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Minimal Siegel modular threefolds

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Abstract

The starting point of this paper is the maximal extension Γ_t^* of Γ_t, the subgroup of $\mathrm{Sp}_4(\mathbb{Q})$ which is conjugate to the paramodular group. Correspondingly we call the quotient $\mathscr{A}_t^* = \Gamma_t^* \backslash \mathbb{H}_2$ the minimal Siegel modular threefold. The space $\mathscr{A}_t^*$ and the intermediate spaces between $\mathscr{A}_t = \Gamma_t \backslash \mathbb{H}_2$ which is the space of $(1,t)$-polarized abelian surfaces and $\mathscr{A}_t^*$ have not yet been studied in any detail. Using the Torelli theorem we first prove that $\mathscr{A}_t^*$ can be interpreted as the space of Kummer surfaces of $(1,t)$-polarized abelian surfaces and that a certain degree 2 quotient of $\mathscr{A}_t$ which lies over $\mathscr{A}_t^*$ is a moduli space of lattice polarized $K3$ surfaces. Using the action of Γ_t^* on the space of Jacobi forms we show that many spaces between $\mathscr{A}_t$ and $\mathscr{A}_t^*$ possess a non-trivial 3-form, i.e. the Kodaira dimension of these spaces is non-negative. It seems a difficult problem to compute the Kodaira dimension of the spaces $\mathscr{A}_t^*$ themselves. As a first necessary step in this direction we determine the divisorial part of the ramification locus of the finite map $\mathscr{A}_t \to \mathscr{A}_t^*$. This is a union of Humbert surfaces which can be interpreted as Hilbert modular surfaces.

Introduction

The moduli space $\mathscr{A}_t$ of abelian surfaces with a $(1,t)$-polarization is the quotient of the Siegel upper half plane $\mathbb{H}_2$ by a subgroup Γ_t of $\mathrm{Sp}_4(\mathbb{Q})$ which is conjugate to the paramodular group $\widetilde{\Gamma}_t$. In Section 1 we define an isomorphism between the symplectic group and the special orthogonal group $\mathrm{SO}(3,2)$ over the integers.

This exhibits $\Gamma_t/\{\pm E_4\}$ as a subgroup of the orthogonal group $\mathrm{SO}(L_t)$ where L_t is the lattice of rank 5 equipped with the form $< 2t > \oplus 2U$ (here U denotes the hyperbolic plane). Let $\widehat{L}_t$ be the dual lattice of L_t. The image of Γ_t in $\mathrm{O}(L_t)$ acts trivially on $\widehat{L}_t/L_t$. The orthogonal group $\mathrm{O}(\widehat{L}_t/L_t)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\nu(t)}$ where

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$\nu(t)$ is the number of prime divisors of t. For every $d||t$ (i.e. $d|t$ and $(d, t/d) = 1$) we construct an element V_d in $\mathrm{Sp}_4(\mathbb{R})$. These elements V_d define a normal extension Γ_t^* of Γ_t of index $2^{\nu(t)}$ such that $\Gamma_t^*/\Gamma_t \cong \mathrm{O}(\widehat{L}_t/L_t)$. It turns out that Γ_t^* is the maximal normal extension of Γ_t as a discrete subgroup of $\mathrm{Sp}_4(\mathbb{R})$. Hence we can consider the moduli spaces $\mathscr{A}_t^* = \Gamma_t^* \backslash \mathbb{H}_2$ as 'minimal' Siegel modular threefolds. In Section 1 we also give a geometric interpretation of the action of V_d on the moduli space $\mathscr{A}_t$. In particular V_t identifies a polarized abelian surface with its dual. It turns out that the space $(\Gamma_t \cup \Gamma_t V_t) \backslash \mathbb{H}_2$ is isomorphic to the moduli space of lattice polarized $K3$ surfaces with a polarization of type $< 2t > \oplus 2E_8(-1)$. Lattice polarized $K3$ surfaces have been studied by Nikulin [**N2**]. They play a role in mirror symmetry for $K3$ surfaces (see Dolgachev [**D**]). We also prove that the variety $\mathscr{A}_t^*$ itself is the space of Kummer surfaces associated to $(1, t)$-polarized abelian surfaces. (For a precise statement see Theorem 1·5.) In particular, this implies that a $(1, t)$-polarized abelian surface and its dual have isomorphic Kummer surfaces. Surprisingly, this result seems not to have been noticed before. At this point one can ask the question which of the Siegel modular varieties lying between $\mathscr{A}_t$ and $\mathscr{A}_t^*$ are moduli spaces of $K3$ surfaces.

In Section 2 we study the action of the elements V_d on the space of Jacobi forms. It turns out that this gives rise to a decomposition of the space of Jacobi forms which was originally found by Eichler and Zagier [**EZ**]. Using lifting results due to the first author this enables us to prove that many moduli spaces lying between $\mathscr{A}_t$ and the minimal Siegel modular threefold $\mathscr{A}_t^*$ are not unirational, resp. have non-negative Kodaira dimension. This method, however, unfortunately does not give us information about the geometric genus or the Kodaira dimension of $\mathscr{A}_t^*$ itself. To determine these seems a fairly hard problem. We hope to come back to it. The general philosophy behind these considerations is the question where these moduli spaces fit into the classification theory of algebraic threefolds. Very little is known about this. For example one can ask, whether the fact that $\mathscr{A}_t^*$ is minimal (in the sense described above) for square free t is in any way reflected in terms of the classification theory of threefolds. To prove results about the Kodaira dimension of the spaces $\mathscr{A}_t^*$ requires a good knowledge of the singularities of these spaces which is a hard problem in itself. Since the singularities of $\mathscr{A}_t$ are known (cf. [**B**]) the problem is equivalent to determining the ramification locus of the finite map $\mathscr{A}_t \to \mathscr{A}_t^*$. As a first step we compute the divisorial part of this ramification locus for square free t (the general case can be treated by the same method). This can be shown to be a finite union of Humbert surfaces. To determine these we reexamine the theory of Humbert surfaces from the point of view of the orthogonal group, which turns out to be a very useful way of studying Humbert surfaces. An example, originally due to Brasch, shows that the ramification locus can also contain curve components. We finally interpret the Humbert surfaces in the ramification locus as Hilbert modular surfaces.

1. *The symplectic and orthogonal groups*

The local isomorphism between the symplectic group $\mathrm{Sp}_4(\mathbb{R})$ and the special orthogonal group $\mathrm{SO}(3, 2)_{\mathbb{R}}$ of signature $(3, 2)$ is well known. In this section we define this isomorphism over $\mathbb{Z}$.

