Ergod. Th. & Dynam. Sys., page 1 of 22 © The Author(s), 2026. Published by Cambridge University Press.
doi:10.1017/etds.2026.10343

On minimal invariant sets and stationary measures for random dynamical systems on the circle

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(*Received* 17 *January* 2025 *UTC and accepted in revised form* 29 *July* 2026 *UTC*)

Abstract. In this paper, we study random dynamical systems (RDSs) of homeomorphisms of the circle without a finite orbit. We characterize the topological dynamics of the associated semigroup by identifying the existence of invariant sets that are finite unions of intervals. We describe the accumulation points of the average orbit of the transfer operator. For each ergodic stationary measure, we establish structural properties of its weight function on the circle. We also establish relationships between the minimal sets of an RDS and its inverse RDS.

Key words: random dynamical systems, local contraction, minimality, proximality, stationary measure
2020 Mathematics Subject Classification: 37H12 (Primary); 37B05, 37E10 (Secondary)

1. *Introduction*

Given a subset $\mathcal{F} \subset \mathrm{Homeo}(\mathbb{S}^1)$, we study the random dynamical system (RDS) associated with $\mathcal{F}$ and the action on the circle of the semigroup $G_{\mathcal{F}}$ generated by $\mathcal{F}$. While the action of the group generated by $\mathcal{F}$ has been extensively studied, the semigroup action considered here requires a different approach; see, for instance, [**Ant84**, **DKN07**, **Nav11**]. Throughout this paper, we assume that there is no Borel probability measure that is invariant under all maps in $\mathcal{F}$. In particular, there is no finite $\mathcal{F}$-invariant set. Since all measures considered here are Borel probability measures, we simply call them measures. By [**Mal17**, Proposition 4.8], there exists an integer d, depending only on $\mathcal{F}$, such that every RDS associated with $\mathcal{F}$ has exactly d ergodic stationary measures. The supports

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of these measures are pairwise disjoint and coincide with the minimal invariant sets. For brevity, we use the term minimal invariant set to mean a non-empty closed $\mathcal{F}$-invariant subset of $\mathbb{S}^1$ that is minimal with respect to inclusion. Recall that a subset $K \subset \mathbb{S}^1$ is $\mathcal{F}$-invariant if $f(K) \subset K$ for every $f \in \mathcal{F}$.

A stationary measure for an RDS reflects the long-term distribution of the orbits under the random system and captures the notion of probabilistic equilibrium. By compactness of $\mathbb{S}^1$ and continuity of the maps in the RDS, for any initial point $x \in \mathbb{S}^1$, the finite-time averages of the iterates of the transfer operator applied to δ_x converge weakly to a stationary measure, denoted by m_x (see [**Fur63**]). We describe the stationary measures that arise in this way for some $x \in \mathbb{S}^1$. By the classical Birkhoff ergodic theorem, each ergodic stationary measure is obtained as such a limit for at least one orbit. The continuity of the map $x \mapsto m_x$ was already established in [**Mal17**]. The results of this paper describe finer properties of this map. To do this, we use the topological structure of $\mathbb{S}^1$ and show the existence of finite families of closed intervals such that the union over any of these families is $\mathcal{F}$-invariant. This was previously established in the case where $\mathcal{F}$ is a finite subset of the group $\mathrm{Homeo}^+(\mathbb{S}^1)$ of orientation-preserving homeomorphisms, see [**KKO18**, Theorem 1.2]. In our more general context, we have the following result.

THEOREM 1. *Let $\mathcal{F} \subset \mathrm{Homeo}(\mathbb{S}^1)$ be a subset for which there is no $\mathcal{F}$-invariant measure on $\mathbb{S}^1$ and let $K_1, \dots, K_d$ be the minimal invariant sets. Assume that $d \geq 2$. Then, there exist pairwise disjoint interval-domains $L_1, \dots, L_d$ for $\mathcal{F}$ such that:*

- *$\partial L_i \subset K_i \subset L_i$ for every $i \in \{1, \dots, d\}$; and*
- *the action of $G_{\mathcal{F}}$ on each connected component of L_i is proximal.*

Here, borrowing the terminology from [**KKO18**], we say that a proper non-empty subset $L \subset \mathbb{S}^1$ is an *interval-domain* for $\mathcal{F}$ if it is an $\mathcal{F}$-invariant closed subset with finitely many connected components. We shall also use the notion of proximality, introduced by Furstenberg [**Fur67**]: the action of $G_{\mathcal{F}}$ on a subset $A \subset \mathbb{S}^1$ is *proximal* if any two points of A can be brought arbitrarily close by a suitable sequence of elements of $G_{\mathcal{F}}$. See also item (e) of Theorem 5 for the precise definition of proximality in our context. Moreover, the interval-domains L_i given by Theorem 1 are minimal with respect to inclusion among the interval-domains L for $\mathcal{F}$ satisfying $\partial L \subset K_i \subset L$. In particular, they are uniquely determined by the minimal invariant sets K_i.

Now, to state the rest of our results, let us introduce some probabilistic concepts involved. For $\mathcal{F} \subset \mathrm{Homeo}(\mathbb{S}^1)$, an RDS associated with $\mathcal{F}$ is given by a probability measure ν on $\mathrm{Homeo}(\mathbb{S}^1)$ such that the support of ν is exactly $\mathcal{F}$. To be more specific, we say that $(\mathcal{F}, \nu)$ is an RDS. The corresponding product probability space $(\mathcal{F}^{\mathbb{N}}, \nu^{\mathbb{N}})$ is invariant and ergodic under the shift map σ. Define the *random walk* $\omega \mapsto (f_\omega^n)$ on the semigroup $G_{\mathcal{F}}$ generated by $\mathcal{F}$ as follows:

$$f_\omega^n = f_n \circ \cdots \circ f_1,$$

with the implicit notation $\omega = (f_n)_{n \in \mathbb{N}} \in \mathcal{F}^{\mathbb{N}}$. We use the notation $f_\omega^0 = \mathrm{id}_{\mathbb{S}^1}$. Consider the *skew-shift map*

$$T : (\omega, x) \mapsto (\sigma\omega, f_1(x))$$

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on $\mathcal{F}^{\mathbb{N}} \times \mathbb{S}^1$. A probability measure μ on $\mathbb{S}^1$ is called *stationary* if the product measure $\nu^{\mathbb{N}} \otimes \mu$ is T-invariant. When $\nu^{\mathbb{N}} \otimes \mu$ is T-ergodic, we say that μ is an *ergodic stationary* measure for $(\mathcal{F}, \nu)$.

As we discuss in §4, by [**Fur63**, **Mal17**], for an RDS $(\mathcal{F}, \nu)$ of circle homeomorphisms without finite orbits, we have that for every x, the sequence

$$\left(\frac{1}{N} \sum_{n=0}^{N-1} \int_{\mathcal{F}^{\mathbb{N}}} \delta_{f_\omega^n(x)} \, d\nu^{\mathbb{N}}(\omega) \right)_{N \in \mathbb{N}} \tag{1}$$

converges in the weak-$*$ topology to a probability measure m_x on $\mathbb{S}^1$, which is stationary for $(\mathcal{F}, \nu)$. Moreover, the map $x \mapsto m_x$ is continuous on $\mathbb{S}^1$.

THEOREM 2. *Let $(\mathcal{F}, \nu)$ be an RDS of circle homeomorphisms without finite orbits. Let d denote the number of ergodic stationary measures $\mu_1, \dots, \mu_d$. For $x \in \mathbb{S}^1$, consider the limit measure m_x on $\mathbb{S}^1$ in the weak-$*$ topology of the sequence in (1). Also, consider the continuous maps $u_1, \dots, u_d \colon \mathbb{S}^1 \to [0, 1]$ such that*

$$m_x = \sum_{i=1}^{d} u_i(x) \mu_i.$$

Then, for every $x \in \mathbb{S}^1$,

$$\#\{j \in \{1, \dots, d\} : u_j(x) > 0\} \leq 2$$

and for every $i \in \{1, \dots, d\}$,

$$\#\{j \in \{1, \dots, d\} \backslash \{i\} : \{u_i > 0\} \cap \{u_j > 0\} \neq \emptyset\} \leq 2.$$

Moreover, for every $i \in \{1, \dots, d\}$:

(i) *both sets $\{u_i = 0\}$ and $\{u_i = 1\}$ are invariant by all $f \in \mathcal{F}$ and can be written as a finite union of closed intervals;*

(ii) *on every connected subset of $\{0 < u_i < 1\}$, the function u_i is monotone, with respect to the orientation of $\mathbb{S}^1$; and*

(iii) *if $u_i|_I \equiv C$ for some connected subset $I \subset \mathbb{S}^1$ and $C \in [0, 1]$, then for every $f \in \mathcal{F}$, there exists $c_f \in [0, 1]$ such that $u_i|_{f(I)} \equiv c_f$.*

As a consequence of item (i), every measure m_x is a convex combination of at most two ergodic stationary measures. In particular, if $d > 2$, then not every stationary measure is of the form m_x for some $x \in \mathbb{S}^1$.

For $\mathcal{F} \subset \mathrm{Homeo}(\mathbb{S}^1)$, consider $\mathcal{F}^- = \{f^{-1} : f \in \mathcal{F}\}$. Also, consider ν^-, the probability measure supported on $\mathcal{F}^-$ naturally induced by ν. We refer to $(\mathcal{F}^-, \nu^-)$ as the *inverse RDS* of $(\mathcal{F}, \nu)$. The following theorem is proved in §7.

THEOREM 3. *Let $(\mathcal{F}, \nu)$ be an RDS of circle homeomorphisms without finite orbits and consider its respective inverse RDS $(\mathcal{F}^-, \nu^-)$. Let d_+ and d_- be the numbers of ergodic stationary measures associated with $(\mathcal{F}, \nu)$ and $(\mathcal{F}^-, \nu^-)$, respectively. Then,*

$$|d_+ - d_-| \leq 1.$$

Moreover, if $\mathcal{F} \subset \mathrm{Homeo}^+(\mathbb{S}^1)$, then $d_+ = d_-$.

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The bound in Theorem 3 is sharp: in Example 1, we exhibit an RDS for which the numbers of ergodic stationary measures of the RDS and of its inverse RDS differ by one.

Since $\mathcal{F}$ has a finite invariant set if and only if $\mathcal{F}^-$ has a finite invariant set, and since the number of ergodic stationary measures coincides with the number of minimal invariant sets in the present setting, we obtain the following topological consequence.

COROLLARY 1. *Let $\mathcal{F} \subset \mathrm{Homeo}(\mathbb{S}^1)$ and $\mathcal{F}^- = \{f^{-1} : f \in \mathcal{F}\}$. Assume that there is no finite set invariant under each map in $\mathcal{F}$ simultaneously. Let d_+ and d_- be the numbers of $\mathcal{F}$-minimal sets and $\mathcal{F}^-$-minimal sets, respectively. Then,*

$$|d_+ - d_-| \leq 1.$$

Moreover, if $\mathcal{F} \subset \mathrm{Homeo}^+(\mathbb{S}^1)$, then $d_+ = d_-$.

