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On mixed local–nonlocal problems with Hardy potential

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In this article, we study the effect of the Hardy potential on existence, uniqueness, and optimal summability of solutions of the mixed local–nonlocal elliptic problem

$$-\Delta u + (-\Delta)^s u - \gamma \frac{u}{|x|^2} = f \text{ in } \Omega, \ u = 0 \text{ in } \mathbb{R}^n \setminus \Omega,$$

where Ω is a bounded domain in $\mathbb{R}^n$ containing the origin and $\gamma > 0$. In particular, we will discuss the existence, non-existence, and uniqueness of solutions in terms of the summability of f and of the value of the parameter γ.

Keywords: existence and regularity of solutions; Hardy potential; mixed local–nonlocal PDEs; hardy inequalties; SOLA solutions

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1. Introduction

Elliptic and parabolic Partial Differential Equations (PDEs) with the Hardy potential $|x|^{-2}$, where $x \in \mathbb{R}^n$ with $n \geq 3$, are nowadays a classical topic of investigation. We refer to the monograph [23] for a detailed description of the mathematical and

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physical reasons that led to the study of such PDEs as well as most of the known result both in the model local and nonlocal case, namely when the leading operator is either $-\Delta$ or $(-\Delta)^s$ with $s \in (0,1)$. In the latter case, the interesting Hardy potential is given by $|x|^{-2s}$. Our aim here is to start the study of problems of the form

$$\begin{cases} \mathcal{L}u - \gamma \dfrac{u}{|x|^2} = f(x) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is an open and bounded set with smooth enough boundary $\partial\Omega$, $0 \in \Omega$, f belongs to some suitable $L^m(\Omega)$ and $\gamma > 0$. It will be soon clarified that the choice of the *local* Hardy potential is the most proper in this case, and this will immediately provide a natural upper bound for γ. Finally, we introduce the operator

$$\mathcal{L} := -\Delta + (-\Delta)^s,$$

where $(-\Delta)^s$ denotes the fractional Laplacian of order $s \in (0,1)$, i.e.,

$$(-\Delta)^s u = C_{n,s} \lim_{\varepsilon \to 0} \int_{\{|x-y| \geq \varepsilon\}} \frac{u(x) - u(y)}{|x-y|^{n+2s}}\, dy,$$

and $C_{n,s} > 0$ is a suitable normalization constant (see, precisely, remark 2.1).

The above mixed local–nonlocal operator is a particular instance of a general class of operators firstly studied in the 60s [11, 12] in connection with the validity of maximum principle and more recently from a probabilistic point of view, see, e.g., [13] and the references therein. Recently, there has been a renewed interest in problems presenting both a local and nonlocal nature, mainly due to their interest in the applications see, e.g., [16]. The most common thread of the more recent contributions concerns the development of a regularity theory making mainly use of purely analytical techniques, see, e.g., [6, 14, 19, 20, 26, 27] and the references therein, so to be able to treat PDEs where the leading operator is $\mathcal{L}$.

As already announced, our interest in this article is to start the study of mixed local–nonlocal PDEs with a singular potential. We just mention that other singular PDEs driven by $\mathcal{L}$ have been considered so far, see, e.g., [5, 9, 18] as well as purely nonlocal PDEs, see, e.g., [4] and the references therein. Coming back to our issues and in order to clarify the choice of the purely local Hardy potential, the starting point is the Hardy inequality, originally proved in [21] in the one-dimensional case and subsequently extended in various directions:

$$C \int_{\mathbb{R}^n} \frac{u^2}{|x|^2}\, dx \leq \int_{\mathbb{R}^n} |\nabla u|^2\, dx, \quad u \in C_0^\infty(\mathbb{R}^n), \tag{1.1}$$

where $C > 0$. As it is customary when dealing with functional inequalities, it is interesting to detect whether there is an optimal constant for which (1.1) holds

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and whether this is ever achieved. These questions are nowadays classical for the inequality (1.1): the best constant is purely dimensional, it is given by

$$\Lambda_n := \frac{(n-2)^2}{4},$$

and it is never achieved. Similarly, there exists an optimal positive constant $\overline{\Lambda}_{n,s}$, depending only on n and s and never achieved, such that

$$\overline{\Lambda}_{n,s} \int_{\mathbb{R}^n} \frac{u^2}{|x|^{2s}}\,dx \le \frac{C_{n,s}}{2}[u]_s^2, \quad u \in C_0^\infty(\mathbb{R}^n), \tag{1.2}$$

where

$$[u]_s^2 := \iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy,$$

denotes the Gagliardo seminorm of u (of order $s \in (0,1)$).

The inequalities (1.1) with $C = \Lambda_n$ and (1.2) are the starting point of our mixed local–nonlocal analysis. Clearly, the following trivially hold

$$\Lambda_n \int_{\mathbb{R}^n} \frac{u^2}{|x|^2}\,dx \le \int_{\mathbb{R}^n} |\nabla u|^2\,dx + \frac{C_{n,s}}{2}[u]_s^2, \quad u \in C_0^\infty(\mathbb{R}^n),$$

and

$$\overline{\Lambda}_{n,s} \int_{\mathbb{R}^n} \frac{u^2}{|x|^{2s}}\,dx \le \int_{\mathbb{R}^n} |\nabla u|^2\,dx + \frac{C_{n,s}}{2}[u]_s^2, \quad u \in C_0^\infty(\mathbb{R}^n).$$

As probably expected, the *best* Hardy potential (in a sense to be specified later on, see after proposition 2.6) to be considered is the one coming from the local Hardy inequality. This leads to the following natural question: does the minimization problem

$$\Lambda_{s,n} := \inf_{u \in C_0^\infty(\mathbb{R}^n)\setminus\{0\}} \frac{\rho(u)^2}{\mathcal{H}(u)} \tag{1.3}$$

admit a minimizer? Here, we have introduced the shorthand notation

$$\rho(u)^2 := \|\nabla u\|^2_{L^2(\mathbb{R}^n)} + \frac{C_{n,s}}{2}[u]_s^2 \quad \text{and} \quad \mathcal{H}(u) := \int_{\mathbb{R}^n} \frac{u^2}{|x|^2}\,dx.$$

Our first result answers to the above question.

THEOREM 1.1. *The number defined by the minimization problem* (1.3) *coincides with the optimal local Hardy constant* Λ_n *and it is never achieved.*

Having in mind applications to mixed local–nonlocal PDEs, we are also interested in studying Hardy-type inequalities in open and bounded sets $\Omega \subset \mathbb{R}^n$. The second main result of the present article is the following.

THEOREM 1.2. *Let* $\Omega \subset \mathbb{R}^n$ *be an open and bounded set such that* $0 \in \Omega$. *Then*

$$\Lambda_{s,n}(\Omega) := \inf\left\{\rho(u)^2 : u \in C_0^\infty(\Omega)\ s.t.\ \mathcal{H}(u) = 1\right\},$$

is independent of Ω, *coincides with* Λ_n *and it is never achieved.*

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In light of similar results proved for mixed local–nonlocal Sobolev inequalities (see [24]), the above fact is not so surprising and it is motivated by the lack of scaling invariance of the functional $\rho(u)$.

The aforementioned results open the way to the study of mixed local–nonlocal PDEs in presence of the classical Hardy potential, namely

$$\begin{cases} -\Delta u + (-\Delta)^s u - \gamma \dfrac{u}{|x|^2} = f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{$\mathrm{D}_{\Gamma,f}$}$$

where $n \geq 3$, $\Omega \subset \mathbb{R}^n$ is an open and bounded set with smooth enough boundary and such that $0 \in \Omega$, $f \in L^m(\Omega)$ for some $m \geq 1$ and $\gamma \in (0, \Lambda_n)$.

The *boundary condition* is the purely nonlocal one and it is quite natural if one is interested in Dirichlet-type boundary conditions, see, e.g., [6] for examples in the case $\gamma = 0$.

Our aim is to study the regularity of the solutions of ($\mathrm{D}_{\Gamma,f}$) in terms of the summability of the source term f: this forces to consider different notions of solutions and it is nowadays quite classical. This line of research is quite old in the case of variational local elliptic operators as well as purely nonlocal ones. We refer to §2.2 for a detailed review of the known results in the case of $-\Delta$, of $(-\Delta)^s$, and $\mathcal{L}$.

Having this background in mind, we state our first result which provides existence, optimal solvability, and gain of integrability of solutions which can be considered of *energy-type* since $f \in L^m(\Omega)$ with $m \geq \frac{2n}{n+2}$.

Remark 1.3. Here, and in all the articles, when we write $f \geq 0$, we mean that $f \gneq 0$ in Ω, i.e. f is not identically zero. Moreover, we point out that the positivity assumption on f is not a restriction, since our problem is linear.

Theorem 1.4. *Let $f \in L^m(\Omega)$ with $m \geq (2^*)' = \frac{2n}{n+2}$. Then, problem ($D_{\Gamma,f}$) admits a unique weak solution $u_f \in \mathcal{X}^{1,2}(\Omega)$ (see §2 for the relevant definition), in the following sense:*

$$\int_{\mathbb{R}^n} \nabla u_f \cdot \nabla v \, dx + \frac{C_{n,s}}{2} \iint_{\mathbb{R}^{2n}} \frac{(u_f(x) - u_f(y))(v(x) - v(y))}{|x-y|^{n+2s}} \, dx \, dy - \gamma \int_{\mathbb{R}^n} \frac{u_f v}{|x|^2} \, dx = \int_\Omega f v \, dx, \quad \text{for every } v \in \mathcal{X}^{1,2}(\Omega). \tag{1.4}$$

Moreover, if $f \geq 0$ then $u > 0$ in Ω. In addition, if f also belongs to $L^m(\Omega)$ with $\frac{2n}{n+2} \leq m < \frac{n}{2}$ and

$$0 < \gamma < \gamma(m) := \frac{n(m-1)(n-2m)}{m^2},$$

then the unique weak solution $u_f \in \mathcal{X}^{1,2}(\Omega)$ of problem ($D_{\Gamma,f}$) is such that

$$u_f \in L^{m^{**}}(\Omega), \quad \text{where } m^{**} := \frac{nm}{n-2m}.$$

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The next result deals with a source term $f \in L^m(\Omega)$ with $1 < m < \frac{2n}{n+2}$. In this case, variational arguments are forbidden but one can rely on the technique leading to *approximated solutions* by means of suitable truncation arguments.

THEOREM 1.5. *Let $f \in L^m(\Omega)$ with $1 < m < \frac{2n}{n+2}$ be a positive function. If*

$$0 < \gamma < \gamma(m) := \frac{n(m-1)(n-2m)}{m^2},$$

then there exists a positive distributional solution $u_f \in W_0^{1,m^*}(\Omega)$ of ($D_{\Gamma,\mathrm{f}}$), where $m^* := \frac{nm}{n-m}$: namely $u_f/|x|^2 \in L^1(\Omega)$ and it holds

$$\int_{\mathbb{R}^n} \nabla u_f \cdot \nabla v \, dx + \frac{C_{n,s}}{2} \iint_{\mathbb{R}^{2n}} \frac{(u_f(x) - u_f(y))(v(x) - v(y))}{|x-y|^{n+2s}} \, dx \, dy - \gamma \int_{\mathbb{R}^n} \frac{u_f v}{|x|^2} \, dx = \int_\Omega f v \, dx, \quad \text{for every } v \in C_0^\infty(\Omega). \tag{1.5}$$

REMARK 1.6. As we will see in the proof of theorem 1.5, the solution u_f is obtained as the limit of solutions of the regularized problems (3.12). We also prove that such a type of solution (usually referred to as SOLA solution) is unique. Moreover using the fact that $u_f \in W_0^{1,m^*}(\Omega)$, by a density argument it can be shown that (1.5) holds in a stronger sense, that is $u_f \in W_0^{1,m^*}(\Omega)$ and it holds

$$\int_{\mathbb{R}^n} \nabla u_f \cdot \nabla v \, dx + \frac{C_{n,s}}{2} \iint_{\mathbb{R}^{2n}} \frac{(u_f(x) - u_f(y))(v(x) - v(y))}{|x-y|^{n+2s}} \, dx \, dy - \gamma \int_{\mathbb{R}^n} \frac{u_f v}{|x|^2} \, dx = \int_\Omega f v \, dx, \quad \text{for every } v \in W_0^{1,(m^*)'}(\Omega),$$

where $(m^*)' = m^*/(m^*-1)$.

The last result concerns the case of $f \in L^1(\Omega)$, where we cannot ensure (in general) that $f \in \mathbb{X} = \left(\mathcal{X}^{1,2}(\Omega)\right)'$, and thus variational arguments are once again forbidden. As for the local case [2, 23], we are able to provide an optimal solvability criteria. Nevertheless, due to the nature of $\mathcal{L}$, which makes the task of finding explicit solutions quite hard, our condition is somehow implicit. In order to give the existence result, we need to consider solutions of an auxiliary problem. Let $g \in L^\infty(\Omega)$ and let $w \in \mathcal{X}^{1,2}(\Omega)$ be the unique weak solution of

$$\begin{cases} -\Delta w + (-\Delta)^s w = g & \text{in } \Omega, \\ w = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.6}$$

in the sense of (1.4), with $\gamma = 0$. We note that the unique solution w belongs to $L^\infty(\Omega)$. Now, we introduce the notion of solution in this setting, adapting the classical concept of duality.

