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Packing Trees into the Complete Graph

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Let $c \leqslant 0.076122$ and $T_1, T_2, \ldots, T_n$ be a sequence of trees such that $|V(T_i)| \leqslant i - c(i-1)$. We prove that, if for each $1 \leqslant i \leqslant n$ there exists a vertex $x_i \in V(T_i)$ such that $T_i - x_i$ has at least $(1-2c)(i-1)$ isolated vertices, then $T_1, \ldots, T_n$ can be packed into K_n. We also prove that if T is a tree of order $n+1-c'n$, $c' \leqslant \frac{1}{25}(37 - 8\sqrt{21}) \approx 0.0135748$, such that there exists a vertex $x \in V(T)$ and $T - x$ has at least $n(1-2c')$ isolated vertices, then $2n+1$ copies of T may be packed into K_{2n+1}.

In 1976 Gyárfás and Lehel [5] conjectured that every sequence of trees $T_1, T_2, \ldots, T_n$ such that $|V(T_i)| = i$, for all $1 \leqslant i \leqslant n$, can be packed into K_n. This conjecture is usually referred to as the Tree Packing Conjecture. Since that time several partial solutions to this conjecture have been obtained. The interested reader is referred to [4] and the references there for surveys of these results. There are, though, a few results which should now be mentioned. First, Bollobás [3] has shown that any sequence of trees $T_1, \ldots, T_s$, $s \leqslant n/\sqrt{2}$ and $|V(T_i)| = i$, can be packed into K_n. Roditty has also shown [8] that the Tree Packing Conjecture is true provided all of the trees in the sequence except three are stars. For a tree T and a vertex $v \in V(T)$, let $n_1(T, v)$ be the number of vertices of $T - v$ of degree at least 1. Note that $T - v$ is a forest on $n_1(T, v)$ vertices that does not contain any isolated vertices, together with a collection of isolated vertices. In [4], the author proved that, if T_i contains a vertex x_i such that $n_1(T_i, x_i) \leqslant \sqrt{6(i-1)}/4$, then $T_1, \ldots, T_n$ can be packed into K_n. In this paper, we prove that if $c \leqslant 0.076122$ and $T_1, \ldots, T_n$ is a sequence of trees such that for some $x_i \in V(T_i)$, $n_1(T_i, x_i) \leqslant c(i-1)$ and $|V(T_i)| \leqslant i - n_1(T_i, x_i)$, then $T_1, \ldots, T_n$ can be packed into K_n. The method by which this result is proved is somewhat general, and using similar techniques we also prove that if $c' \leqslant \frac{1}{25}(37 - 8\sqrt{21}) \approx 0.0135748$ and $T_1, \ldots, T_{2n+1}$ is a sequence of trees such that, for some $x_i \in V(T_i)$, $n_1(T_i, x_i) \leqslant c'n$ and $|V(T_i)| \leqslant i - n_1(T_i, x_i)$, then $T_1, \ldots, T_{2n+1}$ may be packed into K_{2n+1}. This last result is related to a conjecture of Häggvist [6, Conjecture 2.17], stating that any list of k trees of order $\ell + 1$ can be packed into any 2ℓ-regular graph of order k. Finally, an immediate

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consequence of the preceding results is that, if T is any tree of order $n+1-c'n$ with $n_1(T,x) \leqslant c'n$, then $2n+1$ copies of T can be packed into K_{2n+1}. This corollary is related to Ringel's conjecture [7], which states that, if T is a tree of order $n+1$, then $2n+1$ copies of T may be packed into K_{2n+1}.

Notation is standard. For terms not defined in this paper, see [2]. Let $G_1,\ldots,G_\ell,G$ be graphs. We say that $G_1,\ldots,G_\ell$ *can be packed into* G if there exist inclusions $V(G_i) \subseteq V(G)$, $1 \leqslant i \leqslant \ell$, such that, if $e \in E(G_i)$, then $e \in E(G)$ and $e \notin \cup_{j\neq i}E(G_j)$. The inclusions are said to be a *packing* P. We denote by $P(G_i)$ the subgraph of G that is isomorphic to G_i given by the inclusion. Note that not every edge of G need be an edge of any G_i. We commonly abuse notation by saying an edge $e \in P$ if $e \in E(G_i)$ for some i and, if no confusion will arise, will say $H \subseteq G$ if G contains a subgraph isomorphic to H. For a directed graph D, we denote the in-degree of a vertex k in G by $\deg_D^-(k)$. That is, $\deg_D^-(k) = |\{\vec{xk} : x \in V(D) \text{ and } \vec{xk} \in E(D)\}|$.

1. Tools

As in [4], our approach is motivated by the following simple observation. Suppose that a sequence $T_1,\ldots,T_n$ can be packed into $K_{n+1}-v = K_n$. Let $xy \in E(T_i)$ such that $\deg_{T_i}(y) = 1$. Let $T_i' = (T_i - \{xy\}) \cup \{xv\}$. Then $T_1,\ldots,T_{i-1},T_i',T_{i+1},\ldots,T_n$ can be packed into K_{n+1}, and the edge xy is not an edge used in the packing. We begin with a lemma that will allow us to extend a packing of trees $T_1,\ldots,T_n$ in K_t to a packing of trees $T_1,\ldots,T_{n+1}$ into K_t under appropriate circumstances. Before turning to this result, it will be convenient to develop some of the notation that will be used.

