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ON HYPERBOLIC L-SPACE KNOTS WITH BRAID INDEX FOUR AND TUNNEL NUMBER TWO

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Abstract

There are only three known strongly invertible hyperbolic L-space knots with braid index four and tunnel number two. They are `t09284`, `t10496` and `o9_34409` in the SnapPy census. In this paper, we give the first infinite family of strongly invertible hyperbolic L-space knots with braid index four and tunnel number two that includes `t10496` and `o9_34409`.

2020 *Mathematics subject classification*: primary 57K10; secondary 57K18.

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1. Introduction

A knot in the 3-sphere S^3 is called an *L-space knot* if it admits a positive Dehn surgery yielding an L-space. A typical example is a (positive) torus knot. In this paper, we focus on the braid index of L-space knots. Knots with braid index two are torus knots. It is known that L-space knots with braid index three are either torus knots or braid positive twisted torus knots [15]. (Here, a knot that is the closure of a positive braid is said to be braid positive.) Also, L-space knots with braid index less than four have tunnel number one.

In the SnapPy census [1, 2, 8, 9], there are 632 L-space knots, and there are only three known strongly invertible hyperbolic L-space knots with braid index four and tunnel number two; `t09284`, `t10496` and `o9_34409` [1]. This observation leads to the following question.

QUESTION 1.1 [1, Question 10]. What are the L-space knots of braid index four with tunnel number greater than one?

In a previous paper [4], we gave an infinite family of asymmetric hyperbolic L-space knots with braid index four. We can verify that these knots have tunnel number two. (It is not hard to give an unknotting tunnel system consisting of two arcs. Then, the

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asymmetry excludes the possibility of tunnel number one, because a tunnel number one knot is strongly invertible [17, Lemma 5].)

The purpose of this paper is to give the first infinite family of strongly invertible hyperbolic L-space knots with braid index four and tunnel number two.

THEOREM 1.2. *There are infinitely many strongly invertible, hyperbolic L-space knots with braid index four and tunnel number two.*

For an integer $n \geq 0$, let K_n be the closure of the positive 4-braid

$$[3, 2, 2, 1, 3, 2, 2, 3, 2, (1, 2)^{3n}],$$

where an integer i denotes the standard braid generator σ_i in the braid group B_4 of four strands. Since it is the closure of a positive braid, it is fibred [24, Theorem 2] and it has genus $3n + 3$. Hence, $K_m \neq K_n$ if $m \neq n$. Note that K_0 is the torus knot $T(3, 4)$ and K_1 is the $(2, 9)$-cable of $T(2, 3)$. Also, K_2, K_3 and K_4 are `s682`, `t10496` and `o9_34409`, respectively, in the census.

Theorem 1.2 immediately follows from the following theorem.

THEOREM 1.3. *Let K_n be the knots defined above. Then, K_n has the following properties.*

(1) *K_n is a strongly invertible L-space knot.*
(2) *If $n \geq 1$, then K_n has braid index four.*
(3) *If $n \geq 2$, then K_n is hyperbolic.*
(4) *If $n \geq 3$, then K_n has tunnel number two.*

We have two other candidates of infinite families which generalize `t09284`. They are the closures of 4-braids

$$[(1, 2, 3)^4, 2, 1, 3, 2, 2, 3^{2m}], \quad [(1, 2, 3)^{4n}, 2, 1, 3, 2, 2, 3^6].$$

When $m = 3$ or $n = 1$, each gives `t09284`. However, we could not prove that these knots have tunnel number two.

2. L-space knots

In this section, we prove that our knot K_n is a strongly invertible L-space knot. Let $K_0 \cup c$ be the link shown in Figure 1. By performing $(-1/n)$-surgery along c, K_0 will be changed into K_n.

LEMMA 2.1. *For any $n \geq 0$, K_n is strongly invertible.*

PROOF. Figure 1 shows a strongly invertible position of the link $K_0 \cup c$. The dotted line is the axis of the involution. Performing $(-1/n)$-surgery on c changes K_0 into K_n. This observation confirms that K_n is strongly invertible for any $n \geq 0$. □

LEMMA 2.2. *For the link $K_0 \cup c$, $(8, 0)$-surgery yields a lens space $L(9, 7)$.*

PROOF. This is confirmed by Kirby–Rolfsen calculus. See Figure 2. □

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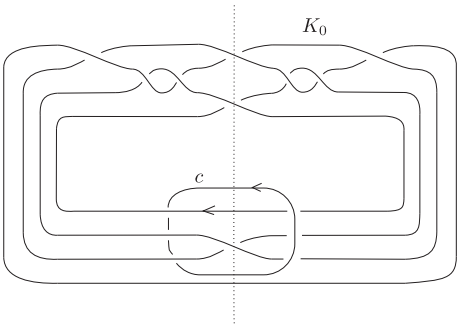


FIGURE 1. The link $K_0 \cup c$. By performing $(-1/n)$-surgery along c, K_0 will be changed into K_n. This diagram also shows a strongly invertible position of the link.