Let us fix a lattice

$$L = e_1\mathbb{Z} \oplus e_2\mathbb{Z} \oplus e_3\mathbb{Z} \oplus e_4\mathbb{Z}.$$

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We identify $l \in L$ with a column-vector in the basis $\{e_i\}$. $L^2 = L \wedge L$ is the lattice of integral bivectors, which is isomorphic to the lattice of integral skew-symmetric matrices. The bivector $e_i \wedge e_j$ corresponds to the elementary skew-symmetric matrix E_{ij}, which has only two non-zero elements $e_{ij} = 1$ and $e_{ji} = -1$. Any linear transformation $g\colon L \to L$ induces a linear map $\wedge^2 g\colon L \wedge L \to L \wedge L$ on the $\mathbb{Z}$-lattice of bivectors. If g is represented with respect to the basis $\{e_i\}$ by the matrix G, then

$$(\wedge^2 g)(X) = GX\,{}^tG \quad \text{for any} \quad X = \sum_{i<j} x_{ij} e_i \wedge e_j \in L \wedge L.$$

One can define a symmetric bilinear form (X, Y) on $L \wedge L$

$$X \wedge Y = (X, Y)\, e_1 \wedge e_2 \wedge e_3 \wedge e_4 \in \wedge^4 L.$$

It is known, that $(X, X) = 2\,\mathrm{Pf}(X)$, where $\mathrm{Pf}(X)$ is the Pfaffian of the matrix X, and $\mathrm{Pf}(MX^tM) = \mathrm{Pf}(X) \det M$.

Definition. The group

$$\widetilde{\Gamma}_t = \{g\colon L \to L \mid \wedge^2 g\,(W_t) = W_t, \quad \text{where} \quad W_t = e_1 \wedge e_3 + te_2 \wedge e_4\} \tag{1·1}$$

is called the integral *paramodular group* of level t.

The lattice $L_t = W_t^{\perp}$ consisting of all elements of $L \wedge L$ orthogonal to W_t has the following basis

$$L_t = (e_1 \wedge e_2,\, e_2 \wedge e_3,\, e_1 \wedge e_3 - te_2 \wedge e_4,\, e_4 \wedge e_1,\, e_4 \wedge e_3)\,\mathbb{Z}^5.$$

We fix this basis for the rest of the paper. The symmetric bilinear form $(\cdot, \cdot)$ defines a quadratic form S of signature $(3, 2)$ on the lattice L_t, which has the following form in the given basis

$$S_t = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2t & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1·2}$$

The group of the real points of the paramodular group is conjugate to $\mathrm{Sp}_4(\mathbb{R})$. Thus the determinant of any element of the paramodular group equals one and $\wedge^2 g$ keeps the bilinear form on $L \wedge L$.

This gives us a homomorphism from the symplectic group in the orthogonal group of the isometries of the lattice L_t

$$\wedge^2\colon \widetilde{\Gamma}_t \to \mathrm{O}(L_t).$$

The paramodular group $\widetilde{\Gamma}_t$ is conjugate to a subgroup of the usual rational symplectic group:

$$\Gamma_t := I_t^{-1} \widetilde{\Gamma}_t I_t = \left\{ \begin{pmatrix} * & * & * & t* \\ t* & * & t* & t* \\ * & * & * & t* \\ * & t^{-1}* & * & * \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Q}) \right\},$$

where all entries $*$ denote integers and $I_t = \mathrm{diag}\,(1, 1, 1, t)$.

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The quotient space

$$\mathscr{A}_t = \Gamma_t \backslash \mathbb{H}_2$$

is the coarse moduli space of abelian surfaces with a polarization of type $(1, t)$.

The composition of the conjugation with the homomorphism $\wedge^2$ defines a homomorphism

$$\Psi\colon \Gamma_t \to \mathrm{O}(L_t) \quad \text{where} \quad \Psi(g) = \wedge^2(I_t\, g\, I_t^{-1}). \tag{1·3}$$

One can extend Ψ to the real symplectic group $\Gamma_t(\mathbb{R}) \cong \mathrm{Sp}_4(\mathbb{R})$.

Let $\widehat{L}_t = \{u \in L_t \otimes \mathbb{Q} \mid \forall\, l \in L_t\ (l, u) \in \mathbb{Z}\}$ be the dual lattice of L_t. The discriminant group

$$A_t := \widehat{L}_t / L_t = (2t)^{-1}\mathbb{Z}/\mathbb{Z} \cong \mathbb{Z}/2t\mathbb{Z}$$

is a finite abelian group equipped with a quadratic form

$$q_t :\ A_t \times A_t \to (2t)^{-1}\mathbb{Z}/2\mathbb{Z} \quad q_t(l, l) \equiv (l, l)_{\widehat{L}_t} \bmod 2\mathbb{Z}$$

(see [**N1**] for a general definition). Any $g \in \mathrm{O}(L_t)$ acts on the finite group A_t. By

$$\widehat{\mathrm{O}}(L_t) = \{g \in \mathrm{O}(L_t) \mid \forall \ell \in \widehat{L}_t \quad g\ell - \ell \in L_t\}$$

we denote the subgroup of the orthogonal group consisting of elements which act identically on the discriminant group.

One can easily prove the next lemma (see [**G1**]).

LEMMA 1·1. *The following relations are valid*

1. $\Psi(\Gamma_t) \subset \widehat{\mathrm{SO}}(L_t) = \widehat{\mathrm{O}}(L_t) \bigcap \mathrm{SO}(L_t)$;
2. $\mathrm{Ker}\,\Psi = \{\pm E_4\}$.

The finite orthogonal group $\mathrm{O}(A_t)$ can be described as follows. For every $d||t$ (i.e. $d|t$ and $(d,\ t/d) = 1$) there exists a unique (mod $2t$) integer ξ_d satisfying

$$\xi_d = -1 \bmod 2d, \quad \xi_d = 1 \bmod 2t/d.$$

All such ξ_d form the group

$$\Xi(t) = \{\, \xi \bmod 2t \mid \xi^2 = 1 \bmod 4t \,\} \cong (\mathbb{Z}/2\mathbb{Z})^{\nu(t)}, \tag{1·4}$$

where $\nu(t)$ is the number of prime divisors of t. It is evident that $\mathrm{O}(A_t) \cong \Xi(t)$.

One can take an element in $\mathrm{SO}(L_t)$ realising the multiplication by ξ_d on A_t. It gives us an element in $\mathrm{Sp}_4(\mathbb{R})$ with integral Ψ-image. For example, for every $d||t$ we can define $x, y \in \mathbb{Z}$ (which are not uniquely determined) such that

$$xd - yt_d = 1 \quad \text{where } t_d = \frac{t}{d}.$$

The matrix

$$\widetilde{V}_d = \begin{pmatrix} dx & -1 & 0 & 0 \\ -yt & d & 0 & 0 \\ 0 & 0 & d & yt \\ 0 & 0 & 1 & dx \end{pmatrix}$$

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is an integral symplectic similitude of degree d. We put

$$V_d = \frac{1}{\sqrt{d}}\widetilde{V}_d \in \mathrm{Sp}_4(\mathbb{R}).$$

V_d has the following Ψ-image

$$\Psi(V_d) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & d & -2yt & y^2 t_d & 0 \\ 0 & -1 & dx + t_d y & -xy & 0 \\ 0 & t_d & -2tx & x^2 d & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1·5}$$

We note here that

$$xd + t_d y = -1 \bmod 2d \quad \text{and} \quad xd + t_d y = 1 \bmod 2t_d,$$

thus V_d induces the multiplication by ξ_d on A_t.