Let $T_1,\ldots,T_{n+r}$ be a sequence of trees that can be packed into K_t, with T_{n+1} a star, and fix such a packing P. Let $x_i \in V(T_i)$, $1 \leqslant i \leqslant n+r$, and $F_i = \{\vec{x_ij} : x_ij \in E(T_i), \deg_{T_i}(j) = 1\}$ if $|V(T_i)| \neq 2$ and $F_i = \{\vec{x_ij}, \vec{jx_i} : x_ij \in E(T_i)\}$ if $|V(T_i)| = 2$. Let $v \in V(T_{n+1})$ such that $n_1(T_{n+1},v) = 0$, and $H = V(T_{n+1}-v)$. We define a digraph D by $V(D) = H$ and

$$E(D) = \{\vec{xy} : x,y \in H \text{ and } \vec{xy} \in F_i \text{ for some } i, 1 \leqslant i \leqslant n+r \text{ with } v \notin V(T_i)\}.$$

Thus $E(D)$ is the set of the directed edges, both of whose endpoints are in $V(D)$, that are contained in some F_i where $v \notin V(T_i)$. Note that no edge of F_{n+1} is contained in D and, if $V(T_i) = \{x,y\}$ and $x,y \in H$, $x \neq v \neq y$, then $\vec{xy} \in E(D)$ and $\vec{yx} \in E(D)$.

Lemma 1.1. *Let $T_1,T_2,\ldots,T_n,T_{n+1},\ldots,T_{n+r}$ be a sequence of trees that can be packed into K_t and fix such a packing P. Assume that T_{n+1} is a star and, for every other $1 \leqslant i \leqslant n+r$, there exists $x_i \in V(T_i)$ such that $n_1(T_i,x_i) \leqslant \ell$. Let T_{n+1}' be a tree and $v \in V(T_{n+1}')$ a vertex such that $n_1(T_{n+1}',v) \leqslant \ell$ and $|V(T_{n+1}')| \leqslant |V(T_{n+1})| - n_1(T_{n+1}',v)$. If there exists $K \subseteq V(D)$ such that, for every $k \in K$, $\deg_D^-(k) \geqslant 3\ell$ and $\hat{K}$, the underlying simple graph of $D[K]$, contains a subgraph isomorphic to the forest obtained from $T_{n+1}'-v$ by deleting isolated vertices, then there exists a packing P' of $T_1,T_2,\ldots,T_n,T_{n+1}',T_{n+2}\ldots,T_{n+r}$ into K_t.*

Proof. Note that, if $\ell = 0$, then T_{n+1}' is isomorphic to (a subgraph of) T_{n+1} and the packing P satisfies the conclusion of this lemma. We thus assume that $\ell \geqslant 1$. Let T be a subdigraph of $D[K]$ such that the underlying simple graph of T is isomorphic to the

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forest obtained from $T'_{n+1} - v$ by removing the isolated vertices. We may and do insist that if $\vec{xy} \in E(D[K])$ and $\vec{yx} \in E(D[K])$, then at most one of these edges is contained in $E(T)$. As $n_1(T'_{n+1}, v) \leqslant \ell$, we have that $|V(T)| \leqslant \ell$ and $|E(T)| \leqslant \ell - 1$. For each edge $e = \vec{x_e y_e} \in E(T)$, let $1 \leqslant i_e \leqslant n+r$ such that $e \in F_{i_e}$. Thus $x_e = x_{i_e}$, unless $|V(T_{i_e})| = 2$, in which case we may, without loss of generality, change our choice of x_{i_e} so that it is x_e. Thus in either case, $x_e = x_{i_e}$. For each edge $e \in E(T)$, choose $w_e \in V(D)$ satisfying the following conditions:

(a) $\vec{w_e x_e} \in E(D)$,
(b) $w_e \notin V(T)$,
(c) $w_e \notin V(T_{i_e})$,
(d) the w_e are all distinct.

Note that, as $\deg^-_D(x) \geqslant 3\ell$ for every $x \in K$, there are at least 3ℓ choices for w_e that satisfy condition (a). As $|V(T)| \leqslant \ell$, there are at least 2ℓ choices for w_e that satisfy conditions (a) and (b). If $\vec{w_e x_e} \in E(D)$, $w_e \in V(T_{i_e})$, and $|V(T_{i_e})| > 2$, then w_e is not an isolated vertex of $T_{i_e} - x_e$ and, as $n_1(T_i, x_i) \leqslant \ell$ for all $1 \leqslant i \leqslant n + r$, there are at most ℓ such vertices. If $\vec{w_e x_e} \in E(D)$, $w_e \in V(T_{i_e})$, and $|V(T_{i_e})| = 2$, then there is at most one such vertex. As $\ell \geqslant 1$ we have that, in either case, there are at most ℓ vertices w_e such that $\vec{w_e x_e} \in E(D)$ and $w_e \in V(T_{i_e})$. Thus there are at least ℓ vertices satisfying conditions (a), (b), and (c) for each edge $e \in E(T)$ and, as $|E(T)| \leqslant \ell - 1$, for each edge $e \in T$ there exists $w_e \in V(D)$ satisfying conditions (a)–(d).