DEFINITION 1.7. *Let $f \in L^1(\Omega)$, and let $u \in L^1(\Omega)$. We say that* u *is a duality solution of ($D_{\Gamma,\mathrm{f}}$), if the following conditions hold:*

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(1) $u \equiv 0$ *a.e. in* $\mathbb{R}^n \setminus \Omega$ *and* $u/|x|^2 \in L^1(\Omega)$*;*
(2) for every $g \in L^\infty(\Omega)$ *and* w *solution to (1.6), we have*

$$\int_\Omega gu\,dx = \gamma \int_\Omega \frac{u}{|x|^2} w\,dx + \int_\Omega fw\,dx.$$

REMARK 1.8. We notice that any function u which solves ($D_{\Gamma,f}$) in the sense of theorems 1.4 and 1.5 is also duality solutions (in the sense of definition 1.7).

With definition 1.7 at hand, we can now state the *optimal-solvability result* for problem ($D_{\Gamma,f}$). Before doing this, we first fix a notation: *we denote by* Φ_Ω *the unique weak solution* in $\mathcal{X}^{1,2}(\Omega)$ of the following problem

$$\begin{cases} \mathcal{L}u = \gamma \dfrac{u}{|x|^2} + 1 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

We explicitly observe that, since the constant function $f \equiv 1$ is positive and bounded on Ω, the existence and uniqueness of Φ_Ω follow from theorem 1.4; moreover, we also have

$$\Phi_\Omega > 0 \text{ a.e. in } \Omega.$$

THEOREM 1.9. *Let* $f \in L^1(\Omega)$, $f \geq 0$ *a.e. in* Ω*. Then, there exists a positive duality solution* $u \in L^1(\Omega)$ *of problem (*$D_{\Gamma,\mathrm{f}}$*) in the sense of definition* 1.7 *if and only if*

$$\int_\Omega f\Phi_\Omega\,dx < \infty. \tag{1.7}$$

In this case, one can also prove that u *satisfies the following properties:*

(1) for every $k \in \mathbb{N}$*, one has* $T_k(u) \in \mathcal{X}^{1,2}(\Omega)$ *(see* (2.11) *for the definition of* T_k*);*
(2) $u|_\Omega$ *is uniformly bounded in* $W_0^{1,p}(\Omega)$ *for every* $p < \frac{n}{n-1}$.

The article is organized as follows: in §2 we firstly describe the natural functional setting associated with $\mathcal{L}$ (see §2.1), we then review all the regularity results connected with our problem (see §2.2), and then we prove both theorems 1.1 and 1.2 (see §2.3). In §3, we move to the study of ($D_{\Gamma,f}$) proving theorems 1.4, 1.5, and 1.9.

2. The functional setting and preliminary results

The aim of this section is to introduce the adequate functional setting for the study of the mixed operator $\mathcal{L} = -\Delta + (-\Delta)^s$ and to collect some known results concerning the $\mathcal{L}$-Dirichlet problem

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$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega, \\ u \equiv 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \tag{D_f}$$

where $\Omega \subseteq \mathbb{R}^n$ is a bounded domain and $f \in L^m(\Omega)$ for some $m \geq 1$. We also study mixed Hardy-type inequality, by proving theorems 1.1 and 1.2.

2.1. The functional setting

Let $s \in (0,1)$ be fixed, and let $\varnothing \neq \Omega \subseteq \mathbb{R}^n$ (with $n \geq 3$) be an *arbitrary* open set, not necessarily bounded. We define the function space $\mathcal{X}^{1,2}(\Omega)$ as the completion of $C_0^\infty(\Omega)$, that is the set of smooth functions with compact support in Ω, with respect to the *mixed global norm*

$$\rho(u) := \left(\|\nabla u\|^2_{L^2(\mathbb{R}^n)} + \frac{C_{n,s}}{2}[u]_s^2 \right)^{1/2}, \qquad u \in C_0^\infty(\Omega),$$

where $[u]_s$ denotes the so-called *Gagliardo seminorm* of u (of order s), that is,

$$[u]_s := \left(\iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \right)^{1/2}.$$

Remark 2.1. A couple of observations concerning the space $\mathcal{X}^{1,2}(\Omega)$ are in order.

(1) The norm $\rho(\cdot)$ is induced by the scalar product (or bilinear form)

$$\mathcal{B}(u,v) := \int_{\mathbb{R}^n} \nabla u \cdot \nabla v\, dx + \frac{C_{n,s}}{2} \iint_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\, dx\, dy,$$

where $\cdot$ denotes the usual scalar product in the Euclidean space $\mathbb{R}^n$, while

$$C_{n,s} := 2^{2s}\pi^{-\frac{n}{2}}\Gamma((n+2s)/2)/|\Gamma(-s)|;$$

thus, $\mathcal{X}^{1,2}(\Omega)$ is endowed with a structure of a (real) *Hilbert space*.

(2) Even if a function $u \in C_0^\infty(\Omega)$ *identically vanishes outside* Ω, it is often still convenient to consider in the definition of $\rho(\cdot)$ the L^2-norm of ∇u *on the whole of* $\mathbb{R}^n$, rather than restricted to Ω (though of course the result would be the same): this is to stress that the elements in $\mathcal{X}^{1,2}(\Omega)$ are functions defined *on the entire space* $\mathbb{R}^n$ and not only on Ω (and this is consistent with the nonlocal nature of the operator $\mathcal{L}$). The benefit of having this global functional setting is that these functions can be *globally approximated on* $\mathbb{R}^n$ (with respect to the norm $\rho(\cdot)$) by smooth functions with compact support in Ω.

In particular, when $\Omega \neq \mathbb{R}^n$, we will see that this *global* definition of $\rho(\cdot)$ implies that the functions in $\mathcal{X}^{1,2}(\Omega)$ naturally satisfy the nonlocal Dirichlet condition

$$u \equiv 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \text{ for every } u \in \mathcal{X}^{1,2}(\Omega). \tag{2.1}$$

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In order to better understand the *nature* of the space $\mathcal{X}^{1,2}(\Omega)$ (and to recognize the validity of (2.1)), we distinguish two cases.

(i) If Ω *is bounded*. In this case, we first recall the following inequality, which expresses the *continuous embedding* of $H^1(\mathbb{R}^n)$ into $H^s(\mathbb{R}^n)$ (see, e.g., [15, proposition 2.2]): there exists a constant $\mathbf{c} = \mathbf{c}(n,s) > 0$ such that, for every $u \in C_0^\infty(\Omega)$, one has

$$[u]_s^2 \leq \mathbf{c}(n,s)\|u\|^2_{H^1(\mathbb{R}^n)} = \mathbf{c}(n,s)\big(\|u\|^2_{L^2(\mathbb{R}^n)} + \|\nabla u\|^2_{L^2(\mathbb{R}^n)}\big). \tag{2.2}$$

This, together with the classical *Poincaré inequality*, implies that $\rho(\cdot)$ and the full H^1-norm in $\mathbb{R}^n$ are *actually equivalent* on the space $C_0^\infty(\Omega)$, and hence

$$\begin{aligned}\mathcal{X}^{1,2}(\Omega) &= \overline{C_0^\infty(\Omega)}^{\,\|\cdot\|_{H^1(\mathbb{R}^n)}} \\ &= \{u \in H^1(\mathbb{R}^n) : u|_\Omega \in H_0^1(\Omega) \text{ and} u \equiv 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}.\end{aligned}$$

(ii) If Ω *is unbounded*. In this case, even if the *embedding inequality* (2.2) is still satisfied, the Poincaré inequality *does not hold*; hence, the norm $\rho(\cdot)$ is no more equivalent to the full H^1-norm in $\mathbb{R}^n$, and $\mathcal{X}^{1,2}(\Omega)$ *is not a subspace of* $H^1(\mathbb{R}^n)$.

On the other hand, by the classical *Sobolev inequality*, there exists a constant $\mathcal{S} = \mathcal{S}_n > 0$, *independent of the open set* Ω, such that

$$\mathcal{S}_n\|u\|^2_{L^{2^*}(\mathbb{R}^n)} \leq \|\nabla u\|^2_{L^2(\mathbb{R}^n)} \leq \rho(u)^2 \qquad \text{for every } u \in C_0^\infty(\Omega). \tag{2.3}$$

As a consequence, we deduce that $\mathcal{X}^{1,2}(\Omega) \hookrightarrow L^{2^*}(\mathbb{R}^n)$.

Remark 2.2. We explicitly notice that, since the *mixed Sobolev-type inequality* (2.3) holds *for every open set* $\Omega \subseteq \mathbb{R}^n$ (bounded or not), we always have

$$\mathcal{X}^{1,2}(\Omega) \hookrightarrow L^{2^*}(\mathbb{R}^n). \tag{2.4}$$

Furthermore, by exploiting the density of $C_0^\infty(\Omega)$ in $\mathcal{X}^{1,2}(\Omega)$, we can extend inequality (2.3) to *every function* $u \in \mathcal{X}^{1,2}(\Omega)$, thereby obtaining

$$\mathcal{S}_n\|u\|^2_{L^{2^*}(\mathbb{R}^n)} \leq \rho(u)^2 = \|\nabla u\|^2_{L^2(\mathbb{R}^n)} + \frac{C_{n,s}}{2}[u]_s^2 \quad \text{for every } u \in \mathcal{X}^{1,2}(\Omega).$$

Due to its relevance in the sequel, we also introduce a distinguished notation for the cone of the non-negative functions in $\mathcal{X}^{1,2}(\Omega)$: we set

$$\mathcal{X}^{1,2}_+(\Omega) = \big\{u \in \mathcal{X}^{1,2}(\Omega) : u \geq 0 \text{ a.e. in } \Omega\big\}.$$

2.2. The $\mathcal{L}$-Dirichlet problem

Now we have introduced the space $\mathcal{X}^{1,2}(\Omega)$, we present some known results concerning *existence and optimal regularity of solutions* for the $\mathcal{L}$-Dirichlet problem (D_f). To this end, we distinguish two different cases.

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(i) $f \in L^m(\Omega)$ for some $m \geq (2^*)' = \dfrac{2n}{n+2}$;

(ii) $f \in L^m(\Omega)$ for some $1 < m < (2^*)'$.

Case (i) - The variational case. In this case, we observe that, by combining the *continuous embedding* (2.4) of $\mathcal{X}^{1,2}(\Omega)$ with the Hölder inequality, we have

$$\int_\Omega |fv|\,dx < +\infty \quad \forall\, v \in \mathcal{X}^{1,2}(\Omega),$$

and thus $f \in \mathbb{X} = (\mathcal{X}^{1,2}(\Omega))'$; in view of this fact, *existence and uniqueness* of solutions for problem (D_f) easily follow from the Lax–Milgram theorem. More precisely, we present the following theorem, which collects most of the published results regarding the existence, uniqueness, and regularity of solutions to problem (D_f).

THEOREM 2.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with smooth boundary, and let $f \in L^m(\Omega)$ for some $m \geq (2^*)'$. Then,* there exists a unique weak solution $u_f \in \mathcal{X}^{1,2}(\Omega)$ *of problem* (D_f)*, in the following sense:*

$$\mathcal{B}(u_f, v) = \int_\Omega f v\,dx \qquad \forall\, v \in \mathcal{X}^{1,2}(\Omega). \tag{2.5}$$

Furthermore, the following properties hold.

(i) *(see [6, theorem 4.7]) If $f \in L^m(\Omega)$ with $m > n/2$, then $u_f \in L^\infty(\mathbb{R}^n)$.*

(ii) *(see [5, theorem 2.3]) If $(2^*)' \leq m \leq n/2$, then $u_f \in L^{m^{**}}(\Omega)$, where*

$$m^{**} = \frac{nm}{n-2m}.$$

(iii) *(see [3, theorem 1.1]) If $f \in L^m(\Omega)$ with* $\mathrm{m} > \mathrm{n}$*, then*

$$u_f \in C^{1,\theta}(\overline{\Omega}) \text{ for some } \theta \in (0,1).$$

(iv) *(see [7, theorem B.1]) If $f \in C^\alpha(\overline{\Omega})$ for some $\alpha \in (0,1)$, and if $2s + \alpha < 1$ (hence, in particular, $0 < s < 1/2$), then $u_f \in C^{2,\alpha}(\overline{\Omega})$.*

(v) *(see, e.g., [6, theorem 1.2] and [8, corollary 3.3 and remark 3.4]) If $f \geq 0$ a.e. in Ω, then we have $u_f \geq 0$ a.e. in Ω; moreover, either $u_f \equiv 0$ or $u_f > 0$ in Ω.*

Case (ii) - The non-variational case. In this case, the inclusion $L^m(\Omega) \subseteq \mathbb{X} = (\mathcal{X}^{1,2}(\Omega))'$ *does not hold*, and we cannot directly apply the Lax–Milgram theorem to study the existence of solutions for problem (D_f). As in the purely local setting, *existence and improved regularity* are proved at the same time by an approximation argument.