The paper is organized as follows. We begin, in §3, by discussing the topological properties of the minimal sets of the semigroup $G_{\mathcal{F}}$. In §4, we prove part of Theorem 2, which can be compared with Theorem 4. In §5, we present results that are used to complete the proof of Theorem 2. We also introduce the notions of neighbors and paths in §5. In §6, we discuss the notion of proximality and construct non-trivial intervals on which the action is proximal. A proof of Theorem 1 is given in §6. Finally, in §7, we study the inverse RDS and prove Theorem 3.

2. *Basic notation*

Throughout the paper, the circle $\mathbb{S}^1$ is endowed with its counterclockwise orientation. We denote by $\mathcal{R} \subset (\mathbb{S}^1)^3$ the canonical ternary cyclic order induced by this orientation: for pairwise distinct points $a, b, c \in \mathbb{S}^1$, we write $[a, b, c] \in \mathcal{R}$ if, starting from a and moving counterclockwise along $\mathbb{S}^1$, one meets b before c.

For $n \geq 3$, we endow the set $\{1, \dots, n\}$ with its natural cyclic order: for pairwise distinct indices $i, j, k \in \{1, \dots, n\}$, we write $[i, j, k]_{\mathrm{cyc}}$ if, starting from i and moving in the order

$$1 \to 2 \to \cdots \to n \to 1,$$

one meets j before k.

An ordered n-tuple $(x_1, \dots, x_n) \in (\mathbb{S}^1)^n$ is said to be *circularly ordered* if the points $x_1, \dots, x_n$ are pairwise distinct and the map

$$i \in \{1, \dots, n\} \longmapsto x_i \in \mathbb{S}^1$$

preserves the cyclic order, that is,

$$[i, j, k]_{\mathrm{cyc}} \quad \Longrightarrow \quad [x_i, x_j, x_k] \in \mathcal{R}$$

for every triple of pairwise distinct indices $i, j, k \in \{1, \dots, n\}$. Equivalently, the points appear in the order $x_1 \to x_2 \to \cdots \to x_n \to x_1$ as the circle is traversed counterclockwise.

We extend the same terminology to finite families of subsets of the circle. Let $n \geq 3$. An ordered n-tuple $(A_1, \dots, A_n)$ of pairwise disjoint subsets of $\mathbb{S}^1$ is called *circularly*

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ordered if, for every choice of points $x_i \in A_i$, $i = 1, \dots, n$, the tuple $(x_1, \dots, x_n)$ is circularly ordered.

Given two distinct points $a, b \in \mathbb{S}^1$, the open interval, or open arc, from a to b is

$$(a, b) := \{x \in \mathbb{S}^1 \setminus \{a, b\} : [a, x, b] \in \mathcal{R}\}.$$

The corresponding closed interval is

$$[a, b] := (a, b) \cup \{a, b\},$$

and we use the convention $[a, a] = \{a\}$.

If $a \neq b$, the interval $[a, b]$ carries the natural order induced by cutting the circle at a. Namely, for two distinct points $x, y \in [a, b]$, we write $x <_{[a,b]} y$ if either $x = a$ or $[a, x, y] \in \mathcal{R}$. We also write $x \leq_{[a,b]} y$ if $x = y$ or $x <_{[a,b]} y$. When the interval is clear from the context, we simply write $x < y$ and $x \leq y$. The same convention will be used on open intervals: if $x, y \in (a, b)$, then $x < y$ whenever $[a, x, y] \in \mathcal{R}$.

Finally, let $I \subset \mathbb{S}^1$ be an interval of the form $[a, b]$ or (a, b), with $a \neq b$, endowed with the order defined above. A map $f \in \mathrm{Homeo}(\mathbb{S}^1)$ is said to be *monotone increasing* on I if $x < y$ implies $f(x) \leq f(y)$ for all $x, y \in I$. Similarly, f is said to be *monotone decreasing* on I if $x < y$ implies $f(y) \leq f(x)$ for all $x, y \in I$. We say that f is *monotone* on I if it is either monotone increasing or monotone decreasing on I.

3. *Minimal invariant sets*

The results in this section are purely topological and rely on the order structure of the circle. Let $\mathcal{F} \subset \mathrm{Homeo}(\mathbb{S}^1)$ and let $\mathcal{K}_\mathcal{F}$ be the collection of $\mathcal{F}$-minimal invariant sets.

For $x \in \mathbb{S}^1$, its *$\mathcal{F}$-orbit* is defined as

$$\mathcal{O}_\mathcal{F}(x) = \{f(x) \colon f \in G_\mathcal{F}\},$$

where $G_\mathcal{F}$ denotes the semigroup generated by $\mathcal{F}$.

The main result of this section is the following.

PROPOSITION 1. *Assume that $\mathcal{F} \subset \mathrm{Homeo}(\mathbb{S}^1)$ and that the collection $\mathcal{K}_\mathcal{F}$ of $\mathcal{F}$-minimal invariant sets is finite. Then, for every $x \in \mathbb{S}^1$, there exist $K_1, K_2 \in \mathcal{K}_\mathcal{F}$, not necessarily distinct, such that*

$$\overline{\mathcal{O}_\mathcal{F}(x)} \cap K = \emptyset \quad \textit{for every } K \in \mathcal{K}_\mathcal{F} \setminus \{K_1, K_2\}.$$

Remark 1. In Proposition 1, we assume that there are finitely many minimal invariant sets, as this is the case needed in the following. However, this assumption can be dropped. Indeed, the existence of finite $\mathcal{F}$-orbits provides enough order structure to control the minimal invariant sets that may be met by the closure of a given orbit: such an orbit can accumulate only on the minimal invariant sets adjacent to it in the circular order. Therefore, the conclusion of Proposition 1 remains valid without assuming that the number of minimal invariant sets is finite.

We now prove Proposition 1 in the finite case by describing the interval-domains determined by the minimal invariant sets.

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Assume from now on that $\mathcal{K}_{\mathcal{F}} = \{K_1, \dots, K_d\}$ is finite. If $d = 1$, we set $D_K = \mathbb{S}^1$ for the unique element $K \in \mathcal{K}_{\mathcal{F}}$. If $d > 1$, then, for each $K \in \mathcal{K}_{\mathcal{F}}$, we define the *K-admissible interval-domain* D_K as the union of all closed intervals $[a, b] \subset \mathbb{S}^1$ such that

$$a, b \in K \quad \text{and} \quad [a, b] \cap K' = \emptyset \quad \text{for every } K' \in \mathcal{K}_{\mathcal{F}} \setminus \{K\}.$$

In particular, $K \subset D_K$.

Before proving Proposition 1, let us establish some topological results.

LEMMA 1. *Assume that $\mathcal{F} \subset \mathrm{Homeo}(\mathbb{S}^1)$ and that $\mathcal{K}_{\mathcal{F}}$ is a finite family. Then, for every $K \in \mathcal{K}_{\mathcal{F}}$, its domain D_K is a finite union of closed intervals.*

Proof. The case $d = 1$ is immediate. Assume $d > 1$, fix $K \in \mathcal{K}_{\mathcal{F}}$, and set

$$K^* := \bigcup_{K' \in \mathcal{K}_{\mathcal{F}} \setminus \{K\}} K'.$$

Since the minimal invariant sets are pairwise disjoint and compact,

$$\delta := \mathrm{dist}(K, K^*) > 0.$$

By definition, $\mathbb{S}^1 \setminus D_K$ is the union of the components I of $\mathbb{S}^1 \setminus K$ that meet K^*. Each such component has length at least δ. Hence, since these components are pairwise disjoint and $\mathbb{S}^1$ has finite length, there are only finitely many of them. Therefore, D_K is a finite union of closed intervals. □

For each $K \in \mathcal{K}_{\mathcal{F}}$, let $\ell_K \in \mathbb{N}$ be the number of connected components of D_K. We write

$$D_K = \bigcup_{j=1}^{\ell_K} I_{K,j}, \quad I_{K,j} = [x_{K,j}, y_{K,j}],$$

where the intervals are indexed according to the circular order in such a way that

$$(x_{K,j}, y_{K,j}, x_{K,j+1}) \in \mathcal{R} \quad \text{for every } j \in \{1, \dots, \ell_K\},$$

with the convention that $x_{K,\ell_K+1} = x_{K,1}$.

LEMMA 2. *Assume that $\mathcal{F} \subset \mathrm{Homeo}(\mathbb{S}^1)$ and $\mathcal{K}_{\mathcal{F}}$ is finite. Then, for all $K \in \mathcal{K}_{\mathcal{F}}$, $j \in \{1, \dots, \ell_K\}$, and $f \in \mathcal{F}$, there is $k_j, r_j \in \{1, \dots, \ell_K\}$ such that $f(I_{K,j}) \subset I_{K,k_j}$ and $f(I_{K,r_j}) \subset I_{K,j}$.*

Proof. Consider $f \in \mathcal{F}$ and $K \in \mathcal{K}_{\mathcal{F}}$. Since $K \subset D_K$, by the $\mathcal{F}$-invariance of K,

$$f(x_{K,j}), f(y_{K,j}) \in f(K) \subset K \subset D_K$$

for all $j = 1, \dots, \ell_K$. If $\ell_K = 1$, the conclusion is immediate with $k_1 = r_1 = 1$.

