Math. Struct. in Comp. Science (2010), *vol.* 20, *pp.* 545–561. © Cambridge University Press 2010
doi:10.1017/S0960129510000034 First published online 27 May 2010

Coherence for monoidal monads and comonads[†]

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Received 13 July 2009; revised 3 November 2009

The goal of this paper is to prove coherence results with respect to relational graphs for monoidal monads and comonads, that is, monads and comonads in a monoidal category such that the endofunctor of the monad or comonad is a monoidal functor (this means that it preserves the monoidal structure up to a natural transformation that need not be an isomorphism). These results are proved first in the absence of symmetry in the monoidal structure, and then with this symmetry. The monoidal structure is also allowed to be given with finite products or finite coproducts. Monoidal comonads with finite products axiomatise a plausible notion of equality of deductions in a fragment of the modal logic S4.

1. Introduction

A monoidal monad is a monad in a monoidal category such that the endofunctor of the monad is a monoidal functor. (The notion of a monoidal functor originated in Eilenberg and Kelly (1966, Section II.1), and the notion of a monoidal monad in Kock (1970; 1972); for historical remarks about the monad and comonad notions, see the notes at the end of Chapter VI of Mac Lane (1998).) We say that a functor is monoidal when it preserves the monoidal structure up to a natural transformation that has as components the arrows

$$\psi_{A,B} : TA \otimes TB \to T(A \otimes B),$$

which need not be isomorphisms, and we also have the arrow $\psi_0 : I \to TI$, which coincides with the unit arrow η_I of the monad; these arrows satisfy the equations given in Section 3 below.

Our goal in this paper is to prove coherence for this and related monad notions with respect to relational graphs; that is, with respect to the category *Rel*, whose arrows are relations between finite ordinals (and not between any sets, which one might have expected from what is perhaps the more common usage of the denotation *Rel*; our category *Rel* is the skeleton of the category of relations between finite sets, but a notation like $Sk(Rel_{fin})$ would be too cumbersome). Sometimes it will be sufficient to have a subcategory of *Rel*, such as the category *Fun* whose arrows are functional relations between finite ordinals, or the simplicial category Δ, whose arrows are order-preserving functional relations between finite ordinals. Coherence states that there is a faithful functor from a freely generated monoidal monad (or a related categorial structure for which we prove coherence) into *Rel* or a subcategory of it. We obtain thereby a characterisation of the freely generated

[†] Work on this paper was supported by the Ministry of Science of Serbia (Grants 144013 and 144029).

https://doi.org/10.1017/S0960129510000034 Published online by Cambridge University Press

monoidal monad (or related categorial structure) in terms of graphs. Such coherence results give very useful procedures for deciding whether a diagram of canonical arrows commutes. (See Došen and Petrić (2004) for a general treatment of coherence in this spirit.)

Before we go on to monoidal monads, we will first consider coherence for the notions of a strong monad originating in Kock (1970; 1972) and Moggi (1991). For monoidal monads we will first prove coherence in the absence of symmetry in the monoidal structure, and then with this symmetry. We will then prove coherence for monoidal monads where the monoidal structure is cartesian (that is, with finite products), or cocartesian (that is, with finite coproducts). This makes up the first part of the paper (Sections 2–6).

A monoidal comonad is a comonad in a monoidal category such that the endofunctor of the comonad is a monoidal functor. This notion is parallel, but not dual, to the notion of a monoidal monad. A dual notion would be a comonoidal comonad, where instead of ψ and ψ_0 we have arrows oriented in the opposite direction. Recent papers on the notion of a monoidal comonad, or the dual notion of a comonoidal monad, which is also called a Hopf monad, opmonoidal monad or bimonad, are Moerdijk (2002), McCrudden (2002), Bruguières and Virelizier (2007) and Pastro and Street (2009). Willerton (2008) and Bruguières and Virelizier (2008), and the references therein, give graphical interpretations for canonical arrows of a comonoidal monad that are different from ours, and more involved; however, coherence is not proved.

The second part of the paper (Sections 7–11) is parallel to the first part; instead of monads we have comonads in the same monoidal contexts, and we prove coherence for these notions. Note that these results are not dual to those in the first part. By duality, we can obtain from the results of both parts of the paper coherence results for comonoidal monads and comonads.

In this paper we rely on the results of Došen and Petrić (2010), where basic coherence results for monoidal endofunctors are proved. Here these endofunctors become endofunctors of monads and comonads. In general, we will assume familiarity with the terminology and notation of Došen and Petrić (2010), though we will repeat some material to make our exposition more self-contained. However, the two papers are distinct and cover different subject matters, so they should not be viewed as forming a single piece of work. For example, there is more than one endofunctor in Došen and Petrić (2010), which is also reflected in the notation (E^I and E versus T); also, while Došen and Petrić (2010) makes much use of strictification, the current paper uses other techniques and only traces of strictification can be found in it.

2. Coherence for strong monads

Let a *left monoidal* endofunctor of a monoidal category $\langle \mathscr{A}, \otimes, I, a, l, r\rangle$ (in the notation of Eilenberg and Kelly (1966, Section II.1)) be a functor T from $\mathscr{A}$ to $\mathscr{A}$ such that the monoidal structure of $\mathscr{A}$ is preserved *locally* by T up to a natural transformation whose components are the arrows

$$\psi^L_{A,B} : TA \otimes B \to T(A \otimes B),$$

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which means that the following equations hold:

$$Ta_{A,B,C} \circ \psi^L_{A\otimes B,C} \circ (\psi^L_{A,B} \otimes \mathbf{1}_C) = \psi^L_{A,B\otimes C} \circ a_{TA,B,C} \qquad (\psi^L a)$$
$$Tr_A \circ \psi^L_{A,I} = r_{TA}. \qquad (\psi^L r)$$

A *right monoidal* endofunctor is defined analogously with respect to a natural transformation whose components are the arrows

$$\psi^R_{A,B} : A \otimes TB \to T(A \otimes B).$$

The equations corresponding to $(\psi^L a)$ and $(\psi^L r)$ are

$$Ta_{A,B,C} \circ \psi^R_{A\otimes B,C} = \psi^R_{A,B\otimes C} \circ (\mathbf{1}_A \otimes \psi^R_{B,C}) \circ a_{A,B,TC} \qquad (\psi^R a)$$
$$Tl_A \circ \psi^R_{I,A} = l_{TA}. \qquad (\psi^R l)$$

A *left strong* monad in a monoidal category $\mathscr{A}$ is a monad $\langle T, \eta, \mu\rangle$ (in the notation of (Mac Lane 1998), Section VI.1) in $\mathscr{A}$ such that T is a left monoidal functor, and, moreover, we have the equations

$$\psi^L_{A,B} \circ (\eta_A \otimes \mathbf{1}_B) = \eta_{A\otimes B} \qquad (\psi^L \eta)$$
$$\psi^L_{A,B} \circ (\mu_A \otimes \mathbf{1}_B) = \mu_{A\otimes B} \circ T\psi^L_{A,B} \circ \psi^L_{TA,B}. \qquad (\psi^L \mu)$$

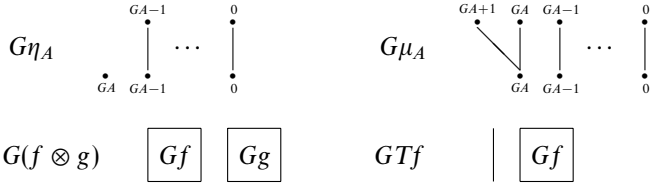
(These equations might be interpreted as saying that η is a *left monoidal* natural transformation from the identity functor, which is left monoidal, to the left monoidal functor T, while μ is a left monoidal natural transformation from the left monoidal functor TT to T.)