It is easy to see, that for all V_d $(d||t)$

$$V_d^2 \in \Gamma_t, \quad V_d \Gamma_t V_d = \Gamma_t,$$

i.e. the V_d are involutions modulo Γ_t. Therefore one can define the following normal extension of the paramodular group Γ_t.

Definition. Γ_t^* is the group generated by the elements of Γ_t and V_d for all $d||t$.

In accordance with Lemma 1·1 any element in $\Psi(V_d \Gamma_t)$ defines the same automorphism of A_t, thus

$$\Gamma_t^*/\Gamma_t \cong \mathrm{O}(\widehat{L}_t/L_t) \cong \Xi(t) \cong (\mathbb{Z}/2\mathbb{Z})^{\nu(t)}. \tag{1·6}$$

The real orthogonal group $\mathrm{O}_{\mathbb{R}}(L_t) = \mathrm{O}(L_t \otimes \mathbb{R})$ acts on a domain lying on a projective quadric, more exactly on

$$\mathbb{PH}_t^3 = \mathbb{PH}_{L_t}^3 = \{Z \in \mathbb{P}(L_t \otimes \mathbb{C}) \mid (Z,Z) = 0,\ (Z,\overline{Z}) < 0\} = \mathbb{PH}_t^+ \cup \overline{\mathbb{PH}}_t^+,$$

where

$$\mathbb{PH}_t^+ = \{Z = {}^t\big((tz_2^2 - z_1 z_3),\, z_3,\, z_2,\, z_1,\, 1\big) \cdot z_0 \in \mathbb{PH}_t^3 \mid \mathrm{Im}\,(z_1) > 0\}. \tag{1·7}$$

This is a classical homogeneous domain of type IV. The condition $(Z,\overline{Z}) < 0$ is equivalent to

$$y_1 y_3 - t y_2^2 > 0 \quad \text{where } y_i = \mathrm{Im}\,(z_i).$$

Taking $z_0 = 1$ one gets the corresponding cylindric domain in the affine coordinates $(z_i)_{1 \leqslant i \leqslant 3}$

$$\mathbb{H}_t^+ = \{Z = {}^t(z_3,\, z_2,\, z_1) \in \mathbb{C}^3 \mid y_1 y_3 - t y_2^2 > 0, \quad y_1 = \mathrm{Im}\,(z_1) > 0\}.$$

The domain $\mathbb{H}_t^+$ for $t = 1$ coincides with the Siegel upper half-plane $\mathbb{H}_2$. For a general t one can define the following isomorphism of the complex domains

$$\psi_t \colon \mathbb{H}_2 \to \mathbb{H}_t^+ \quad \psi_t \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} = {}^t\left(\frac{\tau_3}{t},\, \frac{\tau_2}{t},\, \tau_1\right). \tag{1·8}$$

The linear action of the real orthogonal group $\mathrm{O}_{\mathbb{R}}(L_t)$ on $\mathbb{PH}_t^+$ defines 'fractional-linear' transformations on $\mathbb{H}_t^+$. By $\mathrm{O}_{\mathbb{R}}^+(L_t)$ we denote the subgroup of index 2 of

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the orthogonal group consisting of elements which leave $\mathbb{PH}_t^+$ invariant. (This is the subgroup of the elements with real spin norm equal one.)

PROPOSITION 1·2. *Let t be square free. Then Ψ defines the following isomorphisms*

$$\Psi\colon \Gamma_t^*/\{\pm E_4\} \to \mathrm{SO}^+(L_t),$$

where $\mathrm{SO}^+(L_t) = \mathrm{SO}(L_t) \bigcap \mathrm{O}^+_{\mathbb{R}}(L_t)$, *and*

$$\Psi\colon \Gamma_t/\{\pm E_4\} \to \widehat{\mathrm{SO}}^+(L_t),$$

where $\widehat{\mathrm{SO}}^+(L_t) = \mathrm{SO}^+(L_t) \bigcap \widehat{\mathrm{SO}}(L_t)$. *Moreover the following diagram is commutative*

$$\begin{array}{ccc} \mathbb{H}_2 & \xrightarrow{g} & \mathbb{H}_2 \\ {\scriptstyle \psi_t}\downarrow & & \downarrow{\scriptstyle \psi_t} \\ \mathbb{H}_t^+ & \xrightarrow{\Psi(g)} & \mathbb{H}_t^+. \end{array}$$

Proof. The diagram is commutative for any $g \in \mathrm{Sp}_4(\mathbb{R})$. To prove this one has to calculate the images of standard generators of $\mathrm{Sp}_4(\mathbb{R})$ under Ψ.

It is known that for square free t, the group $\mathrm{P}\Gamma_t^* \cong \Gamma_t^*/\{\pm E_4\}$ is a maximal discrete subgroup of the group of analytic automorphisms of $\mathbb{H}_2$ and $[\Gamma_t^* : \Gamma_t] = 2^{\nu(t)}$ (see for example [**Al**], [**Gu**]). From the description of the finite orthogonal group $\mathrm{O}(A_t)$ given in (1·4) we obtain that $[\mathrm{SO}^+(L_t) : \widehat{\mathrm{SO}}^+(L_t)] = 2^{\nu(t)}$. The statement of the proposition about the isomorphism of the groups follows from the maximality of Γ_t^* and Lemma 1·1. □

The coset $V_t\,\Gamma_t$ (in the case $d = t$ we may take $x = 0, y = -1$) can also be written in the form

$$V_t\,\Gamma_t = \begin{pmatrix} 0 & \sqrt{t^{-1}} & 0 & 0 \\ \sqrt{t} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{t} \\ 0 & 0 & \sqrt{t^{-1}} & 0 \end{pmatrix} \Gamma_t.$$

According to (1·3) and (1·5) $\Psi(V_t)$ defines the multiplication by -1 on $\widehat{L}_t/L_t$, i.e. V_t corresponds to the element $\xi_t = -1$ of $\Xi(t)$ (see 1·4). Therefore

$$-\Psi(V_t) \in \widehat{\mathrm{O}}(L_t).$$

Elements M and $-M \in \mathrm{O}(L_t)$ define the same transformation of the domain $\mathbb{H}_t^+$. Thus we have

COROLLARY 1·3. *Let t be square free. The groups*

$$\Psi(\Gamma_t \bigcup \Gamma_t V_t) \quad \text{and} \quad \mathrm{O}^*(L_t) = \widehat{\mathrm{O}}(L_t) \bigcap \mathrm{O}^+_{\mathbb{R}}(L_t)$$

coincide, if we consider them as groups of analytic transformations of $\mathbb{PH}_t^+$.

PROPOSITION 1·4. *The quotient*

$$(\Gamma_t \bigcup \Gamma_t V_t) \setminus \mathbb{H}_2$$

is isomorphic to the moduli space of polarized $K3$ surfaces with a polarization of type $< 2t > \oplus\, 2E_8(-1)$.

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Proof. It is known that a moduli space of polarized $K3$ surfaces is a quotient of a 19-dimensional homogeneous domain of type IV by an arithmetic group. In the proposition we consider polarized $K3$ surfaces with a condition on the Picard group or equivalently on the lattice of its transcendental cycles. To formulate these conditions we need some definitions (see [**N2**], [**D**]).