THEOREM 2.4 See [5, theorem 2.1] *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with smooth boundary, and let $f \in L^m(\Omega)$ for some $1 < m < (2^*)'$, $f \geq 0$ a.e. in Ω. Then, there exists a positive solution* $\mathrm{u_f}$ *of problem* (D_f)*, in the following sense*

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(a) $u_f \in W_0^{1,m^*}(\Omega)$, *with* $m^* = nm/(n-m)$;
(b) *for every* $v \in W_0^{1,(m^*)'}(\Omega) \cap L^{m'}(\Omega)$ *with* compact support, *we have*

$$\mathcal{B}(u_f, v) = \int_\Omega f v \, dx.$$

REMARK 2.5. A comparison with the local/nonlocal case It is interesting to notice that, on account of theorems 2.3–2.4, the *improved integrability* for the solutions of (D_f) is generated only by $-\Delta$, thus reflecting the merely perturbative role of $(-\Delta)^s$, at least when working on bounded sets. More precisely, let u_f be the solution (to be specified, depending on the summability of f) of

$$\begin{cases} -\Delta u = f \in L^m(\Omega) & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega. \end{cases}.$$

Then, in [25] where more general operators are considered, Stampacchia showed that

(i) if $1 < m < (2^*)'$ then $u_f \in W_0^{1,m^*}(\Omega)$;
(ii) if $(2^*)' \le m \le n/2$ then $u_f \in H_0^1(\Omega) \cap L^{m^{**}}(\Omega)$;
(iii) if $m > n/2$ then $u_f \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

A similar results holds true in the purely nonlocal framework, that is, for the solution u_f of the following *nonlocal Dirichlet problem*

$$\begin{cases} (-\Delta)^s u = f \in L^m(\Omega) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}.$$

In this case, setting

$$m_s^* := \frac{nm}{n-ms} \quad \text{and} \quad m_s^{**} := \frac{nm}{n-2ms},$$

we have that (see [22, theorems 13, 16, 17, 24])

(a) if $1 < m < 2n/(n+2s)$ then the weak-duality solution $u_f \in L^{m_s^{**}}(\Omega)$;
(b) if $2n/(n+2s) \le m \le n/(2s)$ then $u_f \in H_0^s(\Omega) \cap L^{m_s^{**}}(\Omega)$;
(c) if $m > n/(2s)$ then $u \in H_0^s(\Omega) \cap L^\infty(\Omega)$.

2.3. Mixed Hardy-type inequalities

Now, we focus our attention on the mixed Hardy inequalities. In order to do this, we need to introduce some notations and recall some well-known results. Let $n \ge 3$. The classical Hardy inequality states that

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$$\frac{(n-2)^2}{4}\int_{\mathbb{R}^n}\frac{u^2}{|x|^2}\,dx \le \int_{\mathbb{R}^n}|\nabla u|^2\,dx, \quad u\in C_0^\infty(\mathbb{R}^n),$$

and it is known that this constant $\Lambda_n = (n-2)^2/4$ is never achieved, see, e.g., [23]. Clearly,

$$\Lambda_n\int_{\mathbb{R}^n}\frac{u^2}{|x|^2}\,dx \le \int_{\mathbb{R}^n}|\nabla u|^2\,dx \le \rho(u)^2, \quad u\in C_0^\infty(\mathbb{R}^n). \tag{2.6}$$

In particular, due to the mixed nature of the norm $\rho(u)$, it holds

$$\overline{\Lambda}_{n,s}\int_{\mathbb{R}^n}\frac{u^2}{|x|^{2s}}\,dx \le \rho(u)^2, \quad u\in C_0^\infty(\mathbb{R}^n), \tag{2.7}$$

as well, where $\overline{\Lambda}_{n,s}$ denotes the optimal constant for the fractional Hardy inequality, see, e.g., [23, Chapter 9]. Combining (2.6) and (2.7), we get the following.

PROPOSITION 2.6. *For every $p\in[2s,2]$, there exists a positive constant $C>0$, depending only on n and s, such that*

$$C\int_{\mathbb{R}^n}\frac{u^2}{|x|^p}\,dx \le \rho(u)^2, \quad u\in C_0^\infty(\mathbb{R}^n), \tag{2.8}$$

Proof. Let $p\in[2s,2]$ be arbitrarily fixed. Owing to (2.6) and (2.7), we have

$$\begin{aligned}\int_{\mathbb{R}^n}\frac{u^2}{|x|^p}\,dx &= \int_{\{|x|\le 1\}}\frac{u^2}{|x|^p}\,dx + \int_{\{|x|>1\}}\frac{u^2}{|x|^p}\,dx\\ &\le \int_{\{|x|\le 1\}}\frac{u^2}{|x|^2}\,dx + \int_{\{|x|>1\}}\frac{u^2}{|x|^{2s}}\,dx\\ &\le \Big(\frac{1}{\Lambda_n}+\frac{1}{\overline{\Lambda}_{n,s}}\Big)\rho(u)^2 \quad \forall\, u\in C_0^\infty(\mathbb{R}^n),\end{aligned}$$

and this is precisely the desired (2.8). □

We notice that for $p=2$ we can take $C=\Lambda_n$, while for $p=2s$ we can take $C=\overline{\Lambda}_{n,s}$; however, due to the monotonicity of $|x|^{-p}$, it seems that the most interesting case is the one for $p=2$. Let us focus for a moment to the case $p=2$, and let us introduce the shorthand notation

$$\mathcal{H}(u) := \int_{\mathbb{R}^n}\frac{u^2}{|x|^2}\,dx.$$

It is natural to wonder whether the minimization problem

$$\Lambda_{s,n} := \inf_{u\in C_0^\infty(\mathbb{R}^n)\setminus\{0\}}\frac{\rho(u)^2}{\mathcal{H}(u)}$$

admits a minimizer, that is whether $\Lambda_{s,n}$ is achieved or not. The answer is given by theorem 1.1 which we will prove right below:

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Proof of theorem 1.1. Let $u \in C_0^\infty(\mathbb{R}^n)$. By the very definition of $\rho(u)$, we have that

$$\Lambda_n \leq \frac{\|\nabla u\|^2_{L^2(\mathbb{R}^n)}}{\mathcal{H}(u)} \leq \frac{\rho(u)^2}{\mathcal{H}(u)},$$

and therefore, taking the infimum,

$$\Lambda_n \leq \Lambda_{s,n}. \tag{2.9}$$

On the other hand, if $u \in C_0^\infty(\mathbb{R}^n)$, we have that (for $\lambda > 0$),

$$u_\lambda(x) := \lambda^{\frac{n-2}{2}} u(\lambda x) \in C_0^\infty(\mathbb{R}^n),$$

and therefore it can be used as a test function, finding

$$\begin{aligned}\Lambda_{s,n} \leq \frac{\rho(u_\lambda)^2}{\mathcal{H}(u_\lambda)} &= \frac{\|\nabla u\|^2_{L^2(\mathbb{R}^n)} + \lambda^{2s-2}\frac{C_{n,s}}{2}[u]_s^2}{\mathcal{H}(u)} \\ &= \frac{\|\nabla u\|^2_{L^2(\mathbb{R}^n)}}{\mathcal{H}(u)} + \lambda^{2s-2}\frac{C_{n,s}}{2}\frac{[u]_s^2}{\mathcal{H}(u)} \to \frac{\|\nabla u\|^2_{L^2(\mathbb{R}^n)}}{\mathcal{H}(u)}, \quad \text{as } \lambda \to +\infty.\end{aligned}$$

We can then conclude that

$$\Lambda_{s,n} \leq \Lambda_n,$$

which combined with (2.9) proves $\Lambda_{s,n} = \Lambda_n$. The fact that the constant is never achieved follows from the upcoming proposition 2.8. □

Let us now move to the case of bounded sets. Firstly, let us define the set

$$\mathcal{M}(\Omega) := \{u \in C_0^\infty(\Omega) : \mathcal{H}(u) = 1\},$$

and the constants

$$\Lambda_{s,n}(\Omega) := \inf_{u \in C_0^\infty(\Omega)\setminus\{0\}} \frac{\rho(u)^2}{\mathcal{H}(u)} = \inf\left\{\rho(u)^2 : u \in C_0^\infty(\Omega) \cap \mathcal{M}(\Omega)\right\}.$$

We now prove the following proposition leading to the demonstration of theorem 1.2.

PROPOSITION 2.7. *Let $\Omega \subseteq \mathbb{R}^n$ be an open and bounded set such that $0 \in \Omega$. Then*

$$\Lambda_{s,n}(\Omega) = \Lambda_n. \tag{2.10}$$

Proof. Since $\rho(u) \geq \|\nabla u\|_{L^2(\Omega)}$ for every $u \in C_0^\infty(\Omega)$, it holds that

$$\Lambda_{s,n}(\Omega) \geq \Lambda_n.$$

To prove the reverse inequality, we consider $\mathrm{r} > 0$ be such that $B_r(0) \subseteq \Omega$. We now observe that, given any $u \in C_0^\infty(\mathbb{R}^n) \cap \mathcal{M}(\mathbb{R}^n)$, there exists $k_0 = k_0(u) \in \mathbb{N}$ such that

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$$\mathrm{supp}(u) \subseteq B_{kr}(0) \quad \textit{for every } k \geq k_0;$$

as a consequence, setting $u_k(x) := k^{\frac{n-2}{2}} u(kx)$ *(for* $k \geq k_0$*), we readily see that*

$$\mathrm{supp}(u_k) \subseteq B_r(0) \subseteq \Omega \quad \textit{and} \quad \mathcal{H}(u_k) = 1.$$

Taking into account the definition of $\Lambda_{s,n}(\Omega)$*, we find that, for every* $k \geq k_0$*,*

$$\Lambda_{s,n}(\Omega) \leq \rho(u_k)^2 = \|\nabla u_k\|^2_{L^2(\mathbb{R}^n)} + \frac{C_{n,s}}{2}[u_k]^2_s = \|\nabla u\|^2_{L^2(\mathbb{R}^n)} + k^{2s-2}\frac{C_{n,s}}{2}[u]^2_s.$$

From this, letting $k \to \infty$ *(and recalling that* $\mathrm{s} < 1$*), we obtain*

$$\Lambda_{s,n}(\Omega) \leq \|\nabla u\|^2_{L^2(\mathbb{R}^n)}.$$

By the arbitrariness of $u \in C_0^\infty(\mathbb{R}^n) \cap \mathcal{M}(\mathbb{R}^n)$ *and the fact that* Λ_n is independent of the open set Ω*, we finally infer that*

$$\Lambda_{s,n}(\Omega) \leq \inf\left\{\|\nabla u\|^2_{L^2(\mathbb{R}^n)} : u \in C_0^\infty(\mathbb{R}^n) \cap \mathcal{M}(\mathbb{R}^n)\right\} = \Lambda_n,$$

and hence $\Lambda_{s,n}(\Omega) = \Lambda_n$. □

Proposition 2.8. *Let* $\Omega \subseteq \mathbb{R}^n$ *be an open set (not necessarily bounded). Then,* $\Lambda_{s,n}(\Omega)$ *is never achieved.*

Proof. Arguing by contradiction, let us suppose that there exists a function $u_0 \in \mathcal{X}^{1,2}(\Omega)$ such that $\mathcal{H}(u_0) = 1$ (hence, $u_0 \not\equiv 0$) and

$$\rho(u_0)^2 = \|\nabla u_0\|^2_{L^2(\mathbb{R}^n)} + \frac{C_{n,s}}{2}[u_0]^2_s = \Lambda_n.$$

Since $\mathcal{X}^{1,2}(\Omega) \subseteq \mathcal{D}^{1,2}(\Omega)$, we can then infer that

$$\Lambda_n \leq \|\nabla u_0\|^2_{L^2(\Omega)} \leq \|\nabla u_0\|^2_{L^2(\mathbb{R}^n)} + \frac{C_{n,s}}{2}[u_0]^2_s = \rho(u_0)^2 = \Lambda_n,$$

from which we derive that $[u_0]_s = 0$. As a consequence, the function u_0 must be *constant in* $\mathbb{R}^n$, but this is contradiction with the fact that $\mathcal{H}(u_0) = 1$. □

We are, therefore, ready to prove theorem 1.2.

Proof of theorem 1.2. It is enough to combine proposition 2.7 with proposition 2.8. □

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We end this section with the following lemma, showing a connection between the Hardy-type inequality (2.8) and the existence of solutions for the variational inequality

$$\mathcal{L}u \geq \gamma \frac{u}{|x|^2} \quad \text{in } \Omega.$$

In order to state and prove such a result, we first fix a notation: given any $k \in \mathbb{N}$, we set

$$T_k(t) := \max\{\min\{k, t\}, -k\} = \begin{cases} t, & |t| \leq k, \\ k\dfrac{t}{|t|}, & |t| > k. \end{cases} \tag{2.11}$$

LEMMA 2.9. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with smooth boundary, let $\gamma > 0$ be fixed, and let $u \in \mathcal{X}^{1,2}(\Omega)$, $u > 0$, be such that*

$$\mathcal{B}(u, v) \geq \gamma \int_\Omega \frac{uv}{|x|^2}\, dx \qquad \textit{for all } v \in \mathcal{X}^{1,2}_+(\Omega).$$

Then, we have

$$\rho(v)^2 \geq \gamma \int_\Omega \frac{v^2}{|x|^2}\, dx \quad \textit{for all } v \in \mathcal{X}^{1,2}(\Omega).$$

This proof is inspired by [17, lemma B.1]. First of all, let us consider the following truncated problem given by

$$\begin{cases} \mathcal{L}w = T_1\left(\frac{u}{|x|^2}\right) & \text{in } \Omega \\ w = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{2.12}$$

We know that problem (2.12) admits a unique positive weak solution $w \in \mathcal{X}^{1,2}(\Omega)$. Hence, noticing that

$$\mathcal{L}u \geq \gamma \frac{u}{|x|^2} \geq \gamma T_1\left(\frac{u}{|x|^2}\right) = \mathcal{L}w \quad \text{in } \Omega,$$

and since the operator $\mathcal{L}$ is linear, we easily deduce that

$$\begin{cases} \mathcal{L}(w - u) \leq 0 & \text{in } \Omega \\ w - u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

As a consequence, since $v = w - u \in \mathcal{X}^{1,2}(\Omega) \subseteq H^1(\mathbb{R}^n)$, we are entitled to apply the weak maximum principle in [6, theorem 1.2], obtaining

$$u \geq w \qquad \text{in } \mathbb{R}^n.$$

In particular, since $w \in C^{1,\alpha}(\overline{\Omega})$ (see theorem 2.3), we deduce that for every compact set $\mathcal{K} \subset \Omega$ there exists a constant $\vartheta > 0$ such that

$$u \geq w \geq \vartheta > 0 \qquad \text{in } \mathcal{K}, \tag{2.13}$$

and this ensures that $u^{-1} \in L^\infty_{\mathrm{loc}}(\Omega)$.