A *right strong* monad is defined analogously with a right monoidal functor T. The equations corresponding to $(\psi^L \eta)$ and $(\psi^L \mu)$ are

$$\psi^R_{A,B} \circ (\mathbf{1}_A \otimes \eta_B) = \eta_{A\otimes B}, \qquad (\psi^R \eta)$$
$$\psi^R_{A,B} \circ (\mathbf{1}_A \otimes \mu_B) = \mu_{A\otimes B} \circ T\psi^R_{A,B} \circ \psi^R_{A,TB}. \qquad (\psi^R \mu)$$

The notions of left strong and right strong monad are derived from Kock (1970; 1972) in Moggi (1991).

Let $\mathscr{L}^L\mathscr{S}$ be the category of the left strong monad freely generated by an arbitrary set of objects, and let Δ be the simplicial category (see Mac Lane (1998, Section VII.5)); the arrows of Δ are order-preserving functions between finite ordinals. We define a functor G from $\mathscr{L}^L\mathscr{S}$ to Δ by stipulating that GA for A an object $\mathscr{L}^L\mathscr{S}$ is the number of occurrences of T in A, while Gf for an arrow term f of $\mathscr{L}^L\mathscr{S}$ is defined inductively on the complexity of f. If f is $a_{A,B,C}$, l_A, r_A, or $\psi^L_{A,B}$, then Gf is the identity function. We then have clauses corresponding to the following pictures:



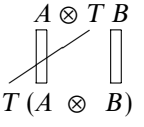
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and $G(g \circ f)$ is the composition of the functions Gf and Gg. We can prove the following proposition.

Proposition 1 ($\mathscr{L}^L\mathscr{S}$-Coherence). The functor G from $\mathscr{L}^L\mathscr{S}$ to Δ is faithful.

Proof. By naturality and the functorial equations, and the equations $(\psi^L\eta)$ and $(\psi^L\mu)$, every arrow term of $f : A \to B$ of $\mathscr{L}^L\mathscr{S}$ is equal to an arrow term $f_2 \circ f_1 : A \to B$ such that η and μ do not occur in $f_1 : A \to C$, and a, l, r and ψ^L do not occur in $f_2 : C \to B$. We can uniquely determine Gf_2 from Gf (since Gf_1 is the identity function, $Gf_2 = Gf$), and from B and Gf_2 we obtain a unique C as a possible source of f_2. By the isomorphism of Δ with the monad freely generated by a single object (see the references in Došen and Petrić (2008, Section 3)), we can conclude that f_2 is uniquely determined by Gf, while f_1 is unique by $\mathscr{L}^L$-Coherence (see Došen and Petrić (2010, Section 5)). □

Let $\mathscr{L}^R\mathscr{S}$ be the category of the right strong monad freely generated by an arbitrary set of objects. It is easy to show that this category is isomorphic to $\mathscr{L}^L\mathscr{S}$; it is a mirror image of $\mathscr{L}^L\mathscr{S}$. Hence $\mathscr{L}^R\mathscr{S}$-Coherence could be a result, exactly analogous to $\mathscr{L}^L\mathscr{S}$-Coherence, about a faithful functor from $\mathscr{L}^R\mathscr{S}$ to the simplicial category Δ. For future use, however, we need another coherence result for $\mathscr{L}^R\mathscr{S}$, which is given with respect to a functor G from $\mathscr{L}^R\mathscr{S}$ to the category *Fun* whose arrows are arbitrary functions between finite ordinals. This functor is defined as G from $\mathscr{L}^L\mathscr{S}$ to Δ except that we have the additional clause for $G\psi^R_{A,B}$ corresponding to the following picture:



In this picture we obtain a crossing if there is a T in A. Hence with this clause we abandon the category Δ, and must also consider functions that are not order-preserving. For this functor G, we can prove the following proposition.

Proposition 2 ($\mathscr{L}^R\mathscr{S}$-Coherence). The functor G from $\mathscr{L}^R\mathscr{S}$ to *Fun* is faithful.

The proof is as for $\mathscr{L}^L\mathscr{S}$-Coherence (Proposition 1), but bearing in mind the following parenthetical remark on determining Gf_2 from Gf. Now Gf_2 is not equal to Gf, but it is still uniquely determined by it. Since Gf_1 is a bijection, and Gf_2 is an order-preserving function, the latter is uniquely determined by knowing for each element of the target of Gf the number of elements of the source of Gf mapped to this element by Gf. Note that we could prove $\mathscr{L}^L\mathscr{S}$-Coherence with respect to a functor from $\mathscr{L}^L\mathscr{S}$ to *Fun* analogous to the functor G from $\mathscr{L}^R\mathscr{S}$ to *Fun*.

3. Coherence for monoidal monads

Let a *locally* monoidal endofunctor of a monoidal category $\langle \mathscr{A}, \otimes, I, a, l, r\rangle$ be a functor T that is both left monoidal and right monoidal, and, moreover, we have the equation

$$Ta_{A,B,C} \circ \psi^L_{A\otimes B,C} \circ (\psi^R_{A,B} \otimes \mathbf{1}_C) = \psi^R_{A,B\otimes C} \circ (\mathbf{1}_A \otimes \psi^L_{B,C}) \circ a_{A,TB,C}. \qquad (\psi^L\psi^R a)$$

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A *monoidal monad* in a monoidal category $\mathscr{A}$ is a monad $\langle T, \eta, \mu\rangle$ in $\mathscr{A}$ both left strong and right strong, such that T is a locally monoidal endofunctor, and, moreover, we have the equation

$$\mu_{A\otimes B} \circ T\psi^L_{A,B} \circ \psi^R_{TA,B} = \mu_{A\otimes B} \circ T\psi^R_{A,B} \circ \psi^L_{A,TB} : TA \otimes TB \to T(A \otimes B). \qquad (\psi^L\psi^R\mu)$$