Let X be a $K3$ surface. Let us take a sublattice $D_t =< 2t > \oplus E_8(-1) \oplus E_8(-1)$ of the lattice

$$L_{K3} = U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1) \cong H^2(X, \mathbb{Z}),$$

where $< 2t >$ $(t \in \mathbb{Z})$ denotes the one-dimensional lattice generated by a vector l such that $l^2 = 2t$, U is the hyperbolic plane with quadratic form $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $E_8(-1)$ is the even unimodular lattice of dimension 8 with the negative definite quadratic form. We note that sign $(D_t) = (1, 16)$ and $D_t^{\perp} \cong U \oplus U \oplus < -2t >= L_t' = L_t(-1)$. (Notation $L_t(-1)$ means that we multiply the quadratic form S_t on the lattice L_t by -1.) We recall (see [**N1**]) that the orthogonal group

$$\mathrm{O}(D_t,\, L_{K3}) = \{\, g\colon L_{K3} \to L_{K3} \mid g\,|_{D_t} \equiv \mathrm{id}\}$$

is isomorphic to the group

$$\widehat{\mathrm{O}}(L_t') = \{\, g\colon L_t' \to L_t' \mid \forall\, l \in \widehat{L}_t' \;\; gl - l \in L_t' \,\}.$$

A marked D_t-polarized $K3$-surface is defined by the following datum (see [**N2**] and [**D**] for more details): a surface X, a fundamental domain $C(M)^+$ of a group generated by some 2-reflections of the lattice D_t acting on a connected component $V(D_t)^+$ of the cone $V(D_t) = \{v \in D_t \otimes \mathbb{R} \mid (v, v) > 0\}$ and an isomorphism of the lattices $\phi\colon H^2(X, \mathbb{Z}) \to L_{K3}$, such that $\phi^{-1}(D_t) \subset \mathrm{Pic}\,(X)$, $\phi^{-1}(V(D_t)^+) \subset V(X)^+$ and $\phi^{-1}(C(M)^+)$ contains at least one numerically effective divisor class. By $V(X)^+$ one denotes the connected component of the cone

$$V(X) = \{v \in H^{1,1}_{\mathbb{R}}(X) \mid (v,\, v) > 0\}$$

containing the cohomology class of a Kähler form on X.

Let us denote by ω_X a holomorphic 2-form which generates $H^{2,0}(X)$. Its image under the isometry ϕ belongs to the following domain in the projective space $\mathbb{P}^4$

$$\phi(\omega_X) \in \mathscr{D}_m = \{v \in \mathbb{P}(L_t' \otimes \mathbb{C})\colon (v, v) = 0, \quad (v, \bar{v}) > 0\}.$$

This domain is an example of the domains of type IV. Its connected components are isomorphic to the domain $\mathbb{H}_t^+$ introduced after (1·7).

The quotient

$$\mathscr{M}(< 2t > \oplus 2E_8(-1)) = \mathrm{O}^*(L_t') \setminus \mathbb{H}_t^+$$

is the moduli space of isomorphism classes of $< 2t > \oplus 2E_8(-1)$-polarized $K3$ surfaces. In accordance with Corollary 1·3

$$\mathscr{M}(< 2t > \oplus 2E_8(-1)) \cong (\Gamma_t \bigcup \Gamma_t V_t) \setminus \mathbb{H}_2. \qquad \square$$

We now want to explain the relationship between the variety $\mathscr{A}_t^* = \Gamma_t^* \setminus \mathbb{H}_2$ and the space of Kummer surfaces associated to $(1, t)$-polarized abelian surfaces. For an abelian surface A we denote by $X = \mathrm{Km}\,(A)$ its associated Kummer surface.

THEOREM 1·5. (i) *Let A, A' be two $(1, t)$-polarized abelian surfaces which define the same point in $\mathscr{A}_t^*$. Then their Kummer surfaces X, X' are isomorphic.*

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(ii) *Assume that the Neron-Severi group of A, resp. A' is generated by the polarization. Then the converse is true*: *If A and A' have isomorphic Kummer surfaces, then A and A' define the same point in $\mathscr{A}_t^*$.*

Proof. We consider the commutative diagram

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{p} & X = \mathrm{Km}(A) \\ {\scriptstyle\sigma}\downarrow & & \downarrow \\ A & \longrightarrow & Y = A/\iota \end{array}$$

where $\tilde{A}$ is the blow-up of A in the 16 points of order 2. By [**BPV**, p. 246] the map

$$\alpha = p_! \circ \sigma^* \colon H^2(A, \mathbb{Z}) \to H^2(X, \mathbb{Z})$$

multiplies the intersection form by 2 and hence is in particular a monomorphism. Let $T \subset H^2(X, \mathbb{Z})$ be the orthogonal complement of the sublattice of $H^2(X, \mathbb{Z})$ generated by the polarization and the nodal classes. Using [**BPV**, corollary VIII·5·6] it follows that $T \cong 2U(2) \oplus < -4t >$. If A is generic, i.e. the Neron-Severi group is generated by the polarization, then T is the transcendental lattice of X.

(i) Assume that A and A' define the same point in $\mathscr{A}_t^*$. Then there exists an element V in Γ_t^* which induces an isomorphism of the lattices T and T' of X and X' respectively. By the Torelli theorem for Kummer surfaces it now suffices to prove the

Claim. The isomorphism $V \colon T \to T'$ can be extended to a Hodge isometry $\widetilde{V} \colon H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$.

Proof of claim. By definition T is a primitive non-degenerate sublattice of the unimodular lattice $H^2(X, \mathbb{Z})$. Its orthogonal complement $T^\perp$ in $H^2(X, \mathbb{Z})$ is an even indefinite lattice of signature $(1, 16)$. The discriminant groups A_T and $A_{T^\perp}$ are isomorphic and have five generators. By [**N1**, theorem 1·14·2] the homomorphism $\mathrm{O}(T^\perp) \to \mathrm{O}(A_{T^\perp}, q_{T^\perp})$ is surjective. Using [**N1**, corollary 1·5·2] one can, therefore, construct an extension of the isomorphism V.

(ii) An isomorphism $f \colon X \to X'$ induces an isomorphism of transcendental lattices. Under the assumption stated this defines an isometry $V \colon T \to T'$ where $T \cong T' \cong 2U(2) \oplus < -4t >= L_t(-2)$, which is an element of the group $\mathrm{O}^+(L_t(-2))$. Since $\mathrm{O}^+(L_t(-2)) = \mathrm{O}^+(L_t)$ it follows from Proposition 1·2 that there is an element $\overline{V} \in \Gamma_t^*$ which defines one of the elements $\pm V$. The element $\overline{V}$ identifies the points defined by A and A' resp. □

Remark. The above theorem justifies it to consider $\mathscr{A}_t^*$ as the moduli space of Kummer surfaces associated to abelian surfaces with a $(1, t)$-polarization.

Remark. In [**D**, example 6·5] Dolgachev considered the space of M_t-polarized abelian surfaces where M_t is the orthogonal complement of $2U(2) \oplus < -4t >$ in L_{K3}. This leads to a subgroup of finite index of $\mathrm{O}^+(L_t)$ and hence a covering space of $\mathscr{A}_t^*$.

