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Electron acoustic–Langmuir solitons in a two-component electron plasma

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Abstract. We investigate the conditions under which 'high-frequency' electron acoustic–Langmuir solitons can be constructed in a plasma consisting of protons and two electron populations: one 'cold' and the other 'hot'. Conservation of total momentum can be cast as a structure equation either for the 'cold' or 'hot' electron flow speed in a stationary wave using the Bernoulli energy equations for each species. The linearized version of the governing equations gives the dispersion equation for the stationary waves of the system, from which follows the necessary – but not sufficient – conditions for the existence of soliton structures; namely that the wave speed must be less than the acoustic speed of the 'hot' electron component and greater than the low-frequency compound acoustic speed of the two electron populations. In this wave speed regime linear waves are 'evanescent', giving rise to the exponential growth or decay, which readily can give rise to non-linear effects that may balance dispersion and allow soliton formation. In general the 'hot' component must be more abundant than the 'cold' one and the wave is characterized by a compression of the 'cold' component and an expansion in the 'hot' component necessitating a potential dip. Both components are driven towards their sonic points; the 'cold' from above and the 'hot' from below. It is this transonic feature which limits the amplitude of the soliton. If the 'hot' component is not sufficiently abundant the window for soliton formation shrinks to a narrow speed regime which is quasi-transonic relative to the 'hot' electron acoustic speed, and it is shown that smooth solitons cannot be constructed. In the special case of a very cold electron population (i.e. 'highly supersonic') and the other population being very hot (i.e. 'highly subsonic') with adiabatic index 2, the structure equation simplifies and can be integrated in terms of elementary transcendental functions that provide the fully non-linear counterpart to the weakly non-linear sech^2-type solitons. In this case the limiting soliton is comprised of an infinite compression in the cold component, a weak rarefaction in the 'hot' electrons and a modest potential dip.

1. Introduction

In a plasma consisting of protons and two electron populations (one 'hot' and the other 'cold') electrostatic waves propagate in two modes. One mode is the classical high-frequency Langmuir wave, exhibiting a cut-off at the total plasma frequency

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and the other is the electron-acoustic mode, which, at low frequencies, propagates at a compound speed intermediate between the hot-acoustic speed and the cold-acoustic speed (Watanabe and Taniuti 1977). Gary and Tokar (1985), subsequently provided a kinetic treatment which demonstrates that the electron acoustic mode is relatively free of Landau damping if the temperature of the 'hot' electrons is very much greater than that of the 'cold' ones. Mace and Hellberg (2001), further investigated the domain of existence of this mode and also, recently, Hellberg et al. (2000) have modelled the 'hot' electron component with a κ-distribution in an attempt to explain laboratory observations of the properties of these waves.

Non-linear aspects of these waves, and other electrostatic waves such as the ion-acoustic and dust-acoustic waves (Rao et al. 1990), have been investigated within the framework of the Korteweg–de Vries and Zakharov–Kuznetov-type equations, particularly in connection with the construction of soliton-type solutions. This approach is analogous to that for weakly non-linear waves in shallow water, in which the independent variables are stretched and the dependent variables expanded in powers of a small parameter measuring the degree of non-linearity (see, e.g. a recent paper by Verheest et al. (2002), for a unified treatment of this method for electrostatic waves). Using the pseudo-potential approach Berthomier et al. (2000), have developed a fully non-linear theory of electron-acoustic solitons in a plasma containing an electron beam in addition to hot and cold electron populations in an analogous way to that of Zank and McKenzie (1988), for ion-acoustic solitons in the presence of a beam, and showed that the beam can give rise to electron-acoustic solitons with a potential hump in certain circumstances.

In this paper we also develop a fully non-linear theory for stationary 'high-frequency' electrostatic waves propagating in a plasma containing both a 'hot' and a 'cold' electron population within a multi-fluid framework. The starting point is total momentum conservation, in which the plasma momentum (dynamic pressures plus thermal pressures) is balanced by the electric stress, which, combined with the Bernoulli energy equations for each species, is cast as the structure equation for the stationary non-linear wave by eliminating the electric field in favour of (one of) the electron flow speeds through one of the electron equations of motion. In this essentially gas-dynamic approach the role played by the electron sonic points is manifest. It is shown that an 'electron acoustic–Langmuir soliton' can be constructed in which the 'cold' electron component is compressed and the 'hot' electrons are rarefied in a potential dip, provided that the 'hot' electrons are sufficiently hot and abundant. In the special case of absolutely cold electrons and very hot electrons (with an adiabatic index of 2) the structure equation simplifies and an exact analytic solution is found, which highlights the aforementioned properties of the wave.

