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RESEARCH ARTICLE

Braid positive surgery diagrams

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Abstract
In this short note, we prove that every closed, oriented, connected 3-manifold arises as Dehn surgery along a braid positive link.

1. Introduction

A classical result by Lickorish and Wallace [10, 18] states that every closed, oriented, connected 3-manifold M arises as *Dehn surgery* along a link L in S^3. That is, M can be obtained by removing an open tubular neighbourhood νL of L from S^3 and gluing back to each torus boundary component of $S^3 \setminus \nu L$ a solid torus by an orientation-preserving diffeomorphism of its boundary. Our motivation in this note stems from the following type of question.

Question 1. *Let $\mathcal{L}$ be a set of links in S^3. Can each closed, oriented, connected 3-manifold be obtained by Dehn surgery along a link $L \in \mathcal{L}$?*

Considering the first homology of a 3-manifold obtained by Dehn surgery directly implies that the number of components of links in $\mathcal{L}$ is unbounded if $\mathcal{L}$ gives a positive answer to Question 1. Question 1 was answered affirmatively for the following sets of links:

1. Links that are fibred, by [17; Theorem 3],
2. Links that are quasipositive, by [16; Proposition 4],
3. Links that are alternating and hyperbolic, by [1; p. 126],
4. Chainmail links, by [13]; see also [2; Theorem 2.2],
5. Generalised L-space links, by [2; Corollary 4.3].[1]

In contrast to these results, not every closed, oriented, connected 3-manifold can be obtained by Dehn surgery along a torus link since Dehn surgeries along torus links are never hyperbolic [9, 12]. Moreover, the answer to Question 1 is negative for the set of Montesinos links by [8] and thus also for pretzel links. Question 1 also has a negative answer for two-bridge links since these have at most two components.

In this short note, we show the following.

Theorem 2. *Every closed, oriented, connected 3-manifold M can be obtained by Dehn surgery along a braid positive link L in S^3. Moreover, L can be chosen to have at most $s(M) + 1$ many components.*

[1]Note that even though L-space knots are fibred, not every (generalised) L-space link is fibred; see [11; Example 3.9].

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Here, we denote by $s(M)$ the *surgery number* of M, which is the minimal number of components of any link in S^3 needed to describe M as Dehn surgery along that link; see, for example, [5]. Moreover, recall that a link is called *braid positive* if it arises as the closure of a positive braid, that is, it is a braid represented by a finite product of the Artin generators of the braid group [4] (see Section 3).

Braid positive links are fibred [17], positive and strongly quasipositive, so we can also add, for example, the set of positive, fibred links or the set of strongly quasipositive, fibred links to the above list.

Conventions. All 3-manifolds are assumed to be smooth and oriented and are considered up to orientation-preserving diffeomorphism. For us, a link in a 3-manifold M is a smooth, closed 1-dimensional submanifold of M. Surgery coefficients (see Section 2) are measured with respect to the Seifert framing.

2. Surgery diagrams and Rolfsen twists

The two main ingredients in the proof of Theorem 2 are the so-called Rolfsen twist [15] and Garside's theory of braids [7]. To make this note self-contained, we will explain Rolfsen twists in this section and a direct consequence of Garside's work needed in our proof in Section 3. Readers who are already familiar with these concepts can skip ahead to Section 4 for the proof of Theorem 2.

First, let us briefly review the basics of Dehn surgery. For more details, the reader is referred, for example, to [14]. Let $L = L_1 \cup \ldots \cup L_m$ be a link in S^3 of m components. The boundary of $S^3 \setminus \nu L$ is orientation-preservingly diffeomorphic to a disjoint union of m tori $\bigcup_{j=1}^m S^1 \times S^1$. The diffeomorphism type of the 3-manifold obtained by Dehn surgery along L, that is, by gluing m solid tori $\bigcup_{j=1}^m S^1 \times D^2$ to $S^3 \setminus \nu L$, is determined by the image of $\bigcup_{j=1}^m \{1\} \times \partial D^2$ under an orientation-preserving diffeomorphism

$$\bigcup_{j=1}^{m} \partial(S^1 \times D^2) \longrightarrow S^3 \setminus \nu L.$$

Given a meridian μ_j and a Seifert longitude λ_j of the link component L_j, the image curves in $\partial(\overline{\nu L_j})$ can be expressed as $p_j\mu_j + q_j\lambda_j$ for coprime integers p_j, q_j for every $j \in \{1, \ldots, m\}$. The quotient

$$p_j/q_j \in \mathbb{Q} \cup \{\infty\}$$

is called the *surgery coefficient* of the link component L_j. By the above discussion, the diffeomorphism type of the 3-manifold obtained by Dehn surgery on the link L is determined by a diagram of L where each component L_j of L is labelled by p_j/q_j. We call L together with the labellings p_j/q_j, $i \in \{1, \ldots, m\}$, a *surgery diagram*.