Assume $\ell_K > 1$. For $j = 1, \dots, \ell_K$, there exist $k_j^-, k_j^+ \in \{1, \dots, \ell_K\}$ such that $f(x_{K,j}) \in I_{K,k_j^-}$ and $f(y_{K,j}) \in I_{K,k_j^+}$. By definition, for all

$$a, b \in \{x_{K,1}, y_{K,1}, x_{K,2}, y_{K,2}, \dots, x_{K,\ell_K}, y_{K,\ell_K}\},$$

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with $b \neq y_{K,j}$, $a \neq x_{K,j}$, we have that both intervals $[a, x_{K,j}]$ and $[y_{K,j}, b]$ intersect $\bigcup_{K' \in \mathcal{K}_\mathcal{F} \setminus \{K\}} K$. Indeed, for some $i = 1, \ldots, \ell_K$, we have $[y_{K,i}, x_{K,j}] \subset [a, x_{K,j}]$ and $[y_{K,j}, x_{K,i}] \subset [y_{K,j}, b]$. By $\mathcal{F}$-invariance, the intervals $f([a, x_{K,j}])$ and $f([y_{K,j}, b])$ also intersect $\bigcup_{K' \in \mathcal{K}_\mathcal{F} \setminus \{K\}} K'$. In particular, for all $I \in \{I_{K,j}\}_{j=1}^{\ell_K}$, $f([a, x_{K,j}]) \not\subset I$ and $f([y_{K,j}, b]) \not\subset I$. Therefore, for all $i, j \in \{1, \ldots, \ell_K\}, i \neq j$,

$$k_i^+, k_i^- \notin \{k_j^+, k_j^-\},$$

and so,

$$\{k_j^- : j = 1, \ldots, \ell_K\} = \{k_j^+ : j = 1, \ldots, \ell_K\} = \{1, \ldots, \ell_K\}.$$

So, we must necessarily have equality $k_j^- = k_j^+$ for all $j = 1, \ldots, \ell_K$. Thus, for each j, the image $f(I_{K,j})$ is contained in some component I_{K,k_j}. Since the map $j \mapsto k_j$ is a permutation of $\{1, \ldots, \ell_K\}$, for every j, there exists r_j such that $k_{r_j} = j$. Hence, $f(I_{K,r_j}) \subset I_{K,j}$. □

LEMMA 3. *Assume that $\mathcal{F} \subset \mathrm{Homeo}(\mathbb{S}^1)$ and $\mathcal{K}_\mathcal{F}$ is finite. Let $K_1, K_2 \in \mathcal{K}_\mathcal{F}$. Let $y_1 \in K_1$ and $x_2 \in K_2$ be such that $(y_1, x_2) \cap K = \emptyset$ for all $K \in \mathcal{K}_\mathcal{F}$. Then, for every $f \in \mathcal{F}$ and $K \in \mathcal{K}_\mathcal{F} \setminus \{K_1, K_2\}$, we have $f([y_1, x_2]) \cap K = \emptyset$.*

Proof. Consider $K_1, K_2 \in \mathcal{K}_\mathcal{F}$, $y_1 \in K_1$, and $x_2 \in K_2$ be such that $(y_1, x_2) \cap K = \emptyset$ for all $K \in \mathcal{K}_\mathcal{F}$. Let us proceed by contradiction. Given $f \in \mathcal{F}$, if $f([y_1, x_2]) \cap K \neq \emptyset$ for some $K \in \mathcal{K}_\mathcal{F} \setminus \{K_1, K_2\}$, then there exists a connected component I of D_K (that is, $I \in \{I_{K,j}\}_{j=1}^{\ell_K}$) such that $I \subset f([y_1, x_2])$. By Lemma 2, there exists $J \in \{I_{K,j}\}_{j=1}^{\ell_K}$ such that $f(J) \subset I \subset f([y_1, x_2])$. Since f is a homeomorphism, $J \subset [y_1, x_2]$. Hence, $(y_1, x_2) \cap K \neq \emptyset$, which contradicts the hypotheses. The lemma is proven. □

Finally, we are ready to prove Proposition 1.

Proof of Proposition 1. Let $x \in \mathbb{S}^1$. If $x \in K$, for some $K \in \mathcal{K}_\mathcal{F}$, it is clear that for all $\omega \in \mathcal{F}^\mathbb{N}$, $f_\omega^n(x) \in K$ for all $n \in \mathbb{N}$. Hence, the proposition follows by taking $K_1 = K_2 = K$. If $x \notin \bigcup_{K \in \mathcal{K}_\mathcal{F}} K$, there exist $K_1, K_2 \in \mathcal{K}_\mathcal{F}$ and $y_1 \in K_1$, $x_2 \in K_2$, such that $(y_1, x) \cap K = (x, x_2) \cap K = \emptyset$ for all $K \in \mathcal{K}_\mathcal{F}$. By Lemma 3, we get that for all $f \in \mathcal{F}$, $f(y_1, x_2) \cap K = \emptyset$ for all $K \in \mathcal{K}_\mathcal{F} \setminus \{K_1, K_2\}$. Therefore, the cluster set of the sequence $(f_\omega^n(x))_{n \in \mathbb{N}}$ does not intersect any $K \in \mathcal{K}_\mathcal{F} \setminus \{K_1, K_2\}$. The proposition is proven. □

4. *Weight maps*

Throughout this section, $(\mathcal{F}, \nu)$ is an RDS of circle homeomorphisms without finite orbits. Let $K_1, \ldots, K_d$ be the corresponding topological supports of the ergodic stationary measures $\mu_1, \ldots, \mu_d$. In our context, $\mathcal{K}_\mathcal{F} = \{K_1, \ldots, K_d\}$ is the collection of the closed $\mathcal{F}$-minimal invariant sets.

The random process $(X_n)_{n \geq 0}$ defined by

$$X_n(\omega) = f_n(X_{n-1}),$$

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with $\omega = (f_n)_{n\in\mathbb{N}} \in \mathcal{F}^{\mathbb{N}}$ and X_0 being a random variable taking values in $\mathbb{S}^1$, is a Markov chain on the probability space $(\mathcal{F}^{\mathbb{N}}, \nu^{\mathbb{N}})$ with state space $\mathbb{S}^1$. By compactness of $\mathbb{S}^1$, we can apply [**Fur63**, Lemma 2.5] to conclude that, for every $x \in \mathbb{S}^1$, every weak-$*$ cluster point of the sequence of probability measures in (1) is a stationary probability measure for the RDS. In particular, when $d = 1$, for each $x \in \mathbb{S}^1$, the sequence in (1) converges to the unique stationary measure in the weak-$*$ topology.

Define the *weight maps* $u_i : \mathbb{S}^1 \to [0, 1]$, $1 \le i \le d$, by

$$u_i(x) = \nu^{\mathbb{N}}(\omega \in \mathcal{F}^{\mathbb{N}} \colon \mathrm{cl}(\{f_\omega^n(x) \colon n \ge 0\}) = K_i), \tag{2}$$

where for a set $A \subset \mathbb{S}^1$, $\mathrm{cl}(A)$ denotes the set of cluster values of A. By invariance of the sets in $\mathcal{K}_{\mathcal{F}}$, we have $u_i = \delta_{i,j}$ on K_j. By [**Mal17**, Proposition 4.9], for every $x \in \mathbb{S}^1$,

$$\sum_{i=1}^{d} u_i(x) = 1, \tag{3}$$

and

$$u_i(x) = \nu^{\mathbb{N}}\left(\omega \in \mathcal{F}^{\mathbb{N}} \colon \frac{1}{N}\sum_{n=0}^{N-1} \delta_{f_\omega^n(x)} \text{weakly-}* \text{ converges to } \mu_i\right),$$

where $f_\omega^0(x) = x$. Moreover, for every $x \in \mathbb{S}^1$, the sequence in (1) converges in the weak-$*$ topology to the probability measure m_x appearing in Theorem 2.

The weight maps also satisfy the following invariance property:

$$u_i(x) = \int_{\mathcal{F}} u_i(f(x))\, d\nu(f), \quad x \in \mathbb{S}^1,\ i = 1, \dots, d. \tag{4}$$

LEMMA 4. *Let $(\mathcal{F}, \nu)$ be an RDS of circle homeomorphisms without finite orbits. Let d be the number of ergodic stationary measures and consider the weight maps u_i, $i = 1, \dots, d$ as in (2). Then, for all $i \in \{1, \dots, d\}$, the sets*

$$\{u_i = 0\} \quad \textit{and} \quad \{u_i = 1\}$$

are $\mathcal{F}$-invariant.

Proof. Let us show the $\mathcal{F}$-invariance of $\{u_i = 1\}$. For $\{u_i = 0\}$, the proof is analogous. We need to prove that for all $f \in \mathcal{F}$,

$$f(\{x \in \mathbb{S}^1 \colon u_i(x) = 1\}) \subset \{x \in \mathbb{S}^1 \colon u_i(x) = 1\}. \tag{5}$$

In fact, using (4), for x such that $u_i(x) = 1$, we get

$$1 = u_i(x) = \int_{\mathcal{F}} u_i(f(x))\, d\nu(f).$$

Since $0 \le u_i \le 1$, we get $u_i(f(x)) = 1$ for ν-almost every $f \in \mathcal{F}$. Using the continuity of u_i and the fact that the support of ν is exactly $\mathcal{F}$, we get $u_i(f(x)) = 1$ for all $f \in \mathcal{F}$ and so (5) holds. □

Let us establish a first result using what we discussed in §3.

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PROPOSITION 2. *Let $(\mathcal{F}, \nu)$ be an RDS of circle homeomorphisms without finite orbits. Let d be the number of ergodic stationary measures. Let $\mathcal{K}_{\mathcal{F}} = \{K_1, \dots, K_d\}$ be the collection of the closed $\mathcal{F}$-minimal invariant sets. Consider u_i as in (2) for $i = 1, \dots, d$. Then, for every $x \in \mathbb{S}^1$, there exist $i_1, i_2 \in \{1, \dots, d\}$, $x_1 \in K_{i_1}$, $x_2 \in K_{i_2}$, such that $x \in [x_1, x_2]$, and for all $j \in \{1, \dots, d\} \backslash \{i_1, i_2\}$, we have $u_j(x) = 0$ and $[x_1, x_2] \cap K_j = \emptyset$.*

Proof. When $d = 1$, the result is evident by taking $i_1 = i_2 = 1$ for all $x \in \mathbb{S}^1$ because $u_1 \equiv 1$. Now, consider $d \geq 2$. Since the collection $\mathcal{K}_{\mathcal{F}} = \{K_1, \dots, K_d\}$ of the closed $\mathcal{F}$-minimal invariant sets is a finite set, we can apply Proposition 1 to get that for every $x \in \mathbb{S}^1$, there exist $i_1, i_2 \in \{1, \dots, d\}$ such that for all $\omega \in \mathcal{F}^{\mathbb{N}}$, the cluster set of the sequence $(f_\omega^n(x))_{n \in \mathbb{N}}$ does not intersect any element of $\mathcal{K}_{\mathcal{F}} \backslash \{K_{i_1}, K_{i_2}\}$. In particular, we can find $x_1 \in K_{i_1}$, $x_2 \in K_{i_2}$, such that $x \in [x_1, x_2]$ and $[x_1, x_2] \cap K_j = \emptyset$ for all $j \in \{1, \dots, d\} \backslash \{i_1, i_2\}$. By (3), using the definition of the maps u_i, we get that $u_{i_1}(x) + u_{i_2}(x) = 1$ (case $i_1 \neq i_2$) or $u_{i_1}(x) = 1$ (case $i_1 = i_2$). Hence, $u_j(x) = 0$ for all $j \in \{1, \dots, d\} \backslash \{i_1, i_2\}$. □

The main result of this section establishes more precisely the properties of the collections of closed intervals mentioned in Theorem 2. Beyond this, we show that if i and j are neighbors (see §5 for a definition), then the functions u_i and u_j are monotone on any interval contained in $\{u_i > 0\} \cap \{u_j > 0\}$.

