3. It was proved in [3] that the Minkowski dimension D of F is the unique solution to $P(-D\xi) = 0$; thus, $d + \delta = D$. Using the bounded distortion property [22, Lemma 2.3.1], we all in all obtain, in the nonlattice situation,

$$\lambda_d(F_{\mathrm{e}^{-t}} \cap O) \sim \mathrm{e}^{t(D-d)} \lim_{m\to\infty} \sum_{\omega\in\Sigma^m} \frac{h_{-D\xi}(\omega x)}{\int \xi \,\mathrm{d}\mu_{-D\xi}} \int_{-\infty}^{\infty} \mathrm{e}^{-T(D-d)} \lambda_d(F_{\mathrm{e}^{-T}} \cap \phi_\omega \Gamma)\,\mathrm{d}T.$$

This shows that ξ being nonlattice implies the existence of the Minkowski content which can be determined explicitly. The lattice and the average cases can be treated similarly.

In the special case that F is self-similar, the approximation arguments are not needed and we can assume that $m = 0$. Furthermore, $h_{-D\xi} = \mathbf{1}_{\Sigma_A}$ and ξ is constant on cylinder sets of length 1, taking the values $-\log(r_i)$ with $r_i := \|\phi_i'\|_\infty$ and $\mu_{-D\xi}([i]) = r_i^D$, where $\|\cdot\|_\infty$ denotes the supremum norm. Therefore, for self-similar sets F, we obtain

$$\lambda_d(F_{\mathrm{e}^{-t}} \cap O) \sim \mathrm{e}^{t(D-d)} \frac{1}{-\sum_{i=1}^M \log(r_i) r_i^D} \int_{-\infty}^{\infty} \mathrm{e}^{-T(D-d)} \lambda_d(F_{\mathrm{e}^{-T}} \cap \Gamma)\,\mathrm{d}T$$

if ξ is nonlattice. If ξ is lattice then the respective formula is more involved, but we can deduce from Theorem 3.1(iii) that

$$\lim_{t\to\infty} t^{-1} \int_0^t \mathrm{e}^{-T(D-d)} \lambda_d(F_{\mathrm{e}^{-t}} \cap O)\,\mathrm{d}T = \frac{\int_{-\infty}^{\infty} \mathrm{e}^{-T(D-d)} \lambda_d(F_{\mathrm{e}^{-T}} \cap \Gamma)\,\mathrm{d}T}{-\sum_{i=1}^M \log(r_i) r_i^D}. \tag{4.4}$$

Example 4.1. The following self-conformal sets are depicted in Figures 1 and 2. We will discuss the Minkowski content for the set depicted in Figure 1(a), and restrict ourselves to the definitions and verification of the main conditions for the sets depicted in Figure 1(b) and (c). In all the examples, $d = 2$.

(a) The Sierpiński gasket F is the self-similar set associated with the IFS $\Phi := \{\phi_1, \phi_2, \phi_3 : \mathbb{R}^2 \to \mathbb{R}^2\}$ given by $\phi_1(x) = x/2$, $\phi_2(x) = x/2 + (\frac{1}{2}, 0)$, and $\phi_3(x) = x/2 + (\frac{1}{4}, \sqrt{3}/4)$. Its Minkowski dimension is $D = \log(3)/\log(2)$. The open triangle O with vertices $(0, 0)$, $(1, 0)$, $(\frac{1}{2}, \sqrt{3}/2)$ is a feasible open set for Φ. The set Γ is the equilateral triangle with vertices $(\frac{1}{2}, 0)$, $(\frac{1}{4}, \sqrt{3}/4)$, $(\frac{3}{4}, \sqrt{3}/4)$, and

$$\lambda_2(F_{\mathrm{e}^{-t}} \cap \Gamma) = \begin{cases} \dfrac{3}{2}\mathrm{e}^{-t} - 3\sqrt{3}\mathrm{e}^{-2t}, & t > -\log\left(\sqrt{\dfrac{3}{12}}\right), \\ \dfrac{\sqrt{3}}{16}, & t \le -\log\left(\sqrt{\dfrac{3}{12}}\right). \end{cases}$$

Thus, $\lambda_2(F_{\mathrm{e}^{-t}} \cap \Gamma) = \mathfrak{o}(\mathrm{e}^{t(D-2)})$. We have $r_i := \|\phi_i'\|_\infty = \frac{1}{2}$ for $i \in \{1, 2, 3\}$. Hence, the system is lattice and (4.4) yields

$$\lim_{t\to\infty} t^{-1} \int_0^t \mathrm{e}^{-T(D-2)} \lambda_2(F_{\mathrm{e}^{-T}} \cap O)\,\mathrm{d}T = \frac{\sqrt{3}^{-(D+1)}}{\log(2)} \left[\frac{1}{2-D} + \frac{2}{D-1} - \frac{1}{D}\right] \approx 1.8126.$$

(b) The set F depicted in Figure 1(b) is the image of the Sierpiński gasket under the complex Möbius transform $g_1 : \mathbb{C} \to \mathbb{C}$, $z \mapsto -\mathrm{i}z/(\sqrt{3}z - \sqrt{3} - \mathrm{i})$. An associated IFS is $\{g_1 \circ \phi_i \circ g_1^{-1} \mid i \in \{1, 2, 3\}\}$ with ϕ_1, ϕ_2, and ϕ_3 as in (a). The maps $g_1 \circ \phi_i \circ g_1^{-1}$ are contractive conformal $\mathcal{C}^{1+\alpha}$-diffeomorphisms, but not similitudes. Indeed, F is self-conformal but not self-similar. It is known that the Minkowski dimension is stable under

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bi-Lipschitz maps; see [8, Chapter 3.2]. Thus, $D = \log(3)/\log(2)$. Since $\partial\Gamma \subseteq F$ is one-dimensional, we have $\lambda_2(F_{e^{-t}} \cap \Gamma) = \mathfrak{O}(e^{-t})$ as $t \to \infty$ with the big Landau symbol $\mathfrak{O}$, whence $\lambda_2(F_{e^{-t}} \cap \Gamma) = \mathfrak{o}(e^{t(D-d)})$. (For $f, g\colon \mathbb{R} \to \mathbb{R}$, we write $f = \mathfrak{O}(g)$ as $t \to \infty$ if $\limsup_{t\to\infty} |f(t)/g(t)| < \infty$.)

(c) In Figure 1(c) the image of the Sierpiński gasket under the complex Möbius transform $g_2\colon \mathbb{C} \to \mathbb{C}$, $z \mapsto (1 - \sqrt{3}\mathrm{i})z/((12 + 10\sqrt{3}\mathrm{i})z - 11 - 11\sqrt{3}\mathrm{i})$ is depicted. Analogous to part (b), an associated IFS is given by $\{g_2 \circ \phi_i \circ g_2^{-1} \mid i \in \{1, 2, 3\}\}$ and $\lambda_2(F_{e^{-t}} \cap \Gamma) = \mathfrak{O}(e^{-t})$.

Remark 4.1. The setting of self-similar sets has been studied, e.g. in [7], [10], [21], and [31]. Self-conformal subsets of $\mathbb{R}$ and limit sets of graph-directed systems in $\mathbb{R}$, including Fuchsian groups of Schottky type, are treated in [13] and [14], where the results of [19] were applied. However, the setting of [19] is too restrictive for higher-dimensional Euclidean spaces $\mathbb{R}^d$, making necessary the renewal theorems developed in the present paper. Note that Lalley's renewal theorems have already been applied in [19] to problems in hyperbolic geometry.

5. Proofs of the renewal theorems and their corollaries

To prove the renewal theorems (Theorems 3.1 and 3.2), we pursue an analytic approach using Fourier–Laplace transforms. In the present setting, the Fourier–Laplace transforms naturally lead to the Ruelle–Perron–Frobenius operator. Thus, in Section 5.1 we present essentials from complex Ruelle–Perron–Frobenius theory. These are used to prove the renewal theorems in Section 5.2.

5.1. Analytic properties of the Ruelle–Perron–Frobenius operator

Here we collect results from [19], [24], and [26] concerning analytic properties of the operator-valued function $z \mapsto (I - \mathcal{L}_{\eta+z\xi})^{-1}$, where $\xi, \eta \in \mathcal{F}_\alpha(\Sigma_A, \mathbb{R})$ are fixed and $z \in \mathbb{C}$.

Let $\mathcal{B}(\mathcal{F}_\alpha(\Sigma_A))$ denote the set of bounded linear operators on $\mathcal{F}_\alpha(\Sigma_A)$ to $\mathcal{F}_\alpha(\Sigma_A)$. As $\mathcal{F}_\alpha(\Sigma_A)$ endowed with $\|\cdot\|_\alpha := |\cdot|_\alpha + \|\cdot\|_\infty$ is a Banach space, also $(\mathcal{B}(\mathcal{F}_\alpha(\Sigma_A)), \|\cdot\|_{\mathrm{op}})$ is a Banach space [12, p. 150]. Here $\|B\|_{\mathrm{op}} := \sup_{g\in\mathcal{F}_\alpha(\Sigma_A), \|g\|_\alpha=1} \|Bg\|_\alpha$ denotes the *operator norm* of $B \in \mathcal{B}(\mathcal{F}_\alpha(\Sigma_A))$. A function $f\colon D \to \mathcal{B}(\mathcal{F}_\alpha(\Sigma_A))$ defined on an open domain $D \subset \mathbb{C}$ is called *holomorphic* if, for all $z \in D$, there exists $l(z) \in \mathcal{B}(\mathcal{F}_\alpha(\Sigma_A))$ such that $\lim_{h\to 0} \|h^{-1}(f(z+h) - f(z)) - l(z)\|_{\mathrm{op}} = 0$. Following convention, we interchangeably use the terms holomorphic and *analytic*. For clarity, we write $\rho := \eta + z\xi$ and sometimes $\rho(z) := \eta + z\xi$ if we want to stress dependence on z. The operator $\mathcal{L}_{\rho(z)}$ is a bounded linear operator on the Banach space $(\mathcal{F}_\alpha(\Sigma_A), \|\cdot\|_\alpha)$. We write $\mathrm{spec}(\mathcal{L}_{\rho(z)}) := \{\lambda \in \mathbb{C} \mid \mathcal{L}_{\rho(z)} - \lambda I \text{ is not invertible}\}$ for its *spectrum* and $\mathrm{spr}(\mathcal{L}_{\rho(z)})$ for its *spectral radius*, i.e. the radius of the smallest closed disc centred at the origin which contains $\mathrm{spec}(\mathcal{L}_{\rho(z)})$. By the *spectral radius formula*, $\lim_{n\to\infty} \|\mathcal{L}^n_{\rho(z)}\|_{\mathrm{op}}^{1/n} = \mathrm{spr}(\mathcal{L}_{\rho(z)})$. For the following statement, let $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ respectively denote the real and imaginary parts of $z \in \mathbb{C}$.