Our next aim is to interpret the involutions V_d geometrically. Because of Proposition 1·2 the element V_d induces a map from $\mathscr{A}_t^*$ to itself.

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Let (A, H) be a $(1, t)$-polarized abelian surface. The polarization H defines an isogeny

$$\begin{aligned} \lambda_H : \quad & A \to \widehat{A} = \mathrm{Pic}^0 A \\ & x \mapsto T_x^* \mathscr{L} \otimes \mathscr{L}^{-1} \end{aligned}$$

where $\mathscr{L}$ is a line bundle representing H and T_x denotes translation by x. The map λ_H only depends on H, not on the line bundle $\mathscr{L}$. There is a (non-canonical) isomorphism $\ker \lambda_H \cong \mathbb{Z}_t \times \mathbb{Z}_t$. For every divisor d of t there is a unique subgroup $G(d) \subset \ker \lambda_H$ which is isomorphic to $\mathbb{Z}_d \times \mathbb{Z}_d$. This subgroup defines a quotient

$$\lambda_d : \quad A \to A/G(d) = A'.$$

If A is given by the period matrix

$$\Omega = \begin{pmatrix} 1 & 0 & \tau_1 & \tau_2 \\ 0 & t & \tau_2 & \tau_3 \end{pmatrix}, \quad \tau = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \in \mathbb{H}_2$$

then A' is given by

$$\Omega' = \begin{pmatrix} d & 0 & d\tau_1 & \tau_2 \\ 0 & t_d & \tau_2 & \tau_3/d \end{pmatrix}.$$

The abelian surface A' carries a uniquely determined polarization H' with

$$dH = \lambda_d^*(H').$$

The polarization H' is of type $e \cdot (1, t/e^2)$ where $e = (d, t_d)$. Altogether this shows that we have a morphism of moduli spaces

$$\begin{aligned} \Phi = \Phi(d) : \quad & \mathscr{A}_t \to \mathscr{A}_{t/e^2} \\ & (A, H) \mapsto (A', H'). \end{aligned}$$

If $d = t$ we obtain as a special case the map

$$\begin{aligned} \Phi(t) : \quad & \mathscr{A}_t \to \mathscr{A}_t \\ & (A, H) \mapsto (\widehat{A}, \widehat{H}) \end{aligned}$$

which maps an abelian surface to its dual polarized abelian surface.

Proposition 1·6. *Let d be a divisor of t with $(d, t_d) = 1$. Then the map*

$$\Phi(d) : \quad \mathscr{A}_t \to \mathscr{A}_t$$

is the map induced by V_d.

Proof. For $\tau = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \in \mathbb{H}_2$ we have the following formula for the action

$$V_d < \tau >= \begin{pmatrix} x & -1 \\ -yt_d & d \end{pmatrix} \begin{pmatrix} d\tau_1 & \tau_2 \\ \tau_2 & \tau_3/d \end{pmatrix} \begin{pmatrix} x & -yt_d \\ -1 & d \end{pmatrix}.$$

Now consider the matrix

$$\begin{pmatrix} 1 & t_d & 0 & 0 \\ y & xd & 0 & 0 \\ 0 & 0 & x & -yt_d \\ 0 & 0 & -1 & d \end{pmatrix} \in \mathrm{SL}_4(\mathbb{Z}).$$

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This matrix transforms the symplectic form $de_1 \wedge e_3 + t_d e_2 \wedge e_4$ into W_t (see (1.1)). The claim now follows from the equality

$$\begin{pmatrix} x & -1 \\ -yt_d & d \end{pmatrix}\begin{pmatrix} d & 0 & d\tau_1 & \tau_2 \\ 0 & t_d & \tau_2 & \tau_3/d \end{pmatrix}\begin{pmatrix} 1 & t_d & 0 & 0 \\ y & xd & 0 & 0 \\ 0 & 0 & x & -yt_d \\ 0 & 0 & -1 & d \end{pmatrix} = \begin{pmatrix} 1 & 0 & V_d < \tau > \\ 0 & t & \end{pmatrix}. \quad \square$$

Remark. In view of Theorem 1.5 this shows in particular that a $(1, t)$-polarized abelian surface and its dual polarized abelian surface have isomorphic Kummer surfaces. Surprisingly this seems not to have been noticed before.

2. *Nonunirationality of some quotient spaces*

We have already shown that the Siegel modular threefolds $\mathscr{A}_t/ < V_t >$ and $\mathscr{A}_t^*$ are moduli spaces of $K3$ surfaces. There are $2^{\nu(t)} - 2$ other threefolds between $\mathscr{A}_t$ and $\mathscr{A}_t^*$. Let us take for example two primes $p \neq q$ and let $t = pq$. The involutions V_p and V_q give rise to the following moduli spaces

$$\mathscr{A}_{pq}^{(p)} = \mathscr{A}_{pq}/ < V_p >,\ \mathscr{A}_{pq}^{(q)} = \mathscr{A}_{pq}/ < V_q >,\ \mathscr{A}_{pq}^{*} = \mathscr{A}_{pq}/ < V_p, V_q > = \Gamma_{pq}^{*}\backslash \mathbb{H}_2$$

resp. a commutative diagram

$$\begin{array}{ccccc} & & \mathscr{A}_{pq} & & \\ & \swarrow & & \searrow & \\ \mathscr{A}_{pq}^{(p)} & & & & \mathscr{A}_{pq}^{(q)} \\ & \searrow & & \swarrow & \\ & & \mathscr{A}_{pq}^{*} & & \end{array}$$

where all maps are 2:1. Using the modular forms constructed in [**G1**] we can obtain information about the geometric genus of some of these moduli spaces.

By $J_{k,t}^{\mathrm{cusp}}$ we denote the space of Jacobi cusp forms of weight k and index t. In [**G1**] a lifting was constructed which associates to a Jacobi form $\Phi \in J_{k,t}^{\mathrm{cusp}}$ a cusp form $F_\Phi \in \mathscr{M}_k(\widehat{\Gamma}_t)$ of weight k with respect to the group

$$\widehat{\Gamma}_t = \left\{ \begin{pmatrix} * & t* & * & * \\ * & * & * & *t^{-1} \\ * & t* & * & * \\ t* & t* & t* & * \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Q}) \right\},$$

where all entries $*$ denote integers.

The groups Γ_t and $\widehat{\Gamma}_t$ are conjugate. Indeed if $C_t = \mathrm{diag}\,(1, t^{-1}, 1, t)$ then $\widehat{\Gamma}_t = C_t \Gamma_t C_t^{-1}$. For

$$\widehat{V}_t = {}^t V_t = \begin{pmatrix} 0 & \sqrt{t} & 0 & 0 \\ \sqrt{t^{-1}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{t^{-1}} \\ 0 & 0 & \sqrt{t} & 0 \end{pmatrix}$$

it was proved in [**G1**, formula (2·8)] that

$$F_\Phi(Z) = (-1)^k F_\Phi|_k \widehat{V}_t(Z).$$

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or equivalently

$$F_\Phi\begin{pmatrix}\tau_1 & \tau_2\\ \tau_2 & \tau_3\end{pmatrix} = F_\Phi\begin{pmatrix}t\tau_3 & \tau_2\\ \tau_2 & t^{-1}\tau_1\end{pmatrix}.$$

In this section we describe the behaviour of F_Φ with respect to the group

$$\widehat{\Gamma}_t^* = <\widehat{\Gamma}_t,\ \widehat{V}_d \mid d||t>,$$

where $\widehat{V}_d = C_t V_d C_t^{-1}$.