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Now we have established (2.13), we can easily complete the proof of the lemma. Indeed, let $\varphi \in C_0^\infty(\Omega)$ be arbitrarily fixed, and let

$$\psi = u^{-1}\varphi^2.$$

By (2.13) (and since $\operatorname{supp}(\varphi) \subset \Omega$), we have $\psi \in L^\infty(\Omega)$; moreover,

$$(u(x)-u(y))(\psi(x)-\psi(y)) \le |\varphi(x)-\varphi(y)|^2.$$

Indeed, by the very definition of ψ, we have

$$\begin{aligned}
(u(x)-u(y))(\psi(x)-\psi(y)) &= \varphi(x)^2+\varphi(y)^2 - u(x)\frac{\varphi^2(y)}{u(y)} - u(y)\frac{\varphi^2(x)}{u(x)}\\
&= (\varphi(x)-\varphi(y))^2 + 2\varphi(x)\varphi(y) - u(x)\frac{\varphi^2(y)}{u(y)} - u(y)\frac{\varphi^2(x)}{u(x)}\\
&= (\varphi(x)-\varphi(y))^2 + 2\varphi(x)\varphi(y) - u(x)u(y)\frac{\varphi^2(y)}{u^2(y)} - u(x)u(y)\frac{\varphi^2(x)}{u^2(x)}\\
&= (\varphi(x)-\varphi(y))^2 - u(x)u(y)\Big\{\frac{\varphi^2(x)}{u^2(x)} + \frac{\varphi^2(y)}{u^2(y)} - 2\frac{\varphi(x)}{u(x)}\frac{\varphi(y)}{u(y)}\Big\}\\
&= (\varphi(x)-\varphi(y))^2 - u(x)u(y)\Big(\frac{\varphi(x)}{u(x)} - \frac{\varphi(y)}{u(y)}\Big)^2\\
&\le (\varphi(x)-\varphi(y))^2,
\end{aligned} \tag{2.14}$$

where we have also used the fact that $u>0$ a.e. in Ω.

From now on we argue as in [17]. In order to prove the result, we take ψ as test function in

$$\int_{\mathbb{R}^n} \nabla u\cdot\nabla\psi\,dx + \frac{C_{n,s}}{2}\iint_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(\psi(x)-\psi(y))}{|x-y|^{n+2s}}\,dxdy \ge \gamma\int_\Omega \frac{u\psi}{|x|^2}\,dx.$$

Noticing that $\nabla u\cdot\nabla\psi = |\nabla\varphi|^2 - u^2|\nabla(u^{-1}\varphi)|^2 \le |\nabla\varphi|^2$, and using (2.14), we deduce

$$\int_{\mathbb{R}^n} |\nabla\varphi|^2\,dx + \frac{C_{n,s}}{2}\iint_{\mathbb{R}^{2n}} \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^{n+2s}}\,dxdy \ge \gamma\int_\Omega \frac{\varphi^2}{|x|^2}\,dx.$$

□

3. The $\mathcal{L}$-Dirichlet problem involving the Hardy potential

Taking into account all the results recalled and/or established so far, we now aim to study *existence and improved-integrability* of the solutions for the $\mathcal{L}$-Dirichlet problem ($\mathrm{D}_{\Gamma,f}$) defined in §1, that is

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$$\begin{cases} \mathcal{L}u - \gamma \dfrac{u}{|x|^2} = f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{$\mathrm{D}_{\Gamma,f}$}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is an open, regular, and bounded set with $0 \in \Omega$, and

$$f \in L^m(\Omega) \text{ for some } m \geq 1 \quad \text{and} \quad \gamma \in (0, \Lambda_n)\,.$$

To this end, motivated by the discussion in §2, we distinguish the following three cases, according to the values of m.

(i) $f \in L^m(\Omega)$ for some $m \geq (2^*)' = \dfrac{2n}{n+2}$;

(ii) $f \in L^m(\Omega)$ for some $1 < m < (2^*)'$;

(iii) $f \in L^1(\Omega)$.

Remark 3.1. We explicitly stress that the restriction $0 < \gamma < \Lambda_n$, generally, is motivated by the results in the previous section. Indeed, if we assume that there exists a *positive weak solution* of problem ($\mathrm{D}_{\Gamma,f}$) for some $\lambda > 0$ and some $f \geq 0$, say $u_f \in \mathcal{X}^{1,2}_+(\Omega)$, then

$$\mathcal{B}(u_f, v) = \gamma \int_\Omega \frac{u_f v}{|x|^2}\,dx + \int_\Omega f v\,dx \geq \gamma \int_\Omega \frac{u_f v}{|x|^2}\,dx \quad \text{for all } v \in \mathcal{X}^{1,2}_+(\Omega);$$

this, together with lemma 2.9 and theorem 1.1, implies that

$$\Lambda_n = \inf_{u \in C_0^\infty(\Omega)\setminus\{0\}} \frac{\rho(u)^2}{\mathcal{H}(u)} \geq \gamma.$$

Hence, the last fact motivates non-existence of positive energy solution of problem ($\mathrm{D}_{\Gamma,f}$) in the case $\gamma > \Lambda_n$ and f sufficiently regular.

On the other hand, if we consider the case $\gamma = \Lambda_n$ in ($\mathrm{D}_{\Gamma,f}$), there is a different approach that can be borrowed, as pointed out in [23]. Thanks to a more general version of the Hardy inequality, the improved Hardy inequality (see, e.g., [28]), one can define an Hilbert space $\mathcal{H}(\Omega)$ larger than $\mathcal{X}^{1,2}(\Omega)$ where it is possible to find a unique positive solution $u_f \in \mathcal{H}(\Omega)$ to ($\mathrm{D}_{\Gamma,f}$) if f belongs to the dual space $\mathcal{H}'(\Omega)$ (e.g., when $f \in L^m(\Omega)$ with $m > 2n/(n+2)$).

3.1. Case (i) - $f \in L^m(\Omega)$ with $m \geq (2^*)'$.

As already pointed out in §2, in this case, we have $f \in \mathbb{X} = (\mathcal{X}^{1,2}(\Omega))'$; as a consequence, we can investigate existence and uniqueness of solutions for (D_f) using the Lax–Milgram theorem, obtaining the following result.

Theorem 3.2. *Let $f \in L^m(\Omega)$ with $m \geq (2^*)'$. Then, problem ($\mathrm{D}_{\Gamma,f}$) admits a unique weak solution $u_f \in \mathcal{X}^{1,2}(\Omega)$, in the following sense:*

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$$\mathcal{B}(u,v) - \gamma \int_{\mathbb{R}^n} \frac{uv}{|x|^2}\,dx = \int_\Omega fv\,dx, \quad \text{for every } v \in \mathcal{X}^{1,2}(\Omega). \tag{3.1}$$

In particular, if $f \equiv 0$, the unique weak solution is the trivial one; while, if $f \geq 0$ a.e. in Ω, then $u_f \geq 0$ a.e. in Ω. Indeed, $u_f > 0$ a.e. in Ω.

Proof. The first assertion follows from Lax–Milgram theorem: in fact, it suffices to notice that the bilinear form defined as

$$\mathcal{B}(u,v) - \gamma \int_{\mathbb{R}^n} \frac{uv}{|x|^2}\,dx$$

is coercive and continuous on the Hilbert space $\mathcal{X}^{1,2}(\Omega)$. The second assertion easily follows noticing that $u \equiv 0$ is a weak solution to ($\mathrm{D}_{\Gamma,f}$) when $f \equiv 0$.

Finally, we turn to prove $u_f \geq 0$ a.e. in Ω when $f \geq 0$. To this end, we first observe that, since $u_f \in \mathcal{X}^{1,2}(\Omega)$, we have

$$v := (u_f)^- = \max\{-u_f, 0\} \in \mathcal{X}^{1,2}(\Omega).$$

We are then entitled to use this function v as a test function in (3.1), obtaining

$$\begin{aligned}
0 \leq \int_\Omega fv\,dx &= \mathcal{B}(u_f,v) - \gamma \int_{\mathbb{R}^n} \frac{u_f v}{|x|^2}\,dx \\
&= \int_{\mathbb{R}^n} \nabla u_f \cdot \nabla v\,dx + \frac{C_{n,s}}{2} \iint_{\mathbb{R}^{2n}} \frac{(u_f(x)-u_f(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy - \gamma \int_{\mathbb{R}^n} \frac{u_f v}{|x|^2}\,dx \\
&= -\Big(\int_\Omega |\nabla v|^2\,dx - \gamma \int_\Omega \frac{v^2}{|x|^2}\,dx \Big) + \frac{C_{n,s}}{2} \iint_{\mathbb{R}^{2n}} \frac{(u_f(x)-u_f(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy \\
&\text{(since } (u_f(x)-u_f(y))(v(x)-v(y)) \leq -|v(x)-v(y)|^2\text{)} \\
&\leq -\Big(\rho(v)^2 - \gamma \int_\Omega \frac{v^2}{|x|^2}\,dx \Big) \\
&\text{(using theorem 1.1)} \\
&\leq -\left(1 - \frac{\gamma}{\Lambda_n}\right) \rho(v)^2
\end{aligned}$$

From this, since $0 < \gamma < \Lambda_n$, we conclude that $v = (u_f)^- = 0$ a.e. in Ω, and thus

$$u_f \geq 0 \quad \text{a.e. in } \Omega. \tag{3.2}$$

To show that $u_f > 0$ a.e. in Ω we now observe that, since u_f is a weak solution of ($\mathrm{D}_{\Gamma,f}$), we can write

$$\mathcal{B}(u_f,v) = \int_\Omega g(x,u_f)v\,dx \quad \forall\ v \in \mathcal{X}^{1,2}(\Omega),$$

where $g(x,t) = f(x) + \lambda\, t/|x|^2$; thus, since $f > 0$ a.e. Ω, we have

$$g(x,t) \geq 0 \text{ for a.e. } x \in \Omega \text{ and } t \geq 0.$$

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Recalling (3.2), we can apply the strong maximum principle arguing as in the proof of [8, corollary 3.3], showing that $u_f > 0$ a.e. in Ω (as $f \not\equiv 0$). □

Similarly to the case $\gamma = 0$ discussed in §2, see theorem 2.3-(ii), also for the unique solution u_f of problem $(\mathrm{D}_{\Gamma,f})$ we have an *improved-integrability result*.

THEOREM 3.3. *Let $f \in L^m(\Omega)$ with $\frac{2n}{n+2} \le m < \frac{n}{2}$, be a positive function. If*

$$0 < \gamma < \gamma(m) := \frac{n(m-1)(n-2m)}{m^2},$$

then the unique weak solution $u_f \in \mathcal{X}^{1,2}(\Omega)$ of problem $(\mathrm{D}_{\Gamma,f})$ (whose existence is guaranteed by theorem 3.2) is such that

$$u_f \in L^{m^{**}}(\Omega), \quad \text{where } m^{**} := \tfrac{nm}{n-2m}.$$

Proof. As it is customary for such kind of problems, for every $k \in \mathbb{N}$, we consider the unique weak solution $\varphi_k \in \mathcal{X}^{1,2}(\Omega) \cap L^\infty(\mathbb{R}^n)$ of the regularized problem

$$\begin{cases} \mathcal{L}u = \gamma \dfrac{\varphi_{k-1}}{|x|^2 + 1/k} + f_k(x) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{3.3}$$

where $\varphi_0 := 0$ and $f_k(x) := T_k(f(x))$. Owing to lemma A.1, we know that

(i) φ_k is well defined for every $k \ge 1$, and $\{\varphi_k\}_k$ is non-decreasing;
(ii) $\varphi_k \to u_f$ as $k \to +\infty$ strongly in $L^p(\Omega)$ for every $p \in [1, 2^*)$, where $u_f \in \mathcal{X}^{1,2}(\Omega)$ is the unique weak solution of problem $(\mathrm{D}_{\Gamma,f})$ (in the sense of theorem 3.2).