Let us remark that a link component L_j with surgery coefficient $\infty = 1/0$ means that we identify the meridian of the attached solid torus with the meridian of L_j. The corresponding surgery simply glues in a solid torus in exactly the same way as the removed tubular neighbourhood of L_j. Therefore, a component with surgery coefficient $\infty = 1/0$ can be removed from a surgery diagram without changing the diffeomorphism type of the surgered manifold.

Let us now turn to Rolfsen twists. To that end, let $L = L_1 \cup \ldots \cup L_m$ be a surgery diagram with an unknotted component, say L_1, with surgery coefficient p_1/q_1. For any integer $N \in \mathbb{Z}$, an *N-fold Rolfsen* twist along L_1 creates out of L a new surgery diagram L' as follows:

- We cut the exterior of L_1, which is a solid torus, along a Seifert disk of L_1 and re-glue it with an N-fold twist (where $N = 1$ corresponds to one right-handed full twist) to get a new link $L' = L'_1 \cup \ldots \cup L'_m$ (the components of which are in natural bijection with the components of L),

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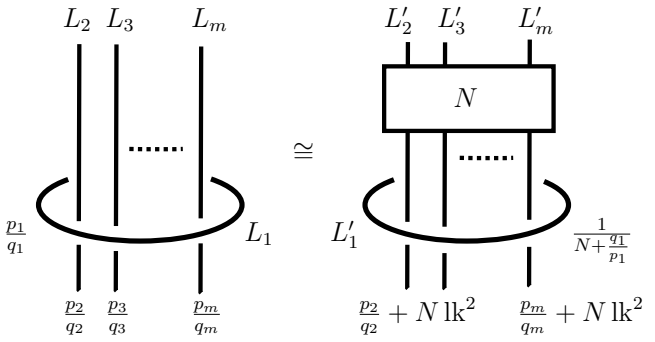


Figure 1. *An N-fold Rolfsen twist.*

- the surgery coefficient of L'_1 is $\frac{1}{N+\frac{q_1}{p_1}}$ and the surgery coefficients of the other components L'_j are given by $\frac{p_j}{q_j} + N\,\mathrm{lk}\,(L_j, L_1)^2$, where lk denotes the linking number for a choice of orientation of L.

We refer to Figure 1 for a visualisation.

Lemma 3 (Rolfsen twist [15]). *If L and L' are two surgery diagrams that differ by a Rolfsen twist, then the 3-manifolds obtained by Dehn surgery along L and L', respectively, are orientation-preservingly diffeomorphic.*

Proof. The process of cutting the exterior of L_1 along a Seifert disk of L_1 and re-glueing it with an N-fold twist describes a diffeomorphism of the exterior of L to the exterior of L'. It is straightforward to check that this diffeomorphism acts on the boundary tori as

$$\mu_1 \longmapsto \mu'_1 + N\lambda'_1$$
$$\lambda_1 \longmapsto \lambda'_1$$
$$\mu_j \longmapsto \mu'_j$$
$$\lambda_j \longmapsto N\,\mathrm{lk}\,(L_j, L_1)^2\mu'_j + \lambda'_j$$

and thus the surgery coefficients transform as claimed. □

3. Garside theory

The second key ingredient in our proof of Theorem 2 is Garside's theory of braids [7]. A *braid* on k strands is a collection of k disjoint, properly embedded arcs in the cylinder $[0, 1] \times D^2$ that run monotonically in the $[0, 1]$-direction from k fixed points in $\{0\} \times D^2$ to $\{1\} \times D^2$. Braids up to ambient isotopy of $[0, 1] \times D^2$ fixing $\{0, 1\} \times D^2$ pointwise form a group called the *braid group* B_k. The group operation is defined by stacking braids. Closing a braid by identifying $\{0\} \times D^2$ and $\{1\} \times D^2$, connecting the endpoints of the arcs and identifying the resulting solid torus with the tubular neighbourhood of an unknot in S^3 produces an oriented link in S^3; see Figure 2.[2] By Alexander's theorem, every link arises in this way [3]. We refer to [6] for more details on braids and their closures.

The braid group B_k is generated by the *Artin generators* [4] denoted $\sigma_1, \ldots, \sigma_{n-1}$, where σ_i corresponds to a positive half-twist exchanging two neighbouring strands with indices i and $i + 1$; see Figure 2.

[2] Note that there are infinitely many ways to identify a solid torus with the tubular neighbourhood of a knot. Here we choose the identification that maps $S^1 \times \{0\}$ onto the Seifert longitude of the unknot.