An alternative definition of monoidal monad is obtained by stipulating that in a monoidal category $\mathscr{A}$ we have a monad $\langle T, \eta, \mu\rangle$ and a natural transformation whose components are the arrows

$$\psi_{A,B} : TA \otimes TB \to T(A \otimes B),$$

together with the arrow $\psi_0 : I \to TI$, which satisfy the equations

$$Ta_{A,B,C} \circ \psi_{A\otimes B,C} \circ (\psi_{A,B} \otimes \mathbf{1}_{TC}) = \psi_{A,B\otimes C} \circ (\mathbf{1}_{TA} \otimes \psi_{B,C}) \circ a_{TA,TB,TC} \qquad (\psi a)$$

$$Tl_A \circ \psi_{I,A} \circ (\psi_0 \otimes \mathbf{1}_{TA}) = l_{TA} \qquad (\psi l)$$

$$Tr_A \circ \psi_{A,I} \circ (\mathbf{1}_{TA} \otimes \psi_0) = r_{TA} \qquad (\psi r)$$

$$\psi_0 = \eta_I \qquad (\psi_0\eta)$$

$$\psi_{A,B} \circ (\eta_A \otimes \eta_B) = \eta_{A\otimes B} \qquad (\psi\eta)$$

$$\psi_{A,B} \circ (\mu_A \otimes \mu_B) = \mu_{A\otimes B} \circ T\psi_{A,B} \circ \psi_{TA,TB}. \qquad (\psi\mu)$$

The first three of these equations say that T is a monoidal functor, while the last three equations, together with the monad equation $\mu_I \circ T\eta_I = \mathbf{1}_{TI}$, ensure that η and μ are monoidal natural transformations in the sense of Eilenberg and Kelly (1966, Section II.1) – see also Mac Lane (1998, Section XI.2). Coherence for monoidal endofunctors is proved in Došen and Petrić (2010, Section 4).

It is clear that ψ_0 is superfluous as a primitive in this alternative definition. We will consider it as defined by $(\psi_0\eta)$.

With $\psi_{A,B}$ being defined as either of the two sides of the equation $(\psi^L\psi^R\mu)$, and with

$$\psi^L_{A,B} =_{df} \psi_{A,B} \circ (\mathbf{1}_{TA} \otimes \eta_B)$$
$$\psi^R_{A,B} =_{df} \psi_{A,B} \circ (\eta_A \otimes \mathbf{1}_{TB}),$$

we can show that the two definitions of a monoidal monad amount to the same notion. Both of these definitions come from Kock (1970; 1972).

Let $\mathscr{LS}$ be the category of the monoidal monad, with ψ^L and ψ^R primitive, freely generated by an arbitrary set of objects. We define the functor G from $\mathscr{LS}$ to the category *Fun* by combining what we described in the preceding section for the functors G from $\mathscr{L}^L\mathscr{S}$ to Δ and $\mathscr{L}^R\mathscr{S}$ to *Fun*. We can prove the following proposition.

Proposition 3 ($\mathscr{LS}$-Coherence). The functor G from $\mathscr{LS}$ to *Fun* is faithful.

Proof. By naturality and the functorial equations, and the equations $(\psi^L\eta)$, $(\psi^L\mu)$, $(\psi^R\eta)$ and $(\psi^R\mu)$, every arrow term of $f : A \to B$ of $\mathscr{LS}$ is equal to an arrow term $f_2 \circ f_1 : A \to B$ such that η and μ do not occur in $f_1 : A \to C$, and a, l, r, ψ^L and ψ^R do not occur in $f_2 : C \to B$. We can uniquely determine Gf_2 from Gf, and, as in the proof of $\mathscr{L}^L\mathscr{S}$-Coherence (Proposition 1) and $\mathscr{L}^R\mathscr{S}$-Coherence (Proposition 2), from B

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and Gf_2 we obtain a unique C as a possible source of f_2. As in the previous proofs, we can conclude that f_2 is uniquely determined by Gf. However, Gf_1 is not thereby uniquely determined, as the two sides of $(\psi^L\psi^R\mu)$ show. Hence f_1 is not unique either.

Let the normal form of the proof of $\mathscr{L}$-Coherence of Došen and Petrić (2010, Section 5) for f_1 be $g_m \cdots g_1$ (this normal form is given in an equivalent strictified category, where the a, l and r arrows are identity arrows). This means first that $g_m \cdots g_1$ is developed and that g_i for $i \in \{1,\ldots\}$ is either a ψ^L-factor or a ψ^R-factor. To every g_i, we assign a finite ordinal $\tau(g_i)$, which is intuitively the place where the proper T of g_i will end up in the codomain of g_m. The other details of this normal form are not essential for us here.

If in this normal form for a ψ^R-factor g_i and a ψ^L-factor g_{i+1} we have $\tau(g_{i+1}) = \tau(g_i) + 1$ and the function Gf_2 has the same value when it is applied to $\tau(g_i)$ and $\tau(g_{i+1})$, then we rely essentially on the equation $(\psi^L\psi^R\mu)$ to permute g_i with g_{i+1}. Proceeding in this manner, we obtain an arrow term f_1' such that the permutation Gf_1' has the least possible number of inversions. Then, by $\mathscr{L}$-Coherence, f_1' is unique. □

4. Coherence for symmetric monoidal monads

A *locally linear* endofunctor is a locally monoidal endofunctor T in a symmetric monoidal category that satisfies the equation

$$Tc_{A,B} \circ \psi^L_{A,B} = \psi^R_{B,A} \circ c_{TA,B}. \qquad (\psi^L\psi^R c)$$

From this equation, which says that T preserves c locally up to ψ^L and ψ^R, we immediately obtain a definition of ψ^R in terms of ψ^L, and *vice versa*.

A *symmetric* monoidal monad is a monoidal monad in a symmetric monoidal category whose endofunctor is a locally linear monoidal endofunctor. In the language of monoidal monads with ψ primitive, the equation $(\psi^L\psi^R c)$ of symmetric monoidal monads is replaced by the equation

$$Tc_{A,B} \circ \psi_{A,B} = \psi_{B,A} \circ c_{TA,TB} \qquad (\psi c)$$

(see Došen and Petrić (2010, Section 6)).

Let $\mathscr{L}_c\mathscr{S}$ be the category of the symmetric monoidal monad, with ψ^L and ψ^R primitive, freely generated by an arbitrary set of objects. We define the functor G from $\mathscr{L}_c\mathscr{S}$ to the category *Fun* by first stipulating that GA is the number of occurrences of generating objects, that is, propositional letters, in A plus the number of occurrences of T in A. Up to this point we have taken GA to be just the number of occurrences of T in A, though we could just as well have counted occurrences of generating objects, but that would have been superfluous. The remainder of the definition of G is the same as the definition of the functor G from $\mathscr{L}\mathscr{S}$ to *Fun*, with the additional standard clause for $Gc_{A,B}$ corresponding to the picture



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We can prove the following proposition.