Let now

$$\alpha := \frac{m(n-2)}{n-2m}, \tag{3.4}$$

and notice the following:

(a) $m \ge \frac{2n}{n+2}$ is equivalent to $\alpha \ge 2$;
(b) denoting by $m' = \frac{m}{m-1}$ the usual Hölder conjugate exponent of m, one has

$$\frac{2^*\alpha}{2} = \frac{nm}{n-2m} = m^{**} = (\alpha - 1)m'.$$

Since $\varphi_k \in \mathcal{X}^{1,2}_+(\Omega)$ for every $k \in \mathbb{N}$, we can use $v_k := \varphi_k^{\alpha-1} (= |\varphi_k|^{\alpha-2}\varphi_k)$, as test function in the variational formulation of (3.3), finding

$$\begin{aligned} \mathcal{B}\left(\varphi_k, \varphi_k^{\alpha-1}\right) &= \gamma \int_\Omega \frac{\varphi_{k-1}\varphi_k^{\alpha-1}}{(|x|^2 + 1/k)}\, dx + \int_\Omega f_k \varphi_k^{\alpha-1}\, dx \\ &\le \gamma \int_\Omega \frac{\varphi_k^\alpha}{|x|^2}\, dx + \int_\Omega f_k \varphi_k^{\alpha-1}\, dx, \end{aligned} \tag{3.5}$$

where we used the increasing monotonicity of $\{\varphi_k\}_k$.

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Let us now consider the left hand side of (3.5). We have

$$\begin{aligned}&\mathcal{B}\left(\varphi_k, \varphi_k^{\alpha-1}\right)\\&= \int_\Omega \nabla\varphi_k \cdot \nabla(\varphi_k^{\alpha-1})\, dx + \frac{C_{n,s}}{2}\iint_{\mathbb{R}^{2n}} \frac{(\varphi_k(x)-\varphi_k(y))(\varphi_k^{\alpha-1}(x)-\varphi_k^{\alpha-1}(y))}{|x-y|^{n+2s}}\, dx\, dy\\&=: (L) + (NL).\end{aligned}$$

The nonlocal part can be treated as in the proof of [1, theorem 4.2]. Let us give the details for sake of completeness: first, exploiting the algebraic inequality [1, Equation (17)], i.e., for $s_1, s_2 \geq 0$ and $a > 0$ it holds

$$(s_1 - s_2)(s_1^a - s_2^a) \geq \frac{4a}{(a+1)^2}\left(s_1^{\frac{a+1}{2}} - s_2^{\frac{a+1}{2}}\right)^2,$$

we deduce that

$$(\varphi_k(x) - \varphi_k(y))(\varphi_k^{\alpha-1}(x) - \varphi_k^{\alpha-1}(y)) \geq \frac{4(\alpha-1)}{\alpha^2}\left(\varphi_k^{\alpha/2}(x) - \varphi_k^{\alpha/2}(y)\right)^2,$$

and therefore

$$(NL) \geq \frac{4(\alpha-1)}{\alpha^2}\left[\varphi_k^{\alpha/2}\right]_s^2. \tag{3.6}$$

In the local part, we have

$$\nabla\varphi_k \cdot \nabla(\varphi_k^{\alpha-1}) = \frac{4(\alpha-1)}{\alpha^2}\left|\nabla\varphi_k^{\alpha/2}\right|^2,$$

and therefore

$$(L) = \frac{4(\alpha-1)}{\alpha^2}\int_\Omega \left|\nabla\varphi_k^{\alpha/2}\right|^2 dx. \tag{3.7}$$

Hence, combining (3.7) and (3.6), we get

$$\mathcal{B}\left(\varphi_k, \varphi_k^{\alpha-1}\right) \geq \frac{4(\alpha-1)}{\alpha^2}\mathcal{B}\left(\varphi_k^{\alpha/2}, \varphi_k^{\alpha/2}\right) = \frac{4(\alpha-1)}{\alpha^2}\rho\left(\varphi_k^{\alpha/2}\right)^2. \tag{3.8}$$

We now move to the right hand side of (3.5). By Hölder inequality, we find that

$$\int_\Omega f_k \varphi_k^{\alpha-1}\, dx \leq \|f_k\|_{L^m(\Omega)}\left(\int_\Omega \varphi_k^{(\alpha-1)m'}\, dx\right)^{1/m'}, \tag{3.9}$$

where m' is the Hölder conjugate exponent of m defined in (b). It remains to deal with the last term: by the mixed Hardy inequality with best constant, see theorem 1.1, we get

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$$\gamma \int_\Omega \frac{\varphi_k^\alpha}{|x|^2}\,dx = \gamma \int_\Omega \frac{\left(\varphi_k^{\alpha/2}\right)^2}{|x|^2}\,dx \le \frac{\gamma}{\Lambda_n}\rho\left(\varphi_k^{\alpha/2}\right)^2. \tag{3.10}$$

Combining (3.5) with (3.8), (3.9), and (3.10), we get

$$\left(\frac{4(\alpha-1)}{\alpha^2} - \frac{\gamma}{\Lambda_n}\right)\rho\left(\varphi_k^{\alpha/2}\right)^2 \le \|f_k\|_{L^m(\Omega)}\left(\int_\Omega \varphi_k^{(\alpha-1)m'}\,dx\right)^{1/m'}. \tag{3.11}$$

As in the purely local case, in order to have a meaningful a priori estimate, we need to have

$$\gamma < \frac{4(\alpha-1)\Lambda_n}{\alpha^2} = \frac{n(m-1)(n-2m)}{m^2} = \gamma(m),$$

as assumed. In order to conclude the proof, we can follow the argument in [10]. By (2.3), and recalling (b), we get

$$\rho\left(\varphi_k^{\alpha/2}\right)^2 \ge \mathcal{S}_n\|\varphi_k^{\alpha/2}\|^2_{L^{2^*}(\Omega)} = \mathcal{S}_n\left(\int_\Omega \varphi_k^{(2^*\alpha)/2}\,dx\right)^{2/2^*} \overset{b)}{=} \mathcal{S}_n\left(\int_\Omega \varphi_k^{(\alpha-1)m'}\,dx\right)^{2/2^*},$$

which, combined with (3.11), gives

$$\begin{aligned}&\mathcal{S}_n\left(\frac{4(\alpha-1)}{\alpha^2} - \frac{\gamma}{\Lambda_n}\right)\left(\int_\Omega \varphi_k^{(\alpha-1)m'}\,dx\right)^{2/2^*-1/m'}\\ &\quad= \mathcal{S}_n\left(\frac{4(\alpha-1)}{\alpha^2} - \frac{\gamma}{\Lambda_n}\right)\|\varphi_k\|_{L^{m^{**}}(\Omega)} \le \|f_k\|_{L^m(\Omega)},\end{aligned}$$

and thus

$$\frac{\mathcal{S}_n}{\Lambda_n}\,(\gamma(m)-\gamma)\,\|\varphi_k\|_{L^{m^{**}}(\Omega)} \le \|f_k\|_{L^m(\Omega)}.$$

Now, by using the Fatou lemma, it is sufficient to pass to the limit for $k \to +\infty$ and note that φ_k converges to the solution u_f of problem ($\mathrm{D}_{\Gamma,f}$). From this, we deduce the thesis. □

We are now ready to prove theorem 1.4.

Proof of theorem 1.4. It is enough to combine theorem 3.2 with theorem 3.3. □

3.2. Case (ii) - $f \in L^m(\Omega)$ with $1 < m < (2^*)'$

As already observed in §2, in this case, we cannot ensure that $f \in \mathbb{X} = (\mathcal{X}^{1,2}(\Omega))'$, and thus we *cannot apply* the Lax–Milgram theorem to study existence and uniqueness of solutions of problem ($\mathrm{D}_{\Gamma,f}$). Hence, we prove *existence and improved integrability* at the same time with a truncation argument.

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Proof of theorem 1.5. Similarly to the proof of theorem 3.3, for every $k \in \mathbb{N}$, we consider the unique weak solution $\varphi_k \in \mathcal{X}_+^{1,2}(\Omega) \cap L^\infty(\mathbb{R}^n)$ of the regularized problem

$$\begin{cases} \mathcal{L}u = \gamma \dfrac{\varphi_{k-1}}{|x|^2 + 1/k} + f_k(x) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{3.12}$$

where $\varphi_0 \equiv 0$ and $f_k(x) := \min\{f(x), k\}$. By arguing exactly as in the *incipit* of the proof of lemma A.1, we easily see that φ_k is well defined for every $k \geq 1$, and the sequence $\{\varphi_k\}_k$ is non-decreasing. We then define the number $\alpha \in (1, 2)$ as in (3.4), and we use

$$v_k := (\varphi_k + \varepsilon)^{\alpha-1} - \varepsilon^{\alpha-1}, \qquad \varepsilon > 0,$$

as test function in the variational formulation of (3.12). Recalling that

$$(\varphi_k + \varepsilon)(x) - (\varphi_k + \varepsilon)(y) = \varphi_k(x) - \varphi_k(y),$$

and making the same computations performed in the proof of theorem 3.3, we get

$$\mathcal{B}(\varphi_k, v_k) = \mathcal{B}\left(\varphi_k, (\varphi_k + \varepsilon)^{\alpha-1} - \varepsilon^{\alpha-1}\right) \geq \frac{4(\alpha-1)}{\alpha^2} \rho \left((\varphi_k + \varepsilon)^{\alpha/2} - \varepsilon^{\alpha/2}\right)^2, \tag{3.13}$$

and, using the monotonicity of $\{\varphi_k\}_k$ and the mixed Hardy inequality, we also find

$$\begin{aligned} &\gamma \int_\Omega \frac{\varphi_k[(\varphi_k + \varepsilon)^{\alpha-1} - \varepsilon^{\alpha-1}]}{|x|^2}\, dx + \int_\Omega f_k[(\varphi_k + \varepsilon)^{\alpha-1} - \varepsilon^{\alpha-1}]\, dx \\ &= \gamma \int_\Omega \frac{\left((\varphi_k + \varepsilon)^{\alpha/2} - \varepsilon^{\alpha/2}\right)^2}{|x|^2}\, dx \\ &\quad + \gamma \int_\Omega \frac{\varphi_k[(\varphi_k + \varepsilon)^{\alpha-1} - \varepsilon^{\alpha-1}] - [(\varphi_k + \varepsilon)^{\alpha/2} - \varepsilon^{\alpha/2}]^2}{|x|^2}\, dx \\ &\quad + \int_\Omega f_k[(\varphi_k + \varepsilon)^{\alpha-1} - \varepsilon^{\alpha-1}]\, dx \\ &\leq \frac{\gamma}{\Lambda_n} \rho \left((\varphi_k + \varepsilon)^{\alpha/2} - \varepsilon^{\alpha/2}\right)^2 \\ &\quad + \gamma \int_\Omega \frac{\varphi_k[(\varphi_k + \varepsilon)^{\alpha-1} - \varepsilon^{\alpha-1}] - [(\varphi_k + \varepsilon)^{\alpha/2} - \varepsilon^{\alpha/2}]^2}{|x|^2}\, dx \\ &\quad + \int_\Omega f_k[(\varphi_k + \varepsilon)^{\alpha-1} - \varepsilon^{\alpha-1}]\, dx. \end{aligned} \tag{3.14}$$

Combining (3.13) and (3.14), we finally get

$$\begin{aligned} \left(\frac{4(\alpha-1)}{\alpha^2} - \frac{\gamma}{\Lambda_n}\right) \rho \left((\varphi_k + \varepsilon)^{\alpha/2} - \varepsilon^{\alpha/2}\right)^2 &\leq \int_\Omega f_k[(\varphi_k + \varepsilon)^{\alpha-1} - \varepsilon^{\alpha-1}]\, dx \\ &\quad + \gamma \int_\Omega \frac{\varphi_k[(\varphi_k + \varepsilon)^{\alpha-1} - \varepsilon^{\alpha-1}] - [(\varphi_k + \varepsilon)^{\alpha/2} - \varepsilon^{\alpha/2}]^2}{|x|^2}\, dx. \end{aligned} \tag{3.15}$$

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Hence, using (2.3), we get

$$\rho\left((\varphi_k+\varepsilon)^{\alpha/2}-\varepsilon^{\alpha/2}\right)^2 \geq \mathcal{S}_n\|(\varphi_k+\varepsilon)^{\alpha/2}-\varepsilon^{\alpha/2}\|^2_{L^{2^*}(\Omega)} = \mathcal{S}_n\left(\int_\Omega [(\varphi_k+\varepsilon)^{\alpha/2}-\varepsilon^{\alpha/2}]^{2^*}\,dx\right)^{2/2^*}.$$

Now, for $k\in\mathbb{N}$ fixed, thanks to the Lebesgue's dominated convergence theorem we can pass to the limit for ε that goes to 0, and thus we obtain

$$\mathcal{S}_n\left(\frac{4(\alpha-1)}{\alpha^2}-\frac{\gamma}{\Lambda_n}\right)\left(\int_\Omega \varphi_k^{(2^*\alpha)/2}\,dx\right)^{2/2^*} \leq \int_\Omega f_k\varphi_k^{\alpha-1}\,dx. \tag{3.16}$$

Hence, applying Hölder inequality in the right hand side of (3.16), we deduce

$$\mathcal{S}_n\left(\frac{4(\alpha-1)}{\alpha^2}-\frac{\gamma}{\Lambda_n}\right)\left(\int_\Omega \varphi_k^{(2^*\alpha)/2}\,dx\right)^{2/2^*} \leq \|f_k\|_{L^m(\Omega)}\left(\int_\Omega \varphi_k^{(\alpha-1)m'}\,dx\right)^{\frac{1}{m'}}.$$