Theorem 5.1. ([26].) *Let $\alpha \in (0, 1)$ and $\xi, \eta \in \mathcal{F}_\alpha(\Sigma_A)$. Suppose that $z \in \mathbb{C} \setminus \mathbb{R}$.*

(i) *If, for some $b \in \mathbb{R}$, the function $(\mathrm{Im}(z)\xi - b)/(2\pi)$ is cohomologous to an integer-valued function, then $e^{ib}\gamma_{\eta+\mathrm{Re}(z)\xi}$ is a simple eigenvalue of $\mathcal{L}_{\eta+z\xi}$, and the rest of the spectrum is contained in a disc centred at 0 of radius strictly less than $\gamma_{\eta+\mathrm{Re}(z)\xi}$.*

(ii) *Otherwise, the entire spectrum of $\mathcal{L}_{\eta+z\xi}$ is contained in a disc centred at 0 of radius strictly less than $\gamma_{\eta+\mathrm{Re}(z)\xi}$.*

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Below we present useful results that follow from [19] and Theorem 5.1. Note that in [19] only the special case in which $\eta \equiv 0$ is covered. However, the proofs work in the same way when $\eta \in \mathcal{F}_\alpha(\Sigma_A, \mathbb{R})$ is arbitrary. Therefore, we omit the proofs and refer the reader to the respective proofs in [19].

Results in regular perturbation theory [12, Sections 4.3 and 7.1] imply that $z \mapsto \gamma_{\rho(z)}$, $z \mapsto h_{\rho(z)}$, and $z \mapsto \nu_{\rho(z)}$ extend to holomorphic functions in a neighbourhood of $\mathbb{R}$ such that $\gamma_{\rho(z)} \neq 0$, $\mathcal{L}_{\rho(z)} h_{\rho(z)} = \gamma_{\rho(z)} h_{\rho(z)}$, $\mathcal{L}^*_{\rho(z)} \nu_{\rho(z)} = \gamma_{\rho(z)} \nu_{\rho(z)}$, and $\nu_{\rho(z)}(h_{\rho(z)}) = \nu_0(h_{\rho(z)}) = 1$ [19, p. 27].

Proposition 5.1. ([19, Propositions 7.1 and 7.2].) *Fix $\alpha \in (0, 1)$. Let $\xi, \eta \in \mathcal{F}_\alpha(\Sigma_A, \mathbb{R})$, and let $-\delta$ be the unique real zero of $t \mapsto P(\eta + t\xi)$. Then*

(i) *$z \mapsto (I - \mathcal{L}_{\rho(z)})^{-1}$ is holomorphic in the half-plane $\mathrm{Re}(z) < -\delta$,*

(ii) *$z \mapsto (I - \mathcal{L}_{\rho(z)})^{-1}$ has a simple pole at $z = -\delta$ and, for $\chi \in \mathcal{F}_\alpha(\Sigma_A)$,*

$$(I - \mathcal{L}_{\rho(z)})^{-1}\chi = \gamma_{\rho(z)}(1 - \gamma_{\rho(z)})^{-1} \int \chi \, \mathrm{d}\nu_{\rho(z)} \cdot h_{\rho(z)} + (I - \mathcal{L}''_{\rho(z)})^{-1}\chi \tag{5.1}$$

for z in some punctured neighbourhood of $z = -\delta$, where

$$\mathcal{L}''_{\rho(z)} \coloneqq \mathcal{L}_{\rho(z)} - \mathcal{L}'_{\rho(z)} \quad \text{with} \quad \mathcal{L}'_{\rho(z)}\chi \coloneqq \gamma_{\rho(z)} \int \chi \, \mathrm{d}\nu_{\rho(z)} \cdot h_{\rho(z)}.$$

Moreover, $z \mapsto (I - \mathcal{L}''_{\rho(z)})^{-1}$ is a holomorphic operator-valued function in a neighbourhood of $z = -\delta$.

The factor $\gamma_{\rho(z)}$ of the first summand of (5.1) is missing in [19]. However, the relevant z-value is $-\delta$, and $\gamma_{-\delta} = 1$.

We are interested in the residue of $z \mapsto (I - \mathcal{L}_{\rho(z)})^{-1}$ at the simple pole $z = -\delta$. For this, we use the facts that the topological pressure function $t \mapsto P(\eta + t\xi)$ is real analytic for $t \in \mathbb{R}$ and $\xi, \eta \in \mathcal{F}_\alpha(\Sigma_A, \mathbb{R})$, and that it satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} P(\eta + t\xi) = \int \xi \, \mathrm{d}\mu_{\eta + t\xi}, \qquad t \in \mathbb{R}. \tag{5.2}$$

The analyticity of $z \mapsto (I - \mathcal{L}_{\rho(z)})^{-1}$ can be proved with methods of analytic perturbation theory as presented in [12]. This method of proof is due to [27]. Furthermore, since $z \mapsto \gamma_{\rho(z)}$ has an analytic continuation to a neighbourhood of $\mathbb{R}$ and $P(\xi) = \log(\gamma_\xi)$ for real valued $\xi \in \mathcal{C}(\Sigma_A, \mathbb{R})$, we can extend P analytically by setting $P(\rho(z)) \coloneqq \log(\gamma_{\rho(z)})$. 'Formally this definition can only be made modulo $2\pi\mathrm{i}$ since log is multiple valued, although we shall ask that $P(\xi)$ be real-valued when ξ is real-valued' [24, p. 31]. In this way, (5.2) extends to a neighbourhood of $\mathbb{R}$. Combined with (5.2), Proposition 5.1(ii) yields the following corollary since $z \mapsto \gamma_{\rho(z)}$, $z \mapsto \int \chi \, \mathrm{d}\nu_{\rho(z)}$, and $z \mapsto h_{\rho(z)}$ are continuous at $z = -\delta$.

Corollary 5.1. *Let $\xi \in \mathcal{C}(\Sigma_A, \mathbb{R})$ and $\chi \in \mathcal{F}_\alpha(\Sigma_A, \mathbb{R})$. Then, for $x \in \Sigma_A$, the residue of $(I - \mathcal{L}_{\eta + z\xi})^{-1}\chi(x)$ at $z = -\delta$ is*

$$-\frac{\int \chi \, \mathrm{d}\nu_{\eta - \delta\xi}}{\int \xi \, \mathrm{d}\mu_{\eta - \delta\xi}} h_{\eta - \delta\xi}(x).$$

The residue is given in [19, p. 27]; however, with a different sign.

Proposition 5.2. ([19, Proposition 7.3].) *If ξ is nonlattice then $z \mapsto (I - \mathcal{L}_{\eta + z\xi})^{-1}$ is holomorphic in a neighbourhood of every z on the line $\mathrm{Re}(z) = -\delta$ except for $z = -\delta$.*

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Proposition 5.3. ([19, Proposition 7.4].) *If ξ is integer valued but not cohomologous to any function valued in a proper subgroup of the integers, then $z \mapsto (I - \mathcal{L}_{\eta+z\xi})^{-1}$ is $2\pi\mathrm{i}$ periodic, and holomorphic at every z on the line $\mathrm{Re}(z) = -\delta$ such that $\mathrm{Im}(z)/(2\pi)$ is not an integer.*

5.2. Proofs of the renewal theorems

5.2.1. *Proof of the renewal theorem, Theorem 3.1.* As the statement of Theorem 3.1 suggests, we need to distinguish between the cases of ξ being lattice and nonlattice. We start with the nonlattice situation, for which we use a standard smoothing argument to show the desired asymptotic. For a probability density $\Pi\colon \mathbb{R} \to \mathbb{R}$, we consider its Fourier–Laplace transform (characteristic function) given by

$$\widehat{\Pi}(\mathrm{i}\theta) := \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\theta t}\,\Pi(t)\,\mathrm{d}t,$$

and introduce the following class of probability densities:

$$\mathfrak{P} := \{\Pi\colon \mathbb{R} \to \mathbb{R} \mid \Pi \text{ is a probability density, } \Pi(t) = \Pi(-t) \text{ for } t \in \mathbb{R}.\\ \widehat{\Pi}(\mathrm{i}\theta) \text{ is nonnegative, is } \mathcal{C}^\infty, \text{ and has compact support}\}.$$

Note that the function $\widehat{\Pi}\colon \mathbb{R} \to \mathbb{C}$ given by

$$\widehat{\Pi}(\mathrm{i}\theta) := \begin{cases} \exp\left(\dfrac{-\theta^2}{1-\theta^2}\right), & |\theta| \le 1, \\ 0, & \text{otherwise}, \end{cases} \tag{5.3}$$

defines an even probability density $\Pi\colon \mathbb{R} \to \mathbb{R}$ which lies in $\mathfrak{P}$. That Π is a probability density is due to Bochner's theorem; see, e.g. [17, Satz 15.29]. Thus, $\mathfrak{P} \neq \varnothing$. For the following, fix Π as such. As Π is an even probability density, we know that, for all $\varepsilon > 0$, there exists $\tau > 0$ such that $\int_{-\tau}^{\tau} \Pi(t)\,\mathrm{d}t \ge 1 - \varepsilon$. For each $\varepsilon > 0$, fix such $\tau = \tau(\varepsilon)$. Thus, Π_ε, which for $\varepsilon > 0$ is defined by

$$\Pi_\varepsilon(t) := \frac{\tau(\varepsilon)}{\varepsilon}\,\Pi\left(t\,\frac{\tau(\varepsilon)}{\varepsilon}\right),$$

satisfies $\int_{-\varepsilon}^{\varepsilon} \Pi_\varepsilon(t)\,\mathrm{d}t = \int_{-\tau(\varepsilon)}^{\tau(\varepsilon)} \Pi(t)\,\mathrm{d}t \ge 1 - \varepsilon$. Moreover, it can be verified that $\Pi_\varepsilon \in \mathfrak{P}$ for all $\varepsilon > 0$. The smoothing argument is as follows.