As $\vec{w_e x_e} \in E(D)$, then $\vec{w_e x_e} \in F_{j_e}$, where $w_e = x_{j_e}$. As the w_e are distinct and j_e determines w_e, all of the j_e are also distinct. We now wish to modify our packing P of $T_1, \ldots, T_{n+r}$ in K_r in the following fashion. For each w_e, we remove the edge $w_e x_e$ from T_{j_e} and replace it with $w_e v$. (Recall that $v \in V(T_{n+1})$ such that $n_1(T_{n+1}, v) = 0$ so that $v = x_{n+1}$.) We denote the resulting graph by T'_{j_e}. As $E(D)$ is contained in the union of the edges of F_i with both endpoints in $V(D)$ and $v \notin V(T_i)$, we have that $v \notin V(T_{j_e})$ so that T'_{j_e} is a tree and is thus isomorphic to T_{j_e}. For $1 \leqslant i \leqslant n + r$, let $U_i = T_i$ if $i \neq j_e$ for any $e \in T$ and $i \neq n + 1$, $U_{j_e} = T'_{j_e}$ and $U_{n+1} = T_{n+1} - \{w_e : e \in E(T)\}$. We then have a packing P_1 of $U_1, \ldots, U_{n+r}$ into K_t and none of the edges $w_e x_e$ are used in this packing. Note that U_{n+1} is a star of order $|V(T_{n+1})| - |E(T)|$, which is at least $|V(T'_{n+1})|$, and of course, if $i \neq n+1$, then $U_i \cong T_i$.

We now wish to modify the packing P_1 of $U_1, \ldots, U_{n+r}$ in K_t in the following way. For each tree $U_{i_e} = T_{i_e}$, we remove the edge $x_e y_e$ from tree T_{i_e}, and add the edge $w_e x_e$. We denote the resulting graph by U'_{i_e}. As $w_e \notin V(T_{i_e})$, each U_{i_e} is a tree and, of course, $U_{i_e} \cong T_{i_e}$. For $1 \leqslant i \leqslant n + r$, let $W_i = U_i$ if $i \neq i_e$ and $W_i = U'_i$ if $i = i_e$ for any $e \in E(T)$. Thus there is a packing P_2 of $W_1, \ldots, W_{n+r}$ into K_t, and no edge of the underlying simple graph of T is used in this packing. Furthermore, $v = x_{n+1}$ is adjacent in W_{n+1} to every vertex of T. Thus $\hat{T} \cup W_{n+1}$, where $\hat{T}$ is the underlying simple graph of T, contains a subgraph isomorphic to T'_{n+1}, and no edge of $\hat{T} \cup W_{n+1}$ is contained in any tree W_i, $i \neq n + 1$. However, $\hat{T} \cup W_{n+1}$ need not be a tree, as every edge $x_{n+1} w \in E(\hat{T} \cup W_{n+1})$ for every $w \in V(T)$. As there are at most ℓ such edges, deleting all such edges gives a tree of order $|V(T_{n+1})| - \ell$ isomorphic to T'_{n+1}, and no edge of this tree is an edge of any W_i, $1 \leqslant i \leqslant n + r$, $i \neq n + 1$. We thus have a packing P' of $W_1, \ldots, W_n, T'_{n+1}, W_{n+2}, \ldots, W_{n+r}$ into K_t. Since for $i \neq n + 1$, $W_i \cong T_i$, the result follows. □

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Remark. If P is the packing of $T_1,\dots,T_{n+r}$ into K_t given in the hypothesis of Lemma 1.1 and P' is the packing of $T_1,\dots,T_n,T'_{n+1},T_{n+2},\dots,T_{n+r}$ given by the conclusion of Lemma 1.1, then $|E(P(T_i))-E(P'(T_i))|\leqslant \ell$, for $i\neq n+1$. Furthermore, if $x_i\in V(T_i)$ as in Lemma 1.1, then $P(T_i)-x_i$ is isomorphic to $P'(T_i)-x_i$ for all $i\neq n+1$. Finally, observe that $V(P'(T'_{n+1}))\subseteq V(P(T_{n+1}))$.

We will need one more tool before proceeding to the main results of this paper. Our main tool for constructing forests of order $\ell+1$ in the underlying simple graph of D will be to find a subgraph of minimal degree ℓ. For this we will use the following result (see [1, p. xvii]).

Lemma 1.2. *Let ℓ be a positive integer. Suppose that H is a graph of order $n\geqslant \ell+1$. If*

$$|E(H)|\geqslant (\ell-1)\left(n-\frac{\ell}{2}\right)+1,$$

then H contains a subgraph F such that $\delta(F)\geqslant \ell$.

Proof. Inductively delete vertices of degree less or equal to $\ell-1$ from H. □

2. The main results

Theorem 2.1. *Let $c\leqslant 0.076122$ and $T_1,\dots,T_n$ be a sequence of trees and $x_i\in V(T_i)$ vertices such that $n_1(T_i,x_i)\leqslant c(i-1)$ and $|V(T_i)|\leqslant i-n_1(T_i,x_i)$ for all $1\leqslant i\leqslant n$. Then $T_1,\dots,T_n$ can be packed into K_n.*

Proof. We assume that $V(K_n)=\{1,2,\dots,n\}=[n]$. First note that we may assume, by adding extra vertices to T_i and edges between these extra vertices and x_i, that $|V(T_i)|=i-n_1(T_i,x_i)$ for all $1\leqslant i\leqslant n$. We will show by induction on n that we may pack any sequence of trees $T_1,\dots,T_n$ as above into K_n in such a way that $x_i=i$. If $i\leqslant 14$, then $n_1(T_i,x_i)\leqslant c(i-1)\leqslant 13c<1$, so that, for $1\leqslant i\leqslant 14$, each tree T_i must be a star of order at most i. Note that, if $S_1,\dots,S_n$ is a sequence of stars with $|S_i|=i$, then $S_1,\dots,S_n$ can be packed into K_n. This is easily shown by induction on n. If $n=1$, then K_1 is a vertex and S_1 is also a vertex. If $S_1,\dots,S_n$ have been packed into K_n, to form K_{n+1}, we add vertex $n+1$ and an edge from $n+1$ to every element of $[n]$. Removing the edges of K_n, we are left with a star of order $n+1$. We may thus assume that $n\geqslant 14$.