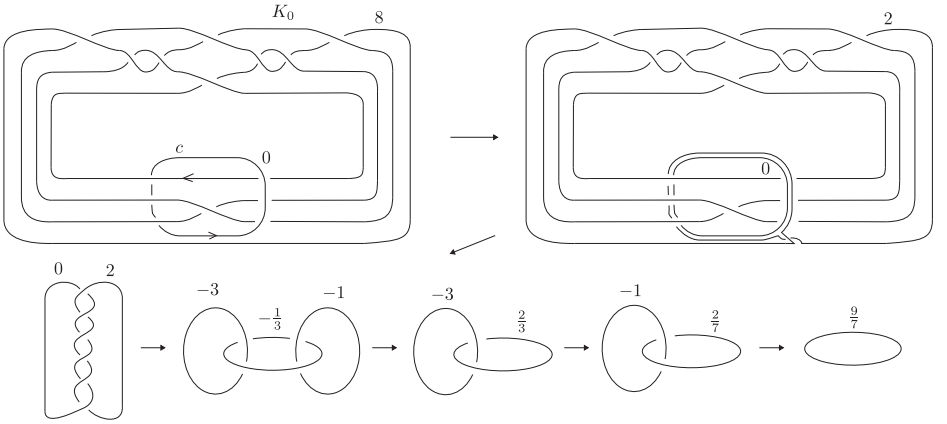


FIGURE 2. Kirby–Rolfsen calculus for $(8,0)$-surgery on $K_0 \cup c$. The result is a lens space $L(9,7)$.

LEMMA 2.3 [3, Theorem 1.13 and Lemma 6.1], [4, Lemma 2.1]. *Let $K \cup c$ be a link of an L-space knot K and an unknot c with linking number $w = \mathrm{lk}(K,c) > 1$. Let K_n be the image of K after $(-1/n)$-surgery on c for integers n. If there exists a slope $r \geq 2g(K) - 1$ such that $(r,0)$-surgery on $K \cup c$ is an L-space, then K_n is an L-space knot for all integers $n \geq 0$.*

LEMMA 2.4. *For each $n \geq 0$, K_n is an L-space knot.*

PROOF. First, K_0 is a torus knot $T(3,4)$ of genus three. By Lemma 2.2, $(8,0)$-surgery on $K_0 \cup c$ yields an L-space, because a lens space is an L-space. Thus, our link $K_0 \cup c$ satisfies the assumption of Lemma 2.3. □

3. Tunnel number

In this section, we prove that K_n has tunnel number two if $n \geq 3$. We need a lemma using the Montesinos trick [16].

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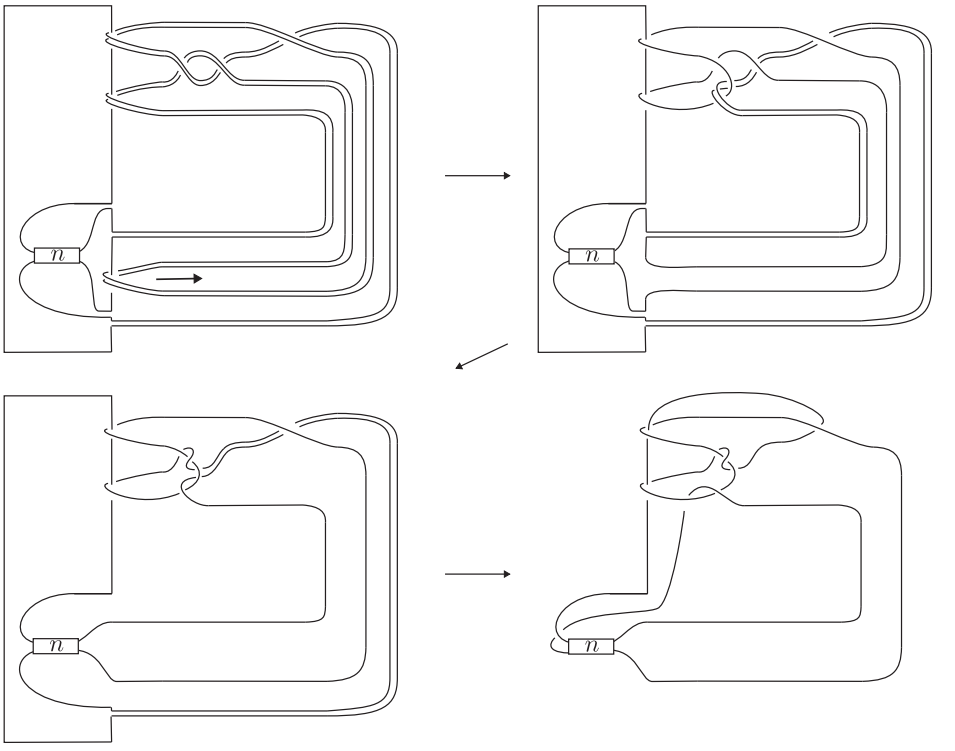