Lemma 5.1. *If, for each sufficiently small $\varepsilon > 0$,*

$$\lim_{r\to\infty} \int_{-\infty}^{\infty} \Pi_\varepsilon(r - T)\mathrm{e}^{-T\delta} N(T, x)\,\mathrm{d}T = G(x) \tag{5.4}$$

uniformly for $x \in \Sigma_A$, then the statement of Theorem 3.1(i) holds.

Proof. In the proof we distinguish between the different cases of (D).
Case (D)(i). This statement follows from the proof of [19, Lemma 8.2].
Case (D)(iii). For $r \in \mathbb{R}$ and $\varepsilon > 0$, condition (B) implies that

$$\left| \int_{-\infty}^{\infty} \Pi_\varepsilon(r - T)\mathrm{e}^{-T\delta} N(T, x)\,\mathrm{d}T - \int_{r-\varepsilon}^{r+\varepsilon} \Pi_\varepsilon(r - T)\mathrm{e}^{-T\delta} N(T, x)\,\mathrm{d}T \right| \le \mathfrak{C}\varepsilon, \tag{5.5}$$

which tends to 0 as $\varepsilon \to 0$ uniformly for $x \in \Sigma_A$. Moreover, we observe that

$$\begin{aligned} \inf_{\tilde{\varepsilon}\in[0,2\varepsilon)} \mathrm{e}^{-(r-\tilde{\varepsilon})\delta} N(r - \tilde{\varepsilon}, x)(1 - \varepsilon) &\le \int_{r-2\varepsilon}^{r} \Pi_\varepsilon(r - \varepsilon - T)\mathrm{e}^{-T\delta} N(T, x)\,\mathrm{d}T \\ &\le \sup_{\tilde{\varepsilon}\in[0,2\varepsilon)} \mathrm{e}^{-(r-\tilde{\varepsilon})\delta} N(r - \tilde{\varepsilon}, x). \end{aligned} \tag{5.6}$$

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These observations imply that

$$\begin{aligned} G(x) &\overset{(5.4)}{=} \lim_{r\to\infty} \int_{-\infty}^{\infty} \Pi_\varepsilon(r-T)\mathrm{e}^{-T\delta} N(T,x)\,\mathrm{d}T \\ &= \liminf_{\varepsilon\searrow 0} \limsup_{r\to\infty} \int_{-\infty}^{\infty} \Pi_\varepsilon(r-T)\mathrm{e}^{-T\delta} N(T,x)\,\mathrm{d}T \\ &\overset{(5.5)}{\le} \liminf_{\varepsilon\searrow 0} \limsup_{r\to\infty} \int_{r-2\varepsilon}^{r} \Pi_\varepsilon(r-\varepsilon-T)\mathrm{e}^{-T\delta} N(T,x)\,\mathrm{d}T \\ &\overset{(5.6)}{\le} \liminf_{\varepsilon\searrow 0} \limsup_{r\to\infty} \sup_{\tilde\varepsilon\in[0,2\varepsilon)} \mathrm{e}^{-(r-\tilde\varepsilon)\delta} N(r-\tilde\varepsilon,x), \end{aligned}$$

and, likewise, that

$$G(x) \ge \limsup_{\varepsilon\searrow 0} \liminf_{r\to\infty} \inf_{\tilde\varepsilon\in[0,2\varepsilon)} \mathrm{e}^{-(r-\tilde\varepsilon)\delta} N(r-\tilde\varepsilon,x).$$

Using (D)(iii) and the inequalities

$$\inf_{\tilde\varepsilon\in[0,2\varepsilon)} \mathrm{e}^{-(r-\tilde\varepsilon)\delta} N(r-\tilde\varepsilon,x) \le \mathrm{e}^{-r\delta} N(r,x) \le \sup_{\tilde\varepsilon\in[0,2\varepsilon)} \mathrm{e}^{-(r-\tilde\varepsilon)\delta} N(r-\tilde\varepsilon,x),$$

we conclude that $G(x) = \lim_{r\to\infty} \mathrm{e}^{-r\delta} N(r,x)$ uniformly for $x \in \Sigma_A$.

Case (D)(ii). For $x \in \Sigma_A$ and $t \in \mathbb{R}$, set

$$g_x(t) \coloneqq \mathrm{e}^{-t\delta} f_x(t).$$

Then $\{|g_x| \mid x \in \Sigma_A\}$ is equi dRi by (D)(ii). Moreover,

$$\begin{aligned} \mathrm{e}^{-t\delta} N(t,x) &\coloneqq \sum_{n=0}^{\infty} \sum_{\{y:\,\sigma^n y = x\}} \chi(y) f_y(t - S_n\xi(y)) \mathrm{e}^{S_n\eta(y)} \mathrm{e}^{-t\delta} \\ &= \sum_{n=0}^{\infty} \sum_{\{y:\,\sigma^n y = x\}} \chi(y) g_y(t - S_n\xi(y)) \mathrm{e}^{S_n(\eta-\delta\xi)(y)}. \end{aligned} \tag{5.7}$$

For showing that (5.4) implies that $\lim_{r\to\infty} \mathrm{e}^{-r\delta} N(r,x) = G(x)$, we consider

$$\begin{aligned} A_\varepsilon(r,x) &\coloneqq \left| \int_{-\infty}^{\infty} \Pi_\varepsilon(r-T)\mathrm{e}^{-T\delta} N(T,x)\,\mathrm{d}T - \mathrm{e}^{-r\delta} N(r,x) \right| \\ &\overset{(5.5)}{\le} \left| \int_{r-2\varepsilon}^{r} \Pi_\varepsilon(r-\varepsilon-T)\mathrm{e}^{-T\delta} N(T,x)\,\mathrm{d}T - \int_{r-2\varepsilon}^{r} \Pi_\varepsilon(r-\varepsilon-T)\mathrm{e}^{-r\delta} N(r,x)\,\mathrm{d}T \right| \\ &\quad + \mathfrak{C}\varepsilon + |\mathrm{e}^{-r\delta} N(r,x)|\varepsilon \\ &\overset{\text{(B)},(5.7)}{\le} \left| \int_{r-2\varepsilon}^{r} \Pi_\varepsilon(r-\varepsilon-T) \sum_{n=0}^{\infty} \sum_{\{y:\,\sigma^n y = x\}} \chi(y) \mathrm{e}^{S_n(\eta-\delta\xi)(y)} \right. \\ &\qquad \left. \times [g_y(T - S_n\xi(y)) - g_y(r - S_n\xi(y))]\,\mathrm{d}T \right| + 2\mathfrak{C}\varepsilon \\ &\le \sum_{n=0}^{\infty} \sum_{\{y:\,\sigma^n y = x\}} \chi(y) \mathrm{e}^{S_n(\eta-\delta\xi)(y)} \\ &\qquad \times \int_{r-2\varepsilon}^{r} \Pi_\varepsilon(r-\varepsilon-T) |g_y(T - S_n\xi(y)) - g_y(r - S_n\xi(y))|\,\mathrm{d}T \\ &\quad + 2\mathfrak{C}\varepsilon. \end{aligned}$$

For the last inequality, we have used the monotone convergence theorem.

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Set

$$d^{2\varepsilon}(t) \coloneqq \sup_{y\in\Sigma_A}\Big(\sup_{\tilde\varepsilon\in[0,2\varepsilon)} g_y(t-\tilde\varepsilon) - \inf_{\tilde\varepsilon\in[0,2\varepsilon)} g_y(t-\tilde\varepsilon)\Big).$$

Since $\{g_x \mid x\in\Sigma_A\}$ is equi dRi, we know that, for sufficiently small $\varepsilon>0$,

$$\sum_{k\in\mathbb{Z}} d^{2\varepsilon}(k\cdot 2\varepsilon) \quad \text{exists and} \quad \lim_{\varepsilon\to 0}\sum_{k\in\mathbb{Z}} d^{2\varepsilon}(k\cdot 2\varepsilon)\cdot 2\varepsilon = 0. \tag{5.8}$$

Therefore, we may deduce the following chain of inequalities:

$$\begin{aligned}
&A_\varepsilon(r,x) - 2\mathfrak{C}\varepsilon\\
&\quad\le \sum_{n=0}^{\infty}\sum_{\{y:\,\sigma^n y=x\}} \chi(y)\mathrm{e}^{S_n(\eta-\delta\xi)(y)} d^{2\varepsilon}(r-S_n\xi(y))\\
&\quad\le \sum_{k\in\mathbb{Z}} d^{2\varepsilon}(k\cdot 2\varepsilon)\|\chi\|_\infty \sum_{n=0}^{\infty}\sum_{\{y:\,\sigma^n y=x\}} \mathrm{e}^{S_n(\eta-\delta\xi)(y)}\mathbf{1}_{((k-1)2\varepsilon,(k+1)2\varepsilon]}(r-S_n\xi(y))\\
&\quad\le c\|\chi\|_\infty \sum_{k\in\mathbb{Z}} d^{2\varepsilon}(k\cdot 2\varepsilon)\underbrace{\sum_{n=0}^{\infty}\sum_{\{y:\,\sigma^n y=x\}} \mu_{\eta-\delta\xi}([y|_n])\mathbf{1}_{((k-1)2\varepsilon,(k+1)2\varepsilon]}(r-S_n\xi(y))}_{=:a_{\varepsilon,k}(r,x)}.
\end{aligned}$$