Let $T_1,\dots,T_n,T_{n+1}$ be a sequence of trees that satisfy the hypothesis. By the induction hypothesis, $T_1,\dots,T_n$ can be packed into $K_n\subset K_{n+1}$ in such a way that $x_i=i$. Let S_{n+1} be the star of order $n+1$. Then $T_1,\dots,T_n,S_{n+1}$ can be packed into K_{n+1} in such a way that, if $1\leqslant i\leqslant n$, then $x_i=i$, and if $v\in V(S_{n+1})$ such that $n_1(S_{n+1},v)=0$, then v is identified with $n+1$. Thus $H=[n]$ and $V(D)=[n]$. Clearly $K_n-\cup_{i=1}^n E(T_i)$ can be decomposed into edges which we will denote by $T_{n+2},\dots,T_{n+r}$. Thus $T_{n+2},\dots,T_{n+r}$ are trees with two vertices each containing one edge of $K_n-\cup_{i=1}^n E(T_i)$. Hence

$$\sum_{i=n+2}^{n+r}|E(T_i)|=|E(K_n)-\cup_{i=1}^n E(T_i)|\geqslant \frac{n(n-1)}{2}-\sum_{i=1}^n[i-1-n_1(T_i,x_i)]=\sum_{i=1}^n n_1(T_i,x_i).$$

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As $|V(T_i)| = i - n_1(T_i, x_i)$ we have that, for $1 \leqslant i \leqslant n$, $|F_i| = i - 1 - 2n_1(T_i, x_i)$. Of course, if $n+2 \leqslant i \leqslant n+r$, then $|F_i| = 2 = 2|E(T_i)|$. Also observe that $n+1 \notin V(T_i)$, $i \neq n+1$. Hence, if $i \neq n+1$, then $F_i \subseteq E(D)$. Hence

$$|E(D)| = \sum_{i=1}^{n} |F_i| + \sum_{i=n+2}^{n+r} |F_i| = \sum_{i=1}^{n} (i - 1 - 2n_1(T_i, x_i)) + 2\sum_{i=1}^{n} n_1(T_i, x, i) = \frac{n(n-1)}{2}.$$

To apply Lemma 1.1, we must show that there exists $K \subseteq V(D)$ such that, for every $k \in K$, $\deg_D^-(k) \geqslant 3\lfloor c(n+1-1)\rfloor = 3\lfloor cn\rfloor$ (that is, we may take the ℓ of Lemma 1.1 to be $\lfloor cn\rfloor$) and the forest obtained from $T_{n+1} - x_{n+1}$ by deleting isolated vertices is contained in the underlying simple graph $\hat{K}$ of $D[K]$, where $x_{n+1} \in V(T_{n+1})$ such that $n_1(T_{n+1}, x_{n+1}) \leqslant cn$ (so that x_{n+1} is identified with $n+1$).

Let $M = \{m \in V(D) : \deg_D^-(m) < 3\lfloor cn\rfloor\}$. Thus the in-degree of a vertex of M in the complement of D is at least $(n-1) - (3\lfloor cn\rfloor - 1) \geqslant n - 3cn$. Then

$$|M|(n - 3cn) \leqslant |E(K_n) - E(D)| \leqslant \frac{n(n-1)}{2}$$

(in the preceding inequality, K_n is viewed as a directed graph with in-degree and out-degree of every vertex equal to $n-1$), and so

$$|M| \leqslant \frac{n(n-1)}{2(n-3cn)} < \frac{n}{2(1-3c)}.$$

Let $K = V(D) - M$. Then, for every $k \in K$, we have $\deg_D^-(k) \geqslant 3\lfloor cn\rfloor$. It thus only remains to show that the underlying simple graph of $\hat{K}$ of $D[K]$ contains the forest obtained from $T_{n+1} - x_{n+1}$ by deleting isolated vertices. We will show that there exists a subgraph of $\hat{K}$ with minimal degree at least $\lfloor cn\rfloor$, which will imply that $\hat{K}$ contains every tree, and thus forest, of order $\lfloor cn\rfloor + 1$. To show this, we will apply Lemma 1.2. We thus need to show that

$$|V(\hat{K})| \geqslant n - \frac{n}{2(1-3c)} \geqslant \lfloor cn\rfloor + 1, \tag{2.1}$$

and

$$|E(\hat{K})| \geqslant (\lfloor cn\rfloor - 1)\left(|V(\hat{K})| - \frac{\lfloor cn\rfloor}{2}\right) + 1. \tag{2.2}$$

Before we verify these inequalities, write $\lfloor cn\rfloor$ as $cn - \epsilon$, where $0 \leqslant \epsilon < 1$. It is then a straightforward computation to show that

$$(cn-1)\left(|V(\hat{K})| - \frac{cn}{2}\right) + 1 \geqslant (cn - \epsilon - 1)\left(|V(\hat{K})| - \frac{cn-\epsilon}{2}\right) + 1$$

provided that $|V(\hat{K})| \geqslant cn - (1+\epsilon)/2$. Thus, if we establish (2.1), it will suffice to verify (2.2) with $\lfloor cn\rfloor$ replaced by cn in that equation.