Eichler and Zagier [**EZ**, section 5], have constructed a decomposition of the space of Jacobi forms

$$J_{k,t}^{\mathrm{cusp}} = \bigoplus_\epsilon J_{k,t}^\epsilon,$$

where ϵ runs over all characters of the group

$$\Xi(t) = \{\,\xi \bmod 2t \mid \xi^2 \equiv 1 \bmod 4t\,\} \cong (\mathbb{Z}/2\mathbb{Z})^{\nu(t)}$$

satisfying $\epsilon(-1) = (-1)^k$. For any d with $d||t$ one can define an operator W_d acting on $J_{k,t}$ in the following way [**EZ**, section 5]. For

$$\Phi(\tau_1,\tau_2) = \sum_{\substack{n,\,l\in\mathbb{Z}\\ 4nt>l^2}} f(n,l)\exp\left(2\pi i(n\tau_1 + l\tau_2)\right) \in J_{k,t}^{\mathrm{cusp}},$$

we put

$$(\Phi \mid W_d)(\tau_1,\tau_2) = \sum_{\substack{n,\,l\in\mathbb{Z}\\ 4nt>l^2}} f(n',l')\exp\left(2\pi i(n\tau_1 + l\tau_2)\right) \in J_{k,t}^{\mathrm{cusp}},$$

where l', n' are determined by

$$l' \equiv -l \bmod 2d, \quad l' \equiv l \bmod 2t/d, \quad 4n't - l'^2 = 4nt - l^2.$$

All W_d are involutions. They form a group isomorphic to $\Xi(t)$. The subspaces $J_{k,t}^\epsilon$ are eigenspaces of the operation W_d, namely

$$J_{k,t}^\epsilon = \{\Phi \in J_{k,t}^{\mathrm{cusp}} \mid \Phi \mid W_d = \epsilon(W_d)\,\Phi\}.$$

Note that if

$$\Phi(\tau_1,\tau_2) = \sum_{\mu \bmod 2t} \varphi_\mu(\tau_1)\theta_{t,\mu}(\tau_1,\tau_2) \in J_{k,t}^\epsilon$$

is the standard decomposition of the Jacobi form Φ with respect to the theta-functions $\theta_{t,\mu}(\tau_1,\tau_2)$ then for $\xi_d \in \Xi(t)$

$$\varphi_{\xi_d\mu}(\tau_1) = \epsilon(\xi_d)\varphi_\mu(\tau_1).$$

THEOREM 2·1. *Let $\Phi \in J_{k,t}^\epsilon$ be a Jacobi form and $F_\Phi \in \mathscr{M}_k(\widehat{\Gamma}_t)$ be its lifting. For any divisor d of t with $(d, t_d) = 1$ the following equality holds*

$$F_\Phi|_k\,\widehat{V}_d = \epsilon(\xi_d)F_\Phi.$$

Proof. Let us recall the definition of the lifting F_Φ in terms of the Fourier expansion [**G2**]. If

$$\Phi(\tau_1,\tau_2) = \sum_{\substack{n,\,l\in\mathbb{Z}\\ 4nt>l^2}} f(n,l)\exp\left(2\pi i(n\tau_1 + l\tau_2)\right) \in J_{k,t}^{\mathrm{cusp}}$$

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then

$$F_\Phi(Z) = \sum_{N \in \mathfrak{A}_t} b(N) \exp\left(2\pi i \operatorname{tr}(NZ)\right),$$

where summation is taken over all positive definite symmetric matrices of the following form

$$N \in \mathfrak{A}_t = \left\{ \begin{pmatrix} n & l/2 \\ l/2 & mt \end{pmatrix} > 0 \mid n, l, m \in \mathbb{Z} \right\}$$

and

$$b\begin{pmatrix} n & l/2 \\ l/2 & mt \end{pmatrix} = \sum_{a \mid (n,l,m)} a^{k-1} f\left(\frac{nm}{a^2}, \frac{l}{a}\right).$$

The action of $\widehat{V}_d$ on F_Φ is given by

$$(F_\Phi|_k \widehat{V}_d)(Z) = F_\Phi(d^{-1} A_d \, Z \, {}^t A_d) \quad \text{where } A_d = \begin{pmatrix} dx & -t \\ -y & d \end{pmatrix}$$

or

$$\begin{aligned}(F_\Phi|_k \widehat{V}_d)(Z) &= \sum_{N \in \mathfrak{A}_t} b(N) \exp\left(2\pi i \operatorname{tr}(d^{-1} \, {}^t A_d N A_d Z)\right) \\ &= \sum_{N \in \mathfrak{A}_t} b(d^{-1} \, {}^t \tilde{A}_d N \tilde{A}_d) \exp\left(2\pi i \operatorname{tr}(NZ)\right)\end{aligned}$$

where $\tilde{A}_d = dA_d^{-1} = \begin{pmatrix} d & t \\ y & dx \end{pmatrix}$. Let $\widetilde{N} = d^{-1} \, {}^t \tilde{A}_d N \tilde{A}_d = \begin{pmatrix} \tilde{n} & \tilde{l}/2 \\ \tilde{l}/2 & \widetilde{m}t \end{pmatrix}$. Clearly $\det N = \det \widetilde{N}$. It is easy to see that the elements n, l, m and $\tilde{n}$, $\tilde{l}$, $\widetilde{m}$ have the same set of common divisors. Moreover

$$\tilde{l} = l(yt_d + dx) + 2(nt + xymt) \quad \text{and} \quad \tilde{l} \equiv \begin{cases} -l & \bmod 2d \\ l & \bmod 2t_d. \end{cases}$$

Hence, by the definition of $J^\epsilon_{k,t}$ we have

$$b(d^{-1} \, {}^t \tilde{A}_d N \tilde{A}_d) = \sum_{a \mid (n,l,m)} a^{k-1} f\left(\frac{\tilde{n}\widetilde{m}}{a^2}, \frac{\tilde{l}}{a}\right) = \epsilon(\xi_d) b(N)$$

which proves the theorem. □

This result has some implications on the Kodaira dimension of Siegel modular varieties which are quotients of $\mathscr{A}_t$. This question has not yet been studied. We consider the following results as a first step towards a more systematic study of the Kodaira dimension of these varieties. Whenever we speak of the *Kodaira dimension* of some moduli space $\mathscr{A}$ we mean the Kodaira dimension of a desingularization of a projective compactification of $\mathscr{A}$.