For the last estimate, we used the Gibbs property (2.1) of $\mu_{\eta-\delta\xi}$ with constant c. For simplicity of presentation, we first treat the case that ξ is strictly positive. In this case there exists $\kappa>0$ such that $\xi(x)\ge\kappa$ for all $x\in\Sigma_A$ since ξ is continuous and Σ_A is compact. As we will later be interested in the limiting behaviour when $\varepsilon\to 0$, we can freely assume that $4\varepsilon\le\kappa$ and set $m\coloneqq\lfloor\kappa/(2\varepsilon)\rfloor$. Note that $m\ge 2$. Let $\sigma^{-n}(x)$ denote the set of pre-images of x under σ^n, that is,

$$\sigma^{-n}(x)\coloneqq\{y\in\Sigma_A \mid \sigma^n y = x\}.$$

Let $y\in\sigma^{-n}(x)$, and assume that $r-S_n\xi(y)\in((k-1)2\varepsilon,(k+1)2\varepsilon]=:I_k(\varepsilon)$. Then $\sigma y\in\sigma^{-(n-1)}(x)$ and

$$\begin{aligned}
r-S_{n-1}\xi(\sigma y) &= r-S_n\xi(y)+\xi(y)\\
&\ge r-S_n\xi(y)+\kappa\\
&> (k-1)2\varepsilon+2m\varepsilon\\
&= ((k-2+m)+1)2\varepsilon,
\end{aligned}$$

whence $r-S_{n-1}\xi(\sigma y)\notin I_{k+q}(\varepsilon)$ for $q\in\{0,\dots,m-2\}$. Now let $\tilde y\in\sigma^{-(n+1)}(x)$ be such that $\sigma\tilde y = y$. Then

$$\begin{aligned}
r-S_{n+1}\xi(\tilde y) &= r-S_n\xi(y)-\xi(\tilde y)\\
&\le r-S_n\xi(y)-\kappa\\
&\le (k+1)2\varepsilon-2m\varepsilon\\
&= ((k-m+2)-1)2\varepsilon,
\end{aligned}$$

whence $r-S_{n+1}\xi(\tilde y)\notin I_{k-q}(\varepsilon)$ for $q\in\{0,\dots,m-2\}$. Therefore,

$$\sum_{q=0}^{m-2} a_{\varepsilon,k+q}(r,x) = \sum_{q=0}^{m-2}\sum_{n=0}^{\infty}\sum_{\{y:\,\sigma^n y=x\}} \mu_{\eta-\delta\xi}([y|_n])\mathbf{1}_{((k+q-1)2\varepsilon,(k+q+1)2\varepsilon]}(r-S_n\xi(y))\le 1 \tag{5.9}$$

as $\mu_{\eta-\delta\xi}$ is a probability measure on Σ_A.

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Next, we show the existence of $q \in \{0, \dots, m-2\}$ such that, for all $\varepsilon < \kappa/8$,

$$\sum_{k\in\mathbb{Z}} d^{2\varepsilon}(k\cdot 2\varepsilon) a_{\varepsilon,k+q}(r,x) \le \frac{2}{\kappa} 2\varepsilon \sum_{k\in\mathbb{Z}} d^{2\varepsilon}(k\cdot 2\varepsilon). \tag{5.10}$$

Assume that this was not the case. Since, for $\varepsilon < \kappa/8$, we have $1 < 2(\kappa - 4\varepsilon)/\kappa < 2\varepsilon(m-1)\cdot 2/\kappa$, it follows that

$$\begin{aligned}\sum_{k\in\mathbb{Z}} d^{2\varepsilon}(k\cdot 2\varepsilon) &< 2\varepsilon(m-1)\frac{2}{\kappa}\sum_{k\in\mathbb{Z}} d^{2\varepsilon}(k\cdot 2\varepsilon)\\ &= \sum_{q=0}^{m-2} 2\varepsilon\frac{2}{\kappa}\sum_{k\in\mathbb{Z}} d^{2\varepsilon}(k\cdot 2\varepsilon)\\ &\overset{\neg(5.10)}{<} \sum_{q=0}^{m-2}\sum_{k\in\mathbb{Z}} d^{2\varepsilon}(k\cdot 2\varepsilon) a_{\varepsilon,k+q}(r,x)\\ &\overset{(5.9)}{\le} \sum_{k\in\mathbb{Z}} d^{2\varepsilon}(k\cdot 2\varepsilon).\end{aligned}$$

As $\sum_{k\in\mathbb{Z}} d^{2\varepsilon}(k\cdot 2\varepsilon)$ exists for sufficiently small ε, it follows that (5.10) must hold for some $q \in \{0, \dots, m-2\}$.

Note that $a_{\varepsilon,k}(r - 2q\varepsilon, x) = a_{\varepsilon,k+q}(r,x)$, and so, for this q,

$$\begin{aligned}\limsup_{r\to\infty} A_\varepsilon(r,x) &= \limsup_{r\to\infty} A_\varepsilon(r-2q\varepsilon,x)\\ &\le c\|\chi\|_\infty \limsup_{r\to\infty}\sum_{k\in\mathbb{Z}} d^{2\varepsilon}(k\cdot 2\varepsilon)\underbrace{a_{\varepsilon,k}(r-2q\varepsilon,x)}_{=a_{\varepsilon,k+q}(r,x)} + 2\mathfrak{C}\varepsilon\\ &\overset{(5.10)}{\le} c\|\chi\|_\infty \frac{2}{\kappa} 2\varepsilon\sum_{k\in\mathbb{Z}} d^{2\varepsilon}(k\cdot 2\varepsilon) + 2\mathfrak{C}\varepsilon.\end{aligned} \tag{5.11}$$

All in all we have, for each sufficiently small $\varepsilon > 0$,

$$\begin{aligned}\limsup_{r\to\infty}|G(x) - \mathrm{e}^{-r\delta}N(r,x)| &\le \limsup_{r\to\infty}\left|G(x) - \int_{-\infty}^{\infty}\Pi_\varepsilon(r-T)\mathrm{e}^{-T\delta}N(T,x)\,\mathrm{d}T\right|\\ &\quad + \limsup_{r\to\infty}\left|\int_{-\infty}^{\infty}\Pi_\varepsilon(r-T)\mathrm{e}^{-T\delta}N(T,x)\,\mathrm{d}T - \mathrm{e}^{-r\delta}N(r,x)\right|\\ &\overset{(5.4)}{=} \limsup_{r\to\infty} A_\varepsilon(r,x)\\ &\overset{(5.11)}{\le} c\|\chi\|_\infty\frac{2}{\kappa}2\varepsilon\sum_{k\in\mathbb{Z}} d^{2\varepsilon}(k\cdot 2\varepsilon) + 2\mathfrak{C}\varepsilon.\end{aligned}$$

This implies that

$$\limsup_{r\to\infty}|G(x) - \mathrm{e}^{-r\delta}N(r,x)| \le \limsup_{\varepsilon\to 0}\left(c\|\chi\|_\infty 2\varepsilon\frac{2}{\kappa}\sum_{k\in\mathbb{Z}} d^{2\varepsilon}(k\cdot 2\varepsilon) + 2\mathfrak{C}\varepsilon\right) \overset{(5.8)}{=} 0.$$

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Now we treat the case that ξ is not strictly positive. By assumption, there exists $n \in \mathbb{N}$ for which $S_n\xi$ is strictly positive. Since ξ is continuous and Σ_A is compact, there exists a $\tilde{\kappa} > 0$ for which $S_n\xi \geq \tilde{\kappa}$. For $\ell \in \mathbb{N}$ and $j \in \{0, \dots, n-1\}$, this implies that

$$\begin{aligned} S_{\ell n+j}\xi(x) &= \sum_{i=0}^{\ell n+j-1} \xi(\sigma^i x) \\ &= \sum_{m=0}^{\ell-1} \sum_{i=mn}^{(m+1)n-1} \xi(\sigma^i x) + \sum_{i=\ell n}^{\ell n+j-1} \xi(\sigma^i x) \\ &= \sum_{m=0}^{\ell-1} S_n\xi(\sigma^i(\sigma^{mn}x)) + \sum_{i=\ell n}^{\ell n+j-1} \xi(\sigma^i x) \\ &\geq \ell\tilde{\kappa} + \inf_{x\in\Sigma_A,\, 0\leq i\leq n-1} S_i\xi(x). \end{aligned} \tag{5.12}$$

From this we can conclude that there exists $m^* \in \mathbb{N}$ such that $S_m\xi$ is strictly positive for all $m \geq m^*$. Thus, there exists $\kappa > 0$ with $S_m\xi \geq \kappa$ for all $m \geq m^*$. In the same way as in the case that ξ is strictly positive, we can show that if $y \in \sigma^{-n}(x)$ with $r - S_n\xi(y) \in I_k(\varepsilon)$ then $r - S_{n-m^*}\xi(\sigma^{m^*}y) \notin I_{k+q}(\varepsilon)$ for $q \in \{0, \dots, \lfloor \kappa/(2\varepsilon)\rfloor - 2\}$ whenever $n \geq m^*$ and $r - S_{n+m^*}\xi(\tilde{y}) \notin I_{k-q}(\varepsilon)$ for all $q \in \{0, \dots, \lfloor \kappa/(2\varepsilon)\rfloor - 2\}$, where $\tilde{y} \in \sigma^{-(n+m^*)}(x)$ is such that $\sigma^{m^*}\tilde{y} = y$. This implies that

$$\sum_{q=0}^{\lfloor \kappa/(2\varepsilon)\rfloor-2} \sum_{n=0}^{\infty} \sum_{\{y:\, \sigma^n y=x\}} \mu_{\eta-\delta\xi}([y|_n]) \mathbf{1}_{((k+q-1)2\varepsilon,(k+q+1)2\varepsilon]}(r - S_n\xi(y)) \leq 2m^*.$$