Considering inequality (2.1), note that as $d < 0.10$ and $n \geqslant 14$, we have that $9cn < n-1$. Thus $2(n-1-3dn) > 4(n-1)/3$, and hence

$$|V(\hat{K})| \geqslant n - \frac{n(n-1)}{2(n-1-3dn)} > \frac{n}{4}.$$

As $c < 0.10$ and $n \geqslant 14$, $n/4 \geqslant cn + 1$ and (2.1) holds.

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Turning to inequality (2.2), first observe that the only edges not in the underlying simple graph of D are those edges that are in some T_i, $1 \leqslant i \leqslant n$, but not in F_i. As $|F_i| = |T_i| - n_1(T_i, x_i)$, the underlying simple graph of D will contain all but at most $\sum_{i=1}^{n} n_1(T_i, x_i) = \sum_{i=1}^{n} c(i-1) = cn(n-1)/2$ edges. Thus

$$|E(\hat{K})| \geqslant \frac{|V(\hat{K})|(|V(\hat{K})|-1)}{2} - \frac{cn(n-1)}{2}.$$

We need only show that

$$|V(\hat{K})|(|V(\hat{K})|-1) - cn(n-1) \geqslant 2(cn-1)\left(|V(\hat{K})| - \frac{cn}{2}\right) + 2 \tag{2.3}$$

whenever $|V(\hat{K})| \geqslant n - \frac{n}{2(1-3c)} = n \cdot \frac{1-6c}{2-6c}$. Equation (2.3) is equivalent to

$$(|V(\hat{K})| - cn)^2 - cn^2 \geqslant 2 - |V(\hat{K})|.$$

As $|V(\hat{K})| \geqslant n/4$ and $n \geqslant 14$, $2 - |V(\hat{K})| \leqslant 0$. It thus suffices to show that

$$(|V(\hat{K})| - cn - \sqrt{c}n)(|V(\hat{K})| - cn + \sqrt{c}n) \geqslant 0. \tag{2.4}$$

Equation (2.4) is a quadratic in $|V(\hat{K})|$ and will hold provided that $|V(\hat{K})|$ is greater than or equal to the largest root of equation (2.4). We thus need that $|V(\hat{K})| \geqslant cn + \sqrt{c}n$ whenever $|V(\hat{K})| \geqslant n \cdot \frac{1-6c}{2-6c}$. It thus suffices to show that

$$n \cdot \frac{1-6c}{2-6c} \geqslant cn + \sqrt{c}n. \tag{2.5}$$

Elementary computations will show that inequality (2.5) holds provided that

$$6c^2 + 6c^{3/2} - 8c - 2\sqrt{c} \geqslant 0. \tag{2.6}$$

Inequality (2.6) will hold provided that c is less than or equal to the smallest positive root of the left-hand side of inequality (2.6). Tedious though it be, elementary calculation (or a symbolic computation program such as *Mathematica*) will show that (2.6) holds provided that $c \leqslant 0.076122$. The result then follows by induction. □

A weak form of the above result generalizes the result of Bollobás mentioned earlier, that if $s \leqslant \sqrt{2}n/2$ and $T_1, \ldots, T_s$ is a sequence of trees such that $|V(T_i)| = i$, then $T_1, \ldots, T_s$ can be packed into K_n, in the case where the trees being packed are 'almost' stars.

Corollary 2.2. *Let $c \leqslant 0.076122$, let $s \leqslant (1-c)n$ be an integer, and $T_1, \ldots, T_s$ be a sequence of trees such that $|V(T_i)| = i$ and there exists $x_i \in V(T_i)$ such that, for $1 \leqslant i \leqslant s$, we have $n_1(T_i, x_i) \leqslant c(i-1)/(1-c)$. Then $T_1, \ldots, T_s$ can be packed into K_n.*

Proof. Let $T_1, \ldots, T_s$ satisfy the hypothesis. Let $T'_1, \ldots, T'_{n-s}$ be a sequence of trees such that $|V(T'_i)| \leqslant i - c(i-1)$, and there exist $x_i \in V(T'_i)$ such that $n_1(T'_i, x_i) \leqslant c(i-1)$, $1 \leqslant i \leqslant n-s$. For $1 \leqslant i \leqslant s$, let $T'_{n-s+i} = T_i$. Clearly, if $T'_1, \ldots, T'_n$ can be packed into K_n, then $T_1, \ldots, T_s$ can be packed into K_n. By Theorem 2.1, it suffices to show that $(n-s+i) - c(n-s+i-1) \geqslant i$ and $c(n-s+i-1) \geqslant c(i-1)/(1-c)$ for $1 \leqslant i \leqslant s$. Let $m = \lfloor (1-c)n \rfloor$ and $r = \lceil cn \rceil$. Note that $r + m = n$, $s \leqslant m$ and $n - s \geqslant r$. We will show

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that $(r+i)-c(r+i-1) \geqslant i$ and $c(r+i-1) \geqslant c(i-1)/(1-c)$ for all $1 \leqslant i \leqslant m$. As $i \leqslant m < m+1$, $i < (1-c)n+1$ so that $ci < cn(1-c)+c$. Hence

$$\begin{aligned} i < cn(1-c)+i(1-c)+c &\leqslant \lceil cn \rceil (1-c)+i(1-c)+c \\ &= (r+i)(1-c)+c \\ &= (r+i)-c(r+i-1). \end{aligned}$$

Again, as $i \leqslant m \leqslant (1-c)n$, $i/(1-c) \leqslant n$ and so

$$\frac{i}{1-c} - i = \frac{ci}{1-c} \leqslant cn \leqslant \lceil cn \rceil = r.$$

Thus $i/(1-c) \leqslant r+i$ and $ci/(1-c) \leqslant c(r+i)$. As $0 < c \leqslant 1$, $-c/(1-c) \leqslant -c$ so that $c(i-1)/(1-c) \leqslant c(r+i-1)$. □

Using techniques analogous to those Bollobás used in [3], it is a straightforward exercise to show the following.