Before stating the next theorem, let us fix some notation. For each $i \in \{1, \dots, d\}$, we denote by D_i the K_i-admissible interval-domain defined in §3. Thus, D_i is the maximal subset of $\mathbb{S}^1$ that can be written as a union of closed intervals with endpoints in K_i and is disjoint from K_j for every $j \neq i$.

We shall also use the following consequence of the definition of the weight maps:

$$K_i \subset D_i \subset \{u_i = 1\}.$$

Indeed, $K_i \subset D_i$ by the definition of D_i, whereas the inclusion $D_i \subset \{u_i = 1\}$ follows from Lemma 2 and the equality in (3).

THEOREM 4. *Let $(\mathcal{F}, \nu)$ be an RDS of circle homeomorphisms without finite orbits. Let d be the number of ergodic stationary measures and let $K_1, \dots, K_d$ be the corresponding topological supports of these measures. For $i = 1, \dots, d$, let u_i be the weight map defined in (2).*

Then, for every $i \in \{1, \dots, d\}$, we have the following.

(a) *The set $\{u_i = 1\}$ has finitely many connected components.*

(b) *If I is a connected component of $\{u_i = 1\}$ and $f \in \mathcal{F}$, then $f(I)$ is contained in a connected component of $\{u_i = 1\}$.*

(c) *If I and J are two distinct connected components of $\{u_i = 1\}$, and if $x \in I$, $y \in J$, then both connected intervals $[x, y]$ and $[y, x]$ in $\mathbb{S}^1$ intersect $\bigcup_{j \neq i} K_j$.*

Moreover, assume that $d \geq 2$. Let $i, j \in \{1, \dots, d\}$, $i \neq j$. Let I be a connected component of $\{u_i = 1\}$ and let J be a connected component of $\{u_j = 1\}$. Suppose that there exist $y_i \in I \cap K_i$ and $x_j \in J \cap K_j$ such that

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$$(y_i, x_j) \cap \bigcup_{r=1}^{d} K_r = \emptyset.$$

Then, u_i is monotone decreasing on $[y_i, x_j]$ and u_j is monotone increasing on $[y_i, x_j]$.

Remark 2. Note that $(y_i, x_j) \subset \{u_i > 0\} \cap \{u_j > 0\}$. Moreover, the set $\{u_i > 0\} \cap \{u_j > 0\}$ is a finite union of open intervals with an extremal point in $\{u_i = 1\}$ and the other extremal point in $\{u_j = 1\}$.

Proof of Theorem 4. When $d = 1$, the result is immediate, since $\{u_1 = 1\} = \mathbb{S}^1$. Thus, assume that $d \geq 2$.

Fix $i \in \{1, \dots, d\}$. Recall that $u_i : \mathbb{S}^1 \to [0, 1]$ is continuous. Hence, the proof that $\{u_i = 1\}$ is a finite union of closed intervals is analogous to the proof of Lemma 1. This proves item (a).

To prove item (b), we use the $\mathcal{F}$-invariance of $\{u_i = 1\}$ established in Lemma 4. The conclusion then follows by arguing analogously to the proof of Lemma 2.

Finally, item (c) follows from Lemma 3 and the fact that every connected component of $\{u_i = 1\}$ intersects K_i.

To prove the second part of Theorem 4, consider $i, j \in \{1, \dots, d\}$, $i \neq j$ and $I = [x_i, y_i] \subset \{u_i = 1\}$, $J = [x_j, y_j] \subset \{u_j = 1\}$ such that

$$(y_i, x_j) \cap \bigcup_{r=1}^{d} K_r = \emptyset.$$

By Proposition 2, for $x \in [y_i, x_j]$, we have $u_i(x) + u_j(x) = 1$. Let us show that u_i is a monotone decreasing function on $[y_i, x_j]$. Note that for $f \in \mathcal{F}$, there exist $I(f) = [x_i(f), y_i(f)] \subset \{u_i = 1\}$, $J(f) = [x_j(f), y_j(f)] \subset \{u_j = 1\}$ such that $f(I) \subset I(f)$, $f(J) \subset J(f)$, and

$$(y_i(f), x_j(f)) \cap \bigcup_{r=1}^{d} K_r = \emptyset \quad \text{or} \quad (y_j(f), x_i(f)) \cap \bigcup_{r=1}^{d} K_r = \emptyset.$$

The last fact will depend on whether f preserves orientation or not. Therefore, for $x \in [y_i, x_j]$ and $\omega \in \mathcal{F}^{\mathbb{N}}$ such that $\mathrm{cl}((f_\omega^n(x))_{n\geq 0}) = K_i$, we get that for all $y \in [y_i, x]$, we also have $\mathrm{cl}((f_\omega^n(y))_{n\geq 0}) = K_i$. Hence,

$$\{\omega \in \mathcal{F}^{\mathbb{N}} : \mathrm{cl}((f_\omega^n(x))_{n\geq 0}) = K_i\} \subset \{\omega \in \mathcal{F}^{\mathbb{N}} : \mathrm{cl}((f_\omega^n(y))_{n\geq 0}) = K_i\} \tag{6}$$

for every $x \in [y_i, x_j]$ and $y \in [y_i, x]$. Consequently, for $x, y \in [y_i, x_j]$, (y_i, x, y, x_j) is circularly ordered, we have that

$$u_i(x) \leq u_i(y).$$

This implies the desired conclusion. The monotonicity of u_j is an immediate consequence of $u_i + u_j = 1$ on $[y_i, x_j]$. □

Let us prove item (iii) in Theorem 2 as a corollary of Theorem 4.

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COROLLARY 2. *Let $(\mathcal{F}, \nu)$ be an RDS of circle homeomorphisms without finite orbits. Let d be the number of ergodic stationary measures. Consider u_i as in (2) for $i = 1, \dots, d$. If $u_i|_I \equiv C$ for some interval $I \subset \mathbb{S}^1$ and $C \in [0, 1]$, then for every $f \in \mathcal{F}$, there exists $c_f \in [0, 1]$ such that $u_i|_{f(I)} \equiv c_f$.*

Proof. Consider $i \in \{1, \dots, d\}$ and $I = [a, b]$ such that $u_i|_I \equiv C$ for some $C \in [0, 1]$. By Lemma 4, when $C \in \{0, 1\}$, it is enough to consider $c_f = C$ for all $f \in \mathcal{F}$.

Assume $C \in (0, 1)$. Since $C < 1$, $I \cap K_j = \emptyset$ for all $j \in \{1, \dots, d\}$. Take $i_1, i_2 \in \{1, \dots, d\}$, $i_1 \neq i_2$, $x_1 \in K_{i_1}$, $x_2 \in K_{i_2}$ such that $I \subset [x_1, x_2]$, and for all $j \in \{1, \dots, d\}$,

$$(x_1, x_2) \cap K_j = \emptyset.$$

Since $C > 0$, either $i_1 = i$ or $i_2 = i$. Consider $c_1, c_2 > 0$ such that $c_1 + c_2 = 1$, $u_{i_1}|_I = c_1$, and $u_{i_2}|_I = c_2$. By the invariance of the weight maps in (4), we get for $x, y \in I$, $x \leq y$,

$$\begin{aligned}
&\int_{\mathcal{F}} \nu^{\mathbb{N}}(\omega \in \mathcal{F}^{\mathbb{N}} \colon \mathrm{cl}((f_\omega^n(f(x)))_{n\geq 0}) = K_{i_1})\, d\nu(f) \\
&\quad = u_{i_1}(x) = c_1 = u_{i_1}(y) \\
&\quad = \int_{\mathcal{F}} \nu^{\mathbb{N}}(\omega \in \mathcal{F}^{\mathbb{N}} \colon \mathrm{cl}((f_\omega^n(f(y)))_{n\geq 0}) = K_{i_1})\, d\nu(f). \qquad (7)
\end{aligned}$$

By the continuity of the maps in $\mathcal{F}$ and the argument leading to (6), for all $f \in \mathcal{F}$, we have

$$\{\omega \in \mathcal{F}^{\mathbb{N}} \colon \mathrm{cl}((f_\omega^n(f(y)))_{n\geq 0}) = K_{i_1}\} \subset \{\omega \in \mathcal{F}^{\mathbb{N}} \colon \mathrm{cl}((f_\omega^n(f(x)))_{n\geq 0}) = K_{i_1}\}. \qquad (8)$$

By (7) and (8), we get that for ν-almost every $f \in \mathcal{F}$,

$$\nu^{\mathbb{N}}(\omega \in \mathcal{F}^{\mathbb{N}} \colon \mathrm{cl}((f_\omega^n(f(x)))_{n\geq 0}) = K_{i_1}) = \nu^{\mathbb{N}}(\omega \in \mathcal{F}^{\mathbb{N}} \colon \mathrm{cl}((f_\omega^n(f(y)))_{n\geq 0}) = K_{i_1}),$$

that is, $u_{i_1}(f(x)) = u_{i_1}(f(y))$. By the continuity of u_{i_1} and the fact that ν is supported on $\mathcal{F}$, we conclude $u_{i_1}(f(x)) = u_{i_1}(f(y))$ for all $f \in \mathcal{F}$. Since x and y are arbitrary in I, we get $u_{i_1}|_{f(I)} = c_{1,f}$ for some $c_{1,f} \in [0, 1]$. Analogously, we can show that for all $f \in \mathcal{F}$, we have $u_{i_2}|_{f(I)} = c_{2,f} = 1 - c_{1,f}$. To conclude this demonstration, it only remains to consider $c_f = c_{k,f}$ for all $f \in \mathcal{F}$ and $k \in \{1, 2\}$ such that $i = i_k$. □

We say that a finite set $P = \{x_0, x_1, \dots, x_{n_P}\}$, $n_P \geq 2$ is a *partition* of $\mathbb{S}^1$ if $x_{n_P} = x_0$, $x_i \neq x_{i+1}$, and P appears in the order $x_1 \to x_2 \to \cdots \to x_{n_P}$ when traversing the circle $\mathbb{S}^1$ counterclockwise. In particular, when $n_P \geq 2$, we have that $(x_1, \dots, x_{n_P})$ and $(x_0, x_1, \dots, x_{n_P-1})$ are circularly ordered. When $n_P = 2$, P is a set of the form $\{a, b, a\}$ with $a \neq b$. Let $\mathcal{P}$ denote the set of partitions P of $\mathbb{S}^1$.

The *total variation* of a real-valued function $\varphi : \mathbb{S}^1 \to \mathbb{R}$, defined on $\mathbb{S}^1$, is the quantity

$$V(\varphi) = \sup_{P \in \mathcal{P}} \sum_{i=0}^{n_P - 1} |\varphi(x_i) - \varphi(x_{i+1})|.$$

A function $\varphi : \mathbb{S}^1 \to \mathbb{R}$ is said to be of *bounded variation* (BV function) on $\mathbb{S}^1$ if its total variation is finite.