The rest of the proof follows in the same way as in the case that ξ is strictly positive. □

Proof of Theorem 3.1(i). For $x \in \Sigma_A$, consider the Fourier–Laplace transform of $t \mapsto \mathrm{e}^{-t\delta}N(t, x)$ at $z \in \mathbb{C}$:

$$L(z, x) := \int_{-\infty}^{\infty} \mathrm{e}^{zT}\mathrm{e}^{-T\delta}N(T, x)\,\mathrm{d}T.$$

This approach is inspired by Lalley [19, Theorem 1], whose proof we extend by some technical twists which are necessary in our more general setting. Conditions (B) and (C) imply that $L(\cdot, x)\colon \mathbb{C} \to \mathbb{C}$, $z \mapsto L(z, x)$ is well defined and analytic on $\{z \in \mathbb{C} \mid -s < \mathrm{Re}(z) < 0\}$. What is more, for small enough $\varepsilon > 0$, conditions (B) and (C) imply that $L(\cdot, x)$ converges absolutely and uniformly on $\{z \in \mathbb{C} \mid -s + \varepsilon \leq \mathrm{Re}(z) \leq -\varepsilon\}$. Now, in every such region, using (B) as well as the monotone and dominated convergence theorems, we obtain

$$\begin{aligned} L(z, x) &= \sum_{n=0}^{\infty} \sum_{\{y:\, \sigma^n y=x\}} \mathrm{e}^{S_n(\eta+(z-\delta)\xi)(y)}\chi(y) \int_{-\infty}^{\infty} \mathrm{e}^{(z-\delta)T} f_y(T)\,\mathrm{d}T \\ &= \sum_{n=0}^{\infty} \mathcal{L}^n_{\eta+(z-\delta)\xi}\left(\chi \int_{-\infty}^{\infty} \mathrm{e}^{(z-\delta)T} f_\cdot(T)\,\mathrm{d}T\right)(x), \end{aligned}$$

where $f_\cdot(T)\colon \Sigma_A \to \mathbb{R}$, $x \mapsto f_x(T)$. Note that (A)–(C) imply that $\int_{-\infty}^{\infty} \mathrm{e}^{(z-\delta)T} f_x(T)\,\mathrm{d}T$ exists if $-s < \mathrm{Re}(z) \leq 0$, since $\chi(x)|f_x(t)| \leq N^{\mathrm{abs}}(t, x)$. Thus, by Theorem 5.1(ii), the

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spectral radius formula (Section 5.1), and the fact that $\gamma_{\eta-\delta\xi} = 1$ (see Proposition 2.1), the above series converges for $-s < \mathrm{Re}(z) < 0$, and we obtain

$$L(z,x) = (I - \mathcal{L}_{\eta+(z-\delta)\xi})^{-1}\left(\chi \int_{-\infty}^{\infty} \mathrm{e}^{(z-\delta)T} f_\cdot(T)\,\mathrm{d}T\right)(x).$$

By Proposition 5.2, the operator-valued function $z \mapsto (I - \mathcal{L}_{\eta+(z-\delta)\xi})^{-1}$ is holomorphic at every z on the line $\mathrm{Re}(z) = 0$ except for $z = 0$, which is a simple pole by Proposition 5.1. Thus, according to Corollary 5.1, the residue of $z \mapsto L(z,x)$ at $z = 0$ is

$$-\frac{\int_{\Sigma_A} \chi(y) \int_{-\infty}^{\infty} \mathrm{e}^{-T\delta} f_y(T)\,\mathrm{d}T\,\mathrm{d}\nu_{\eta-\delta\xi}(y)}{\int \xi\,\mathrm{d}\mu_{\eta-\delta\xi}} h_{\eta-\delta\xi}(x) = -G(x),$$

where G is as in Theorem 3.1(i). Hence, $L(z,x)$ has the representation

$$L(z,x) = q(z,x) - \frac{G(x)}{z}, \tag{5.13}$$

where $q(\cdot,x)\colon \mathbb{C} \to \mathbb{C}$, $z \mapsto q(z,x)$ is holomorphic in a region containing the strip $\{z \in \mathbb{C} \mid -s+\varepsilon \le \mathrm{Re}(z) \le 0\}$ with sufficiently small $\varepsilon > 0$. Conditions (B) and (C) and Lebesgue's dominated convergence theorem now imply that, for every $\varepsilon \in (0,1]$,

$$\int_{-\infty}^{\infty} \Pi_\varepsilon(r-T)\mathrm{e}^{-T\delta} N(T,x)\,\mathrm{d}T = \lim_{\beta \searrow 0} \int_{-\infty}^{\infty} \Pi_\varepsilon(r-T)\mathrm{e}^{-T(\delta+\beta)} N(T,x)\,\mathrm{d}T. \tag{5.14}$$

Using the inverse Fourier–Laplace transform $\Pi_\varepsilon(t) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\theta t}\widehat{\Pi_\varepsilon}(\mathrm{i}\theta)\,\mathrm{d}\theta/(2\pi)$ and the fact that the integral on the left-hand side of (5.14) exists, we can convert the integral from the right-hand side of (5.14) for sufficiently small $\beta > 0$ as follows:

$$\begin{aligned}
&\int_{-\infty}^{\infty} \Pi_\varepsilon(r-T)\mathrm{e}^{-\beta T - T\delta} N(T,x)\,\mathrm{d}T \\
&\quad= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\theta(r-T)}\widehat{\Pi_\varepsilon}(\mathrm{i}\theta)\frac{\mathrm{d}\theta}{2\pi}\mathrm{e}^{-\beta T - T\delta} N(T,x)\,\mathrm{d}T \\
&\quad= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{(\mathrm{i}\theta-\beta)T}\mathrm{e}^{-T\delta} N(T,x)\widehat{\Pi_\varepsilon}(\mathrm{i}\theta)\mathrm{e}^{-\mathrm{i}\theta r}\frac{\mathrm{d}\theta}{2\pi}\,\mathrm{d}T \\
&\quad= \int_{-\infty}^{\infty} L(\mathrm{i}\theta-\beta,x)\widehat{\Pi_\varepsilon}(\mathrm{i}\theta)\mathrm{e}^{-\mathrm{i}\theta r}\frac{\mathrm{d}\theta}{2\pi} \\
&\quad\overset{(5.13)}{=} \int_{-\infty}^{\infty}\left(q(\mathrm{i}\theta-\beta,x) + \frac{G(x)(\mathrm{i}\theta+\beta)}{\theta^2+\beta^2}\right)\widehat{\Pi_\varepsilon}(\mathrm{i}\theta)\mathrm{e}^{-\mathrm{i}\theta r}\frac{\mathrm{d}\theta}{2\pi}.
\end{aligned} \tag{5.15}$$

The measures given by $\beta\,\mathrm{d}\theta/\pi(\theta^2+\beta^2)$ converge weakly to the Dirac point mass at 0 as $\beta \to 0$ [19, p. 31]. Moreover, the imaginary part on the right-hand side of (5.15) can be ignored since the left-hand side is real. Using the facts that $\widehat{\Pi_\varepsilon}(\mathrm{i}\theta)$ is real and that $\widehat{\Pi_\varepsilon}(0) = 1$ for all $\varepsilon \in (0,1]$, we obtain

$$\begin{aligned}
&\lim_{\beta\searrow 0}\int_{-\infty}^{\infty} \Pi_\varepsilon(r-T)\mathrm{e}^{-\beta T - T\delta} N(T,x)\,\mathrm{d}T \\
&\quad= \mathrm{Re}\left(\int_{-\infty}^{\infty} q(\mathrm{i}\theta,x)\widehat{\Pi_\varepsilon}(\mathrm{i}\theta)\mathrm{e}^{-\mathrm{i}\theta r}\frac{\mathrm{d}\theta}{2\pi} + \frac{G(x)}{2} + G(x)\int_{-\infty}^{\infty}\widehat{\Pi_\varepsilon}(\mathrm{i}\theta)\frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}\theta r}}{\theta}\frac{\mathrm{d}\theta}{2\pi}\right).
\end{aligned} \tag{5.16}$$

We separately treat the two integrals on the right-hand side of (5.16) and begin with the first integral. Recall that $\widehat{\Pi_\varepsilon}(\mathrm{i}\theta) = \widehat{\Pi}(\mathrm{i}\theta\varepsilon/\tau(\varepsilon))$ is $\mathcal{C}^\infty$ and has compact support, which is contained

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in $[-\tau(\varepsilon)/\varepsilon, \tau(\varepsilon)/\varepsilon] =: [-S, S]$. Also, recall that $q(\cdot, x)$ is analytic in a neighbourhood of $[-\mathrm{i}S, \mathrm{i}S]$ and continuous in x. As mentioned in [19, p. 31f.], the Cauchy integral formula for derivatives implies that $\mathrm{d}q(z, x)/\mathrm{d}z|_{z=\mathrm{i}\theta}$ is uniformly continuous in θ, whence bounded on $[-S, S] \times \Sigma_A$. Thus, $\mathrm{d}q(z, x)/\mathrm{d}z$ is bounded on $[-\mathrm{i}S, \mathrm{i}S] \times \Sigma_A$. Integration by parts now implies that

$$\begin{aligned}\int_{-S}^{S} q(\mathrm{i}\theta, x)\widehat{\Pi}_\varepsilon(\mathrm{i}\theta)\mathrm{e}^{-\mathrm{i}\theta r}\frac{\mathrm{d}\theta}{2\pi} \\ = \mathrm{i}q(\mathrm{i}\theta, x)\widehat{\Pi}_\varepsilon(\mathrm{i}\theta)\frac{\mathrm{e}^{-\mathrm{i}\theta r}}{2\pi r}\Big|_{\theta=-S}^{S} + \mathrm{i}\int_{-S}^{S}\frac{\mathrm{d}}{\mathrm{d}\theta}(q(\mathrm{i}\theta, x)\widehat{\Pi}_\varepsilon(\mathrm{i}\theta))\frac{\mathrm{e}^{-\mathrm{i}\theta r}}{2\pi r}\,\mathrm{d}\theta. \end{aligned} \tag{5.17}$$