Theorem 2.3. *Let $c \leqslant 2(2-\sqrt{3}) \approx 0.535898$. Every sequence of trees $T_0, \ldots, T_{2n}$ such that $|V(T_i)| \leqslant cn$ can be packed into K_{2n+1}.* □

As before, we may strengthen the preceding result for those trees that are 'almost' stars.

Theorem 2.4. *Let $c = \frac{1}{25}(37-8\sqrt{21}) \approx 0.0135748$. Let $T_0, \ldots, T_{2n}$ be a sequence of trees and $x_i \in V(T_i)$ be such that $n_1(T_i, x_i) \leqslant cn$ and $|V(T_i)| \leqslant n+1-n_1(T_i, x_i)$, $0 \leqslant i \leqslant 2n$. Then $T_0, \ldots, T_{2n}$ can be packed into K_{2n+1}.*

Proof. We denote $V(K_{2n+1})$ by $\{0, 1, \ldots, 2n\} = \mathbf{Z}_{2n+1}$, the integers modulo $2n+1$. We will first adopt a canonical packing of $2n+1$ stars of order $n+1$ into K_{2n+1}, that will be modified in the remainder of the proof. Let S_0 be the subgraph of K_{2n+1} with $V(S_0) = \{0, \ldots, n\}$ and $E(S_0) = \{(0,i) : 1 \leqslant i \leqslant n\}$. Clearly S_0 is a star of order $n+1$. Let $\tau : \mathbf{Z}_{2n+1} \to \mathbf{Z}_{2n+1}$ by $\tau(i) \equiv i+1 \pmod{2n+1}$. Let $S_i = \tau^i(S_0)$, $1 \leqslant i \leqslant 2n$. One can then verify that $S_0, \ldots, S_{2n}$ is a packing of $2n+1$ stars into K_{2n+1}, which we denote by P, or see [9, Theorem 7] (this is, in different terminology, just observing that a star of order $n+1$ is *graceful* and then that K_{2n+1} can be cyclically decomposed into $2n+1$ copies of a graceful tree). The result thus holds for those n for which $cn < 1$, as the trees under consideration are then stars. We may thus assume that $n \geqslant 74$. Note that P is a tight packing so that $\cup_{i=0}^{2n} E(S_i) = E(K_{2n+1})$. We will show by induction on $0 \leqslant i-1 \leqslant 2n$ that there is a packing P' of $T_0, \ldots, T_{i-1}, S_i, \ldots, S_{2n}$ into K_{2n+1} that satisfies the following conditions:

(1) if $0 \leqslant j \leqslant i-1$, then $x_j = j$, and if $i \leqslant j \leqslant 2n$, then $j \in V(P'(S_j))$ and $n_1(P'(S_j), j) = 0$,
(2) $|E(P(S_j)) \cap E(P'(S_j))| \geqslant n - 2\sqrt{cn} - cn$, $i \leqslant j \leqslant 2n$,
(3) if $e = xy \in E(P'(T_j) - j)$, then $x, y \in V(P(S_j))$, $0 \leqslant j \leqslant i-1$.

We begin with the packing P of $S_0, \ldots, S_{2n}$ of $2n+1$ stars into K_{2n+1} as above. If $i = 0$, then let $H = \{1, \ldots, n\}$. Note that $0 \notin V(S_j)$, for $1 \leqslant j \leqslant n$. By Lemma 1.1, we must show that D contains a subdigraph K such that, if $k \in K$, then $\deg_D^-(k) \geqslant 3cn$ and the forest obtained from $T_0 - x_0$ by deleting isolated vertices is contained in the underlying

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simple graph $\hat{K}$ of $K[D]$. As $0 \notin V(S_j)$, for $1 \leqslant j \leqslant n$, we have that D is a transitive tournament. That is, a tournament such that vertex i has in-degree $i-1$ (with the labelling we are using). Thus there are at most $3cn$ vertices of D with in-degree less than $3cn$. Let $K = \{v \in V(D) : \deg_D^-(v) \geqslant 3cn\}$. Then $|K| \geqslant n - 3cn$ and, as the underlying simple graph of D is the complete graph, the underlying simple graph of $K[D]$ is the complete graph. We must show that the underlying simple graph of $K[D]$ contains the forest obtained from $T_0 - x_0$ after isolated vertices are removed. We need only show that $|K| \geqslant cn$. As $|K| \geqslant n - 3cn$, we need that $n \geqslant 4cn$, which is true as $c < 1/4$. Thus by Lemma 1.1 there exists a packing P' of $T_0, S_1, \ldots, S_{2n}$ into K_{2n+1}. By the remark following Lemma 1.1, $|E(P(S_j)) \cap E(P'(S_j))| \geqslant n - cn$, for $1 \leqslant i \leqslant 2n$, and if $e = xy \in E(T_0)$, then $x, y \in V(P(S_0))$. Finally, as $n_1(P(S_j), j) = 0$ for every $j \in \mathbf{Z}_{2n}$, we have that $n_1(P'(S_j), j) = 0$, $1 \leqslant j \leqslant 2n$ and $|E(P(S_j)) \cap E(P'(S_j))| \geqslant n - cn$, $1 \leqslant i \leqslant 2n$. Thus conditions (1), (2) and (3) hold.