Using the definition of α in (3.4) and recalling that $\frac{\alpha}{2}\,2^* = \frac{nm}{n-2m} = m^{**} = (\alpha-1)m'$, we deduce

$$\mathcal{S}_n\left(\frac{4(\alpha-1)}{\alpha^2}-\frac{\gamma}{\Lambda_n}\right)\|\varphi_k\|_{L^{m^{**}}(\Omega)} \leq \|f_k\|_{L^m(\Omega)} \leq \|f\|_{L^m(\Omega)}. \tag{3.17}$$

Hence, by (3.17), we deduce

$$\|\varphi_k\|_{L^{m^{**}}(\Omega)} \leq \mathcal{K}(n,\alpha,\gamma,f) =: \mathcal{K}. \tag{3.18}$$

Exploiting the mixed Hardy inequality (see theorem 1.2) to the term $\rho\left((\varphi_k+\varepsilon)^{\alpha/2}-\varepsilon^{\alpha/2}\right)^2$, we have

$$\rho\left((\varphi_k+\varepsilon)^{\alpha/2}-\varepsilon^{\alpha/2}\right)^2 \geq \Lambda_n\int_\Omega \frac{[(\varphi_k+\varepsilon)^{\alpha/2}-\varepsilon^{\alpha/2}]^2}{|x|^2}\,dx. \tag{3.19}$$

Going back to (3.15), and using (3.19), we get

$$\begin{aligned}
&\Lambda_n\left(\frac{4(\alpha-1)}{\alpha^2}-\frac{\gamma}{\Lambda_n}\right)\int_\Omega \frac{[(\varphi_k+\varepsilon)^{\alpha/2}-\varepsilon^{\alpha/2}]^2}{|x|^2}\,dx\\
&\leq \left(\frac{4(\alpha-1)}{\alpha^2}-\frac{\gamma}{\Lambda_n}\right)\rho\left((\varphi_k+\varepsilon)^{\alpha/2}-\varepsilon^{\alpha/2}\right)^2\\
&\leq \int_\Omega f_k[(\varphi_k+\varepsilon)^{\alpha-1}-\varepsilon^{\alpha-1}]\,dx\\
&+\gamma\int_\Omega \frac{\varphi_k[(\varphi_k+\varepsilon)^{\alpha-1}-\varepsilon^{\alpha-1}]-[(\varphi_k+\varepsilon)^{\alpha/2}-\varepsilon^{\alpha/2}]^2}{|x|^2}\,dx.
\end{aligned} \tag{3.20}$$

Let k be fixed, by letting $\varepsilon\to 0$, we obtain

$$\Lambda_n\left(\frac{4(\alpha-1)}{\alpha^2}-\frac{\gamma}{\Lambda_n}\right)\int_\Omega \frac{\varphi_k^\alpha}{|x|^2}\,dx \leq \int_\Omega f_k\cdot\varphi_k^{\alpha-1}\,dx. \tag{3.21}$$

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Making use of Hölder inequality in the right hand side of (3.21), we deduce that

$$\begin{aligned}\Lambda_n\left(\frac{4(\alpha-1)}{\alpha^2}-\frac{\gamma}{\Lambda_n}\right)\int_\Omega \frac{\varphi_k^\alpha}{|x|^2}\,dx &\le \int_\Omega f_k\cdot\varphi_k^{\alpha-1}\,dx\\ &\le \left(\int_\Omega f_k^m\,dx\right)^{1/m}\left(\int_\Omega \varphi_k^{(\alpha-1)m'}\,dx\right)^{1/m'}\\ &\le \|f\|_{L^m(\Omega)}\cdot\|\varphi_k\|_{L^{m^{**}}(\Omega)}^{m^{**}/m'}.\end{aligned}\tag{3.22}$$

As a consequence of (3.22), taking into account (3.18), we are able to conclude that

$$\int_\Omega \frac{\varphi_k^\alpha}{|x|^2}\,dx \le \mathcal{E}\|f\|_{L^m(\Omega)},\tag{3.23}$$

where $\mathcal{E} := \mathcal{K}^{m^{**}/m'}\Lambda_n^{-1}(4(\alpha-1)/\alpha^2-\gamma/\Lambda_n)^{-1}$ is a positive constant which does not depend on k.

Our aim is to show that the sequence $\{\varphi_k\}_k$ is bounded in $W_0^{1,m^*}(\Omega)$. In order to prove this fact, from (3.15), we have

$$\begin{aligned}&\left(\frac{4(\alpha-1)}{\alpha^2}-\frac{\gamma}{\Lambda_n}\right)\left(\frac{\alpha}{2}\right)^2\int_\Omega\frac{|\nabla\varphi_k|^2}{(\varphi_k+\varepsilon)^{2-\alpha}}\,dx\\ &\quad\le\left(\frac{4(\alpha-1)}{\alpha^2}-\frac{\gamma}{\Lambda_n}\right)\rho\left((\varphi_k+\varepsilon)^{\alpha/2}-\varepsilon^{\alpha/2}\right)^2\\ &\quad\le\gamma\int_\Omega\frac{\varphi_k[(\varphi_k+\varepsilon)^{\alpha-1}-\varepsilon^{\alpha-1}]-[(\varphi_k+\varepsilon)^{\alpha/2}-\varepsilon^{\alpha/2}]^2}{|x|^2}\,dx\\ &\qquad+\int_\Omega f_k[(\varphi_k+\varepsilon)^{\alpha-1}-\varepsilon^{\alpha-1}]\,dx\\ &\quad(\text{since } 0<\alpha-1<1)\\ &\quad\le\gamma\int_\Omega\frac{\varphi_k[\varphi_k^{\alpha-1}+\varepsilon^{\alpha-1}-\varepsilon^{\alpha-1}]}{|x|^2}\,dx+\int_\Omega f_k[\varphi_k^{\alpha-1}+\varepsilon^{\alpha-1}-\varepsilon^{\alpha-1}]\,dx\\ &\quad=\gamma\int_\Omega\frac{\varphi_k^\alpha}{|x|^2}\,dx+\int_\Omega f_k\cdot\varphi_k^{\alpha-1}\,dx.\end{aligned}\tag{3.24}$$

If we fix $\varepsilon = R$ in (3.24), using (3.17) and (3.23), we get

$$\int_\Omega\frac{|\nabla\varphi_k|^2}{(\varphi_k+R)^{2-\alpha}}\,dx \le \mathcal{C}(m,n,\gamma,\mathcal{S}_n) =: \mathcal{C}.$$

Thanks to Hölder inequality with exponents $(2/m^*, 2/(2-m^*))$, we get

$$\begin{aligned}\int_\Omega|\nabla\varphi_k|^{m^*}\,dx &\le \int_\Omega\frac{|\nabla\varphi_k|^{m^*}}{(\varphi_k+R)^{\frac{(2-\alpha)m^*}{2}}}(\varphi_k+R)^{\frac{(2-\alpha)m^*}{2}}\,dx\\ &\le \mathcal{C}^{\frac{m^*}{2}}\left(\int_\Omega(\varphi_k+R)^{\frac{(2-\alpha)m^*}{2-m^*}}\,dx\right)^{\frac{2-m^*}{2}}.\end{aligned}$$

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Finally, we have

$$\int_\Omega |\nabla \varphi_k|^{m^*}\,dx \le \mathcal{C}^{\frac{m^*}{2}} \left(\int_\Omega (\varphi_k + R)^{m^{**}}\,dx\right)^{\frac{2-m^*}{2}},$$

which implies the boundedness of $\{\varphi_k\}_k$ in $W_0^{1,m^*}(\Omega)$. Since $m>1$, we deduce that there exists u such that (up to subsequences)

$$\varphi_k \rightharpoonup u \text{in } W_0^{1,m^*}(\Omega) \text{ and a.e. in } \Omega,\ \frac{\varphi_k}{|x|^2} \to \frac{u}{|x|^2} \text{ strongly in } L^1(\Omega).$$

We recall that φ_k weakly solves (3.12), namely

$$\int_{\mathbb{R}^n} \nabla\varphi_k \cdot \nabla v\,dx + \frac{C_{n,s}}{2}\iint_{\mathbb{R}^{2n}} \frac{(\varphi_k(x)-\varphi_k(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dxdy = \gamma\int_\Omega \frac{\varphi_{k-1}}{|x|^2+1/k} v\,dx + \int_\Omega f_k(x) v\,dx,$$

for every $v \in C_0^\infty(\Omega)$.

We note that, for any $v \in C_0^\infty(\Omega)$, the following

$$W_0^{1,m^*}(\Omega) \ni g \mapsto \int_{\mathbb{R}^n} \nabla g \cdot \nabla v\,dx + \frac{C_{n,s}}{2}\iint_{\mathbb{R}^{2n}} \frac{(g(x)-g(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dxdy$$

is a bounded linear functional in $W_0^{1,m^*}(\Omega)$. Then for every $v \in C_0^\infty(\Omega)$, we obtain

$$\int_{\mathbb{R}^n} \nabla\varphi_k \cdot \nabla v\,dx + \frac{C_{n,s}}{2}\iint_{\mathbb{R}^{2n}} \frac{(\varphi_k(x)-\varphi_k(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dxdy \longrightarrow \int_{\mathbb{R}^n} \nabla u \cdot \nabla v\,dx + \frac{C_{n,s}}{2}\iint_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dxdy \text{ as } k \to +\infty.$$

Collecting the last information on the operator, the convergence of φ_k discussed before, and using the Lebesgue's dominated convergence theorem in the right hand side of the regularized problem (3.12), we can pass to the limit in it, showing that u is a solution (as defined in (1.4)) belonging to $W_0^{1,m^*}(\Omega)$. The solution u is positive in Ω being $\{\varphi_k\}_k$ non-decreasing.

Now we want to prove that this solution $u \in W_0^{1,m^*}(\Omega)$, obtained as limit of solutions of the regularized problems (3.12), is unique. Consider another sequence of functions $\{g_k\}_k$ converging to f in $L^m(\Omega)$. Then let us consider the following regularized problem

$$\begin{cases} \mathcal{L} w_k = \gamma \dfrac{w_{k-1}}{|x|^2+1/k} + g_k(x) & \text{in } \Omega, \\ w_k = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Hence, as before we can prove that, up to subsequences, w_k converges to a solution w belonging to $W_0^{1,m^*}(\Omega)$. Hence, arguing on the function $\varphi_k - w_k$, we deduce that it solves

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$$\begin{cases} \mathcal{L}(\varphi_k - w_k) = \gamma \dfrac{(\varphi_{k-1} - w_{k-1})}{|x|^2 + 1/k} + f_k(x) - g_k(x) & \text{in } \Omega, \\ \varphi_k - w_k = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Arguing as above, we can derive an analogous estimate to (3.17) for the difference function

$$\mathcal{S}_n \left(\frac{4(\alpha - 1)}{\alpha^2} - \frac{\gamma}{\Lambda_n} \right) \|\varphi_k - w_k\|_{L^{m^{**}}(\Omega)} \leq \|f_k - g_k\|_{L^m(\Omega)}.$$

Passing to the limit for k that goes to $+\infty$, we easily deduce that

$$\|u - w\|_{L^{m^{**}}(\Omega)} = 0,$$

which immediately implies the uniqueness. □

3.3. Case (iii) - $f \in L^1(\Omega)$

Taking into account all the results established so far, we end this section by proving an *optimal condition* for the existence of a positive duality solution of problem (3) when $\beta \in (0, \Lambda_n)$ and f is just a *positive* L^1 *function* in the sense of definition 1.7.

Proof of theorem 1.9. (*Necessity of* (1.6)). We first assume that there exists a positive duality solution u of problem (3), and we show that the 'integrability condition' (1.6) is fulfilled by datum f.

To this end (and similarly to the proof of theorems 3.3–1.5), for every $k \geq 1$, we consider the unique weak solution $\phi_k \in \mathcal{X}_+^{1,2}(\Omega) \cap L^\infty(\mathbb{R}^n)$ of the problem

$$\begin{cases} \mathcal{L}u = g_k & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{3.25}$$

where $\phi_0 \equiv 0$ and, to simplify the notation, we have set

$$g_k(x) = \gamma \frac{\phi_{k-1}}{|x|^2 + 1/k} + 1.$$

Notice that, owing to lemma A.1 (with $f \equiv 1$), we have

(a) ϕ_k is well defined for every $k \geq 1$, and the sequence $\{\phi_k\}_k$ is non-decreasing;
(b) $\phi_k \to \Phi_\Omega$ as $k \to +\infty$ in $L^p(\Omega)$, for every $1 \leq p < 2^*$.

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Now, choosing $w := \phi_k$ and $g := g_k$ in definition 1.7 (notice that this is legitimate, since we have $g_k \in L^\infty(\Omega)$ and $\phi_k \in \mathcal{X}^{1,2}_+(\Omega) \cap L^\infty(\mathbb{R}^n)$ solves (3.25)), we get

$$\begin{aligned}\int_\Omega f\phi_k\,dx &= \int_\Omega u\left(g_k - \gamma\frac{\phi_k}{|x|^2}\right)dx\\ &\quad\text{(by definition of } g_k)\\ &= \int_\Omega u\left(1+\gamma\frac{\phi_{k-1}}{|x|^2+1/k} - \gamma\frac{\phi_k}{|x|^2}\right)dx\\ &\quad\text{(by monotonicity of } \phi_k)\\ &\le \int_\Omega u\left[1+\gamma\phi_k\left(\frac{1}{|x|^2+1/k} - \frac{1}{|x|^2}\right)\right]dx\\ &\quad\text{(since } \phi_k \text{ is positive)}\\ &\le \int_\Omega u\,dx = \|u\|_{L^1(\Omega)} < \infty.\end{aligned} \tag{3.26}$$

From this, by passing to the limit as $k \to \infty$ in (3.26) with the aid of the monotone convergence theorem (and recalling (b)), we immediately obtain the desired (1.6).