FIGURE 3. The diagram after the tangle replacements (top left) and its deformation. The box with integer n contains right handed horizontal n half twists.

LEMMA 3.1. *If $n \geq 3$, then $(9n+9)$-surgery on K_n yields a graph manifold M which is the union of two Seifert fibred spaces over the disk with two exceptional fibres of indices $(3,3)$ and $(2, n-1)$, respectively. In addition, the graph manifold M has Heegaard genus at least three.*

PROOF. For the link $K_0 \cup c$ given in Figure 1, consider $(9, -1/n)$-surgery. Performing n-twists on c shows that it represents $(9n+9)$-surgery on K_n.

Take the quotient under the involution along the axis depicted there. Then, we have a diagram as shown in Figure 3 (top left) after the tangle replacements. (Since K_0 has writhe 9 in the diagram of Figure 1, the tangle surgery for the quotient of K_0 is done by the 0-tangle.)

After a series of deformations in Figures 3 and 4, the resulting link (right in Figure 4) consists of two Montesinos tangles $[-1/3, 1/3]$ and $[-1/2, -1/(n-1)]$. By taking the double branched cover, which gives M, we have the first conclusion.

To confirm that this graph manifold M has Heegaard genus at least three, we follow the argument in [1, Section 2.3], which is based on the classification by Kobayashi [12] of closed orientable 3-manifolds with Heegaard genus two that contain

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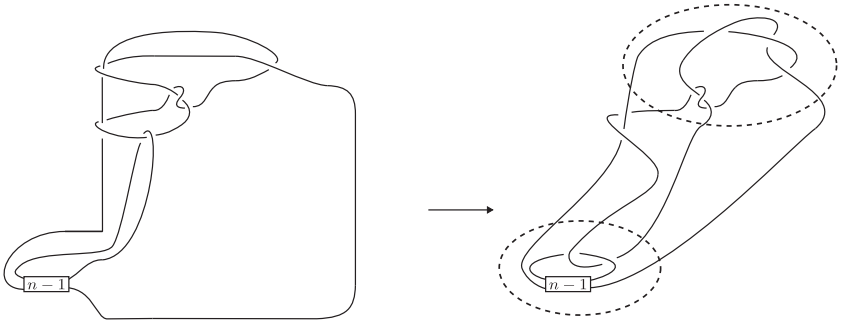


FIGURE 4. Further deformation. The resulting link (right) consists of two Montesinos tangles $[-1/3, 1/3]$ and $[-1/(n-1), -1/2]$.

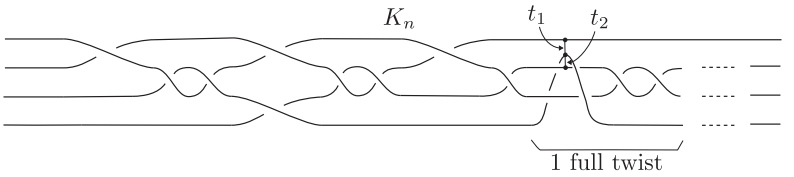


FIGURE 5. An unknotting tunnel system $\{t_1, t_2\}$ for K_n.

an essential torus. It claims that if a graph manifold obtained from gluing two Seifert fibred manifolds along an incompressible boundary torus has Heegaard genus two and contains a unique essential torus, then the regular fibres of the two pieces on the common torus intersect just once.