We now assume that $T_0, \ldots, T_{i-1}, S_i, \ldots, S_{2n}$ can be packed into K_{2n+1} in such a way that conditions (1), (2) and (3) are satisfied, with P' such a packing. This packing need not be a tight packing, so, as in the previous result, we let $T_{2n+1}, \ldots, T_{2n+r}$ denote the trees of order 2 whose union is $K_{2n+1} - ((\cup_{j=0}^{i-1} P'(T_j)) \cup (\cup_{j=i}^{2n} P'(S_j)))$. In this proof, we will not be able to exploit the fact that if $j \geqslant 2n+1$ then $F_j = \{\vec{x_j y}, \vec{y x_j} : x_j y \in E(T_j)\}$, so for this proof, we will assume that $F_j = \{\vec{x_j y} : x_j y \in E(P(T_j))$ and $\deg_{P(T_j)}(y) = 1\}$, $2n+1 \leqslant j \leqslant 2n+r$. Let $H = \{j \in V(P(S_i)) : x_i j \in E(P(S_i)) \cap E(P'(S_i))\}$. By condition (2) we have that $|H| \geqslant n - 2\sqrt{c}n - cn$. We claim that, if $j \in H$, then $i \notin V(P'(S_j))$ if $i < j \leqslant 2n$ and $i \notin V(P'(T_j))$, if $0 \leqslant j \leqslant i-1$.

Suppose that $j \in H$ and $i \in V(P'(S_j))$, $i < j \leqslant 2n$. By condition (1), i is an isolated vertex of $P'(S_j) - j$. Thus $ij \in E(P'(S_j))$ and, by the definition of H, $ij \in E(P'(S_i))$ as well, a contradiction. If $0 \leqslant j \leqslant i-1$, then by the immediately preceding argument we may assume that i is not an isolated vertex of $P'(T_j - j)$. Thus there exists $e \in E(P'(T_j)) - j$ such that $e = iy$ for some y. By condition (3), $i \in V(P(S_j))$, and so $i \in \{j+k \pmod{2n+1} : 0 \leqslant k \leqslant n\}$. Also, as $j \in H$, we have that $j \in V(P(S_i))$, so that $j \in \{i+k \pmod{2n+1} : 0 \leqslant k \leqslant n\}$. Thus $i, j \in \{j+k \pmod{2n+1} : 0 \leqslant k \leqslant n\} \cap \{i+k \pmod{2n+1} : 0 \leqslant k \leqslant n\}$. The only possible way for this to occur is if $i = j$, a contradiction, establishing the claim.

Define a digraph D by $V(D) = \{h \in H : |E(P(S_h)) \cap E(P'(S_h))| \geqslant n - 2\sqrt{c}n\}$, with $E(D)$ those edges e of $\cup_{j=0}^{2n+r} F_j$ such that both endpoints of e are in $V(D)$. Let L be the directed graph defined by $V(L) = \mathbf{Z}_{2n+1}$ and $E(L) = \cup_{j=0}^{2n} \{\vec{jk} : k$ is adjacent to j in $P(S_j)\}$. Then $L[V(D)]$ is a transitive tournament. By the proof of Lemma 1.1, for each edge $e \in \cup_{j=0}^{i-1}(P'(T_j) - j)$, there are at most two edges of L that are not contained in $\cup_{j=0}^{2n} F_j$. (One edge is contained in $P'(T_j) - j$, one edge has its direction reversed, and one edge is no longer used in the packing, so it is put back in with the correct direction as an F_ℓ, $2n+1 \leqslant \ell \leqslant 2n+r$.) Thus

$$\begin{aligned} |E(D) \cap E(L)| &\geqslant \frac{|V(D)|(|V(D)|-1)}{2} - 2\sum_{j=0}^{i-1} |E(P'(T_j)) - j)| \\ &\geqslant \frac{|V(D)|(|V(D)|-1)}{2} - 2 \cdot i \cdot cn \\ &\geqslant \frac{|V(D)|(|V(D)|-1)}{2} - 4cn^2. \end{aligned}$$

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Let $N = H - V(D)$. As, for each $h \in N$, at least $2\sqrt{cn}$ edges of $P(S_i)$ are not edges of $P'(S_i)$, we have that

$$|N| \cdot 2\sqrt{cn} \leqslant 2\sum_{j=0}^{i-1} |E(P'(T_j) - j)| \leqslant 2 \cdot i \cdot cn \leqslant 4cn^2.$$

Hence $|N| \leqslant 2\sqrt{cn}$ and $|V(D)| \geqslant |H| - |N|$ so that $|V(D)| \geqslant n - 4\sqrt{cn} - cn$. We remark that the number $n - 2\sqrt{cn} - cn$ of condition (2) was chosen to maximize $|V(D)|$.

To apply Lemma 1.1 we must again show that there exists $K \subset V(D)$ such that $\deg_D^-(k) \geqslant 3cn$ (so that $\deg_D^-(k) \geqslant \lfloor 3cn \rfloor$) for every $k \in K$ and the underlying simple graph $\hat{K}$ of $D[K]$ contains the forest obtained from $T_i - x_i$ by deleting isolated vertices.