2. The equations of the system

It is convenient to formulate the problem of stationary progressive waves of the form $f(x+Ut)$ in the wave frame in which the system appears steady ($\partial/\partial t = 0$) and the protons and electrons stream along the x-axis with speed U at $x = -\infty$. In this frame the mass flux $M_i = m_i n_i u_i$ of each species is conserved and equals $m_i n_{io} U$, where n_{io} is the unperturbed species density at $x = -\infty$ and the suffix i takes the labels c (cold electrons), h (hot electrons) and p (protons). The total

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momentum flux in the direction of propagation is also conserved, giving

$$U^2 \sum_{i=p,c,h} m_i n_{io} P_i(u_i) = \frac{\varepsilon_0 E^2}{2} = \frac{\varepsilon_0}{2}\left(\frac{d\phi}{dx}\right)^2, \tag{1}$$

in which $E(\phi)$ is the electric field (potential). The (normalized) particle momentum function $P_i(u_i)$ for each species is given by

$$P_i(u_i) = (u_i - 1) + \frac{1}{\gamma_i M_i^2}\left(\frac{1}{u_i^{\gamma_i}} - 1\right), \tag{2}$$

in which the speeds u_i have been normalized to the wave speed and adiabatic flow has been assumed (the pressure $p_i \propto n_i^{\gamma_i} \propto u_i^{-\gamma_i}$), so that the species Mach number M_i is

$$M_i = \frac{U^2}{c_{io}^2}, \quad c_{io}^2 = \frac{\gamma_i p_{io}}{m_i n_{io}}, \tag{3}$$

where c_{io} is the electron acoustic speed of each component.

The function $P_i(u_i)$ represents the change in the momentum flux at any point in the transition from its initial value, with the first term, $(u_i - 1)$, on the RHS of (2) representing the change in dynamic pressure, and the second term the change in the thermal pressure. Each species possesses an energy integral of the Bernoulli form

$$\varepsilon_i(u_i) \equiv \frac{1}{2}\left(u_i^2 - 1\right) + \frac{1}{(\gamma_i - 1)\, M_i^2}\left(\frac{1}{u_i^{\gamma_i - 1}} - 1\right) = \frac{e_i \phi}{m_i U^2}, \tag{4}$$

with $e_i = e$ for protons and $e_i = -e$ for electrons. Here the first term is the change in kinetic energy and the second the change in enthalpy. Equation (4) implies, of course, the overall energy relations

$$m_p \varepsilon_p(u_p) = -m_e \varepsilon_e(u_c) = -m_e \varepsilon_h(u_h). \tag{5a}$$

In a 'high-frequency' wave the 'massive' protons undergo very small changes in speed, i.e. $u_p - 1 \ll 1$, so that their momentum and energy functions may be approximated by the first term in their Taylor expansion about $u_p = 1$,

$$P_p(u_p) \doteqdot (u_p - 1)\left(1 - \frac{1}{M_p^2}\right) \doteqdot \varepsilon(u_p). \tag{6}$$

Moreover, by virtue of the energy equation (5) and the approximation (6), the flux of the proton momentum may be written as

$$U^2 m_p n_{po} P_p(u_p) = U^2 m_p n_{po} \varepsilon_p(u_p) = U^2 m_p n_{po}\left[-\varepsilon_e(u_e)\frac{m_e}{m_p}\right],$$

$$= -U^2\left[m_e n_{co} \varepsilon_c(u_c) + m_e n_{ho} \varepsilon_h(u_h)\right], \tag{7a}$$

with the last step following from charge neutrality at $x = -\infty$, where $n_{po} = n_{co} + n_{ho}$.

It is important to emphasize that although the protons are 'sluggish' in a 'high-frequency' stationary wave, their momentum flux cannot be neglected in the wave

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frame. Studies which adopt the view that the protons merely supply charge, and that their momentum flux may be neglected, are certainly seriously flawed if such an assumption is carried over into the wave frame (e.g. Mace and Hellberg 2002; Verheest et al. 2002).