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From Theorem 4, we can conclude the bounded variation of the weight maps using their continuity.

COROLLARY 3. *Let $(\mathcal{F}, \nu)$ be an RDS of circle homeomorphisms without finite orbits. Let d be the number of ergodic stationary measures. Then, for $i = 1, \dots, d$, the map u_i defined by (2) has bounded variation.*

Proof. If $d = 1$, the corollary follows easily because u_1 is constant, implying that $V(u_1) = 0$. Assume $d \geq 2$ and take $i \in \{1, \dots, d\}$. By Theorem 4, we can find a circularly ordered tuple $(a_0, a_1, \dots, a_m)$, with $m = 4k - 1$ for some $k \in \mathbb{N}$ such that for $j = 1, \dots, k$:

- $u_i = 1$ on $[a_{4j-4}, a_{4j-3}]$;
- u_i is monotone decreasing on $[a_{4j-3}, a_{4j-2}]$;
- $u_i = 0$ on $[a_{4j-2}, a_{4j-1}]$; and
- u_i is monotone increasing on $[a_{4j-1}, a_{4j}]$, where $a_{4k} = a_0$.

Therefore, for any $P \in \mathcal{P}$, the sum

$$\sum_{j=0}^{n_P-1} |u_i(x_j) - u_i(x_{j+1})|$$

can be separated into $m + 1 = 4k$ sums with half of them being 0 and the other half (corresponding to the monotonicity intervals of u_i) being telescopic sums. Thus, using $0 \leq u_i \leq 1$,

$$\sum_{j=0}^{n_P-1} |u_i(x_j) - u_i(x_{j+1})| \leq 2k.$$

Since k does not depend on the partition P, we can conclude the bounded variation of u_i. □

5. *Neighbors*

Throughout this section, we use the notation introduced in §4. For $i, j \in \{1, \dots, d\}$, we say that i and j are neighbors if $\{u_i > 0\} \cap \{u_j > 0\} \neq \emptyset$. See Lemma 6 for a non-trivial upper bound on the number of neighbors for i. Define

$$\mathcal{E} \subset \{1, \dots, d\} \times \{1, \dots, d\} \tag{9}$$

as the set of pairs (i, j) such that i and j are neighbors. That is, $(i, j) \in \mathcal{E}$ if and only if there is $x \in \mathbb{S}^1$ such that $u_i(x) > 0$ and $u_j(x) > 0$.

Define $\mathcal{S}_\nu$ as the collection of stationary measures that are weak-$*$ limits of the sequence (1) for some $x \in \mathbb{S}^1$.

Using the topological facts shown in §3, let us prove the following results, which immediately imply the first part of Theorem 2.

LEMMA 5. *Let $(\mathcal{F}, \nu)$ be an RDS of circle homeomorphisms without finite orbits. Let d be the number of ergodic stationary measures $\mu_1, \dots, \mu_d$. Then,*

$$\mathcal{S}_\nu = \{t\mu_i + (1-t)\mu_j : t \in [0, 1],\ (i, j) \in \mathcal{E}\}.$$

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Proof. Let $K_i = \operatorname{supp}(\mu_i)$, $i = 1, \dots, d$. We first prove the inclusion

$$\mathcal{S}_\nu \subset \{t\mu_i + (1-t)\mu_j : t \in [0,1],\ (i,j) \in \mathcal{E}\}. \tag{10}$$

Take $x \in \mathbb{S}^1$. By [**Mal17**, Proposition 4.11], the sequence in (1) converges in the weak-$*$ topology to $m_x = \sum_{i=1}^{d} u_i(x)\mu_i$. Since for every $n \geq 0$, the support of the n-step transition measure starting from x is contained in $\mathcal{O}_\mathcal{F}(x)$, we must have

$$\operatorname{supp}(m_x) \subset \overline{\mathcal{O}_\mathcal{F}(x)}.$$

If $u_i(x) > 0$, then $K_i \subset \operatorname{supp}(m_x)$. Consequently, $K_i \subset \overline{\mathcal{O}_\mathcal{F}(x)}$. Thus, every minimal set K_i with positive weight $u_i(x)$ intersects $\overline{\mathcal{O}_\mathcal{F}(x)}$. By Proposition 1, the set $\overline{\mathcal{O}_\mathcal{F}(x)}$ can intersect at most two minimal sets. Therefore, at most two coefficients in the convex decomposition of m_x are non-zero. It follows that, for each $x \in \mathbb{S}^1$,

$$m_x = t\mu_i + (1-t)\mu_j$$

for some $t \in [0,1]$ and some pair $(i,j) \in \mathcal{E}$. Hence,

$$m_x \in \{t\mu_i + (1-t)\mu_j : t \in [0,1],\ (i,j) \in \mathcal{E}\}.$$

Since, by definition, every element of $\mathcal{S}_\nu$ is of the form m_x for some $x \in \mathbb{S}^1$, we obtain the inclusion in (10).

We now prove the reverse inclusion. Let $(i,j) \in \mathcal{E}$. If $i = j$, by the Birkhoff ergodic theorem, we have $\mu_i \in \mathcal{S}_\nu$. Let us assume $i \neq j$.

By the definition of $\mathcal{E}$, there exists a point $x \in \mathbb{S}^1$ such that $\overline{\mathcal{O}_\mathcal{F}(x)}$ intersects both K_i and K_j. Moreover, by Proposition 1, there is no $l \in \{1, \dots, d\}\backslash\{i,j\}$ such that $\overline{\mathcal{O}_\mathcal{F}(x)}$ intersects K_l. Take $x_i \in K_i$ and $x_j \in K_j$ such that $x \in (x_i, x_j)$ and $(x_i, x_j) \cap K = \emptyset$ for $K \in \{K_1, \dots, K_d\}\backslash\{K_i, K_j\}$. By the continuity of the weight maps, we must have for all $t \in [0,1]$,

$$t\mu_i + (1-t)\mu_j \in \mathcal{S}_\nu.$$

Combining both inclusions, we obtain the desired equality. □

Now, for $i \in \{1, \dots, d\}$, consider

$$V_i = \{j \in \{1, \dots, d\} : j \neq i,\ (i,j) \in \mathcal{E}\}. \tag{11}$$

If $d = 1$, it is clear that $V_1 = \emptyset$. Let us establish the following property of the sets V_i, $i = 1, \dots, d$, which will be key to completing the second inequality in Theorem 2.

LEMMA 6. *Let $(\mathcal{F}, \nu)$ be an RDS of circle homeomorphisms without finite orbits. Let d be the number of ergodic stationary measures. Let $\mathcal{K}_\mathcal{F} = \{K_1, \dots, K_d\}$ be the collection of the closed $\mathcal{F}$-minimal invariant sets. Consider u_i as in (2) for $i = 1, \dots, d$. If $d \geq 2$, then for all $i = 1, \dots, d$,*

$$1 \leq \#V_i \leq 2, \tag{12}$$

where V_i is as in (11). Moreover,

$$\#\{i \in \{1, \dots, d\} : \#V_i = 1\} \leq 2. \tag{13}$$

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Proof of (12). Assume $d \geq 2$. For $i = 1, \dots, d$, let $\mathcal{A}_i$ be the collection of connected components of D_{K_i}. Fix $i = 1, \dots, d$. Note that there exists $j \neq i$ such that

$$\operatorname{dist}\Big(\bigcup_{I \in \mathcal{A}_i} I, \bigcup_{J \in \mathcal{A}_j} J \Big) = \min_{k \neq i} \operatorname{dist}\Big(\bigcup_{I \in \mathcal{A}_i} I, \bigcup_{J \in \mathcal{A}_k} J \Big).$$

From the continuity of the weight maps, we can conclude that $j \in V_i$. Hence, $\#V_i \geq 1$.

Now, let us show that $\#V_i \leq 2$. Take $j \in V_i$ and consider $I' = [a_i', b_i'] \in \mathcal{A}_i$ and $J' = [a_j', b_j'] \in \mathcal{A}_j$ such that:

- $(b_i', a_j') \cap A = \emptyset$ for all $A \in \bigcup_k \mathcal{A}_k$; or
- $(b_j', a_i') \cap A = \emptyset$ for all $A \in \bigcup_k \mathcal{A}_k$.

In any case, by item (c) of Theorem 4, we have that for every $I = [a_i, b_i] \in \mathcal{A}_i$, there exists $f \in \mathcal{F}$ and $J = [a_j, b_j] \in \mathcal{A}_j$ such that $f(I') \subset I$ and $f(J') \subset J$. From Lemma 3, we get:

- $(b_i, a_j) \cap A = \emptyset$ for all $A \in \bigcup_k \mathcal{A}_k$; or
- $(b_j, a_i) \cap A = \emptyset$ for all $A \in \bigcup_k \mathcal{A}_k$.

We have just shown that for all $j \in V_i$ and $I = [a_i, b_i] \in \mathcal{A}_i$, there exists $J = [a_j, b_j] \in \mathcal{A}_j$ such that the two items above are satisfied. Using the topological structure of $\mathbb{S}^1$, we conclude that $\#V_i \leq 2$ since each interval $I \in \mathcal{A}_i$ has only two intervals in $\bigcup_{k \neq i} \mathcal{A}_k$ as neighbors. □

To prove (13), let us first introduce the notion of $\mathcal{F}$-paths over the index set $\{1, \dots, d\}$. According to the conclusions in Theorem 4, we have a finite family of intervals on the circle, say $I_1, \dots, I_k$, such that for each $\ell = 1, \dots, k$, there exists $r(\ell) \in \{1, \dots, d\}$ for which $u_{r(\ell)} = 1$ on I_ℓ. Assume that the enumeration of these intervals is counterclockwise. Then, two different indices i and j are neighbors if, for some ℓ, we have $r(\ell) = i$ and also, $r(\ell - 1) = j$ or $r(\ell + 1) = j$. With these concepts in mind, we define an $\mathcal{F}$-path as any finite sequence on $\{1, \dots, d\}$ of the form $[r(\ell), \dots, r(\ell + n)]$, where $n \geq 0$, $\ell \in \{1, \dots, k\}$, and $r(k + m) = r(m)$, for $m \in \mathbb{N}$. Let us formalize the definition of $\mathcal{F}$-paths.

For $n \in \mathbb{N}$ and $i_1, \dots, i_n \in \{1, \dots, d\}$, we say that $[i_1, \dots, i_n]$ is an *$\mathcal{F}$-path* if for all $k = 1, \dots, n - 1$:

- $i_k \neq i_{k+1}$; and
- there exist $I_k = [a_k, b_k] \in \mathcal{A}_{i_k}$ and $I_{k+1} = [a_{k+1}, b_{k+1}] \in \mathcal{A}_{i_{k+1}}$ such that the interval (b_k, a_{k+1}) does not intersect any $I \in \bigcup_i \mathcal{A}_i$.