Note that a transitive tournament with vertex set $\mathbf{Z}_m$, labelled so that vertex i has in-degree i, has at most $\lfloor 3cn \rfloor$ vertices of in-degree less than $\lfloor 3cn \rfloor$, namely, the vertices $0, 1, \ldots, \lfloor 3cn \rfloor - 1$. If we remove edges from this transitive tournament to obtain a subdigraph with the maximum number of vertices of in-degree less than $\lfloor 3cn \rfloor$, the edges must be removed recursively from vertices with minimal in-degree greater than or equal to $\lfloor 3cn \rfloor$. It is then not difficult to see that we will be removing edges of a transitive tournament of order, say, r (these vertices would, in order, be $\lfloor 3cn \rfloor, \lfloor 3cn \rfloor + 1, \ldots, \lfloor 3cn \rfloor + r - 1$). Furthermore, by arguments above, D has a subdigraph that is a transitive tournament with at most $4cn^2$ directed edges removed. We thus want $r(r-1)/2 \leqslant 4cn^2$, so certainly $r^2 \leqslant 4cn^2$ will do and $r \leqslant 2\sqrt{cn}$. Thus D can contain at most $3cn + 2\sqrt{cn}$ vertices of in-degree less than $3cn$. Removing these vertices from D to obtain the digraph K, we have that $|V(K)| \geqslant n - 6\sqrt{cn} - 4cn$ and $\deg_D^-(k) \geqslant \lfloor 3cn \rfloor$ for every $k \in V(K)$. It thus only remains to be shown that $\hat{K}$, the underlying simple graph of K, contains a forest isomorphic to the forest obtained from $T_i - x_i$ by deleting isolated vertices, and of course, that conditions (1), (2) and (3) hold for this new packing. As before, we will show that $\hat{K}$ contains a subgraph L such that the minimum degree of L is at least cn.

To apply Lemma 1.2, we must show that

$$|V(\hat{K})| \geqslant n - 6\sqrt{cn} - 4cn \geqslant cn + 1 \tag{2.7}$$

and

$$|E(\hat{K})| \geqslant (cn - 1)\left(|V(\hat{K})| - \frac{cn}{2}\right) + 1. \tag{2.8}$$

Instead of verifying equation (2.7) directly, it will be useful later to verify the slightly stronger inequality $n - 6\sqrt{cn} - 4cn \geqslant cn + 2$. This inequality is equivalent to $n(1 - 6\sqrt{c} - 5c) \geqslant 2$. The function $f(c) = 1 - 6\sqrt{c} - 5c$ is decreasing on the interval $(-\infty, 3/5]$, so as $c < 1/64$, it suffices to show $n(1 - 6/8 - 5/64) \geqslant 2$, which easily holds as $n \geqslant 74$. To show that (2.8) holds, first note that, if $e \notin E(\hat{K})$, then $e \in E(T_j - j)$ for some $j \leqslant i - 1$ and there are at most $2cn^2$ such edges. It thus suffices to show that

$$\frac{|V(\hat{K})|(|V(\hat{K})| - 1)}{2} - 2cn^2 \geqslant (cn - 1)\left(|V(\hat{K})| - \frac{cn}{2}\right) + 1 \tag{2.9}$$

whenever $|V(\hat{K})| \geqslant n - 6\sqrt{cn} - 4cn$. Equation (2.9) is equivalent to

$$(|V(\hat{K})| - cn)^2 - 4cn^2 \geqslant cn - |V(\hat{K})| + 2. \tag{2.10}$$

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As $|V(\hat{K})| \geqslant cn + 2$, the left-hand side of equation (2.9) is at most 0, so equation (2.9) will hold provided $(|V(\hat{K}| - cn)^2 \geqslant 4cn^2$. This inequality will be true provided that $|V(\hat{K})| \geqslant cn + 2\sqrt{c}n$. We thus need only show that $|V(\hat{K})| \geqslant cn + 2\sqrt{c}n$ whenever $|V(\hat{K})| \geqslant n - 6\sqrt{c}n - 4cn$. It thus suffices to show that $n - 6\sqrt{c}n - 4cn \geqslant cn + 2\sqrt{c}n$. This inequality is equivalent to $1 - 8\sqrt{c} - 5c \geqslant 0$, which will hold provide that $c \leqslant (37 - 8\sqrt{21})/25$. Thus there exists a packing P'' of $T_0, \ldots, T_i, S_{i+1}, \ldots, S_{2n}$ into K_{2n+1}, by Lemma 1.1.

By the remark following Lemma 1.1, we have $j \in V(P''(S_j)), 0 \leqslant j \leqslant 2n$ and, for $j \geqslant i+1$, $n_1(P''(S_j), j) = 0$. As every vertex of H is a vertex of $P(S_i)$, if $e = xy \in E(P'(T_i) - i)$, then $x, y \in V(P(S_i))$. As $P''(T_j) = P'(T_j)$, $0 \leqslant j \leqslant i - 1$, we have that if $e = xy \in E(P'(T_j) - j)$ then $x, y \in V(P(S_j))$. Finally, as $V(D) = \{h \in H : |E(P(S_h)) \cap E(P'(S_h))| \geqslant n - 2\sqrt{cn}\}$, we have that $|E(P''(S_j))| \geqslant n - 2\sqrt{cn} - cn$ by the remark following Lemma 1.1 and the choice of ℓ of Lemma 1.1 to be $\lfloor cn \rfloor$. Thus conditions (1), (2) and (3) hold, and the result follows by induction. □

Clearly we also have the following result related to Ringel's conjecture.

Corollary 2.5. *Let* $c \leqslant \frac{1}{25}(37 - 8\sqrt{21}) \approx 0.0135748$ *and* T *a tree of order* $n + 1 - cn$ *such that there exists* $x \in V(T)$ *and* $n_1(T, x) \leqslant cn$. *Then* $2n + 1$ *copies of* T *can be packed into* K_{2n+1}. □

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