Using this result, (7a), (1) may then be written in the symmetrical form

$$\frac{\overline{E}^2}{2} = \frac{1}{l_c^2}\left[P_c(u_c) - \varepsilon_c(u_c)\right] + \frac{1}{l_h^2}\left[P_h(u_h) - \varepsilon_h(u_h)\right] \equiv R_i(u_i), \quad \text{say}, \tag{8a}$$

where the length scales l_i are defined as the wave speed divided by the electron plasma frequency

$$\frac{1}{l_i} = \frac{\omega_{p_i}}{U}, \quad \omega_{p_i}^2 = \frac{e^2 n_{io}}{\varepsilon_0 m_i}, \quad i = c, h. \tag{8b}$$

If any additional heavier species are present, their momentum fluxes must also be included in the same way as we have done here for the protons. Interestingly the right-hand side of (8a) remains intact because, for the heavier species, i, say,

$$\frac{m_i}{Z_i}\varepsilon_i(u_i) = -m_e\varepsilon_e(u_c) = -m_e\varepsilon_h(u_h), \tag{5b}$$

so that (7a) generalizes to

$$\begin{aligned} U^2\sum_{i,p} m_i n_{io} P_i(u_i) &\approx U^2 \sum_{i,p} m_i n_{io}\varepsilon_i(u_i) \\ &= U^2 \sum_{i,p} m_i n_{io}\left[-\frac{m_e}{m_i} Z_i \varepsilon_e(u_e)\right] \\ &= -U^2 \sum_{i,p}\left[m_e n_{eo}\varepsilon_e(u_e) + m_e n_{ho}\varepsilon_h(u_h)\right] \end{aligned} \tag{7b}$$

by virtue of charge neutrality, $\sum_{i,p} Z_i n_{io} = n_e + n_p$, at $x = \pm\infty$. Thus, although the masses of the protons and heavier species disappear from (8a), their momentum fluxes are not negligible, being of the same order as the electron momentum fluxes, as demonstrated by (7).

The normalized electric field $\overline{E}$ may be expressed in terms of either the cold or hot electron speed from the equation of motion

$$\overline{E} = u_i\left(1 - \frac{1}{M_i^2 u_i^{\gamma_i+1}}\right)\frac{du_i}{dx}, \quad i = c \text{ or } h. \tag{9}$$

Substituting this expression for $\overline{E}$, (8a) becomes a first-order differential equation for either the cold or hot electron flow speed, u_i, where one may be eliminated in favour of the other through the energy relation $\varepsilon_c(u_c) = \varepsilon_h(u_h)$. This is the structure equation for the flow field in the stationary wave. In this gas dynamic form it is immediately apparent that the electron sonic points, where

$$u_i = \frac{1}{M_i^{2/(\gamma_i+1)}}, \tag{10}$$

corresponds to the local electron flow speed matching its acoustic speed, can play a crucial role in limiting the amplitude of the wave by choking the flow before a possible equilibrium point, given by a zero of the right-hand side of (8a), can be reached from the initial point ($u_i = 1$, the beginning of the wave).

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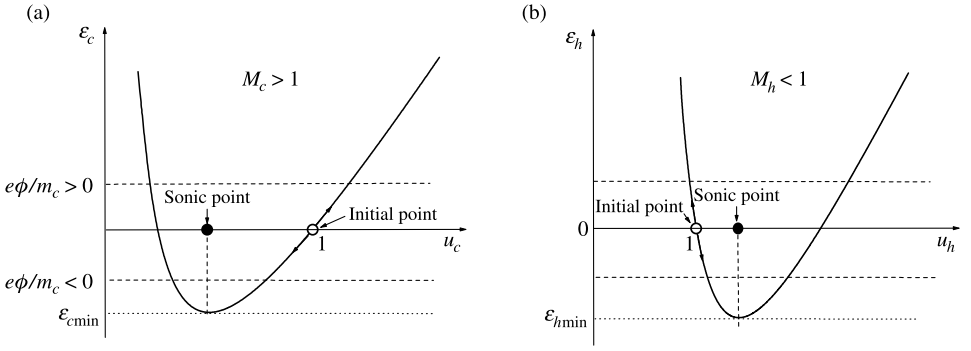


Figure 1. The electron energy functions $\varepsilon_{c,h}(u_{c,h})$ as a function of flow speed for (a) cold 'supersonic' electrons, $M_h < 1$ and (b) hot 'subsonic' electrons, $M_h < 1$. The intersections between the energy curve and the horizontal potential lines show how the electrons are decelerated/accelerated differently according to whether the initial speeds are supersonic/subsonic.

The electron sonic point (either for the cold or hot species) is also the point at which the Bernoulli-like energy function $\varepsilon_e(u_e)$ (and the momentum function $P_e(u_e)$) takes a minimum (negative) value $\varepsilon_{\min}$ The energy equations for the electrons (4) ($i = c, h$) are readily interpreted graphically in Fig. 1, which shows the energy function curves intersected by the horizontal potential line $e\phi/m_eU^2$ for (a) supersonic ($M_c > 1$, cold electrons) and (b) subsonic ($M_h < 1$, hot electrons). Thus, in a potential dip ($\phi < 0$) the hot (subsonic) electrons are accelerated ($u_h > 1$) and driven towards their sonic points, whereas the cold (supersonic) electrons are decelerated ($u_c < 1$) and driven towards their sonic point. The converse holds for a potential hill ($\phi > 0$) with both hot and cold electrons being driven away from their sonic points. As we shall see subsequently, soliton formation requires a compression in the cold species necessitating a potential dip and a rarefaction in the hot electron population.