As the support of $\widehat{\Pi}_\varepsilon$ is contained in $[-\mathrm{i}S, \mathrm{i}S]$ and $\widehat{\Pi}_\varepsilon$ is $\mathcal{C}^\infty$, the first term on the right-hand side of (5.17) equals 0 for all $r > 0$. For the second term on the right-hand side of (5.17), the definition of $\widehat{\Pi}$ given in (5.3) and the fact that $\widehat{\Pi}_\varepsilon(\mathrm{i}\theta) = \widehat{\Pi}(\mathrm{i}\theta\varepsilon/\tau(\varepsilon))$ imply that $\widehat{\Pi}_\varepsilon(\mathrm{i}\theta)$ and $\mathrm{d}\widehat{\Pi}_\varepsilon(\mathrm{i}\theta)/\mathrm{d}\theta$ are uniformly bounded for $\varepsilon \in (0, 1]$. This shows that the second term on the right-hand side of (5.17) converges to 0 uniformly for $\varepsilon \in (0, 1]$ and $x \in \Sigma_A$ as $r \to \infty$. Thus,

$$\lim_{r\to\infty} \mathrm{Re}\left(\int_{-\infty}^{\infty} q(\mathrm{i}\theta, x)\widehat{\Pi}_\varepsilon(\mathrm{i}\theta)\mathrm{e}^{-\mathrm{i}\theta r}\frac{\mathrm{d}\theta}{2\pi}\right) = 0 \tag{5.18}$$

uniformly for $x \in \Sigma_A$. Now, we consider the second integral on the right-hand side of (5.16):

$$\mathrm{Re}\left(\int_{-\infty}^{\infty}\widehat{\Pi}_\varepsilon(\mathrm{i}\theta)\frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}\theta r}}{\theta}\frac{\mathrm{d}\theta}{2\pi}\right) = \int_{-S}^{S}\widehat{\Pi}_\varepsilon(\mathrm{i}\theta)\frac{\sin(\theta r)}{\theta}\frac{\mathrm{d}\theta}{2\pi} = \int_{0}^{S}\widehat{\Pi}_\varepsilon(\mathrm{i}\theta)\frac{\sin(\theta r)}{\theta}\frac{\mathrm{d}\theta}{\pi}.$$

Using the sine integral $\mathtt{Si}(t) := \int_0^t \sin(\theta)\,\mathrm{d}\theta/\theta$ and $\lim_{t\to\infty}\mathtt{Si}(t) = \pi/2$, we infer that

$$\lim_{r\to\infty}\int_0^S \frac{\sin(\theta r)}{\theta}\frac{\mathrm{d}\theta}{\pi} = \lim_{r\to\infty}\int_0^{rS}\frac{\sin(\theta)}{\theta}\frac{\mathrm{d}\theta}{\pi} = \frac{1}{\pi}\lim_{r\to\infty}\mathtt{Si}(rS) = \frac{1}{2},$$

and remark that inserting the factor $\widehat{\Pi}_\varepsilon(\mathrm{i}\theta)$ in the above integrand does not change this limit, since $\widehat{\Pi}_\varepsilon$ is uniformly continuous and $\widehat{\Pi}_\varepsilon(0) = 1$. Thus, for the third term on the right-hand side of (5.16), we obtain

$$\lim_{r\to\infty}\mathrm{Re}\left(G(x)\int_{-\infty}^{\infty}\widehat{\Pi}_\varepsilon(\mathrm{i}\theta)\frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}\theta r}}{\theta}\frac{\mathrm{d}\theta}{2\pi}\right) = \frac{G(x)}{2} \tag{5.19}$$

uniformly for $x \in \Sigma_A$. Combining (5.14), (5.16), (5.18), and (5.19), it follows that

$$\lim_{r\to\infty}\int_{-\infty}^{\infty}\Pi_\varepsilon(r - T)\mathrm{e}^{-T\delta}N(T, x)\,\mathrm{d}T = G(x)$$

uniformly for $x \in \Sigma_A$. An application of Lemma 5.1 yields the desired result. □

Proof of Theorem 3.1(ii). In the lattice situation we work with discrete Fourier–Laplace transforms inspired by [19, Proof of Theorem 2] and again insert a few technical twists to fit our more general framework. Conditions (B) and (C) imply that, for fixed $\beta \in [0, a)$ and $x \in \Sigma_A$, the function $\widehat{N}^\beta(\cdot, x)$ given by

$$\widehat{N}^\beta(z, x) := \sum_{\ell=-\infty}^{\infty}\mathrm{e}^{\ell z}N(a\ell + \beta - \psi(x), x)$$

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is well defined and analytic on $\mathcal{Z} := \{z \in \mathbb{C} \mid -a(s+\delta) < \operatorname{Re}(z) < -a\delta\}$. Note that $S_n\xi = S_n\zeta + \psi - \psi \circ \sigma^n$ and recall that $S_n\zeta \in a\mathbb{Z}$ for all $n \in \mathbb{N}$. Thus, (B) and (C) allow for the following conversions for $z \in \mathcal{Z}$:

$$\begin{aligned}
\widehat{N}^\beta(z,x) &= \sum_{n=0}^{\infty} \sum_{\{y:\, \sigma^n y = x\}} \chi(y) \mathrm{e}^{S_n\eta(y)} \sum_{\ell=-\infty}^{\infty} \mathrm{e}^{\ell z} f_y(a\ell + \beta - \psi(x) - S_n\xi(y)) \\
&= \sum_{n=0}^{\infty} \sum_{\{y:\, \sigma^n y = x\}} \chi(y) \mathrm{e}^{S_n\eta(y)} \sum_{\ell=-\infty}^{\infty} \mathrm{e}^{(\ell + a^{-1}S_n\zeta(y))z} f_y(a\ell + \beta - \psi(y)) \\
&= \sum_{n=0}^{\infty} \sum_{\{y:\, \sigma^n y = x\}} \chi(y) \mathrm{e}^{S_n(\eta + a^{-1}z\zeta)(y)} \sum_{\ell=-\infty}^{\infty} \mathrm{e}^{\ell z} f_y(a\ell + \beta - \psi(y)) \\
&= \sum_{n=0}^{\infty} \mathcal{L}^n_{\eta + a^{-1}z\zeta} \left(\chi \sum_{\ell=-\infty}^{\infty} \mathrm{e}^{\ell z} f_\cdot(a\ell + \beta - \psi) \right)(x)
\end{aligned}$$

with $f_\cdot(t)\colon \Sigma_A \to \mathbb{R}$, $x \mapsto f_x(t)$ as before. For $z \in \mathcal{Z}$, we have $\operatorname{Re}(a^{-1}z) < -\delta$. Moreover, observe that $h_{\eta - \delta\zeta} = \mathrm{e}^{-\delta\psi} h_{\eta - \delta\xi}$ and $\gamma_{\eta - \delta\zeta} = \gamma_{\eta - \delta\xi}$. Thus, Proposition 2.1 yields $\gamma_{\eta + \operatorname{Re}(a^{-1}z)\zeta} < 1$. Theorem 5.1 (both parts) and the spectral radius formula (see Section 5.1) now imply that, for every $z \in \mathcal{Z}$,

$$\widehat{N}^\beta(z,x) = (I - \mathcal{L}_{\eta + a^{-1}z\zeta})^{-1} \left(\chi \sum_{\ell=-\infty}^{\infty} \mathrm{e}^{\ell z} f_\cdot(a\ell + \beta - \psi) \right)(x).$$

Note that $\| \chi \sum_{\ell=-\infty}^{\infty} \mathrm{e}^{\ell z} f_\cdot(a\ell + \beta - \psi) \|_\infty$ is finite for all $z \in \mathcal{Z}$ because of conditions (B) and (C).

Because $a^{-1}\zeta$ is integer valued but not cohomologous to any function valued in a proper subgroup of the integers, we can apply Proposition 5.3. Thus, $z \mapsto (I - \mathcal{L}_{\eta + a^{-1}z\zeta})^{-1}$ is $2\pi\mathrm{i}$ periodic and holomorphic at every z on the line $\operatorname{Re}(z) = -a\delta$ such that $\operatorname{Im}(z)/(2\pi)$ is not an integer. Therefore, $z \mapsto (I - \mathcal{L}_{\eta + a^{-1}z\zeta})^{-1}$ has an isolated singularity at $z = -a\delta$ and is holomorphic at each $z = -a\delta + \mathrm{i}\theta$ for $0 < |\theta| \le \pi$. By Proposition 5.1, the singularity of $\widehat{N}^\beta(z,x)$ at $z = -a\delta$ is

$$\frac{\gamma_{\eta + a^{-1}z\zeta}}{1 - \gamma_{\eta + a^{-1}z\zeta}} \underbrace{\int_{\Sigma_A} \chi(y) \sum_{\ell=-\infty}^{\infty} \mathrm{e}^{\ell z} f_y(a\ell + \beta - \psi(y)) \, \mathrm{d}\nu_{\eta + a^{-1}z\zeta}(y) h_{\eta + a^{-1}z\zeta}(x)}_{=: E_{x,\beta}(z)}.$$

Since the function $E_{x,\beta}$ is continuous at $-a\delta$, we deduce from (5.2) that the singularity of $\widehat{N}^\beta(z,x)$ at $z = -a\delta$ is a simple pole with residue

$$C_\beta(x) := -\frac{a}{\int \zeta \, \mathrm{d}\mu_{\eta - \delta\zeta}} E_{x,\beta}(-a\delta).$$

It follows that $\widehat{N}^\beta(\cdot, x)\colon \mathbb{C} \to \mathbb{C}$, $z \mapsto \widehat{N}^\beta(z,x)$ is meromorphic in

$$\tilde{\mathcal{Z}}(\varepsilon) := \{z \in \mathbb{C} \mid -a(\delta + s) < \operatorname{Re}(z) < -a\delta + \varepsilon,\ 0 \le \operatorname{Im}(z) \le \pi\}$$

for some $\varepsilon > 0$, and that the only singularity in this region is a simple pole at $-a\delta$ with residue $C_\beta(x)$. Additionally, by (C), $\sum_{\ell=-\infty}^{-1} \mathrm{e}^{\ell z} N(a\ell + \beta - \psi(x), x)$ is finite for $\operatorname{Re}(z) > -a(\delta + s)$.