Let us make some observations about $\mathcal{F}$-paths.

Remark 3. Each $\mathcal{F}$-path with length greater than $\#(\bigcup_i \mathcal{A}_i)$ passes through all the indices in $\{1, \dots, d\}$. Also, note that i and j are neighbors if and only if $[i, j]$ or $[j, i]$ is an $\mathcal{F}$-path. Further, if $d = 1$, the unique $\mathcal{F}$-path is $[1]$ and if $d = 2$, the $\mathcal{F}$-paths are as follows: $[i_1, i_2, i_1, \dots, i_1, i_2, i_1]$ or $[i_1, i_2, i_1, \dots, i_2, i_1, i_2]$.

LEMMA 7. *Let $(\mathcal{F}, \nu)$ be an RDS of circle homeomorphisms without finite orbits. Let d be the number of ergodic stationary measures. Then, for every $i, j, k, a, b, c \in \{1, \dots, d\}$,*

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with $i \neq j$, $j \neq k$, and $k \neq i$, at least one of the triples $[a, i, a]$, $[b, j, b]$, or $[c, k, c]$ is not an $\mathcal{F}$-path.

Proof. Assume $d \geq 3$. For $d < 3$, the conclusion is clear. We argue by contradiction. Take $i, j, k, a, b, c \in \{1, \ldots, d\}$, with $i \neq j$, $j \neq k$, and $k \neq i$, such that $[a, i, a]$, $[b, j, b]$, and $[c, k, c]$ are $\mathcal{F}$-paths, that is, $\#V_i = \#V_j = \#V_k = 1$. Using (12), we can conclude that $\#V_a = 2$. In fact, if $\#V_a = 1$, then $[i, a, i, a, i]$ is an $\mathcal{F}$-path and, in particular, for any $n \in \mathbb{N}$, any n-tuple of the form $[i, a, i, a, \ldots, i]$ is an $\mathcal{F}$-path, which is not possible by Remark 3 and the fact $d \geq 3$. Analogously, we have $\#V_a = \#V_b = \#V_c = 2$. Using the circle structure, we get that for $n \in \mathbb{N}$ and any two $\mathcal{F}$-paths $[i_1, \ldots, i_n]$ and $[j_1, \ldots, j_n]$ with $i_1 = j_n = a$ and $i_n = j_1 = b$, we have $i_m = j_{n-m+1}$ for $m = 1, \ldots, n$. Further, we have $i_m = k$ for some $m = 1, \ldots, n$. Since we must conclude the same for $\mathcal{F}$-paths starting at a (or b) and ending at c, we can conclude that $a = b$. However, this contradicts that $i \neq j$, since $\#V_a = 2$. Therefore, at least one of the triples $[a, i, a]$, $[b, j, b]$, or $[c, k, c]$ is not an $\mathcal{F}$-path. □

Proof of (13) in Lemma 6. Since $\#V_i = 1$ implies that $[j, i, j]$ is an $\mathcal{F}$-path for some $j \neq i$, we can apply Lemma 7 to get (13). □

We already have proven Theorem 2, as shown below.

Proof of Theorem 2. Proposition 2 and Lemma 6 imply the first part of the theorem. Item (i) follows from Lemma 4 and Theorem 4. The second part of Theorem 4 implies item (ii). Item (iii) was already established in Corollary 2. □

6. *Proximal action*

An important concept in the study of RDSs is *proximality*. Introduced by Furstenberg in his seminal paper [**Fur67**], it has since been employed by many researchers in the field. Proximality measures the degree of similarity between two orbits in a dynamical system. Specifically, two points are proximal if their images can be brought arbitrarily close along a suitable sequence of iterates or semigroup elements. See item (e) of Theorem 5 for further details.

Let us state the main result of this section, which further illustrates the usefulness of the collections of intervals defined in Theorem 2.

THEOREM 5. *Let $(\mathcal{F}, \nu)$ be an RDS of circle homeomorphisms without finite orbits. Assume the number of ergodic stationary measures is $d > 1$. Let $K_1, \ldots, K_d$ be their corresponding topological supports. For $i = 1, \ldots, d$, let u_i be the weight map defined in (2) and let $\mathcal{A}_i$ be the collection of connected components of $\{u_i = 1\}$.*

Then, for every $i \in \{1, \ldots, d\}$, there exists a finite collection $\mathcal{B}_i$ of pairwise disjoint closed intervals such that:

(a) *either $\#\mathcal{B}_i = \#\mathcal{A}_i$ or $\#\mathcal{B}_i = 2\#\mathcal{A}_i$;*

(b) *we have the inclusions*

$$K_i \subset \bigcup_{B \in \mathcal{B}_i} B \subset \bigcup_{A \in \mathcal{A}_i} A = \{u_i = 1\};$$

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(c) *every $A \in \mathcal{A}_i$ contains either one or two intervals of $\mathcal{B}_i$;*
(d) *for every $B_1, B_2 \in \mathcal{B}_i$, there exists $f \in \mathcal{F}$ such that $f(B_1) \subset B_2$;*
(e) *the semigroup $G_{\mathcal{F}}$ generated by $\mathcal{F}$ acts proximally on each $I \in \mathcal{B}_i$, that is, for every $x, y \in I$, there exists a sequence $(g_n)_{n\in\mathbb{N}}$ in $G_{\mathcal{F}}$ such that*

$$\lim_{n\to\infty} \operatorname{dist}(g_n(x), g_n(y)) = 0.$$

Theorem 5 may be viewed as a probabilistic version of Theorem 1. Indeed, if for each minimal set K_i, we consider its domain D_{K_i}, defined in §3, instead of the set $\{u_i = 1\}$, then the same argument gives the corresponding topological statement.

Remark 4. Consider an RDS $(\mathcal{F}, \nu)$ as in Theorem 5. Since $d > 1$ and there are no finite orbits, there is no probability measure invariant under all elements of $G_{\mathcal{F}}$. Applying [**Mal17**, Proposition 4.18] and using the finiteness of the collection $\mathcal{B}_i$, we can conclude that the action of $G_{\mathcal{F}}$ is synchronizing on each interval $I \in \mathcal{B}_i$, that is, for every $x, y \in I$ and $\nu^{\mathbb{N}}$-almost every sequence $\omega \in \mathcal{F}^{\mathbb{N}}$, we have

$$\lim_{n\to\infty} \operatorname{dist}(f_\omega^n(x), f_\omega^n(y)) = 0.$$

COROLLARY 4. *Assume that $(\mathcal{F}, \nu)$ is an RDS without finite orbits, with $\mathcal{F} \subset \mathrm{Homeo}^+(\mathbb{S}^1)$. Also, assume that the number d of ergodic stationary measures is greater than 1. For $i \in \{1, \dots, d\}$, consider the finite collection of disjoint closed intervals $\mathcal{A}_i$ as in Theorem 5. Then, the semigroup $G_{\mathcal{F}}$ generated by $\mathcal{F}$ acts proximally on each $I \in \mathcal{A}_i$.*

In the proof of Theorem 5, it becomes apparent that the primary elements used are the topological properties of the collections $\mathcal{A}_i$. Specifically, for a fixed $i \in \{1, \dots, d\}$, we employ the $\mathcal{F}$-invariance properties obtained in Theorem 4. Furthermore, we use the fact that almost every random orbit starting from an extremal point of any interval within $\mathcal{A}_i$ converges to the minimal K_i. Consequently, in the case $d = 1$, when the closed minimal set K is contained within a finite union of non-trivial closed intervals, we can follow the same approach as in the proof of Theorem 5 to establish the subsequent result.

COROLLARY 5. *Let $(\mathcal{F}, \nu)$ be an RDS of circle homeomorphisms without finite orbits. Let the number d be the number of ergodic stationary measures and assume $d = 1$. Let $K \subset \mathbb{S}^1$ be the closed minimal set. If there exists a finite collection $\mathcal{A}$ of closed intervals such that $\bigcup_{I\in\mathcal{A}} I \subsetneq \mathbb{S}^1$ and for any $I, J \in \mathcal{A}$ and any $f \in \mathcal{F}$, there exist $I', J' \in \mathcal{A}$ such that $f(I) \subset I'$ and $f(J') \subset J$, then, $K \subset \bigcup_{I\in\mathcal{A}} I$ and there exists a finite collection of disjoint closed intervals $\mathcal{B}$ such that:*
(a) *$\#\mathcal{B} = \#\mathcal{A}$ or $\#\mathcal{B} = 2\#\mathcal{A}$;*
(b) *$K \subset \bigcup_{I\in\mathcal{B}} I \subset \bigcup_{I\in\mathcal{A}} I$;*
(c) *each interval in $\mathcal{A}$ contains either one or two intervals in $\mathcal{B}$;*
(d) *for all $I, J \in \mathcal{B}$, there exists $f \in \mathcal{F}$ such that $f(I) \subset J$;*
(e) *the semigroup $G_{\mathcal{F}}$ generated by $\mathcal{F}$ acts proximally on each $I \in \mathcal{B}$.*

Before showing Theorem 5, let us establish some useful results. In what follows, as $x, y \in [a, b]$, $x \neq y$, we write $x < y$ whenever (a, x, y, b) is circularly ordered.

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LEMMA 8. *Assume that $(\mathcal{F}, \nu)$ is an RDS with $\mathcal{F} \subset \mathrm{Homeo}(\mathbb{S}^1)$ without finite orbits. Let the number d be the number of ergodic stationary measures and assume $d \geq 2$. Let $G_{\mathcal{F}}$ be the semigroup generated by $\mathcal{F}$. Fix $i \in \{1, \ldots, d\}$. Let $\mathcal{A}_i$ be as in Theorem 4. Given $I = [a, b] \in \mathcal{A}_i$, let $S_I \subset G_{\mathcal{F}}$ denote the semigroup of the maps f in $G_{\mathcal{F}}$ such that $f(I) \subset I$. Let $S_I^+ \subset S_I$ be the semigroup of orientation-preserving maps in S_I. Set*

$$\hat{a} = \sup\{f(a) \colon f \in S_I^+\}$$

and

$$\hat{b} = \inf\{f(b) \colon f \in S_I^+\}.$$

Then either:

(i) *$\hat{a} \geq \hat{b}$ and $G_{\mathcal{F}}$ acts proximally on $[a, b]$; or*

(ii) *$\hat{a} < \hat{b}$ and $G_{\mathcal{F}}$ acts proximally on $[a, \hat{a}]$ and $[\hat{b}, b]$. In this case, we also have $\hat{a}, \hat{b} \in K_i$ and $(\hat{a}, \hat{b}) \cap K_i = \emptyset$.*

In case (ii), for each $f \in S_I$, we have either:

(iii) *f preserves orientation, $f([a, \hat{a}]) \subset [a, \hat{a}]$, and $f([\hat{b}, b]) \subset [\hat{b}, b]$; or*

(iv) *f does not preserve orientation, $f([a, \hat{a}]) \subset [\hat{b}, b]$, and $f([\hat{b}, b]) \subset [a, \hat{a}]$.*

Proof. Using that S_I^+ is a semigroup, we get that for $g \in S_I^+$,

$$\begin{aligned} g(\hat{a}) &= g(\sup\{f(a) \colon f \in S_I^+\}) \\ &= \sup\{g(f(a)) \colon f \in S_I^+\} \\ &\leq \sup\{f(a) \colon f \in S_I^+\} = \hat{a}, \end{aligned}$$

that is, $g(\hat{a}) \leq \hat{a}$. Analogously, for $g \in S_I^+$, we get $g(\hat{b}) \geq \hat{b}$.