3. The structure equation for the Langmuir soliton

Equation (8a) for the cold electron speed may be written as

$$u_c\left(1 - \frac{1}{M_c^2 u_c^{\gamma_e+1}}\right)\frac{du_c}{dx} = \pm\sqrt{2R(u_c)}, \tag{8c}$$

where we presume the energy relation (5) has been used to eliminate u_h in favour of u_c. Near the initial point $u_i = 1 + \delta_i$, $\delta_i \ll 1$, ($i = c, h$) (8c) linearizes to

$$\frac{d\delta_c}{dx} = \pm\delta_c\sqrt{\frac{1}{l_c^2\left(M_c^{-2} - 1\right)} + \frac{1}{l_h^2\left(M_h^{-2} - 1\right)}}, \tag{9a}$$

in which we have used the linearized energy relation

$$\delta_c\left(1 - M_c^{-2}\right) = \delta_h\left(1 - M_h^{-2}\right). \tag{9b}$$

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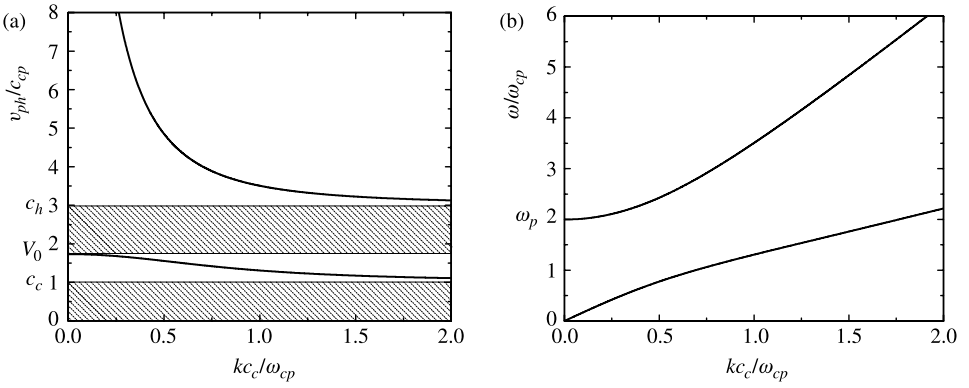


Figure 2. The dispersion equation for the electron acoustic–Langmuir waves. (a) The phase velocity shows two gaps (hatched areas) in which solitons can be formed and (b) the (ω, k) diagnostic diagram.

Hence (9a) possesses exponential ('evanescent') solutions $\delta_c \propto \exp(\pm\kappa x)$ where the exponent κ is

$$\kappa^2 = \frac{1}{l_c^2\left(M_c^{-2}-1\right)} + \frac{1}{l_h^2\left(M_h^{-2}-1\right)}, \tag{11a}$$

$$= \frac{\omega_{pc}^2}{\left(c_c^2-U^2\right)} + \frac{\omega_{ph}^2}{\left(c_h^2-U^2\right)}. \tag{11b}$$

A necessary condition for the existence of a soliton solution is $\kappa^2 > 0$ which requires the wave speed U to lie in the ranges:

$$(i) \quad V_o < U < c_h, \tag{12a}$$

$$V_o = \sqrt{\frac{c_h^2\omega_{pc}^2 + c_c^2\omega_{ph}^2}{\left(\omega_{pc}^2+\omega_{ph}^2\right)}} \tag{12b}$$

or

$$(ii) \quad U < c_c. \tag{12c}$$

Here V_o is the 'compound' phase speed of the low-frequency, small-wavenumber acoustic mode, which is more clearly evident in the propagating version of the dispersion relation obtained from (11) by replacing κ by ik (where k is the wavenumber) and U by the phase velocity $V_{ph} = \omega/k$ (where ω is the frequency). The diagnostic diagram (ω–k plot) is shown in Fig. 2(a) and the phase velocity V_{ph} versus k in Fig. 2(b). The speed regimes (12) correspond to the two 'gaps' (shown as hatched) in the phase velocity curves.