Sufficiency of (1.6). We now assume that f satisfies condition (1.6), and we prove that there exists a solution u of problem (3), further satisfying properties (1)–(2).

To this end, for every fixed $k \ge 1$, we consider once again the unique variational solution $\varphi_k \in \mathcal{X}^{1,2}_+(\Omega) \cap L^\infty(\mathbb{R}^n)$ of the approximated problem

$$\begin{cases}\mathcal{L}u = \gamma\frac{\varphi_{k-1}}{|x|^2+1/k} + f_k & \text{in } \Omega,\\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega,\end{cases} \tag{3.27}$$

where $\varphi_0 \equiv 0$ and $f_k = T_k(f)$ (here, $T_k(\cdot)$ is the truncation operator defined in (2.11)). By arguing as in the *incipit* of the proof of lemma A.1, we see that φ_k is well defined for every $k \ge 1$, and the sequence $\{\varphi_k\}_k$ is non-decreasing; hence, we can define

$$u(x) = \lim_{n\to\infty}\varphi_k(x) = \sup\{\varphi_k(x) : n \in \mathbb{N}\} \ge 0.$$

We then prove that this u is a duality solution of (3), further satisfying (1)–(2).

(a) *u is a duality solution of* (3). First of all, we show that $u \in L^1(\Omega)$. To this end, it suffices to observe that, using φ_k as a test function for the (variational) equation solved by Φ_Ω and vice versa, see (3.1), from the assumed condition (1.6) we have

$$\begin{aligned}\int_\Omega \varphi_k\,dx &= -\gamma\int_\Omega \frac{\Phi_\Omega\varphi_k}{|x|^2}\,dx + \mathcal{B}(\Phi_\Omega,\varphi_k)\\ &\quad\text{(since } \varphi_k \text{ solves (3.27))}\\ &= \gamma\int_\Omega \Phi_\Omega\Big(\frac{\varphi_{k-1}}{|x|^2+1/k} - \frac{\varphi_k}{|x|^2}\Big)dx + \int_\Omega f_k\Phi_\Omega\,dx\\ &\le \int_\Omega f_k\Phi_\Omega\,dx \le \int_\Omega f\Phi_\Omega\,dx = \mathbf{c} < \infty;\end{aligned} \tag{3.28}$$

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this, together with the monotone convergence theorem, proves at once that $u \in L^1(\Omega)$ (in particular, $u < \infty$ a.e. in Ω), and $\varphi_k \to u$ in $L^1(\Omega)$.

Taking into account definition 1.7, we now turn to prove that $u/|x|^2 \in L^1(\Omega)$. To this end, let $\phi_1 \in \mathcal{X}^{1,2}_+(\Omega) \cap L^\infty(\Omega)$ be the unique solution of

$$\begin{cases} \mathcal{L}u = 1 & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Moreover, let $r > 0$ be such that $B_r(0) \Subset \Omega$. Owing to the weak Harnack inequality in [19, theorem 8.1], we can find a constant $C > 0$, depending on ϕ_1 and on r, such that

$$\phi_1 \geq C \text{ a.e. in } B_r(0);$$

as a consequence, using φ_k as a test function for the equation solved by ϕ_1 and vice versa, from the above (3.28) we obtain the following estimate

$$\begin{aligned} \int_\Omega \frac{\varphi_{k-1}}{|x|^2 + 1/k}\, dx &\leq \frac{1}{C} \int_{B_r(0)} \frac{\phi_1 \varphi_{k-1}}{|x|^2 + 1/k}\, dx + \frac{1}{r^2} \int_{\Omega \setminus B_r(0)} \varphi_{k-1}\, dx \\ &\text{(since } \phi_1,\, \varphi_{k-1} \geq 0 \text{ a.e. in } \Omega) \\ &\leq \frac{1}{C} \int_\Omega \frac{\phi_1 \varphi_{k-1}}{|x|^2 + 1/k}\, dx + \frac{1}{r^2} \int_\Omega \varphi_{k-1}\, dx \\ &\text{(using } \phi_1 \text{ as a test function for the equation solved by } \varphi_k) \\ &\leq \frac{1}{C} \left(\mathcal{B}(\varphi_k, \phi_1) - \int_\Omega f_k \phi_1\, dx \right) + \frac{1}{r^2} \int_\Omega \varphi_{k-1}\, dx \\ &\text{(using } \varphi_k \text{ as a test function for the equation solved by } \phi_1) \\ &\leq \frac{1}{C} \int_\Omega \varphi_k\, dx + \frac{1}{r^2} \int_\Omega \varphi_{k-1}\, dx \leq \mathbf{c} < \infty, \end{aligned} \tag{3.29}$$

where $\mathbf{c} > 0$ is a constant independent of k. From this, again by the monotone convergence theorem, we infer that

$$\frac{u}{|x|^2} \in L^1(\Omega) \quad \text{and} \quad \frac{\varphi_{k-1}}{|x|^2 + 1/k} \to \frac{u}{|x|^2} \text{ in } L^1(\Omega).$$

Gathering all these facts, we can easily prove that u is a duality solution of problem (3) in the sense of definition 1.7: in fact, we already know that u, $u/|x|^2 \in L^1(\Omega)$; moreover, since φ_k solves (3.27), we get

$$\int_\Omega \mathcal{B}(\varphi_k, \psi)\, dx = \int_\Omega \gamma \frac{\varphi_{k-1}\psi}{|x|^2 + 1/k} dx + \int_\Omega f_k \psi\, dx \quad \forall\, \psi \in \mathcal{X}^{1,2}(\Omega) \cap L^\infty(\Omega).$$

If, in particular, ψ is a solution of (1.5), we have

$$\int_\Omega g\varphi_k = \int_\Omega \gamma \frac{\varphi_{k-1}\psi}{|x|^2 + 1/k} dx + \int_\Omega f_k \psi\, dx.$$

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Then, letting $k \to \infty$ (and observing that $f_k \to f$ in $L^1(\Omega)$), by the above considerations, we conclude that u is a solution of (3).

(b) *u satisfies* (1)–(2). We first prove that validity of property (1). To this end, we arbitrarily fix $j, k \in \mathbb{N}$ and we use $T_k(\varphi_j)$ as a test function for the equation solved by φ_j: taking into account the computation in (3.29), we get

$$\begin{aligned}\int_\Omega |\nabla T_k(\varphi_j)|^2\,dx \le \mathcal{B}(\varphi_j, T_k(\varphi_j)) &= \gamma \int_\Omega \frac{\varphi_{j-1}\,T_k(\varphi_j)}{|x|^2+1/j}\,dx + \int_\Omega f_j\,T_k(\varphi_j)\,dx \\ &\text{(since } \varphi_{j-1},\, f_j \ge 0 \text{ and } 0 \le T_k(\varphi_k) \le k) \\ &\le k\Big(\gamma \int_\Omega \frac{\varphi_{j-1}}{|x|^2+1/j}\,dx + \int_\Omega f\,dx\Big) \\ &\text{(using (3.29), and since } f \in L^1(\Omega)) \\ &\le \mathbf{c}_k < \infty,\end{aligned}$$

and this shows that the sequence $\{T_k(\varphi_j)\}_j$ is bounded in $\mathcal{X}^{1,2}(\Omega)$. Then, by the Sobolev embedding theorem, there exists a function $g \in \mathcal{X}^{1,2}(\Omega)$ such that (up to a subsequence)

(i) $T_k(\varphi_j) \rightharpoonup g$ in $\mathcal{X}^{1,2}(\Omega)$ as $j \to \infty$;

(ii) $T_k(\varphi_j) \to g$ strongly in $L^2(\Omega)$ and pointwise a.e. in Ω.

On the other hand, since we also have $T_k(\varphi_j) \to T_k(u)$ pointwise a.e. in Ω as $j \to \infty$ (recall that, by definition, u is the pointwise limit of φ_j), we conclude that

$$g = T_k(u),$$

and this proves that $T_k(u) \in \mathcal{X}^{1,2}(\Omega)$ for every $k \in \mathbb{N}$, as desired. □

We then turn to prove the validity of property (2). To this end, we fix $\beta < 1/2$ (to be conveniently chosen later on), and we use the function $w_k = 1-(1+\varphi_k)^{2\beta-1}$ as a test function for the equation solved by φ_k (see, precisely, (3.27)): this gives

$$\mathcal{B}(\varphi_k, w_k) = \gamma \int_\Omega \frac{\varphi_{k-1} w_k}{|x|^2+1/k}\,dx + \int_\Omega f_k w_k\,dx. \tag{3.30}$$

We now observe that, since $\beta < 1/2$, we have $w_k \le 1$; as a consequence, for the second term of the right hand side of the above (3.30), we have

$$\int_\Omega f_k w_k\,dx \le \int_\Omega f_k\,dx \le \|f\|_{L^1(\Omega)}.$$

For the second term, thanks to (3.29), we have

$$\gamma \int_\Omega \frac{\varphi_{k-1} w_k}{|x|^2+1/k}\,dx \le \gamma \int_\Omega \frac{\varphi_{k-1}}{|x|^2+1/k}\,dx \le \mathbf{c}.$$

Combining this last two information, we deduce

$$\|f\|_{L^1(\Omega)} + \mathbf{c} \ge \mathcal{B}(\varphi_k, w_k).$$

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Now, let us compute the bilinear form $\mathcal{B}(\varphi_k, w_k)$:

$$\begin{aligned}\mathcal{B}(\varphi_k, w_k) =&|2\beta-1| \int_\Omega \frac{|\nabla \varphi_k|^2}{(1+\varphi_k)^{2(1-\beta)}}\,dx \\ &+ \frac{C_{n,s}}{2} \iint_{\mathbb{R}^{2n}} \frac{(\varphi_k(x)-\varphi_k(y))((1+\varphi_k(y))^{2\beta-1}-(1+\varphi_k(x))^{2\beta-1})}{|x-y|^{n+2s}}.\end{aligned}$$

We note that the nonlocal part of the bilinear form, i.e. the second term on the right hand side of the previous equation, is non-negative. This assertion follows from the following couple of facts

$$\begin{cases}\varphi_k(x)-\varphi_k(y)\geq 0 \;\Rightarrow (1+\varphi_k(y))^{2\beta-1}-(1+\varphi_k(x))^{2\beta-1}\geq 0,\\ \varphi_k(x)-\varphi_k(y)< 0 \;\Rightarrow (1+\varphi_k(y))^{2\beta-1}-(1+\varphi_k(x))^{2\beta-1}< 0.\end{cases}$$

Hence, we are able to deduce that

$$\|f\|_{L^1(\Omega)}+\mathbf{c}\geq \mathcal{B}(\varphi_k, w_k)\geq \frac{|2\beta-1|}{\beta^2}\int_\Omega |\nabla(1+\varphi_k)^\beta|^2\,dx.$$

Thus from the previous estimate, we get

$$\int_\Omega \frac{|\nabla \varphi_k|^2}{(1+\varphi_k)^{2(1-\beta)}}\,dx \leq \frac{\|f\|_{L^1(\Omega)}+\mathbf{c}}{|2\beta-1|}. \tag{3.31}$$

On the other hand, for any given $p<2$, we can write

$$\int_\Omega |\nabla \varphi_k|^p\,dx = \int_\Omega \frac{|\nabla \varphi_k|^p}{(1+\varphi_k)^{2(1-\beta)\frac{p}{2}}}(1+\varphi_k)^{2(1-\beta)\frac{p}{2}}\,dx.$$

Applying the Hölder inequality with exponents $(2/p, 2/(2-p))$ on the right hand side, and the Sobolev inequality on the left hand side, we obtain

$$\begin{aligned}\mathcal{S}_n^p\left(\int_\Omega \varphi_k^{p^*}\,dx\right)^{\frac{p}{p^*}} &\leq \int_\Omega |\nabla \varphi_k|^p\,dx \\ &\leq \left(\int_\Omega \frac{|\nabla \varphi_k|^2}{(1+\varphi_k)^{2(1-\beta)}}\,dx\right)^{\frac{p}{2}}\left(\int_\Omega (1+\varphi_k)^{\frac{2p(1-\beta)}{2-p}}\,dx\right)^{1-\frac{p}{2}}.\end{aligned}$$

Thanks to (3.31), we get

$$\mathcal{S}_n^p\left(\int_\Omega \varphi_k^{p^*}\,dx\right)^{\frac{p}{p^*}} \leq \int_\Omega |\nabla \varphi_k|^p\,dx \leq C|\Omega|^{1-\frac{p}{2}}+C\left(\int_\Omega \varphi_k^{\frac{2p(1-\beta)}{2-p}}\,dx\right)^{1-\frac{p}{2}}, \tag{3.32}$$

where $C=\left[(\|f\|_{L^1(\Omega)}+\mathbf{c})/|2\beta-1|\right]^{p/2}$, and it is independent on k.