Proposition 4 ($\mathscr{L}_c\mathscr{S}$-Coherence). The functor G from $\mathscr{L}_c\mathscr{S}$ to *Fun* is faithful.

Proof. The proof follows along the same lines as the proof of $\mathscr{L}\mathscr{S}$-Coherence (Proposition 3), so we will only give a brief sketch. Every arrow term of $f : A \to B$ of $\mathscr{L}_c\mathscr{S}$ is equal to an arrow term $f_2 \circ f_1 : A \to B$ such that η and μ do not occur in $f_1 : A \to C$, and c, a, l, r, ψ^L and ψ^R do not occur in $f_2 : C \to B$. As in the proof of Proposition 3, f_2 is uniquely determined by Gf, but f_1 is not unique.

Let the normal form of the proof of $\mathscr{L}_c$-Coherence of Došen and Petrić (2010, Section 6) for f_1 be $g_m \cdots g_1$ (this normal form is given in an equivalent strictified category where the a, l and r arrows are identity arrows). The details of this normal form are not essential for us here. We mention only that we have Ψ-factors, which are ψ^L-factors or ψ^R-factors sometimes preceded by a c-factor, and that we have a function τ analogous to what we had in the proof of Proposition 3. For the remainder of the proof we proceed as in the proof of Proposition 3, but using $\mathscr{L}_c$-Coherence. Note that only Ψ-factors g_i and g_{i+1} such that $\tau(g_{i+1}) = \tau(g_i)+1$ may be responsible for useless inversions; c-factors cannot be responsible for that. □

5. Coherence for cartesian monoidal monads

A *cartesian* monoidal monad (which should not be confused with the cartesian monads of Leinster (2003, Section 4.1)) is a symmetric monoidal monad in a cartesian category (by which we mean a monoidal category whose monoidal structure is given by finite products), which, moreover, satisfies the equation

$$T\Delta_A = \psi_{A,A} \circ \Delta_{TA}, \tag{$\psi\Delta$}$$

where $\Delta_A : A \to A \otimes A$ is a component of the diagonal natural transformation of the cartesian structure. The equation ($\psi\Delta$) says intuitively that T preserves Δ up to ψ. In the terminology of Došen and Petrić (2010, Section 7), T is a conjunctive relevant endofunctor.

Note that in the definition of cartesian monoidal monad we do not assume the equation

$$T¡_A = \eta_I \circ ¡_{TA}, \tag{$\psi¡$}$$

where $¡_A : A \to I$ is the unique arrow from A to the terminal object (empty product) I. The equation ($\psi¡$) says intuitively that T preserves ¡ up to ψ, but we cannot assume that equation if we are guided by coherence, as the following pictures show:



(here $G¡_B$ is, of course, the empty relation, as in $G\eta_I$).

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We will now deal with coherence for the notion of a cartesian monoidal monad. Let $\mathscr{CS}$ be the category of the cartesian monoidal monad freely generated by an arbitrary set of objects. We define the functor G from $\mathscr{CS}$ to the category *Rel*, whose arrows are arbitrary relations between finite ordinals, by adding to the definition of G from $\mathscr{L}_c\mathscr{S}$ to *Fun* the clause for $G\Delta_A$ that corresponds to the picture



and the clause that says that $G¡_A$ is the empty relation between GA and Ø (see Došen and Petrić (2010, Sections 7 and 8)).

As an auxiliary result for the proof of $\mathscr{CS}$-Coherence, we first establish a lemma for which we need the notions of a diversified object and scope presented in Došen and Petrić (2010, Section 2). An object that is a propositional formula is diversified when every generator (which means generating object, that is, propositional letter, or generating functor) occurs in it at most once; for E^iC a subformula of D, the scope in D of the outermost occurrence of E^i in E^iC is the set of all the generators in C. The category $\mathscr{L}_c$ is the free symmetric monoidal category with a family of locally linear endofunctors (see the beginning of the preceding section and Došen and Petrić (2010, Section 6). Here is our auxiliary lemma.

Lemma 1 ($\mathscr{L}_c$-Theoremhood Lemma). For A and B diversified objects, there is an arrow $f : A \to B$ of $\mathscr{L}_c$ if and only if the generators of A and B coincide, and for every generating functor E^i of A the scope of E^i in A is a subset of the scope of E^i in B.

Proof. We proceed as in the proof of the $\mathscr{M}_c$-Theoremhood Lemma of Došen and Petrić (2010, Section 7, Lemma 1), except for the assumption about the form of A in the induction step, which is now

$$D_1 \otimes A_1 \otimes \cdots \otimes D_{i-1} \otimes A_{i-1} \otimes D_i \otimes E^1A_i \otimes D_{i+1} \otimes A_{i+1} \otimes \cdots \otimes D_n \otimes A_n \otimes D_{n+1},$$

with the union of the generators of $A_1, \ldots, A_n$ making the scope of E^1 in B. □

Let $\mathscr{L}_c\mu$ be the category defined like $\mathscr{L}_c$, with ψ^l and ψ^R primitive, except that we have in addition the primitive arrow terms $\mu^i_A : E^iE^iA \to E^iA$ for which we assume the naturality and associativity equations of monads, and the equations $(\psi^L\mu)$ and $(\psi^L\psi^R\mu)$ with μ and T replaced by μ^i and E^i, respectively. There are no essential differences between this category and $\mathscr{L}_c\mathscr{S}$, in which η is absent. From the proof of $\mathscr{L}_c\mathscr{S}$-Coherence (Proposition4), we may easily infer $\mathscr{L}_c\mu$-*Coherence* in the following form:

For all arrow terms $f, g : A \to B$ of $\mathscr{L}_c\mu$ with B diversified we have $f = g$ in $\mathscr{L}_c\mu$.

(This is because from '$f, g : A \to B$ of $\mathscr{L}_c\mu$ with B diversified' we can infer $Gf = Gg$.) We define ψ^i in $\mathscr{L}_c\mu$ in the same way as we did in Section 3.

We have introduced the category $\mathscr{L}_c\mu$ to formulate the following lemma without complications involving graphs.

Lemma 2 ($\mathscr{L}_c\mu$-Theoremhood Lemma). For A diversified on generating objects and B diversified, there is an arrow $f : A \to B$ of $\mathscr{L}_c$ if and only if the generators of A and

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B coincide, and for every generating functor E^i of B, the union of the scopes of the occurrences of E^i in A is a subset of the union of the scope of E^i in B with the set $\{E^i\}$.

Proof. The union with the set $\{E^i\}$ is mentioned above because an occurrence of E^i may be in the scope of another occurrence of E^i in A. Such an occurrence of E^i is said to be *nested*.