COROLLARY 2·2. *Let $p \neq q$ be primes $\geqslant 5$. Then the Kodaira dimension of at least one of the spaces $\mathscr{A}^{(p)}_{pq}$ or $\mathscr{A}^{(q)}_{pq}$ is $\geqslant 0$.*

Proof. The Eichler–Zagier decomposition gives a decomposition

$$J_{3,\,pq} = J^{(+,-)}_{3,\,pq} \oplus J^{(-,+)}_{3,\,pq}.$$

If $p, q \geqslant 5$ then there exists a cusp form in $J_{3,\,pq}$ and hence in $J^{(+,-)}_{3,\,pq}$ or $J^{(-,+)}_{3,\,pq}$. By

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[**G1**], [**G2**] this can be lifted to a weight 3 cusp form with respect to $< \Gamma_t, V_p >$, resp. $< \Gamma_t, V_q >$. By Freitag's extension theorem this defines a differential form on any desingularization of a projective compactification of $\mathscr{A}_{pq}^{(p)}$, resp. $\mathscr{A}_{pq}^{(q)}$. This gives the result. □

For any integer t let us take a character of the group $\Xi(t)$ isomorphic to the orthogonal group of the discriminant group of L_t (see (1·4))

$$\epsilon\colon \Xi(t) \to \{\pm 1\}.$$

We define a set $U(\epsilon) = \{ V_d \in \Gamma_t^* |\ \epsilon(\xi_d) = 1 \}$ and a subgroup

$$\Gamma_t^\epsilon = < \Gamma_t,\ \xi_d \mid \xi_d \in U(\epsilon) > \subset \Gamma_t^*$$

of Γ_t^*. Theorem 2·1 and the method of the proof of Corollary 2·2 gives us the next result.

Corollary 2·3. *If* $\dim (J_{3,t}^\epsilon) > 0$, *then the Kodaira dimension of the quotient space*

$$\mathscr{A}_t^\epsilon = \Gamma_t^\epsilon \backslash \mathbb{H}_2$$

is nonnegative.

Remark. Corollary 2·3 gives information about $\mathscr{A}_t^\epsilon$ only if $V_t \notin U(\epsilon)$. If $\epsilon(V_t) = \epsilon(-1) = 1$, then $\dim (J_{3,t}^\epsilon) = 0$.

Corollary 2·4. *Let* $t \geqslant 21$ $(t \neq 24, 30, 36)$ *and let its number of prime divisors* $\nu(t) \geqslant 2$. *Then there exists a finite quotient of* $\mathscr{A}_t$ *of degree* $2^{\nu(t)-1}$ *which is not unirational.*

Proof. For any integer t from the corollary the dimension of $J_{3,t}^{\mathrm{cusp}}$ is positive. Thus there is a character ϵ of $\Xi(t)$ such that $\epsilon(\xi_t) = -1$ and $\dim J_{3,t}^\epsilon > 0$. $\Xi(t) \cong (\mathbb{Z}/2\mathbb{Z})^{\nu(t)}$ therefore $[\Gamma_t^\epsilon\colon \Gamma_t] = 2^{\nu(t)-1}$. □

Using dimension formulae for the spaces $J_{k,t}^\epsilon$ one can obtain more precise results. It is easy to get an exact dimension formula using the trace formula of the operator W_d on the space $J_{k,t}^{\mathrm{cusp}}$ given in [**SZ**]. By definition of W_d we have

$$\operatorname{tr}(W_d,\ J_{3,t}^{\mathrm{cusp}}) = \sum_\epsilon \epsilon(\xi_d) \dim (J_{3,t}^\epsilon),$$

where the sum is taken over all characters of $\Xi(t)$. Therefore

$$\dim (J_{3,t}^\epsilon) = 2^{-\nu(t)} \sum_{d||t} \epsilon(\xi_d) \operatorname{tr}(W_d,\ J_{3,t}^{\mathrm{cusp}}).$$

For weight 3 the trace formula of W_d on $J_{3,t}^{\mathrm{cusp}}$ (we recall that $d|t$ and $(d, t_d) = 1$, where $t_d = t/d$) proved in [**SZ**, theorem 1] can be reduced to the following expression

$$\begin{aligned}\operatorname{tr}(W_d,\ J_{3,t}^{\mathrm{cusp}}) = {} & \frac{1}{4}\sum_{e|d} H_{t_d}(-4e) - \frac{1}{4}\sum_{e'|t_d} H_d(-4e') + \tfrac{3}{2}(H_d(0) - H_{t_d}(0)) \\ & + \tfrac{1}{2}\big(\delta_2(t_d) H_d(-4) - \delta_2(d) H_{t_d}(-4)\big) + \big(\delta_3(t_d) H_d(-3) - \delta_3(d) H_{t_d}(-3)\big) \\ & + \tfrac{1}{4}\big((Q(t_d), 2) Q(d) - (Q(d), 2) Q(t_d)\big).\end{aligned}$$

We denote by $Q(n)$ the greatest integer whose square divides n; $\delta_a(b) = 1$ or 0 if $a|b$

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or $a \not\mid b$ and $H_n(\Delta)$ is a generalization of the Hurwitz–Kronecker class number, i.e $H_1(0) = -\frac{1}{12}$ and $H_1(\Delta)$ for $\Delta < 0$ is the number of equivalence classes with respect to $\mathrm{SL}_2(\mathbb{Z})$ of integral, positive definite, binary quadratic forms of discriminant Δ, counting forms equivalent to a multiple of x^2+y^2 (resp. x^2+xy+y^2) with multiplicity $\frac{1}{2}$ (resp. $\frac{1}{3}$). For $n \geqslant 2$ with $(n, \Delta) = a^2 b$ and square free b

$$H_n(\Delta) = \begin{cases} a^2 b \left(\dfrac{\Delta/a^2b^2}{n/a^2b}\right) H_1(\Delta/a^2b^2) & \text{if } a^2b^2 | \Delta \\ 0 & \text{otherwise,} \end{cases}$$

where $\left(\frac{\cdot}{\cdot}\right)$ is the generalized Kronecker symbol.

We note that the trace formula has the simplest form for a square free t coprime to 6:

$$\mathrm{tr}\,(W_d,\ J^{\mathrm{cusp}}_{3,\,t}) = \frac{1}{4}\left(\sum_{e|d}\left(\frac{-4e}{t_d}\right) H_1(-4e) - \sum_{e'|t_d}\left(\frac{-4e'}{d}\right) H_1(-4e')\right) + \frac{t_d - d}{8}.$$

Example 2·5. The calculation gives us only thirteen different threefolds of type $\mathscr{A}^{\epsilon}_t$ with t having only two prime divisors ($t = p^a q^b$), whose geometric genus could be equal to zero. They correspond to the following *trivial* subspaces of $J^{\mathrm{cusp}}_{3,t}$ of type $J^{-+}_{3,\,p^aq^b}$ (this notation means that $J^{-+}_{3,\,p^aq^b} = J^{\epsilon}_{3,\,p^aq^b}$ with $\epsilon(\xi_{p^a}) = -1$ and $\epsilon(\xi_{q^b}) = 1$):

$$J^{-+}_{3,\,2.11},\ J^{-+}_{3,\,2.13},\ J^{-+}_{3,\,2.17},\ J^{-+}_{3,\,2.19},\ J^{-+}_{3,\,2.25},\ J^{-+}_{3,\,2.27},$$
$$J^{-+}_{3,\,3.7},\ J^{-+}_{3,\,3.13},\ J^{-+}_{3,\,3.16},\ J^{-+}_{3,\,5.7},\ J^{-+}_{3,\,5.8},\ J^{-+}_{3,\,7.4},\ J^{-+}_{3,\,7.8}.$$

We may add to this list twenty threefolds $\mathscr{A}_t$ with $t = 1, \ldots, 12, 14, 15, 16, 18, 20, 24, 30, 36$ (see [**G2**]) whose geometric genus could be zero. According to classical results and new results of M. Gross and S. Popescu (see [**GP**]) it really is for $t = 1, \ldots, 12, 14, 16, 18, 20$.