The second condition for the existence of a soliton is that the function $R_i(u_i)$ ($i = c$ or p), apart from having a double positive zero at $u_i = 1$ as guaranteed by relations (12), must also have a zero located nearer to the initial point than the sonic point, otherwise the flow becomes choked (i.e. $du_i/dx = \infty$ occurs before $du_i/dx = 0$) and a smooth soliton cannot be constructed. To see how this condition can be met, note that the momentum–energy difference functions, $(P-\varepsilon)_i$, are of the form shown in Fig. 3 for (a) $M_h < 1$, hot electrons and (b) $M_c > 1$, cold

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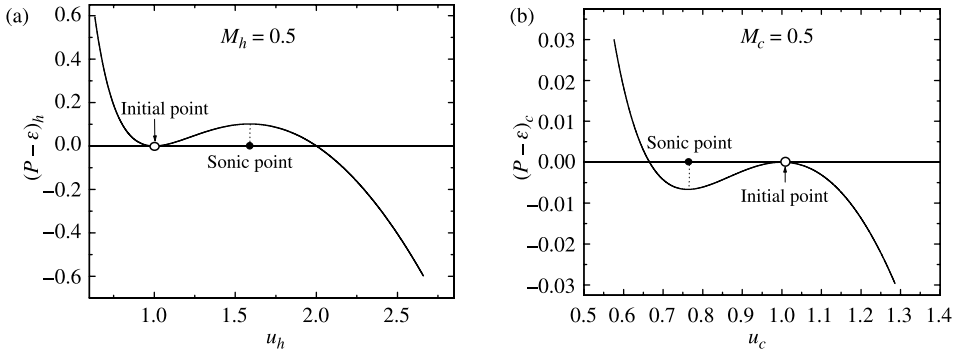


Figure 3. The $(P-\varepsilon)_i$ curves for (a) subsonic (hot) and (b) supersonic (cold) electrons.

electrons. In each case the sonic point is placed between the simple zero of $(P-\varepsilon)$ and the double zero at the initial point, since $(P-\varepsilon)$ has a turning point at the sonic point as follows from

$$\frac{\partial}{\partial u_i}(P-\varepsilon)_i = (1-u_i)\left(1-\frac{1}{M_i^2 u_i^{\gamma_i+1}}\right). \tag{13}$$

Hence for the total particle momentum function $R_c(u_c)$, given by

$$R_c(u_c) = \frac{1}{l_c^2}(P-\varepsilon)_c + \frac{1}{l_h^2}(P-\varepsilon)_h \tag{14}$$

to have a zero corresponding to a possible equilibrium point (the centre of the soliton) a judicious mixture of hot and cold electrons are required, since $(P-\varepsilon)_c < 0$, $(M_c > 1)$ and $(P-\varepsilon)_h > 0$, $(M_h < 1)$, in the relevant flow speed regime. At this stage it is possible to rule out soliton formation in the lower Mach number regime $(M_c < 1)$ (condition 12c), because if $(M_h \ll 1)$, $(P-\varepsilon)_h > 0$ for all u_h, whereas $(P-\varepsilon)_c$ only goes negative beyond its sonic point u_{cs} $(= 1/M_c^{2/(\gamma_c+1)} > 1)$. Therefore, any possible equilibrium point of the total momentum function R must be further from the initial point than the sonic point, which intervenes to choke the cold flow, thereby preventing the formation of a smooth soliton structure.

The location of the zeros of $R(u_c)$ is assisted by its derivative

$$\frac{\partial R}{\partial u_c} = \left(1-\frac{1}{M_c u_c^{\gamma_c+1}}\right)\left[\frac{1}{l_c^2}(1-u_c)+\frac{1}{l_h^2}(1-u_h)\frac{n_c}{n_h}\right], \tag{15a}$$

or, alternatively

$$\frac{\partial R}{\partial u_h} = \left(1-\frac{1}{M_h u_h^{\gamma_h+1}}\right)\left[\frac{1}{l_c^2}(1-u_c)\frac{n_h}{n_c}+\frac{1}{l_h^2}(1-u_h)\right]. \tag{15b}$$

Hence $R(u_i)$ has turning points at the sonic point $(u_i = M_i^{-2/(\gamma_i+1)})$ and the zero of the square brackets, namely at

$$u_h = \left[1+\left(\frac{n_c}{n_h}\right)_o\left(\frac{u_c-1}{u_c}\right)\right]^{-1}, \tag{16}$$

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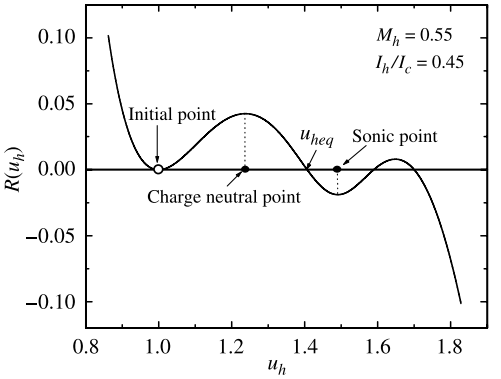


Figure 4. The total plasma momentum function $R(u_h)$ as a function of the hot electron speed. The equilibrium point (centre of the wave) must lie between the charge neutral point and the sonic point for a soliton to exist.