To begin, we proceed for this proof as in the proof of the $\mathscr{M}_c$-Theoremhood Lemma of Došen and Petrić (2010, Section 7, Lemma 1) up until the assumption about the form of A in the induction step of the main induction. This form is now like the form mentioned in the proof of the $\mathscr{L}_c$-Theoremhood Lemma (Lemma 1), except that some of the A_j for $j \neq i$ may be replaced by E^1A_j.

Then we have an auxiliary induction on the number m of nested occurrences of E^1 in A in order to prove that there is an arrow f' of $\mathscr{L}_c\mu$ from A to the formula A' obtained from A by deleting all the nested occurrences of E^1. In the basis of this auxiliary induction, when $m = 0$, we have an identity arrow. In the induction step of this auxiliary induction, since E^1 is not in the scope of any E in B, we have a subformula of A like $E^1(C_1 \otimes E^1C_2 \otimes C_3)$. Let A'' be obtained from A by replacing this subformula with $E^1(C_1 \otimes C_2 \otimes C_3)$. It is clear that we have an arrow $g : A \to A''$ of $\mathscr{L}_c\mu$, and by the induction hypothesis of the auxiliary induction, we have a desired arrow $f'' : A'' \to A'$. So we have a desired arrow $f' : A \to A'$.

Then, as in the $\mathscr{M}_c$-Theoremhood and $\mathscr{L}_c$-Theoremhood Lemmas, we have a desired arrow from A' to

$$E^1(A_1 \otimes \cdots \otimes A_n) \otimes D_1 \otimes \cdots \otimes D_{n+1},$$

and we proceed as before for the remainder of the proof. □

We can then prove the following lemma, which is analogous to Došen and Petrić (2010, Section 7, Lemma 2). In the formulation of this technical lemma, we use some notation and terminology which we will not need later. Let us just say that we work in an equivalent strictified category where the a, l and r arrows are identity arrows, and where H the functor from the strictified category to the unstrictified category, with the help of which it is established that these are equivalent categories. We have that B is a part (proper or not) of an object of $A[B]$, and $A[B']$ is obtained from this object by replacing B by B'. (We replace a single part B by a single B'.) For $f : B \to B'$, let $A[f] : A[B] \to A[B']$ be constructed out of f with identity arrows, $\otimes$ and E in the obvious way.

Lemma 3. For the arrow term

$$f : A[EA_1 \otimes EA_2] \to B$$

of $\mathscr{L}_c\mu$ and with $g : B \to C$ a μ-factor such that the ordinals corresponding to the outermost occurrences of E in EA_1 and EA_2 are i and j, respectively, and $(GHf)(i) \neq (GHf)(j)$, while $(GH(g \circ f))(i) = (GH(g \circ f))(j)$, there exists an arrow term

$$f' : A[E(A_1 \otimes A_2)] \to C$$

of $\mathscr{L}_c\mu$ such that $g \circ f = f' \circ A[\psi_{A_1,A_2}]$.

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Proof. Note first that every arrow term of $\mathscr{L}_c\mu$ is a substitution instance of an arrow term of $\mathscr{L}_c\mu$ with a diversified target. So we may assume that C in the lemma is diversified. That f' exists follows from the assumption that we have $g \circ f$ and from the $\mathscr{L}_c\mu$-Theoremhood Lemma (Lemma 2). That $g \circ f = f' \circ A[\psi_{A_1,A_2}]$ follows from $\mathscr{L}_c\mu$-Coherence. □

We can now prove the following proposition.

Proposition 5 ($\mathscr{CS}$-Coherence). The functor G from $\mathscr{CS}$ to *Rel* is faithful.

Proof. We first establish that every arrow term f of $\mathscr{CS}$ is equal to an arrow term $f_3 \circ f_2 \circ f_1$ such that in the developed arrow term f_1, all the heads of factors are of the form Δ_A or $\mathfrak{i}_A$ with A atomic (see Došen and Petrić (2010, Section 2) for the developed, head, factor and atomic notions), while f_2 is an arrow term of $\mathscr{L}_c\mathscr{S}$ without occurrences of η, and in the developed arrow term f_3, all the heads are of the form η_B. To produce f_1, we rely on naturality and the functorial equations, and on the following equations of cartesian categories:

$$\Delta_I = l_I^{-1},$$
$$\Delta_{A\otimes B} = c^m_{A,A,B,B} \circ (\Delta_A \otimes \Delta_B)$$

(for the arrow $c^m_{A,B,C,D} : (A \otimes B) \otimes (C \otimes D) \to (A \otimes C) \otimes (B \otimes D)$)

$$\mathfrak{i}_I = \mathbf{1}_I$$
$$\mathfrak{i}_{A\otimes B} = l_I \circ (\mathfrak{i}_A \otimes \mathfrak{i}_B)$$

(*cf.* the proofs of Došen and Petrić (2010, Section 7, Lemma 3) and of $\mathscr{C}$-Coherence in Došen and Petrić (2010, Section 8)). Note that nothing can block pushing to the right Δ_A or $\mathfrak{i}_A$ with A atomic. To produce f_3, we rely on naturality and the functorial equations, and on the equations $(\psi^L\eta)$ and $(\psi^R\eta)$ of Section 2.

The rest of the proof is analogous to the proof of Došen and Petrić (2010, Section 7, Lemma 6), which is based on Došen and Petrić (2010, Section 7, Lemma 4). The notion of short circuit is the same, but in the analogue of the proof of Došen and Petrić (2010, Lemma 4), g is now a μ-factor, rather than a ψ-factor.

The fact that Gf uniquely determines Gf_1, Gf_2, Gf_3 and the targets of f_1 and f_2 is established in the same way as at the end of the proof of Došen and Petrić (2010, Section 8, Proposition 5). We may assume that the target of f is $\otimes$-free, so f_1 will be Δ-free. □

6. Coherence for cocartesian monoidal monads

A *cocartesian* monoidal monad is a symmetric monoidal monad in a cocartesian category, by which we mean a monoidal category whose monoidal structure is given by finite coproducts. Moreover, we have the equation

$$\psi_{A,B} = \nabla_{T(A\otimes B)} \circ (T\iota^1_{A,B} \otimes T\iota^2_{A,B}), \qquad (\psi \; def)$$

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where $\nabla_A : A \otimes A \to A$ is a component of the codiagonal natural transformation, while $\iota^1 : A \to A \otimes B$ and $\iota^2 : B \to A \otimes B$ are components of the injection natural transformations of the cocartesian structure.

More simply, we can define a cocartesian monoidal monad as a cocartesian category with a monad in it. The definition of ψ is given by (ψ *def*).