First, the cases where $n = 3$ and 4 are treated in [1, Section 2.3.2]. Thus, we assume $n \geq 5$. We remark that each piece of M admits a unique Seifert fibration. (In fact, this holds when $n \geq 4$.) From Figure 4, we see that the regular fibres of the two pieces on the common torus intersect twice (see the proof of [7, Proposition 12.28]). This implies that M has Heegaard genus at least three. □

LEMMA 3.2. *If $n \geq 3$, then K_n has tunnel number two.*

PROOF. First, the two arcs t_1 and t_2, as shown in Figure 5, give an unknotting tunnel system for K_n. It is straightforward to verify that the exterior of the neighbourhood of $K_n \cup t_1 \cup t_2$ is a handlebody of genus three. Hence, K_n has tunnel number at most two.

If K_n has tunnel number one, then any resulting manifold by Dehn surgery on K_n has Heegaard genus at most two. This is impossible by Lemma 3.1. Hence, K_n has tunnel number two. □

REMARK 3.3. It is well known that a torus knot has tunnel number one, so is K_0. For K_2, which is `s682`, it is not hard to see that it admits an unknotting tunnel. In fact, t_1, shown in Figure 5, gives the unknotting tunnel for K_2. However, K_1, the $(2, 9)$-cable of $T(2, 3)$, has tunnel number two [5, 18].

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4. Hyperbolicity and braid index

In this section, we prove that K_n is hyperbolic when $n \geq 2$. First, we calculate the Alexander polynomial $\Delta_{K_n}(t)$ of K_n.

LEMMA 4.1. *Let $n \geq 1$. Then, the Alexander polynomial $\Delta_{K_n}(t)$ of K_n is*

$$\Delta_{K_n}(t) = (t^{6n+6} - t^{6n+5}) + \sum_{i=0}^{n-1}(t^{3n+3i+5} - t^{3n+3i+4}) + t^{3n+3} - \sum_{i=0}^{n-1}(t^{3i+5} - t^{3i+4}) - t + 1.$$

PROOF. We start from the link $K_0 \cup c$ used in Section 2. The orientation of each component is shown in Figure 1. The multivariable Alexander polynomial $\Delta_{K_0 \cup c}(x, y)$ is

$$(x^8 - x^5 + x^4)y^2 + (x^5 - x^4 + x^3)y + x^4 - x^3 + 1.$$

(We used Kodama's knot [13] for the calculation.)

By performing $(-1/n)$-surgery on c, the link $K_0 \cup c$ is changed into the new link $K_n \cup c_n$, where c_n is the unknot. We set $c = c_0$. Clearly, two links $K_0 \cup c$ and $K_n \cup c_n$ have homeomorphic exteriors. Then, the induced isomorphism of the homeomorphism on their homology groups relates the Alexander polynomials of these two links [10, 20].

Let μ_{K_n} and λ_{K_n} (respectively μ_{c_n} and λ_{c_n}) be the oriented meridian and longitude of K_n (respectively c_n). Then, $\mu_{K_n} = \mu_{K_0}$, $\mu_{c_n} = \mu_c - n\lambda_c$ and $\lambda_c = 3\mu_{K_0}$. These give $\mu_c = \mu_{c_n} + 3n\mu_{K_n}$.

Hence, $\Delta_{K_n \cup c_n}(x, y) = \Delta_{K_0 \cup c}(x, yx^{3n})$. Since $\mathrm{lk}(K_n, c_n) = \mathrm{lk}(K_0, c) = 3$, the Torres condition [25] implies

$$\Delta_{K_n \cup c_n}(x, 1) = \frac{x^3 - 1}{x - 1}\Delta_{K_n}(x).$$

Thus,

$$\begin{aligned}\Delta_{K_n}(t) &= \frac{t-1}{t^3-1}\Delta_{K_n \cup c_n}(t, 1) = \frac{t-1}{t^3-1}\Delta_{K_0 \cup c}(t, t^{3n}) \\ &= \frac{t-1}{t^3-1}(t^{6n+8} - t^{6n+5} + t^{6n+4} + t^{3n+5} - t^{3n+4} + t^{3n+3} + t^4 - t^3 + 1).\end{aligned}$$