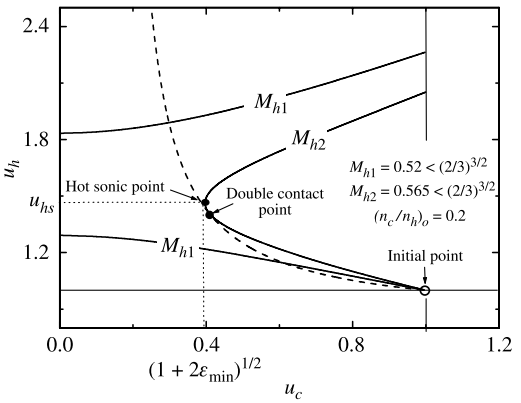


Figure 5. The energy hodograph showing the relation between hot, u_h, and cold, u_c, electron speeds for $M_h \lessgtr (2/3)^{3/2}$ and the charge neutral line (dashed curve) (see text).

which corresponds to the charge neutral point, $n_p = n_c + n_h$, in the transition where the electric field maximizes. Since the nearest possible zeros of $R(u_i)$ to the initial point must be interlaced between the charge neutral point and the sonic point, it follows that the former must be located closer to the initial point than the latter. This favourable situation, leading to soliton formation, is indicated in Fig. 4, where the zero of $R_h(u_h)$, u_{heq} say, lying between the charge neutral point and the sonic point, defines the amplitude of the soliton at its centre. The hot electrons are rarefied ($u_{heq} > 1$) and the cold electrons are compressed ($u_{ceq} < 1$), which by Bernoulli's equations (4) implies a potential dip, $\phi < 0$. The critical situation develops where $R(u_h)$ possesses a double zero at the sonic point, and this point defines the maximum possible amplitude of the soliton. The general signature of the soliton is shown in Fig. 5.

These statements can be placed on a quantitative basis by considering the slightly simpler case of absolutely cold electrons ($M_c = \infty$) and warm electrons ($M_h < 1$).

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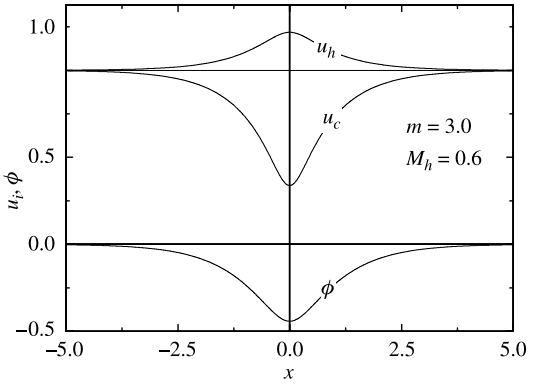


Figure 6. The characteristic signature of the electron acoustic–Langmuir soliton showing the cold component being compressed, the hot component expanded and the associated potential dip.

The energy relations (or hodograph) between u_c and u_h takes the simple form

$$u_c = \sqrt{1 + 2\varepsilon_h(u_h)}, \tag{17a}$$

$$\varepsilon_h(u_h) = (u_h - 1)\left[\frac{(u_h + 1)}{2} - \frac{1}{u_h M_h^2}\right], \tag{17b}$$

where, for algebraic convenience, we have used $\gamma_h = 2$, but this choice has no significant impact on the analysis. The minimum value of ε_h, occurring at the sonic point $u_{hs}^3 = M_h^{-2}$, may be written as

$$\varepsilon_{h\,\min} = -\frac{1}{2}(u_{hs} - 1)^2(2u_{hs} + 1). \tag{18}$$

The energy hodograph (13) is shown in Fig. 5 for $M_h < (2/3)^{3/2}$ and $M_h > (2/3)^{3/2}$ along with the charge neutral curve (16). In the former case ($M_h < (2/3)^{3/2}$, for which the Mach number follows from $2\varepsilon_{h\,\min} = -1$) the hodograph continues to $u_c = 0$, whereas in the latter the hot sonic point intervenes at $u_h = M_h^{-2/3}$, with no real solutions for $u_c < \sqrt{1 + 2\varepsilon_{h\,\min}}$. In the former case, the charge neutral point always intersects the hodograph (in $u_c < 1$, $u_h > 1$) and $\exists$ an equilibrium point, provided that the necessary soliton condition (12) is satisfied, i.e.