In every cocartesian monoidal monad, the functor T preserves ∇ up to ψ in the sense that we have the equation

$$T\nabla_A \circ \psi_{A,A} = \nabla_{TA}.$$

If $!_A : I \to A$ is the unique arrow from the initial object I to A, then T also preserves ! up to ψ_0, which is defined as η_I in the sense that we have the equation

$$T!_A \circ \eta_I = !_{TA}$$

because I is the initial object. For the same reason, we have $\psi_0 = \eta_I = !_{TI}$.

We will now deal with coherence for the notion of a cocartesian monoidal monad. Let $\mathscr{DS}$ be the category of the cocartesian monoidal monad freely generated by an arbitrary set of objects. We define the functor G from $\mathscr{DS}$ to the category *Fun* whose arrows are arbitrary relations between finite ordinals by adding to the definition of G from $\mathscr{L}_c\mathscr{S}$ to *Fun* the clause for $G\nabla_A$ that corresponds to the picture



and the clause that says that $G!_A$ is the empty function from Ø to GA (see Došen and Petrić (2010, Section 9)). We can now prove the following proposition.

Proposition 6 ($\mathscr{DS}$-Coherence). The functor G from $\mathscr{DS}$ to *Fun* is faithful.

Proof. We first show that every arrow term f of $\mathscr{DS}$ is equal to an arrow term $f_2 \circ f_1$ such that f_1 is an arrow term of $\mathscr{L}_c\mathscr{S}$, and in the developed arrow term f_2 every factor is either a ∇-factor or a !-factor such that the index of the head is a propositional letter.

Then we can ascertain that the target of f_1, which is the source of f_2, is uniquely determined by Gf. The graph Gf_2 is uniquely determined by Gf, and so is the graph Gf_1 if in f_1 we get rid of useless crossings (*cf.* Došen and Petrić (2010, Section 7). Then we rely on $\mathscr{L}_c\mathscr{S}$-Coherence of Section 4 (Proposition 4) and the $\mathscr{D}$-Coherence of Došen and Petrić (2010, Section 9) to obtain $\mathscr{DS}$-Coherence. (As a matter of fact, what we need is a rather trivial instance of $\mathscr{D}$-Coherence.) □

7. Coherence for strong comonads

A *left strong* comonad in a monoidal category $\mathscr{A}$ is a comonad $\langle L, \varepsilon, \delta\rangle$ (in the notation of Mac Lane (1998, Section VI.1) in $\mathscr{A}$ such that L is a left monoidal functor (see Section 2), and, moreover, we have the equations

$$\varepsilon_{A\otimes B} \circ \psi^L_{A,B} = \varepsilon_A \otimes \mathbf{1}_B \qquad (\psi^L\varepsilon)$$

$$\delta_{A\otimes B} \circ \psi^L_{A,B} = L\psi^L_{A,B} \circ \psi^L_{LA,B} \circ (\delta_A \otimes \mathbf{1}_B). \qquad (\psi^L\delta)$$

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(These equations might be interpreted as saying that ε is a left monoidal natural transformation from L to the identity functor, while δ is a left monoidal natural transformation from L to LL.)

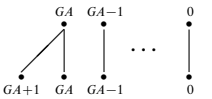
A *right strong* comonad is defined analogously to a right monoidal functor L. The equations corresponding to $(\psi^L\varepsilon)$ and $(\psi^L\delta)$ are

$$\varepsilon_{A\otimes B}\circ\psi^R_{A,B} = \mathbf{1}_A\otimes\varepsilon_B \qquad (\psi^R\varepsilon)$$

$$\delta_{A\otimes B}\circ\psi^R_{A,B} = L\psi^R_{A,B}\circ\psi^R_{A,LB}\circ(\mathbf{1}_A\otimes\delta_B). \qquad (\psi^R\delta)$$

Let $\mathscr{L}^L\mathscr{S}^{co}$ be the category of the left strong comonad freely generated by an arbitrary set of objects. We define a functor G from $\mathscr{L}^L\mathscr{S}^{co}$ to the category Δ^{op} as the functor G from $\mathscr{L}^L\mathscr{S}$ to the simplicial category Δ of Section 2, with the clauses for $G\eta_A$ and $G\mu_A$ replaced by dual clauses, which correspond to the following pictures:

$G\varepsilon_A$ 

$G\delta_A$ 

We can prove the following proposition by proceeding in the same ways as for the proof of $\mathscr{L}^L\mathscr{S}$-Coherence (Proposition 1) in Section 2.

Proposition 7 ($\mathscr{L}^L\mathscr{S}^{co}$-Coherence). The functor G from $\mathscr{L}^L\mathscr{S}^{co}$ to Δ^{op} is faithful.

We can prove coherence results for right strong comonads analogous to the two versions of $\mathscr{L}^R\mathscr{S}$-Coherence in Section 2.

8. Coherence for monoidal comonads

A *monoidal* comonad in a monoidal category $\mathscr{A}$ is a comonad $\langle L,\varepsilon,\delta\rangle$ (in the notation of Mac Lane (1998, Section VI.1) in $\mathscr{A}$ together with a natural transformation whose components are the arrows

$$\psi_{A,B} : LA\otimes LB \to L(A\otimes B),$$

together with the arrow $\psi_0 : I \to LI$, such that L with ψ and ψ_0 is a monoidal functor (which means that we have the equations (ψa), (ψl) and (ψr) with T replaced by L), and, moreover, we have the equations

$$\varepsilon_{A\otimes B}\circ\psi_{A,B} = \varepsilon_A\otimes\varepsilon_B \qquad (\psi\varepsilon)$$

$$\delta_{A\otimes B}\circ\psi_{A,B} = L\psi_{A,B}\circ\psi_{LA,LB}\circ(\delta_A\otimes\delta_B) \qquad (\psi\delta)$$

$$\varepsilon_I\circ\psi_0 = \mathbf{1}_I \qquad (\psi_0\epsilon)$$

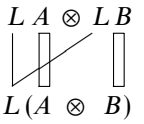
$$\delta_I\circ\psi_0 = L\psi_0\circ\psi_0, \qquad (\psi_0\delta)$$

which say that ε and δ are monoidal natural transformations. (References for the notion of a monoidal comonad are given in Section 1.)

Let $\mathscr{M}\mathscr{S}^{co}$ be the category of the monoidal comonad freely generated by an arbitrary set of objects. We define the functor G from $\mathscr{M}\mathscr{S}^{co}$ to *Rel* using the clause for $G\psi_{A,B}$

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corresponding to the following picture:



We also have the clause that says that $G\psi_0$ is the empty relation from Ø to $\{\emptyset\}$, the clauses for $G\varepsilon_A$ and $G\delta_A$ given above, and the remaining clauses as in Section 2.