Here,

$$t^{6n+8} - t^{6n+5} = t^{6n+5}(t^3 - 1), \quad t^{6n+4} - t^{3n+4} = t^{3n+4}(t^{3n} - 1)$$

and

$$t^{3n+5} + t^{3n+3} + t^4 = (t^{3n+5} + t^{3n+4} + t^{3n+3}) - t^{3n+4} + t^4 = t^{3n+3}(t^2 + t + 1) - t^4(t^{3n} - 1).$$

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Therefore,

$$\begin{aligned}\Delta_{K_n}(t) &= t^{6n+5}(t-1) + t^{3n+4}(t-1)\sum_{i=0}^{n-1} t^{3i} + t^{3n+3} - t^4(t-1)\sum_{i=0}^{n-1} t^{3i} - t + 1 \\ &= (t^{6n+6} - t^{6n+5}) + \sum_{i=0}^{n-1}(t^{3n+3i+5} - t^{3n+3i+4}) + t^{3n+3} - \sum_{i=0}^{n-1}(t^{3i+5} - t^{3i+4}) - t + 1,\end{aligned}$$

as desired. □

Recall the *formal semigroup* $\mathcal{S}$ of an L-space knot K [26]. It is a set of nonnegative integers defined from the formal power series expansion

$$\frac{\Delta_K(t)}{1-t} = \sum_{s\in\mathcal{S}} t^s \in \mathbb{Z}[[t]].$$

LEMMA 4.2. *Let $n \geq 2$. The formal semigroup of K_n is*

$$\begin{aligned}\mathcal{S} = \{0, 4, 7, 10, \ldots, 3n-2, 3n+1\} \cup \{3n+3\} \cup \{3n+5, 3n+6, \ldots, 6n-1, 6n\} \\ \cup \{6n+2, 6n+3, 6n+4\} \cup \{6n+6, 6n+7, \ldots\}.\end{aligned}$$

In particular, $\mathcal{S}$ is not a semigroup if $n \geq 2$.

PROOF. The first conclusion immediately follows from Lemma 4.1. If $n \geq 2$, then $4 \in \mathcal{S}$, but $8 \notin \mathcal{S}$. Hence, $\mathcal{S}$ is not closed under addition. □

We remark that $\mathcal{S} = \{0, 4, 6, 8, 9, 10\} \cup \{12, 13, \ldots\}$ when $n = 1$. This is closed under addition (see [6, Example 1.10] and [26, Theorem 1.4]).

LEMMA 4.3. *If $n \geq 2$, then K_n is hyperbolic.*

PROOF. By Lemma 4.2, the formal semigroup $\mathcal{S}$ of K_n is not a semigroup. Hence, K_n is not a torus knot, since the formal semigroup of a torus knot is a genuine semigroup [6, Example 1.10].

Suppose that K_n is a satellite knot for a contradiction. First, by Lemma 2.4, K_n is prime because an L-space knot is prime [14, Theorem 1.2]. Since K_n is the closure of a 4-braid, it has bridge index at most four. Hence, the companion is a 2-bridge knot and the pattern has wrapping number two by [23].

In addition, the companion is an L-space knot and the pattern is braided [3, Theorems 1.17 and 7.2]. This implies that the companion is a 2-bridge torus knot [21, Theorem 1.5] and the pattern is 2-cabled. Then, K_n is an iterated torus L-space knot, so its formal semigroup is closed under addition [26, Theorem 1.4]. This contradicts Lemma 4.2. Hence, K_n is hyperbolic. □

LEMMA 4.4. *If $n \geq 1$, then K_n has braid index four.*

PROOF. We know that K_1 is the $(2, 9)$-cable of the trefoil, so it has bridge index four. Thus, K_1 has braid index four.

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Next, consider K_2, which is s682. The Morton–Franks–Williams inequality [11, 19] gives a lower bound on the braid index in terms of its HOMFLY-PT polynomial. For the $P(v, z)$ version of the HOMFLY-PT polynomial, let d_+ and d_- be the max and min degrees of v. Then, $(d_+ - d_-)/2 + 1$ gives the lower bound for the braid index. For K_2, $d_+ = 24$ and $d_- = 18$ by SageMath [22]. This shows that K_2 has braid index four.

Assume $n \geq 3$. If K_n has braid index at most three, then it has tunnel number one as mentioned in Section 1. By Lemma 3.2, K_n has tunnel number two. Hence, it has braid index four. □

PROOF OF THEOREM 1.3. This follows from Lemmas 2.1, 2.4, 3.2, 4.3 and 4.4. □

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