$$\frac{1}{1 + (n_h/n_c)_o} < M_h^2 < 1, \tag{19}$$

which for $M_h < (2/3)^{3/4}$ requires

$$\left(\frac{n_h}{n_c}\right)_0 > \left(\frac{3}{2}\right)^{3/2} - 1 \simeq 2.75. \tag{20}$$

Hence, in this 'very subsonic' case, soliton formation requires the hot electron population to be about three times more numerous than the cold population. This requirement on $(n_h/n_c)_o$ may possibly be less stringent if $M_h > (2/3)^{3/2}$ ($\simeq 0.55$) but the existence of the equilibrium point ($R(u_h) = 0$) now means that the charge neutral curve (16) must intersect the hodograph before the sonic point of the hot

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electron is reached. This gives

$$\left(\frac{n_h}{n_c}\right)_o > \frac{u_h}{(u_h-1)}\left(\frac{1}{\sqrt{1+2\varepsilon_h}}-1\right), \tag{21a}$$

$$\gtrapprox \left(\frac{1}{M_h^2}-1\right)-\frac{3}{2}(u_h-1)\left[1-\left(\frac{1}{M_h^2}-1\right)^2\right], \tag{21b}$$

$$\gtrapprox \frac{1}{2}\left(\frac{1}{M_h^2}-1\right), \quad \text{for} \quad u_h=\frac{1}{M_h^{2/3}}, \quad M_h \cong 1. \tag{21c}$$

This condition can be made precise by requiring that the charge neutral curve and the energy hodograph have a point of double contact before the hot sonic point is attained (see Fig. 5), and these requirements would yield the $((n_h/n_c)_o,\ M_h)$ parameter space in which solitons can exist when $(2/3)^{3/2} < M_h < 1$. However, as we shall see in the weakly nonlinear case, the intervention of the hot sonic point rules out soliton formation as $M_h \to 1$.

4. An exact soliton solution

In the case of very subsonic (hot) electrons, $M_h \ll 1$ (with $\gamma_h = 2$), and highly supersonic (cold) electrons, $M_c \to \infty$, the total particle momentum function R becomes

$$R = -\frac{1}{2l_c^2}(u_c-1)^2+\frac{1}{2l_h^2}\frac{(u_h-1)^2}{M_h^2u_h^2}, \tag{22}$$

which, along with the energy hodograph

$$u_h = \left[1+\frac{M_h^2}{2}(u_c^2-1)\right]^{-1}, \tag{23}$$

yields the following simple form for the structure equation (8c):

$$u_c\frac{du_c}{dx} = \pm\frac{(u_c-1)}{l_c}\sqrt{\left(\frac{n_h}{n_c}\right)_o M_h^2\left(\frac{u_c+1}{2}\right)^2-1}. \tag{24}$$

This equation integrates, in terms of elementary transcendental functions, to yield

$$\frac{x+\text{const}}{l_c} = \pm\left[\frac{2}{m}\ln\left(\frac{\zeta+\sqrt{\zeta^2-1}}{m+\sqrt{m^2-1}}\right)+\frac{1}{\sqrt{m^2-1}}\ln\left(\frac{\sqrt{\frac{\zeta-1}{\zeta+1}}-\sqrt{\frac{m^2-1}{m^2+1}}}{\sqrt{\frac{\zeta-1}{\zeta+1}}+\sqrt{\frac{m^2-1}{m^2+1}}}\right)\right], \tag{25}$$

where

$$\zeta \equiv \frac{m(u_c+1)}{2}, \tag{26a}$$

$$m^2 \equiv M_h^2\left(\frac{n_h}{n_c}\right)_o. \tag{26b}$$

The beginning of the wave ($x \to -\infty$) is at $u_c = 1$ ($\zeta = m$) and the centre at $\zeta = 1$, which yields the amplitude (clearly evident in 24)

$$u_{ceq} = \frac{2}{m}-1, \tag{27a}$$

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where for $u_{ceq} < 1$ requires,

$$1 < m < 2. \tag{27b}$$

The structure (24), and its analytic solution (25), are, in fact, very good approximations for any γ provided that the hot electrons are sufficiently subsonic, i.e. $M_h \ll 1$. The soliton signature is shown in Fig. 6.