For the proof of $\mathscr{M}\mathscr{S}^{co}$-Coherence (Proposition 9), we need the following notion of a normal form. An arrow term $f_2 \circ f_1$ of $\mathscr{M}\mathscr{S}^{co}$ is in normal form when every factor of the developed arrow term $f_1 : A \to C$ is an ε-factor or a δ-factor, and $f_2 : C \to B$ is an arrow term of the category $\mathscr{M}$, which is the free monoidal category with a single monoidal endofunctor (see Došen and Petrić (2010, Section 3); the family of monoidal endofunctors of $\mathscr{M}$ is here taken to be the singleton $\{L\}$). It is easy to see that the equations of $\mathscr{M}\mathscr{S}^{co}$ mean that every arrow term is equal to an arrow term in normal form. To show that $G(f_2 \circ f_1)$ uniquely determines Gf_1, Gf_2 and C, we rely on a general proposition (Proposition 8) on decomposing an arbitrary binary relation between finite ordinals into three functions.

To formulate this proposition, let $<_l$ be the lexicographical order on $n \times m$; that is, for $x_1, x_2 \in n$ and $y_1, y_2 \in m$, we have

$$(x_1, y_1) <_l (x_2, y_2) \text{ iff } (x_1 < x_2 \text{ or } (x_1 = x_2 \text{ and } y_1 < y_2)).$$

We call this the *left* lexicographical order; the *right* lexicographical order $<_r$ is defined by

$$(x_1, y_1) <_r (x_2, y_2) \text{ iff } (y_1 < y_2 \text{ or } (y_1 = y_2 \text{ and } x_1 < x_2)).$$

Let $\langle \nu, \mu, \beta \rangle$ be a triple of functions $\nu : k \to n$, $\mu : k \to m$ and $\beta : k \to k$, for β a bijection, such that for $z \in k$ and

$$l_{\nu,\mu,\beta}(z) =_{df} (\nu(z), \mu(\beta(z)))$$
$$r_{\nu,\mu,\beta}(z) =_{df} (\nu(\beta^{-1}(z)), \mu(z))$$

we have for every $u, v \in k$

$$\text{if } u < v, \text{ then } (l_{\nu,\mu,\beta}(u) <_l l_{\nu,\mu,\beta}(v) \text{ and } r_{\nu,\mu,\beta}(u) <_r r_{\nu,\mu,\beta}(v)). \qquad (*)$$

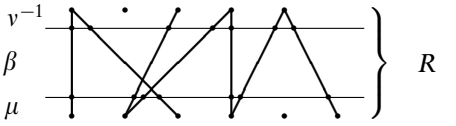
An alternative condition equivalent to (*) is to say that $l_{\nu,\mu,\beta}(z)$ is the $(z+1)$-th pair in the $<_l$ ordering of the image of $l_{\nu,\mu,\beta}$, and analogously for r. The image of $l_{\nu,\mu,\beta}$ coincides with the image of $r_{\nu,\mu,\beta}$; it coincides also with the set of ordered pairs $\mu \circ \beta \circ \nu^{-1}$, whose cardinality is k.

We say triples of functions such as $\langle \nu, \mu, \beta \rangle$ above are *coordinated*. Saying that a triple of functions is coordinated amounts to saying that ν and μ are order-preserving and that in k there are no analogues of the short circuits and useless crossings of Došen and Petrić (2010, Section 7). We can now formulate our general decomposition proposition.

Proposition 8 (Decomposition Proposition). For every relation $R \subseteq n \times m$, there is a unique coordinated triple of functions $\langle \nu, \mu, \beta \rangle$ such that $R = \mu \circ \beta \circ \nu^{-1}$. The domain of ν, μ and β is the cardinality of R.

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Here $\circ$ on the right-hand side is composition of relations, and v^{-1} is the relation converse to the function v. We use $|R|$ to denote the cardinality of the set of ordered pairs R. The Decomposition Proposition is illustrated by the following example:



This example, makes the truth of the proposition pretty obvious, but we will still give a formal proof. To do this, we let the bijection $l_R : |R| \to R$ be defined by

$$l_R(z) \text{ is the } (z+1)\text{-th ordered pair of } R \text{ in the ordering } <_l.$$

We define the bijection $r_R : |R| \to R$ via $<_r$ analogously.

Given R, consider the following functions:

$$\begin{aligned} v_R &=_{df} \quad p^1 \circ l_R : |R| \to n \\ \mu_R &=_{df} \quad p^2 \circ r_R : |R| \to m \\ \beta_R &=_{df} r_R^{-1} \circ l_R : |R| \to |R|, \end{aligned}$$

where p^1 and p^2, respectively, are the first and second projection with domain $n \times m$.

To show that l_{v_R,μ_R,β_R} and r_{v_R,μ_R,β_R} satisfy $(*)$, it is enough to verify that the following lemma holds.

Lemma 4. $l_{v_R,\mu_R,\beta_R} = l_R$ and $r_{v_R,\mu_R,\beta_R} = r_R$.

Proof. For the first equation we have

$$\begin{aligned} (v_R(z), \mu_R(\beta_R(z))) &= (p^1(l_R(z)), p^2(r_R(r_R^{-1}(l_R(z))))) \\ &= (p^1(l_R(z)), p^2(l_R(z))) \\ &= l_R(z), \end{aligned}$$

The second equation is proved similarly. □

Since we also have $R = \mu_R \circ \beta_R \circ v_R^{-1}$, the triple $\tau(R) = \langle v_R, \mu_R, \beta_R \rangle$ is a coordinated triple of functions such as we require for the Decomposition Proposition.

To show that $\tau(R)$ is unique, we proceed as follows. It is enough to verify, in addition to $R = \mu_R \circ \beta_R \circ v_R^{-1}$, that for every coordinated triple of functions $\langle v, \mu, \beta \rangle$, we have

$$\tau(\mu \circ \beta \circ v^{-1}) = \langle v, \mu, \beta \rangle.$$

To do this, we use the following lemmas.

Lemma 5. $l_{\mu \circ \beta \circ v^{-1}} = l_{v,\mu,\beta}$ and $r_{\mu \circ \beta \circ v^{-1}} = r_{v,\mu,\beta}$.

The proof of this relies on the comment after $(*)$.

Lemma 6. $v_{\mu \circ \beta \circ v^{-1}} = v$, $\mu_{\mu \circ \beta \circ v^{-1}} = \mu$ and $\beta_{\mu \circ \beta \circ v^{-1}} = \beta$.