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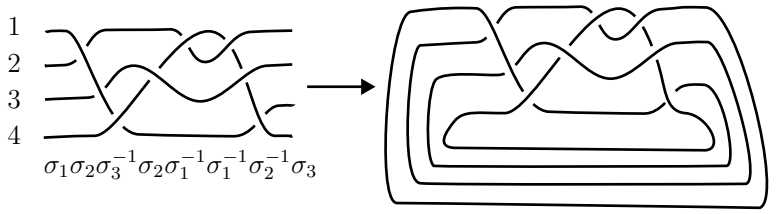


Figure 2. *A braid (left) and its closure (right).*

A braid on k strands is *positive* if it can be represented by a word in B_k that is a product of positive powers of the Artin generators. The closure of a positive braid is called a *braid positive* link. We denote by

$$T_k = (\sigma_1 \dots \sigma_{k-1})^k$$

a full twist in B_k; see, for example, Figure 3. The following lemma is a direct corollary from Garside's work [7].[3] For completeness and to keep our arguments purely geometric, we present a proof.

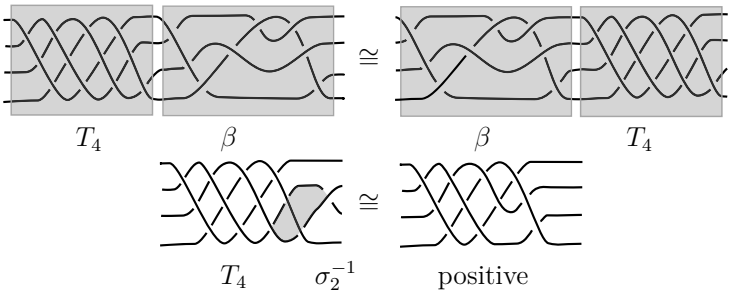


Figure 3. *The first row shows that a full twist T_k commutes with any other braid β on k strands. The second row visualises that a full twist T_k followed by a negative Artin generator σ_i^{-1} yields a positive braid.*

Lemma 4 ([7]). *For each braid β on k strands, there is an integer $N \geq 0$ such that $T_k^N\beta$ is a positive braid on k strands. More concretely, we can choose N to be the number of negative Artin generators in a braid word representative of β.*

Proof. We start with two simple observations. First, the full twist T_k in B_k commutes with any other element β of B_k, that is, $T_k\beta = \beta T_k$. See Figure 3 for an illustration. Second, for any Artin generator σ_i, the braid $T_k\sigma_i^{-1}$ is positive; see Figure 3 again.

If β is a positive braid, we can set $N = 0$. Otherwise, we write β as a word in the Artin generators and distinguish two cases. In the first case, the first letter of β is a positive generator, say $\beta = \sigma_i\beta'$. Using the above observations, we can then write $T_k^N\beta = \sigma_i T_k^N\beta'$. In the second case, if the first letter of β is a negative Artin generator, say $\beta = \sigma_i^{-1}\beta'$, then we write

$$T_k^N\beta = T_k^{N-1}T_k\sigma_i^{-1}\beta' = T_k^{N-1}p\beta' = pT_k^{N-1}\beta',$$

where p is a positive braid. The claim follows using an induction on the length of the braid word β. □

[3] Note that in Garside's notation, we have $T_k = \Delta^2$.

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4. Braid positive surgery diagrams

In this section, we finally prove Theorem 2.

Proof of Theorem 2. Given a closed, oriented, connected 3-manifold M, let L be a link whose number of components realises $s(M)$. We can represent L as the closure of a braid β on, say, k strands. Figure 4(a) represents a braid diagram β of L. Inserting a new component with surgery coefficient $\infty = 1/0$ and performing an N-fold Rolfsen twist, we obtain the diagram shown in Figure 4(b). Figure 4(c) shows the result of an isotopy of the new link diagram performed on the new component. The link L' in Figure 4(c) arises as the closure of a braid β'. Note that β' is a positive braid whenever the braid $\alpha := T_k^N \beta$ on k strands is a positive braid. The theorem thus follows from Lemma 4, choosing N sufficiently large. □

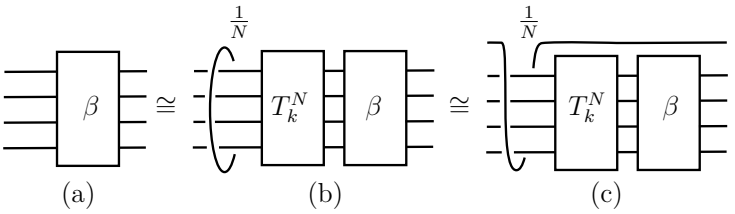


Figure 4. *Transforming a braid into a positive braid by a single surgery.*

Remark 5. *Note that Rudolph's proof for quasipositive links [16], in very brief, inserts a new unknotted component for each negative band generator in a band presentation of a braid representing L, so it increases both the braid index and the number of components of L potentially much more than the procedure described in our above proof. Moreover, his method does not immediately carry over to obtain braid positive links.*

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Conflicts of interest. The authors declare no conflicts of interest.

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