Case $\hat{a} > \hat{b}$. In this case, it is easy to show the proximal action of $G_{\mathcal{F}}$ on I. In fact, take $g \in S_I^+$ such that $\hat{b} < g(a) \leq g(b) \leq b$. Then, consider a sequence $(g_n)_{n\in\mathbb{N}}$ in S_I^+ such that $g_n(b)$ converges to $\hat{b}$ as $n \to \infty$. Hence, for every $x \in I$, $\hat{b} \leq g_n(\hat{b}) \leq g_n(g(a)) \leq g_n(g(x)) \leq g_n(g(b)) \leq g_n(b)$ and so the sequence $(g_n(g(x)))_{n\in\mathbb{N}}$ converges to $\hat{b}$ so that the sequence $(g_n \circ g)_{n\in\mathbb{N}}$ in $S_I^+ \subset G_{\mathcal{F}}$ degenerates the interval I to $\{\hat{b}\}$, which implies what is desired.

Case $\hat{a} = \hat{b}$. For all $g \in S_I^+$, we get

$$g(\hat{b}) \geq \hat{b} = \hat{a} \geq g(\hat{a}),$$

that is, $c = \hat{a} = \hat{b} \in [a, b]$ is a fixed point of every map in S_I^+. Now, let us show that for any $\varepsilon > 0$, there exists $g \in S_I^+$ such that $g([a, b]) \subset (c - \varepsilon, c + \varepsilon)$. By definition of supremum and infimum, given $\varepsilon > 0$, there exist $f \in S_I^+$ and a sequence $(f_n)_{n\in\mathbb{N}}$ such that

$$c \leq f(b) < c + \varepsilon$$

and

$$\lim_{n\to\infty} f_n(a) = c.$$

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By continuity,

$$\lim_{n\to\infty} f \circ f_n(a) = f(c) = c.$$

By definition of c, it is clear that $c \leq f_n(b) \leq b$ for all $n \in \mathbb{N}$ so that for N large enough, we have

$$f \circ f_N(a) \in (c-\varepsilon, c] \quad \text{and} \quad f \circ f_N(b) \in [c, f(b)] \subset [c, c+\varepsilon).$$

Consequently, $g([a,b]) \subset (c-\varepsilon, c+\varepsilon)$ for $g = f \circ f_N$. Using the arbitrariness of ε, we can conclude that the semigroup S_I^+ (and so $G_{\mathcal{F}}$) acts proximally on $I = [a,b]$.

Case $\hat{a} < \hat{b}$. By definition of $\hat{a}$ and $\hat{b}$, it is clear that S_I^+ acts proximally on $[a, \hat{a}]$ and $[\hat{b}, b]$. Since $S_I^+ \subset S_I \subset G_{\mathcal{F}}$, we have that $G_{\mathcal{F}}$ acts proximally on $[a, \hat{a}]$ and $[\hat{b}, b]$. Now, let us show that $f(a), f(b) \in [a, \hat{a}] \cup [\hat{b}, b]$ for all $f \in S_I$. Take $f \in S_I$ and note $f^2 \in S_I^+$. If f is not orientation-preserving, then $f(a) > \hat{b}$, because

$$a \leq f(b) < f(a) < \hat{b}$$

implies

$$f(\hat{b}) < f^2(a) < f^2(b) \leq f(a) < \hat{b},$$

which is a contradiction to the definition of $\hat{b}$. Analogously, $f(b) < \hat{a}$. Therefore, for $f \in S_I$,

$$f(a) \in [a, \hat{a}], \quad f(b) \in [\hat{b}, b]$$

whenever f preserves orientation, or

$$f(a) \in [\hat{b}, b], \quad f(b) \in [a, \hat{a}]$$

whenever f does not preserve orientation. Using Theorem 4, for $f \in G_{\mathcal{F}}$ such that $f(a) \in [a,b]$, we conclude that $f \in S_I$. Since $u_i(a) = 1$, for any $x \in K_i$, we can find $f \in G_{\mathcal{F}}$ such that $f(a)$ is arbitrarily close to x. Hence, we must have $(\hat{a}, \hat{b}) \cap K_i = \emptyset$.

Moreover, $\hat{a}, \hat{b} \in K_i$. Let us show that $\hat{a} \in K_i$; the proof of $\hat{b} \in K_i$ is analogous. Use the finiteness of the collection $\mathcal{A}_i$, the definition of the weight map u_i, and that $u_i(a) = u_i(b) = 1$ to conclude $K_i \cap [a, \hat{a}] \neq \emptyset$. Hence, if $\hat{a} \notin K_i$, then take $x \in [a, \hat{a}) \cap K_i$ such that $(x, \hat{a}) \cap K_i = \emptyset$. Now, by $\mathcal{F}$-invariance of K_i, for all $f \in S_I^+$, $f(x) \in K_i$ and $f(a) \leq f(x) \leq \hat{a}$. However, by definition of x, $f(x) \leq x$ so that $f(a) \leq x < \hat{a}$ for all $f \in S_I^+$, which contradicts the definition of $\hat{a}$.

The lemma is proven. □

Remark 5. Case (ii) in Lemma 8 can occur only when there exists a map in S_I that does not preserve orientation.

Proof of Theorem 5. Consider $i \in \{1, \dots, d\}$ and $\mathcal{A}_i = \{I_1, \dots, I_m\}$ as in Theorem 4. Given $I_j = [a_j, b_j] \in \mathcal{A}_i$, consider $A_j = [a_j, \hat{a}_j]$ and $B_j = [\hat{b}_j, b_j]$, where $\hat{a}_j$ and $\hat{b}_j$ are defined analogously to $\hat{a}$ and $\hat{b}$ in Lemma 8 taking $a = a_j$ and $b = b_j$. In the case $\hat{a}_j \geq \hat{b}_j$, set $\mathcal{B}_{i,j} = \{I_j\}$. In the case $\hat{a}_j < \hat{b}_j$, set $\mathcal{B}_{i,j} = \{A_j, B_j\}$. Consider

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$$\mathcal{B}_i = \bigcup_{j=1}^{m} \mathcal{B}_{i,j}.$$

By definition of $\mathcal{B}_i$, item (c) holds. Items (b) and (e) are consequences of items (i) and (ii) of Lemma 8. Item (d) is a consequence from item (b) of Theorem 4 and items (iii), (iv) of Lemma 8.

To see item (a), it is enough to show that $\#\mathcal{B}_{i,j} = \#\mathcal{B}_{i,k}$ for all $j, k \in \{1, \dots, m\}$. Assume that there exist j, k such that $\#\mathcal{B}_{i,j} = 2$ and $\#\mathcal{B}_{i,k} = 1$. Since $\#\mathcal{B}_{i,j} = 2$, we have $\hat{a}_j < \hat{b}_j$. Set $\delta = \operatorname{dist}(\hat{a}_j, \hat{b}_j) > 0$. Fix $x \in [a_j, \hat{a}_j] \subset [a_j, b_j]$ and $y \in [\hat{b}_j, b_j] \subset [a_j, b_j]$. By item (iv) of Lemma 8, we get $\operatorname{dist}(g(x), g(y)) \geq \delta > 0$ for all g preserving I_j. By item (b) of Theorem 4, there exists $f \in G_{\mathcal{F}}$ such that $f(I_j) \subset I_k$. From the proximal action of $G_{\mathcal{F}}$ on I_k, we can find a sequence $(g_n)_{n\in\mathbb{N}}$ in $G_{\mathcal{F}}$ such that

$$\lim_{n\to\infty} \operatorname{dist}(g_n(f(x)), g_n(f(y))) = 0.$$

Without loss of generality, let us assume that the sequences $(g_n(f(x)))_{n\in\mathbb{N}}$ and $(g_n(f(y)))_{n\in\mathbb{N}}$ converge to a point $z \in I_r$ for some $r \in \{1, \dots, m\}$. Take $h \in \mathcal{F}$ such that $h(I_r) \subset I_j$. Consider $V_z = h^{-1}(h(z) - \varepsilon, h(z) + \varepsilon)$ for some $\varepsilon \in (0, \delta/4)$. Take N such that $g_N(f(x)), g_N(f(y)) \in V_z$. Hence, $g = h \circ g_N \circ f$ preserves I_j. Further, $g(x), g(y) \in (h(z) - \varepsilon, h(z) + \varepsilon)$, which is a contradiction because $\operatorname{dist}(g(x), g(y)) > \delta > 2\varepsilon$. Therefore, there is no i, j such that $\#\mathcal{B}_{i,j} \neq \#\mathcal{B}_{i,k}$, which implies the desired. □

Proof of Corollary 4. Following the notation in the proof of Theorem 5, by Remark 5, for all i and j, we get that $\mathcal{B}_{i,j} = \{I_j\}$. Therefore, for all $i = 1, \dots, d$, we have $\mathcal{B}_i = \mathcal{A}_i$. Applying Theorem 5, we conclude the desired. □

Proof of Theorem 1. For each i, write

$$D_i := D_{K_i}.$$

We repeat the proof of Theorem 5, replacing the set $\{u_i = 1\}$ by the domain D_i. This gives a finite collection $\widehat{\mathcal{B}}_i$ of pairwise disjoint closed intervals with the same properties as the collection $\mathcal{B}_i$ in Theorem 5. In particular,

$$K_i \subset \bigcup_{I\in\widehat{\mathcal{B}}_i} I \subset D_i.$$

Moreover, the intervals in $\widehat{\mathcal{B}}_i$ are mapped into each other by elements of $\mathcal{F}$, and the semigroup $G_{\mathcal{F}}$ acts proximally on each of them.