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Proof. For the first equation we have

$$\begin{aligned} v_{\mu\circ\beta\circ\nu^{-1}}(z) &= p^1(l_{\mu\circ\beta\circ\nu^{-1}}(z)) \\ &= p^1(v(z), \mu(\beta(z))) \quad \text{(by Lemma 5)} \\ &= v(z). \end{aligned}$$

The second equation is derived analogously, while for the third equation we have

$$\begin{aligned} \beta_{\mu\circ\beta\circ\nu^{-1}}(z) &= r^{-1}_{\mu\circ\beta\circ\nu^{-1}}(l_{\mu\circ\beta\circ\nu^{-1}}(z)) \\ &= r^{-1}_{\nu,\mu,\beta}(v(z), \mu(\beta(z))) \quad \text{(by Lemma 5)} \\ &= \beta(z) \end{aligned}$$

since we have

$$\begin{aligned} r_{\nu,\mu,\beta}(\beta(z)) &= v(\beta^{-1}(\beta(z)), \mu(\beta(z))) \\ &= (v(z), \mu(\beta(z))). \end{aligned}$$

□

This concludes the proof of the Decomposition Proposition (Proposition 8). (The Decomposition Proposition could be used to obtain a normal form for arrow terms of the category *Rel* – this is an alternative to the iota normal form of Došen and Petrić (2009, Section 13.)

We can now complete the proof of the following proposition.

Proposition 9 ($\mathscr{M}\mathscr{S}^{co}$-Coherence). The functor G from $\mathscr{M}\mathscr{S}^{co}$ to *Rel* is faithful.

Proof. We rely on the normal form $f_2 \circ f_1$, which we introduced before the Decomposition Proposition (Proposition 8). In f_1 we find what corresponds to ν^{-1} in the Decomposition Proposition, and in f_2 what corresponds to $\mu\circ\beta$. We then rely on coherence for comonads (see Došen and Petrić (2008, Section 3), and references therein) and on $\mathscr{M}$-Coherence of Došen and Petrić (2010, Section 4). □

The normal form of this proof could be refined to $f_4 \circ f_3 \circ f_2 \circ f_1$, where in the developed arrow term f_1 every factor is an ε-factor, in the developed arrow term f_2 every factor is a δ-factor, in the developed arrow term f_4 every factor is a ψ_0-factor, and f_3 is an arrow term of $\mathscr{M}$ without occurrences of ψ_0.

9. Coherence for symmetric monoidal comonads

A *symmetric* monoidal comonad is a monoidal comonad (see the beginning of the preceding section) in a symmetric monoidal category whose endofunctor L is a functor that satisfies the equation (ψc) of Section 4 with T replaced by L; in other words, this endofunctor is a linear endofunctor in the sense of Došen and Petrić (2010, Section 6).

Let $\mathscr{M}_c\mathscr{S}^{co}$ be the category of the symmetric monoidal comonad freely generated by an arbitrary set of objects. We define the functor G from $\mathscr{M}_c\mathscr{S}^{co}$ to *Rel* as G from $\mathscr{M}\mathscr{S}^{co}$ to *Rel*, except that now GA is the number of occurrences of generating objects, that is, propositional letters, in A plus the number of occurrences of L in A (see Section 4).

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Moreover, we have a clause for $Gc_{A,B}$ as in Section 4. We can prove the following proposition.

Proposition 10 ($\mathscr{M}_c\mathscr{S}^{co}$-Coherence). The functor G from $\mathscr{M}_c\mathscr{S}^{co}$ to *Rel* is faithful.

For the proof we proceed as for $\mathscr{M}\mathscr{S}^{co}$-Coherence (Proposition 9) by relying on a normal form $f_2 \circ f_1$, where f_1 is as before and f_2 is an arrow term of the category $\mathscr{M}_c$, which is the free symmetric monoidal category with a single linear endofunctor. We then appeal to $\mathscr{M}_c$-Coherence of Došen and Petrić (2010, Section 6).

10. Coherence for cartesian monoidal comonads

A *cartesian* monoidal comonad is a symmetric monoidal comonad in a cartesian category, which also satisfies the equation ($\psi\Delta$) of Section 5 with T replaced by L.

Let $\mathscr{C}\mathscr{S}^{co}$ be the category of the cartesian monoidal comonad freely generated by an arbitrary set of objects. This category may be taken as axiomatising the equality of deductions in the $\{\Box, \wedge, \top\}$ fragment of the modal logic S4 (*cf.* Došen and Petrić (2008)). We define the functor G from $\mathscr{C}\mathscr{S}^{co}$ to *Rel* as G from $\mathscr{M}_c\mathscr{S}^{co}$ to *Rel* with additional clauses for $G\Delta_A$ and $G¡_A$ as in Section 5. We can prove the following proposition.

Proposition 11 ($\mathscr{C}\mathscr{S}^{co}$-Coherence). The functor G from $\mathscr{C}\mathscr{S}^{co}$ to *Rel* is faithful.

For the proof we proceed as for $\mathscr{C}$-Coherence in Došen and Petrić (2010, Section 8). We again rely on the possibility of assuming that the targets are $\otimes$-free.

11. Coherence for cocartesian monoidal comonads

A *cocartesian* monoidal comonad is a symmetric monoidal comonad in a cocartesian category. Moreover, we have the equation (ψ *def*) of Section 6. The equation

$$\psi_0 = !_{LI} \qquad (\psi_0 \; def)$$

follows from the assumption that I is an initial object (*cf.* the beginning of Section 6), which comes with the assumption that we are in a cocartesian category.

More simply, we can define a cocartesian monoidal comonad as a cocartesian category with a comonad in it. The definitions of ψ and ψ_0 are then given by (ψ *def*) and (ψ_0 *def*).

Let $\mathscr{D}\mathscr{S}^{co}$ be the category of the cocartesian monoidal comonad freely generated by an arbitrary set of objects. We define the functor G from $\mathscr{D}\mathscr{S}^{co}$ to *Rel* as G from $\mathscr{M}_c\mathscr{S}^{co}$ to *Rel* with additional clauses for $G\nabla_A$ and $G!_A$ as in Section 6. We can prove the following proposition.

Proposition 12 ($\mathscr{D}\mathscr{S}^{co}$-Coherence). The functor G from $\mathscr{D}\mathscr{S}^{co}$ to *Rel* is faithful.

Proof. We rely on a normal form $f_2 \circ f_1$ where every factor of the developed arrow term $f_1 : A \to C$ is an ε-factor or a δ-factor, and $f_2 : C \to B$ is an arrow term of the category $\mathscr{D}$, which is the free cocartesian category with a single endofunctor (see Došen and Petrić (2010, Section 9)). To show that $G(f_2 \circ f_1)$ uniquely determines Gf_1, Gf_2 and C, we use the Decomposition Proposition (Proposition 8). We then rely on coherence for

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comonads as in the proof of $\mathscr{MS}^{co}$-Coherence (Proposition 9), and on the $\mathscr{D}$-Coherence of Došen and Petrić (2010, Section 9). □

Acknowledgements

We would like to thank Masahito Hasegawa for taking charge as an editor of both this paper and Došen and Petrić (2010), and for arranging for them to appear together. The formulations at a few points in the present version of the paper are due to an anonymous referee, whom we thank for suggesting improvements to the exposition. (Our original formulations may be found in the arXiv.)

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https://doi.org/10.1017/S0960129510000034 Published online by Cambridge University Press