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Now, we can choose β such that

$$\frac{2p(1-\beta)}{2-p} = p^*,$$

and since $\beta < 1/2$, we have $p < \frac{n}{n-1}$. Thanks to this choice of β, we have

$$\mathcal{S}_n^p \left(\int_\Omega \varphi_k^{p^*} \, dx \right)^{\frac{p}{p^*}} \leq C|\Omega|^{1-\frac{p}{2}} + C \left(\int_\Omega \varphi_k^{p^*} \, dx \right)^{1-\frac{p}{2}},$$

thus we can deduce that φ_k is uniformly bounded in $L^{p^*}(\Omega)$. From (3.32), we also deduce that $\nabla \varphi_k$ is uniformly bounded in $L^p(\Omega)$. Combining the last two information, we get that φ_k is uniformly bounded in $W_0^{1,p}(\Omega)$. Arguing as before, it is possible to show that up to subsequences $\varphi_k \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, for $p < \frac{n}{n-1}$. This closes the proof. □

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References

1 B. Abdellaoui, M. Medina, I. Peral and A. Primo. The effect of the Hardy potential in some Calderón-Zygmund properties for the fractional Laplacian. *J. Differential Equations* **260** (2016), 8160–8206.

2 B. Abdellaoui and I. Peral. A note on a critical problem with natural growth in the gradient. *J. Eur. Math. Soc. (JEMS)* **8** (2006), 157–170.

3 C. A. Antonini and M. Cozzi. Global gradient regularity and a Hopf lemma for quasilinear operators of mixed local-nonlocal type. *J. Differential Equations* **425** (2025), 342–382.

4 R. Arora, P.T. Nguyen and V. Radulescu, *A large class of nonlocal elliptic equations with singular nonlinearities*, preprint. https://arxiv.org/abs/2211.06634.

5 R. Arora and V. Radulescu. Combined effects in mixed local-nonlocal stationary problems, *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, pp. 1–47, Published online 2023. doi:10.1017/prm.2023.801.06701.

6 S. Biagi, S. Dipierro, E. Valdinoci and E. Vecchi. Mixed local and nonlocal elliptic operators: Regularity and maximum principles. *Comm. Partial Differential Equations* **47** (2022), 585–629.

7 S. Biagi, S. Dipierro, E. Valdinoci and E. Vecchi. A Faber-Krahn inequality for mixed local and nonlocal operators. *J. Anal. Math.* **150** (2023), 405–448.

8 S. Biagi, D. Mugnai and E. Vecchi. A Brezis–Oswald approach for mixed local and nonlocal operators. *Commun. Contemp. Math.* **26** (2024), 2250057.

https://doi.org/10.1017/prm.2025.28 Published online by Cambridge University Press

9 S. Biagi and E. Vecchi. Multiplicity of positive solutions for mixed local-nonlocal singular critical problems. *Calc. Var. Partial Differ. Equ.* **63** (2024), 221.

10 L. Boccardo, L. Orsina and I. Peral. A remark on existence and optimal summability of solutions of elliptic problems involving Hardy potential. *Discrete Contin. Dyn. Syst.* **16** (2006), 513–523.

11 J.-M. Bony, P. Courrège and P. Priouret. Semi-groupes de Feller sur une variété à bord compacte et problèmes aux limites intégro-différentiels du second ordre donnant lieu au principe du maximum. *Ann. Inst. Fourier (Grenoble)* **18** (1968), 369–521.

12 C. Cancelier. Problèmes aux limites pseudo-différentiels donnant lieu au principe du maximum. *Comm. Partial Differential Equations* **11** (1986), 1677–1726.

13 Z.-Q. Chen, P. Kim, R. Song and Z. Vondraček. Boundary Harnack principle for $\Delta + \Delta^{\alpha/2}$. *Trans. Amer. Math. Soc.* **364** (2012), 4169–4205.

14 C. De Filippis and G. Mingione. Gradient regularity in mixed local and nonlocal problems. *Math. Ann.* **388** (2024), 261–328.

15 E. Di Nezza, G. Palatucci and E. Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.* **136** (2012), 521–573.

16 S. Dipierro, E. Proietti Lippi and E. Valdinoci. (Non)local logistic equations with Neumann conditions. *Ann. Inst. H. Poincaré C Anal. Non Linéaire.* **40** (2022), 1093–1166. doi:10.4171/AIHPC/57.

17 M. M. Fall and R. Musina. Sharp nonexistence results for a linear elliptic inequality involving Hardy and Leray potentials. *J. Inequal. Appl.* **2011**, 917201.

18 P. Garain. On a class of mixed local and nonlocal semilinear elliptic equation with singular nonlinearity. *J. Geom. Anal.* **33** (2023), 212.

19 P. Garain and J. Kinnunen. On the regularity theory for mixed local and nonlocal quasilinear elliptic equations. *Trans. Amer. Math. Soc.* **375** (2022), 5393–5423.

20 P. Garain and E. Lindgren. Higher Hölder regularity for mixed local and nonlocal degenerate elliptic equations. *Calc. Var.* **62** (2023), 67.

21 G. H. Hardy, J. E. Littlewood and G. Polya. *Inequalities*. Reprint of the 1952 edition (Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1988).

22 T. Leonori, I. Peral, A. Primo and F. Soria. Basic estimates for solutions of a class of nonlocal elliptic and parabolic equations. *Discrete Contin. Dyn. Syst.* **35** (2015), 6031–6068.

23 I. Peral and F. Soria. *Elliptic and parabolic equations involving the Hardy-Leray potential*, De Gruyter Series in Nonlinear Analysis and Applications, Vol. 38 (Walter de Gruyter GmbH, Berlin/Boston, 2021).

24 S. Biagi, S. Dipierro, E. Valdinoci and E. Vecchi, A Brezis-Nirenberg type result for mixed local and nonlocal operators, preprint. https://arxiv.org/abs/2209.07502.

25 G. Stampacchia. Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus. *Ann. Inst. Fourier (Grenoble)* **15** (1965), 189–258.

26 X. Su, E. Valdinoci, Y. Wei and J. Zhang. Regularity results for solutions of mixed local and nonlocal elliptic equations. *Math. Z.* **302** (2022), 1855–1878.

27 X. Su, E. Valdinoci, Y. Wei and J. Zhang. On some regularity properties of mixed local and nonlocal elliptic equations. *J. Differential Equations* **416** (2025), 576–613.

28 J. L. Vazquez and E. Zuazua. The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential. *J. Funct. Anal.* **173** (2000), 103–153.

Appendix A. A convergence lemma

The aim of this appendix is to give an auxiliary result which clarify the convergence of the sequence $\{u_k\}$ in in the proof of theorem 1.5.

LEMMA A.1. Approximated Dirichlet problems *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with smooth enough boundary $\partial\Omega$, and let $f \in L^{\frac{2n}{n+2}}(\Omega)$, $f \geq 0$ a.e. in Ω. For a fixed $\gamma \in (0, \Lambda_n)$, we let $u \in \mathcal{X}_+^{1,2}(\Omega)$ be the unique weak solution of*

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$$\begin{cases} \mathcal{L}u = \gamma \frac{u}{|x|^2} + f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{A.1}$$

Setting $\varphi_0 = 0$, for every $k \geq 1$, we then consider the unique weak solution $\varphi_k \in \mathcal{X}^{1,2}_+(\Omega) \cap L^\infty(\mathbb{R}^n)$ of the following approximated *$\mathcal{L}$-Dirichlet problem*

$$\begin{cases} \mathcal{L}u = \gamma \frac{\varphi_{k-1}}{|x|^2 + 1/k} + f_k & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{A.2}$$

where $f_k = T_k(f) = \min\{k, f\}$. Then,

$$\varphi_k \to u \text{ weakly in } \mathcal{X}^{1,2}(\Omega) \text{ and strongly in } L^p(\Omega) \text{ for every } 1 \leq p < 2^*.$$

Proof. We begin with a couple of preliminary observations. First of all, the existence and the uniqueness of the solution $\varphi_k \in \mathcal{X}^{1,2}_+(\Omega) \cap L^\infty(\mathbb{R}^n)$ follow from theorem 2.3 (and from the recursive structure of problem (A.2)), since the right-hand side

$$g_k(x) = \gamma \frac{\varphi_{k-1}}{|x|^2 + 1/k} + f_k$$

is non-negative and bounded on Ω (as, by induction, the same is true of φ_{k-1}). Furthermore, by using a classical induction argument based on the weak maximum principle for $\mathcal{L}$ (and since the sequence $\{f_k\}_k$ is non-decreasing), we have that

$$\{\varphi_k(x)\}_k \text{ is non-negative and non-decreasing for every } x \in \Omega.$$

As a consequence, we can define

$$\psi(x) = \lim_{k \to \infty} \varphi_k(x) \in (0, +\infty] \text{ for all } x \in \Omega.$$

Now, using φ_k as a test function for the equation solved by u and vice versa, we have

$$\begin{aligned} \int_\Omega f_k \varphi_k \, dx &\leq \int_\Omega f \varphi_k \, dx = \mathcal{B}(u, \varphi_k) - \gamma \int_\Omega \frac{u \varphi_k}{|x|^2} \, dx \\ &\qquad \text{(since } \varphi_k \geq \varphi_{k-1} \text{ and } |x|^2 \leq |x|^2 + 1/k) \\ &\leq \mathcal{B}(\varphi_k, u) - \gamma \int_\Omega \frac{\varphi_{k-1} u}{|x|^2 + 1/k} \, dx \\ &= \int_\Omega f_k u \, dx \leq \int_\Omega f u \, dx; \end{aligned}$$

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hence, by the monotone convergence theorem, we derive that

$$f\psi \in L^1(\Omega).$$

On account of this fact, and using φ_k as a test function for the equation solved by φ_k itself, we obtain the following estimate

$$\begin{aligned}\rho(\varphi_k)^2 = \mathcal{B}(\varphi_k,\varphi_k) &= \gamma\int_\Omega \frac{\varphi_k\varphi_{k-1}}{|x|^2+1/k}\,dx + \int_\Omega f_k\varphi_k\,dx\\ &\text{(since } \varphi_{k-1}\le\varphi_k\le\psi \text{ a.e. in } \Omega)\\ &\le \gamma\int_\Omega \frac{\varphi_k^2}{|x|^2}\,dx + \int_\Omega f\psi\,dx\\ &\text{(by Hardy's inequality)}\\ &\le \frac{\gamma}{\Lambda_n}\rho(\varphi_k)^2 + \|f\psi\|_{L^1(\Omega)}.\end{aligned}$$

This, together with the fact that $\gamma\in(0,\Lambda_n)$, shows that the sequence $\{\varphi_k\}_k$ is *bounded in the Hilbert space* $\mathcal{X}^{1,2}(\Omega)$; hence, recalling that $\varphi_k\to\psi$ pointwise in Ω and using the Sobolev embedding theorem, we deduce that $\psi\in\mathcal{X}^{1,2}(\Omega)$ and

$$\varphi_k\to\psi \text{ weakly in } \mathcal{X}^{1,2}(\Omega) \text{ and strongly in } L^p(\Omega) \text{ for every } 1\le p<2^*.$$

To complete the proof, it now suffices to show that $\psi\in\mathcal{X}^{1,2}(\Omega)$ solves the same equation solved by u; by uniqueness, this will prove that $\psi=u$, as desired. Let then $\varphi\in\mathcal{X}^{1,2}(\Omega)$ be arbitrarily fixed. Since φ_k is a weak solution of (A.2), we can write

$$\mathcal{B}(\varphi_k,\varphi) = \gamma\int_\Omega \frac{\varphi_{k-1}\varphi}{|x|^2+1/k}\,dx + \int_\Omega f_k\varphi\,dx. \tag{A.3}$$

Since $\varphi_k\rightharpoonup\psi$ in $\mathcal{X}^{1,2}(\Omega)$, we have

$$\mathcal{B}(\varphi_k,\varphi)\to\mathcal{B}(\psi,\varphi)\quad\text{as } k\to\infty.$$

Moreover, since $0\le\varphi_k\le\psi$, we have

$$\Big|\frac{\varphi_{k-1}\varphi}{|x|^2+1/k}\Big| \le \frac{\psi|\varphi|}{|x|^2} = g(x)\quad\forall\ k\ge1,$$

and $g\in L^1(\Omega)$ by Hölder's and Hardy's inequality (recall that $\psi,\varphi\in\mathcal{X}^{1,2}(\Omega)$). We are then entitled to apply the Lebesgue dominated convergence theorem, showing that

$$\int_\Omega \frac{\varphi_{k-1}\varphi}{|x|^2+1/k}\,dx \to \int_\Omega \frac{\psi\varphi}{|x|^2}\,dx.$$

Finally, since $0\le f_k\le f$, we also have

$$|f_k\varphi|\le f\varphi = h(x)\quad\forall\ k\ge1,$$

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and $h \in L^1(\Omega)$ by Hölder's inequality (recall that $f \in L^{\frac{2n}{n+2}}(\Omega)$ and $\varphi \in \mathcal{X}^{1,2}(\Omega) \subset L^{2^*}(\Omega)$). Again by the Lebesgue dominated convergence theorem, we then get

$$\int_\Omega f_k \varphi \, dx \to \int_\Omega f \varphi \, dx.$$

Gathering all these facts, we can finally pass to the limit as $k \to \infty$ in the above (A.3), thus deducing that ψ is a weak solution of the same equation solved by u. □

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