We now define

$$L_i := \bigcup_{I\in\widehat{\mathcal{B}}_i} I.$$

Then, L_i is a finite union of pairwise disjoint closed intervals, contains K_i, is contained in D_i, and satisfies all the required properties. This proves the theorem. □

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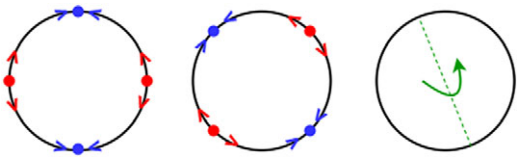


FIGURE 1. Dynamics on the circle of the functions f_1, f_2 and f_3 in Example 1.

7. *The inverse RDS*

Assume that $(\mathcal{F}, \nu)$ is an RDS with $\mathcal{F} \subset \mathrm{Homeo}(\mathbb{S}^1)$. Consider the *inverse RDS* $(\mathcal{F}^-, \nu^-)$ associated with $(\mathcal{F}, \nu)$, that is,

$$\mathcal{F}^- = \{f^{-1} \colon f \in \mathcal{F}\}$$

and

$$\nu^-(\cdot) = \nu(\{f \colon f^{-1} \in \cdot\}).$$

Also, we assume there is no finite orbit for $\mathcal{F}$. Hence, no finite orbit exists for $\mathcal{F}^-$.

Let us begin the discussion on the number of stationary measures of $(\mathcal{F}, \nu)$ and $(\mathcal{F}^-, \nu^-)$ with an example illustrating that such quantities do not necessarily coincide.

Example 1. Consider $\mathcal{F} = \{f_1, f_2, f_3\} \subset \mathrm{Homeo}(\mathbb{S}^1)$, where the maps f_i are defined as follows:

- f_1 is a orientation-preserving homeomorphism with only 4 fixed points 0, 1/4, 1/2, and 3/4, where 0 and 1/2 are attracting points, and 1/4 and 3/4 are repelling points;
- f_2 is a orientation-preserving homeomorphism with only 4 fixed points 1/8, 3/8, 5/8, and 7/8, where 3/8 and 7/8 are attracting points, and 1/8 and 5/8 are repelling points;
- f_3 is a reflection map on the line that passes through the points 7/16 and 15/16. So, f_3 is an isometry fixing only 7/16 and 15/16. Further, $f_3(0) = 7/8$, $f_3(1/4) = 5/8$, $f_3(1/2) = 3/8$, and $f_3(3/4) = 1/8$ (see Figure 1).

We consider $\mathbb{S}^1 = [0, 1)$. Since each point fixed by f_1 is different from any point fixed by f_2, there is no probability measure m on $\mathbb{S}^1$ such that $f_* m = m$ for all $f \in \mathcal{F}$ (and so for all $f \in \mathcal{F}^-$). Note that $f_i([3/4, 1/8]) \subset [3/4, 1/8]$ and $f_i([1/4, 5/8]) \subset [1/4, 5, 8]$ for all $i = 1, 2, 3$. Therefore, it is not difficult to conclude that there are exactly $d_+ = 2$ minimal invariant sets for $\mathcal{F}$. However, using the dynamics of the maps in $\mathcal{F}$, we have that $\mathcal{F}^- = \{f_1^{-1}, f_2^{-1}, f_3^{-1}\}$ has a unique minimal set contained in $[1/2, 7/8] \cup [0, 3/8]$. Let $d_- = 1$. Then, for any probability measure ν supported in $\mathcal{F}$ (that is, with $\nu(\{f_i\}) > 0$ for $i = 1, 2, 3$), the number of ergodic stationary measures for $(\mathcal{F}, \nu)$ is $d_+ = 2$. However, if we consider ν^- by $\nu^-(\{f_i^{-1}\}) = \nu(\{f_i\})$, we have that the number of ergodic stationary measures for $(\mathcal{F}^-, \nu^-)$ is $d_- = 1$.

Let d_+ and d_- denote the number of ergodic stationary measures for $(\mathcal{F}, \nu)$ and $(\mathcal{F}^-, \nu^-)$, respectively. First, let us establish some preliminary results.

LEMMA 9. *Let $(\mathcal{F}, \nu)$ be an RDS of circle homeomorphisms without finite orbits. Let $(\mathcal{F}^-, \nu^-)$ be the inverse RDS. Let d_+ and d_- be the numbers of ergodic stationary measures associated with $(\mathcal{F}, \nu)$ and $(\mathcal{F}^-, \nu^-)$, respectively. If $d_+ = n + 2$, then $d_- \geq n + 1$.*

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Before proving Lemma 9, let us introduce some useful sets. For $i, j \in \{1, \dots, d_+\}$, $i \neq j$, let $C_{i,j}$ be the set of closed intervals I with one of its extremal points in $\bigcup_{J \in \mathcal{A}_i} J$ and the other extremal point in $\bigcup_{J \in \mathcal{A}_j} J$ such that the interior of I does not intersect $\bigcup_{J \in \bigcup_k \mathcal{A}_k} J$. Note that $C_{i,j} = C_{j,i}$ and that it is a finite set. Furthermore, $C_{i,j} \neq \emptyset$ whenever i and j are neighbors.

Remark 6. By item (b) in Theorem 4, for all $I, J \in C_{i,j}$, there exists $f \in \mathcal{F}$ such that $f^{-1}(I) \subset J$ (or $I \subset f(J)$). In other words, the $\mathcal{F}$-invariance of the sets $\mathcal{A}_i$ implies the $\mathcal{F}^{-1}$-invariance of the sets $C_{i,j}$. Hence,

$$f^{-1}\Big(\bigcup_{I \in C_{i,j}} I \Big) \subset \bigcup_{I \in C_{i,j}} I$$

for all $f \in G_{\mathcal{F}}$. Then, for each $(i, j) \in \mathcal{E}$, $i \neq j$, there exists a closed $\mathcal{F}^-$-minimal set $F \subset C_{i,j}$.

Proof of Lemma 9. Assume $d_+ = n + 2$. By Lemma 6, then

$$\#\{i \in \{1, \dots, d_+\} \colon \#V_i = 2\} \geq n.$$

Hence, there are n distinct points $i_1, \dots, i_n$, in $\{1, \dots, d_+\}$ such that $V_{i_k} = 2$ for $k = 1, \dots, n$. Since $d_+ = n + 2 > n$, we have that any n-cycle is not an $\mathcal{F}$-path. (An n-cycle is a path of the form $[j_0, j_1, \dots, j_n]$ with $j_0 = j_n$ and $j_k \neq j_\ell$ for $k, \ell = 1, \dots, n$ and $\ell \neq k$.) Therefore, we can find $i_{n+1}, j_1, \dots, j_n, j_{n+1} \in \{1, \dots, d_+\}$ with $(i_1, j_1), \dots, (i_{n+1}, j_{n+1}) \in \mathcal{E}$ such that for $k \neq \ell$,

$$i_k \neq j_k \quad \text{and} \quad \{i_k, j_k\} \neq \{i_\ell, j_\ell\}.$$

For $k = 1, \dots, n + 1$, consider the set C_{i_k,j_k} as defined above Remark 6. Let $C_k = C_{i_k,j_k}$. Note that for $k \neq \ell$, $C_k \neq C_\ell$ and so

$$\Big(\bigcup_{I \in C_k} I \Big) \cap \Big(\bigcup_{J \in C_\ell} J \Big) = \emptyset.$$

By Remark 6, for $k = 1, \dots, n + 1$, there exists a closed $\mathcal{F}^-$-minimal set $F_k \subset C_k$. Therefore, there exist at least $n + 1$ closed $\mathcal{F}^-$-minimal sets and so $d_- \geq n + 1$. □

LEMMA 10. *Let $(\mathcal{F}, \nu)$ be an RDS of circle homeomorphisms without finite orbits. Let $(\mathcal{F}^-, \nu^-)$ be the inverse RDS. Let d_+ and d_- be the numbers of ergodic stationary measures associated with $(\mathcal{F}, \nu)$ and $(\mathcal{F}^-, \nu^-)$, respectively. If $\mathcal{F} \subset \mathrm{Homeo}^+(\mathbb{S}^1)$, then $d_- \geq d_+$.*

Let us introduce a variation of the sets $C_{i,j}$. Now we will respect the order of the indices. For $i, j \in \{1, \dots, d_+\}$, $i \neq j$, let $C_{[i,j]}$ be the set of closed intervals $I = [a, b]$ with $a \in \bigcup_{J \in \mathcal{A}_i} J$ and $b \in \bigcup_{J \in \mathcal{A}_j} J$ such that (a, b) has no points of $\bigcup_{J \in \bigcup_k \mathcal{A}_k} J$. Note that $C_{[i,j]} \neq C_{[j,i]}$ (provided that one of the two is non-empty) and $C_{[i,j]}$ is a finite set. Further, $C_{[i,j]} \neq \emptyset$ whenever $[i, j]$ is an $\mathcal{F}$-path. By item (b) in Theorem 4,

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$$f^{-1}\Big(\bigcup_{I\in C_{[i,j]}} I\Big)\subset \bigcup_{I\in C_{[i,j]}} I$$

for all $f\in G_{\mathcal{F}}\subset \mathrm{Homeo}^+(\mathbb{S}^1)$. Hence, for each $\mathcal{F}$-path $[i,j]$, there exists a closed $\mathcal{F}^-$-minimal set $F\subset C_{[i,j]}$.

Proof of Lemma 10. If $d_+=1$, it is clear that $d_-\geq d_+$. Assume $d_+\geq 2$. For all $i=1,\dots,d_+$, take $j_i\in\{1,\dots,d_+\}\backslash\{i\}$ such that $[i,j_i]$ is an $\mathcal{F}$-path and so $(i,j_i)\in\mathcal{E}$. As we discussed above, we have for each $\mathcal{F}$-path $[i,j_i]$ (that is, $(i,j_i)\in\mathcal{E}$), there exists a closed $\mathcal{F}^-$-minimal set $F_i\subset C_{[i,j_i]}$. Hence, there exist at least d_+ closed $\mathcal{F}^-$-minimal sets and so $d_-\geq d_+$. □

Now we are ready to prove Theorem 3.

Proof of Theorem 3. By Lemma 9, $d_-\geq d_+-1$. Using that $(\mathcal{F}^-)^-=\mathcal{F}$, we also get $d_+\geq d_--1$. Consequently, $d_++1\geq d_-\geq d_+-1$ and, hence, $|d_+-d_-|\leq 1$.

Now, assume $\mathcal{F}\subset\mathrm{Homeo}^+(\mathbb{S}^1)$. Since $(\mathcal{F}^-)^-=\mathcal{F}$, by Lemma 10, $d_-\geq d_+$ and $d_+\geq d_-$. Therefore, $d_-=d_+$. □

Acknowledgements. This study was financed by grant 2022/10341-9 from the São Paulo Research Foundation (FAPESP) and by the Center of Excellence 'Dynamics, Mathematical Analysis and Artificial Intelligence' at Nicolaus Copernicus University in Toruń.

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https://doi.org/10.1017/etds.2026.10343 Published online by Cambridge University Press