The lower limit on the parameter m (which may be called the soliton number in analogy with the case of oblique solitons in a cold magnetized plasma (McKenzie and Doyle 2001) is merely the necessary soliton condition ($U > V_o$) in the very subsonic limit ($M_h^2 \ll 1$). At the upper limit on the soliton number $m = 2$, the cold species undergoes an infinite compression ($u_c = 0$), while the hot one undergoes a weak rarefaction ($u_h = 1/(1 - M_h^2/2)$) and the associated potential dip is a mere $2kT_h/(n_h/n_c)_o$, since $(n_h/n_c)_o$ is larger than unity by an amount required by $M_h^2(n_h/n_c)_o = 2$ for $M_h^2 \ll 1$.

It is remarkable that the structure (24) and its analytic solution (25), also describes the ion-acoustic soliton (McKenzie 2002) and the dust-acoustic soliton (McKenzie 2002; McKenzie and Doyle 2002), in which the Langmuir soliton number m is replaced in the former case by the ion-acoustic Mach number ($M_{cp} = \sqrt{\gamma_c kT_c/m_p}/U$) and the length scale $l_p = U/\omega_p$, while in the latter it is replaced by the collective dust-acoustic Mach number and the length scale $l = U/\omega_{pd}$. A case of the same mathematical creature under different physical disguises!

5. Weakly non-linear transitions

Our general discussion of the structure equation (8c) in Sec. 3 and the existence of the exact solution in Sec. 4, shows that if the hot electron Mach number is sufficiently small and the hot electrons are sufficiently abundant an electron-acoustic soliton can be constructed. Let us now examine the weakly non-linear case which requires M_h near, but less than, unity and expand the total momentum function $R(u_h)$ in powers of $\delta_h = u_h - 1 \ll 1$, to yield

$$R(u_h) = \frac{\delta_h^2}{l_c^2}\frac{1}{2}\left(\frac{1}{M_h^2} - 1\right)^2 (m-1)\left(1 - \frac{\delta_h}{\delta_{\rm eq}}\right), \tag{28a}$$

where

$$\frac{1}{\delta_{\rm eq}} = \frac{1}{\left(1/M_h^2 - 1\right)M_h^2}\left[\frac{2}{3}(\gamma_h + 1)m + \frac{1}{M_h^2} - (\gamma_h + 2)\right]\frac{1}{(m-1)}, \tag{28b}$$

$$m \equiv \left(\frac{n_h}{n_c}\right)_o \left(\frac{1}{M_h^2} - 1\right)^{-1}. \tag{28c}$$

The momentum function $R(u_h)$ must be balanced by the normalized electric stress, which in this limit approximates to

$$\frac{1}{2}\left[\left(\frac{1}{M_h^2} - 1\right)(1 + \delta_h)\left(1 - \frac{\delta_h}{\delta_s}\right)\left(\frac{d\delta_h}{dx}\right)\right]^2, \tag{29a}$$

where

$$\frac{1}{\delta_s} = \frac{(\gamma_h + 1)}{M_h^2\left(1/M_h^2 - 1\right)}, \tag{29b}$$

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to give the weakly non-linear version of the differential equation (8c) for the structure. In the first instance the necessary soliton condition $m > 1$ must be satisfied. Secondly, the equilibrium point $\delta_h = \delta_{\rm eq}$ must be reached before the sonic point $\delta_h = \delta_s$, requiring

$$\frac{1}{\delta_{\rm eq}} > \frac{1}{\delta_s} \tag{30a}$$

or

$$m < \frac{3}{(\gamma_h + 1)} \left(\frac{1}{M_h^2} - 1 \right), \tag{30b}$$

which violates the necessary condition $m > 1$. Therefore, in this weakly non-linear case the hot sonic point intervenes between the initial point and the possible equilibrium point, thereby choking the flow and preventing the formation of a smooth soliton. This result illustrates the general point made earlier concerning the importance of the sonic point either in preventing smooth soliton formation or in limiting its amplitude.

6. Summary and conclusions

We have developed a fully non-linear theory of the electron acoustic–Langmuir soliton, which, by emphasizing the gas dynamical aspect of the electron flows in the wave, complements the more traditional approach using the pseudo-potential method (e.g. Berthomier et al. 2000), and highlights the crucial roles played by the electron sonic points.

The speed of the soliton must, of necessity, lie in the range $V_o < V < c_h$, which guarantees evanescent-type behaviour at each end. The second requirement for the existence of the soliton comes from an analysis of the total particle momentum function $R_i(u_i)$. This shows that any possible equilibrium point (the centre of the wave) must lie between the charge neutral point and the sonic point (see Fig. 4). This can only occur if the cold electrons are compressed and the hot electrons rarefied, and, therefore, the potential must form a dip. These fundamental properties are highlighted by the exact solution in Sec. 4 and the characteristic signature shown in Fig. 5.

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