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We conclude that there exists $\varepsilon > 0$ such that

$$\sum_{\ell=0}^{\infty} \mathrm{e}^{\ell z} N(a\ell + \beta - \psi(x), x) - \frac{C_\beta(x)}{z + a\delta}$$

is holomorphic in $\tilde{\mathcal{Z}}(\varepsilon)$. Also, observe that $z \mapsto (\mathrm{e}^{z+a\delta} - 1)/(z + a\delta)$ is holomorphic in $\mathbb{C}$. Making the change of variable $\tilde{z} := \mathrm{e}^{z+a\delta}$, it follows that

$$\sum_{\ell=0}^{\infty} \tilde{z}^\ell \mathrm{e}^{-a\ell\delta} N(a\ell + \beta - \psi(x), x) - \frac{C_\beta(x)}{\tilde{z} - 1}$$

and whence

$$L(\tilde{z}, x) := \sum_{\ell=0}^{\infty} \tilde{z}^\ell (\mathrm{e}^{-a\ell\delta} N(a\ell + \beta - \psi(x), x) + C_\beta(x))$$

are respectively holomorphic in $\{\mathrm{e}^{z+a\delta} \mid z \in \tilde{\mathcal{Z}}(\varepsilon)\}$ and $\{\tilde{z} \mid |\tilde{z}| < \mathrm{e}^{\varepsilon}\}$ (compare [19, p. 27]). Since $\mathrm{e}^{\varepsilon} > 1$, the coefficient sequence of the power series of $L(\cdot, x)\colon \mathbb{C} \to \mathbb{C}$, $z \mapsto L(z, x)$ converges to 0 exponentially fast; more precisely,

$$\mathrm{e}^{-an\delta} N(an + \beta - \psi(x), x) + C_\beta(x) = \mathfrak{o}\left(\left(1 + \frac{\mathrm{e}^{\varepsilon} - 1}{2}\right)^{-n}\right)$$

as $n \to \infty$ ($n \in \mathbb{N}$). Thus, for $x \in \Sigma_A$, we have

$$\begin{aligned}
N(t, x) &= N\Bigg(a \underbrace{\left\lfloor \frac{t + \psi(x)}{a} \right\rfloor}_{=:n} + a \underbrace{\left\{ \frac{t + \psi(x)}{a} \right\}}_{=:\beta} - \psi(x), x\Bigg) \\
&\sim -\mathrm{e}^{a\lfloor (t+\psi(x))/a \rfloor \delta} C_{a\{(t+\psi(x))/a\}}(x) \\
&= \mathrm{e}^{t\delta} \mathrm{e}^{-a\{(t+\psi(x))/a\}\delta} \mathrm{e}^{\delta\psi(x)} \frac{a}{\int \zeta \, \mathrm{d}\mu_{\eta - \delta\zeta}} h_{\eta - \delta\zeta}(x) \\
&\quad \times \int_{\Sigma_A} \chi(y) \sum_{\ell=-\infty}^{\infty} \mathrm{e}^{-\ell a\delta} f_y\left(a\ell + a\left\{ \frac{t + \psi(x)}{a} \right\} - \psi(y)\right) \mathrm{d}\nu_{\eta - \delta\zeta}(y) \\
&= \mathrm{e}^{t\delta} \tilde{G}_x(t)
\end{aligned}$$

as $t \to \infty$. Since in all instances where t occurs only the fractional part is involved, it is clear that $\tilde{G}_x$ is periodic with period a, which completes the proof. □

Proof of Theorem 3.1(iii). First, consider the case that ξ is nonlattice. As $\mathrm{e}^{-t\delta} N(t, x)$ is bounded in t by (B), the result from Theorem 3.1(i) implies Theorem 3.1(iii).

Second, consider the case that ξ is lattice. Theorem 3.1(ii) states that $\mathrm{e}^{-t\delta} N(t, x) \sim \tilde{G}_x(t)$ as $t \to \infty$. Since $\mathrm{e}^{-t\delta} N(t, x)$ is bounded in t by (B), and $\tilde{G}_x$ is periodic with period a, we have

$$\begin{aligned}
\lim_{t\to\infty} t^{-1} \int_0^t \mathrm{e}^{-T\delta} N(T, x) \, \mathrm{d}T &= \lim_{t\to\infty} t^{-1} \int_0^{a\lfloor t/a \rfloor} \mathrm{e}^{-T\delta} N(T, x) \, \mathrm{d}T \\
&= \lim_{t\to\infty} t^{-1} \int_0^{a\lfloor t/a \rfloor} \tilde{G}_x(T) \, \mathrm{d}T \\
&= \lim_{t\to\infty} t^{-1} \left\lfloor \frac{t}{a} \right\rfloor \int_0^a \tilde{G}_x(T) \, \mathrm{d}T \\
&= \frac{1}{a} \int_0^a \tilde{G}_x(T) \, \mathrm{d}T.
\end{aligned}$$

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The latter integral transforms as follows:

$$\begin{aligned}
&\int_0^a \tilde{G}_x(T)\,\mathrm{d}T \\
&\quad= \int_0^a \mathrm{e}^{-a\{(T+\psi(x))/a\}\delta}\frac{a\mathrm{e}^{\delta\psi(x)}}{\int\zeta\,\mathrm{d}\mu_{\eta-\delta\zeta}}h_{\eta-\delta\zeta}(x) \\
&\qquad\times\int_{\Sigma_A}\chi(y)\sum_{\ell=-\infty}^{\infty}\mathrm{e}^{-a\ell\delta}f_y\left(a\ell+a\left\{\frac{T+\psi(x)}{a}\right\}-\psi(y)\right)\mathrm{d}\nu_{\eta-\delta\zeta}(y)\,\mathrm{d}T \\
&\quad= \frac{a\mathrm{e}^{\delta\psi(x)}}{\int\zeta\,\mathrm{d}\mu_{\eta-\delta\zeta}}h_{\eta-\delta\zeta}(x) \\
&\qquad\times\int_{\Sigma_A}\chi(y)\sum_{\ell=-\infty}^{\infty}\int_0^a \mathrm{e}^{-(a\ell+a\{(T+\psi(x))/a\})\delta} \\
&\qquad\qquad\times f_y\left(a\ell+a\left\{\frac{T+\psi(x)}{a}\right\}-\psi(y)\right)\mathrm{d}T\,\mathrm{d}\nu_{\eta-\delta\zeta}(y) \\
&\quad= \frac{a\mathrm{e}^{\delta\psi(x)}h_{\eta-\delta\zeta}(x)}{\int\zeta\,\mathrm{d}\mu_{\eta-\delta\zeta}}\int_{\Sigma_A}\chi(y)\mathrm{e}^{-\psi(y)\delta}\sum_{\ell=-\infty}^{\infty}\int_{a\ell-\psi(y)}^{a(\ell+1)-\psi(y)}\mathrm{e}^{-T\delta}f_y(T)\,\mathrm{d}T\,\mathrm{d}\nu_{\eta-\delta\zeta}(y) \\
&\quad= \frac{ah_{\eta-\delta\xi}(x)}{\int\xi\,\mathrm{d}\mu_{\eta-\delta\xi}}\int_{\Sigma_A}\chi(y)\int_{-\infty}^{\infty}\mathrm{e}^{-T\delta}f_y(T)\,\mathrm{d}T\,\mathrm{d}\nu_{\eta-\delta\xi}(y) \\
&\quad= aG(x),
\end{aligned}$$

where for the second to last equality we used the facts that $\mathrm{e}^{\delta\psi}h_{\eta-\delta\zeta}=h_{\eta-\delta\xi}$, $\int\zeta\,\mathrm{d}\mu_{\eta-\delta\zeta}=\int\xi\,\mathrm{d}\mu_{\eta-\delta\xi}$, and $\mathrm{e}^{-\delta\psi}\,\mathrm{d}\nu_{\eta-\delta\zeta}=\mathrm{d}\nu_{\eta-\delta\xi}$. □

5.2.2. *Proof of Theorem 3.2.* In the setting of Theorem 3.2, (D)(ii) is automatically satisfied. Thus, Theorem 3.2 is proved by combining the following three lemmas with Theorem 3.1.

Lemma 5.2. *If* $\{t\mapsto \mathrm{e}^{-t\delta}|f_x(t)| \mid x\in\Sigma_A\}$ *is equi dRi then (A) holds.*

Proof. This is clear since dRi implies Lebesgue integrability. □

Lemma 5.3. *If* $\{t\mapsto \mathrm{e}^{-t\delta}|f_x(t)| \mid x\in\Sigma_A\}$ *is equi dRi and there exist* $\mathfrak{C}', s>0$ *such that* $\mathrm{e}^{-t\delta}|f_x(t)|\le\mathfrak{C}'\mathrm{e}^{st}$ *for* $t<0$ *and* $x\in\Sigma_A$*, then (C) holds, i.e. there exist* $\tilde{\mathfrak{C}}>0$ *and* $t^*\le 0$ *such that* $\mathrm{e}^{-\delta t}N^{\mathrm{abs}}(t,x)\le\tilde{\mathfrak{C}}\mathrm{e}^{st}$ *for* $t\le t^*$.

Proof. In (5.12) we showed that the existence of $n\in\mathbb{N}$ for which $S_n\xi$ is strictly positive implies the existence of $m^*\in\mathbb{N}$ such that $S_m\xi$ is strictly positive for all $m\ge m^*$. Set

$$t^*:=\min\Big\{0,\inf_{x\in\Sigma_A,\,0\le m\le m^*}S_m\xi(x)\Big\}.$$

Then $t-S_m\xi<0$ for all $t<t^*$ and $m\in\mathbb{N}$. Additionally, using the fact that there exist $\mathfrak{C}', s>0$ such that $\mathrm{e}^{-t\delta}|f_x(t)|\le\mathfrak{C}'\mathrm{e}^{st}$ for $t<0$ and $x\in\Sigma_A$, we have, for $t\le t^*$,

$$\begin{aligned}
\mathrm{e}^{-\delta t}N^{\mathrm{abs}}(t,x)&=\sum_{n=0}^{\infty}\sum_{\{y:\,\sigma^n y=x\}}\mathrm{e}^{S_n(\eta-\delta\xi)(y)}|\chi(y)|\mathrm{e}^{-\delta(t-S_n\xi(y))}|f_y(t-S_n\xi(y))| \\
&\le\mathfrak{C}'\mathrm{e}^{st}\sum_{n=0}^{\infty}\sum_{\{y:\,\sigma^n y=x\}}\mathrm{e}^{S_n(\eta-(\delta+s)\xi)(y)}|\chi(y)|
\end{aligned}$$

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$$= \mathfrak{C}' \mathrm{e}^{st} \sum_{n=0}^{\infty} \mathcal{L}^n_{\eta-(\delta+s)\xi} |\chi|(x)$$
$$\leq \mathfrak{C}' \mathrm{e}^{st} \sum_{n=0}^{\infty} \|\mathcal{L}^n_{\eta-(\delta+s)\xi}\|_{\mathrm{op}} \|\chi\|_\infty.$$

By Proposition 2.1 we know that $t \mapsto \gamma_{\eta+t\xi}$ is strictly monotonically increasing and $\gamma_{\eta-\delta\xi} = 1$. Hence, by the spectral radius formula (Section 5.1), the last series converges and the assertion follows. □

Lemma 5.4. *If* $\{t \mapsto \mathrm{e}^{-t\delta}|f_x(t)| \mid x \in \Sigma_A\}$ *is equi dRi then (B) is satisfied, i.e. there exists* $\mathfrak{C} > 0$ *such that* $\mathrm{e}^{-t\delta} N^{\mathrm{abs}}(t, x) \leq \mathfrak{C}$ *for all* $t \in \mathbb{R}$.

Proof. By assumption, there exists $n \in \mathbb{N}$ for which $S_n\xi$ is strictly positive. Fix this n, choose $\kappa > 0$ such that $S_n\xi \geq \kappa$ and consider the nth iterate of the renewal equation, namely,

$$N(t, x) = \sum_{\{y:\, \sigma^n y = x\}} N(t - S_n\xi(y), y)\mathrm{e}^{S_n\eta(y)} + \sum_{i=0}^{n-1} \sum_{\{y:\, \sigma^i y = x\}} \chi(y) f_y(t - S_i\xi(y))\mathrm{e}^{S_i\eta(y)}.$$

The function defined by $M(t, x) := \mathrm{e}^{-t\delta} N(t, x)/h_{\eta-\delta\xi}(x)$ satisfies

$$M(t, x) = \sum_{\{y:\, \sigma^n y = x\}} M(t - S_n\xi(y), y)\mathrm{e}^{S_n(\eta-\delta\xi)(y)} \frac{h_{\eta-\delta\xi}(y)}{h_{\eta-\delta\xi}(x)}$$
$$+ \sum_{i=0}^{n-1} \sum_{\{y:\, \sigma^i y = x\}} \chi(y)\mathrm{e}^{-\delta(t-S_i\xi(y))} f_y(t - S_i\xi(y))\mathrm{e}^{S_i(\eta-\delta\xi)(y)} \frac{1}{h_{\eta-\delta\xi}(x)}. \quad (5.20)$$

Set $g_x(t) := \mathrm{e}^{-t\delta} f_x(t)$ and $g_{i,x}(t) := g_x(t - S_i\xi(x))$ for $x \in \Sigma_A$. Then (5.20) becomes

$$M(t, x) = \sum_{\{y:\, \sigma^n y = x\}} M(t - S_n\xi(y), y)\mathrm{e}^{S_n(\eta-\delta\xi)(y)} \frac{h_{\eta-\delta\xi}(y)}{h_{\eta-\delta\xi}(x)} + \sum_{i=0}^{n-1} \frac{\mathcal{L}^i_{\eta-\delta\xi}(\chi g_{i,\cdot}(t))(x)}{h_{\eta-\delta\xi}(x)}, \quad (5.21)$$

with $g_{i,\cdot}(t)\colon \Sigma_A \to \mathbb{R}$, $x \mapsto g_{i,x}(t)$. Define

$$\overline{M}(t) := \sup_{t' \in (t-\kappa, t],\, x \in \Sigma_A} M(t', x).$$

Then (5.21) implies that

$$\overline{M}(t) \leq \sup_{y \in \Sigma_A} \overline{M}(t - S_n\xi(y)) + \sup_{t' \in (t-\kappa, t]} \sum_{i=0}^{n-1} \left\| \frac{\mathcal{L}^i_{\eta-\delta\xi}(\chi g_{i,\cdot}(t'))}{h_{\eta-\delta\xi}} \right\|_\infty \quad (5.22)$$

as $\sum_{\{y:\, \sigma^n y = x\}} \mathrm{e}^{S_n(\eta-\delta\xi)(y)} h_{\eta-\delta\xi}(y)/h_{\eta-\delta\xi}(x) = 1$. Iterating (5.22) k times and using the abbreviation $I_\kappa(t, x^1, \ldots, x^m) := (t - \sum_{j=1}^m S_n\xi(x^j) - \kappa, t - \sum_{j=1}^m S_n\xi(x^j)]$ yields

$$\overline{M}(t) \leq \sup_{x^1, \ldots, x^k \in \Sigma_A} \left[\overline{M}\left(t - \sum_{j=1}^k S_n\xi(x^j)\right) + \sum_{i=0}^{n-1} \sum_{m=0}^{k-1} \sup_{t' \in I_\kappa(t, x^1, \ldots, x^m)} \left\| \frac{\mathcal{L}^i_{\eta-\delta\xi}(\chi g_{i,\cdot}(t'))}{h_{\eta-\delta\xi}} \right\|_\infty \right],$$

where $\sum_{j=1}^0 S_n\xi(x^j)$ shall be understood to be 0.

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Fix $t \in \mathbb{R}$. As $S_n\xi \geq \kappa > 0$, we can find a $k^* \in \mathbb{N}$ such that, for all $k \geq k^*$ and $x^1, \dots, x^k \in \Sigma_A$, we have $t - \sum_{j=1}^{k} S_n\xi(x^j) \leq t^* \leq 0$ with t^* as in Lemma 5.3. Hence, by Lemma 5.3, there exists a constant $\mathfrak{C}''$ such that

$$\overline{M}(t) \leq \mathfrak{C}'' + \sup_{x^1,x^2,\dots\in\Sigma_A} \sum_{i=0}^{n-1} \sum_{m=0}^{\infty} \sup_{t'\in I_\kappa(t,x^1,\dots,x^m)} \left\| \frac{\mathcal{L}^i_{\eta-\delta\xi}(\chi g_{i,\cdot}(t'))}{h_{\eta-\delta\xi}} \right\|_\infty . \tag{5.23}$$

If $t' \mapsto \|\mathcal{L}^i_{\eta-\delta\xi}(\chi g_{i,\cdot}(t'))/h_{\eta-\delta\xi}\|_\infty$ is dRi then it follows that

$$t - \sum_{j=1}^{m+1} S_n\xi(x^j) \leq t - \sum_{j=1}^{m} S_n\xi(x^j) - \kappa$$

for $x^1, \dots, x^{m+1} \in \Sigma_A$ implies that the series in (5.23) converges for any constellation $x^1, x^2, \dots \in \Sigma_A$ and, thus, is uniformly bounded for $x^1, x^2, \dots \in \Sigma_A$. This proves the assertion. Hence, all that remains to be shown is that the mapping $t' \mapsto \|\mathcal{L}^i_{\eta-\delta\xi}(\chi g_{i,\cdot}(t'))/h_{\eta-\delta\xi}\|_\infty$ is dRi. For this, note that, using the terminology of Definition 3.1,

$$\underline{m}_k(\mathcal{L}^i_{\eta-\delta\xi}(\chi g_{i,\cdot})(x), h) \geq \sum_{\{y:\,\sigma^i y=x\}} \chi(y)\mathrm{e}^{S_i(\eta-\delta\xi)(y)} \underline{m}_{k-S_i\xi(y)/h}(g_y, h),$$

$$\overline{m}_k(\mathcal{L}^i_{\eta-\delta\xi}(\chi g_{i,\cdot})(x), h) \leq \sum_{\{y:\,\sigma^i y=x\}} \chi(y)\mathrm{e}^{S_i(\eta-\delta\xi)(y)} \overline{m}_{k-S_i\xi(y)/h}(g_y, h).$$

Since $\sum_{\{y:\,\sigma^i y=x\}} \chi(y)\mathrm{e}^{S_i(\eta-\delta\xi)(y)} = \mathcal{L}^i_{\eta-\delta\xi}\chi(x)$ is finite, the hypothesis of $\{g_x \mid x \in \Sigma_A\}$ being equi dRi implies that $t' \mapsto \|\mathcal{L}^i_{\eta-\delta\xi}(\chi g_{i,\cdot}(t'))/h_{\eta-\delta\xi}\|_\infty$ is dRi, which completes the proof. □

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