Example 2·6. We have the following quotients of order 4 and 8 which are not unirational. One has

$$\dim J^{\mathrm{cusp}}_{3,\,42} = \dim J^{++-}_{3,\,2.3.7} = 1.$$

Thus for $\mathscr{A}^{(2,3)}_{42} =< V_2, V_3 > \setminus \mathscr{A}_{42}$ we obtain $h^{3,0}(\mathscr{A}^{(2,3)}_{42}) \geqslant 1$.

The geometric genus of all four threefolds of type $< V_a, V_b, V_c > \setminus \mathscr{A}_{210}$, where $a,\ b,\ c \in \{2, 3, 5, 7\}$ is positive.

3. *The Humbert surfaces and the ramification locus*

If one wants to compute the Kodaira dimension of the variety $\mathscr{A}^*_t = \Gamma^*_t \backslash \mathbb{H}_2$ it is essential to know its singularities. Already the singularities of $\mathscr{A}_t$ are very complicated to determine (cf. [**B**]). In view of this result the corresponding result for $\mathscr{A}^*_t$ is essentially equivalent to a description of the ramification locus of the finite map $\mathscr{A}_t \to \mathscr{A}^*_t$, i.e. the locus where the stabilizer of the finite group Γ^*_t/Γ_t is not trivial. This, too, turns out to be a difficult question. As a first step we shall determine the divisorial part of the ramification locus which is a union of a finite number of Humbert surfaces. We shall restrict ourselves to t square free, but our method works in general. Here we study Humbert surfaces from the point of view of the orthogonal group, rather than the symplectic group, which turns out to be a very appropriate approach.

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First we collect some known facts about divisors on the homogeneous domain $\mathbb{PH}_t^+$. For any $v \in L_t \otimes \mathbb{R}$ we set

$$\mathscr{H}_v = \{Z \in \mathbb{PH}_t^+ \mid v.Z = 0\},$$

where $v.u = (v,u)_t$ is the bilinear product corresponding to the quadratic form S_t (see (1·2)).

LEMMA 3·1.

1. $\mathscr{H}_{gv} = g^{-1}\mathscr{H}_v$ *for any* $g \in \mathrm{O}_{\mathbb{R}}^+(L_t)$.
2. *Let* $v \neq 0$, *then* $\mathscr{H}_v \neq \varnothing$ *if and only if* $v^2 > 0$.
3. $\mathscr{H}_v \cap \mathscr{H}_u \neq \varnothing$ *if and only if the matrix* $\begin{pmatrix} v^2 & v.u \\ v.u & u^2 \end{pmatrix}$ *is positive definite.*

Proof. 1. The first property is trivial.

2. The orthogonal group $\mathrm{O}_{\mathbb{R}}^+(L_t)$ acts transitively on $\mathbb{PH}_t^+$. Thus any $Z \in \mathbb{PH}_t^+$ can be reduced to $Z_{\mathbf{i}} = {}^t(1,i,0,i,1)$ in the coordinates (z_i) from 1·7. If $v = (a,b,c,d,e)$ and $v.Z_{\mathbf{i}} = 0$, then $a = -e$ and $b = -d$. Thus $v^2 = 2a^2 + 2b^2 + 2tc^2 > 0$.

Let $L_t \otimes \mathbb{R} = \mathbb{R}v \oplus V$ with $v^2 > 0$. One has $\mathrm{sign}\,(V) = (2,2)$. The group $\mathrm{SO}^+(V) \cong \mathrm{SO}_{\mathbb{R}}^+(2,2)$ is locally isomorphic to $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$. Thus

$$\mathscr{H}_v \cong \mathbb{PH}_V^+ \cong \mathbb{H}_1 \oplus \mathbb{H}_1,$$

where $\mathbb{H}_1$ is the usual upper half-plane. This proves the second statement.

3. Let us suppose that $\mathscr{H}_v \cap \mathscr{H}_u \neq \varnothing$. It follows from $(xu - v)^2 > 0$ that the matrix in 3 is positive definite.

If the symmetric bilinear form on the plane $P = \mathbb{R}v \oplus \mathbb{R}u$ is positive definite, then $\mathrm{sign}\,(P^\perp) = (1,2)$. The group $\mathrm{SO}^+(P^\perp) \cong \mathrm{SO}^+(1,2)_{\mathbb{R}}$ is locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$ and

$$\mathscr{H}_v \cap \mathscr{H}_u \cong \mathbb{PH}_{P^\perp}^+ \cong \mathbb{H}_1. \quad \square$$

Remark. For $l \in L_t$ $(l^2 > 0)$ the group $\widehat{\mathrm{SO}}^+(l^\perp)$ is isomorphic to a subgroup of $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ or to a subgroup of a Hilbert modular group.

Definition. Let $\ell \in \widehat{L}_t$ be a vector in the dual lattice. The *Humbert surface* H_ℓ is defined by

$$H_\ell = \pi\left(\bigcup_{g \in \widehat{SO}^+(L_t)} \mathscr{H}_{g\ell}\right),$$

where $\pi\colon \mathbb{PH}_t^+ \to \widehat{\mathrm{SO}}^+(L_t) \setminus \mathbb{PH}_t^+$ is the natural projection.

$\mathscr{H}_\ell$ depends only on the one dimensional lattice $\mathbb{Z}\ell$, thus we can restrict ourselves to primitive vectors $\ell \in \widehat{L}_t$. The primitivity means that $\ell/d \notin \widehat{L}_t$ for any integer $d > 1$. The first statement of Lemma 3·1 says that there is a one to one correspondence between the $\widehat{\mathrm{SO}}^+(L_t)$-orbits of primitive vectors $\ell \in \widehat{L}_t$ with positive norm and the Humbert surfaces.

It is well known that for any even integral lattice L with two hyperbolic planes (in particular for L_t) the $\widehat{\mathrm{SO}}(L)$-orbit of any $l \in L$ depends only on the norm of l and its canonical image $l^* := l/\mathrm{div}\,(l)$ in the discriminant group $\widehat{L}/L$, where the *divisor*

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$\operatorname{div}(l) \in \mathbb{N}$ of l is the positive generator of the ideal $\{(x,l)_L \mid x \in L\}$. As a corollary we have

LEMMA 3·2. *Let ℓ_1, $\ell_2 \in \widehat{L}_t$ be two primitive vectors with the same image in the discriminant group (i.e. $\ell_1 - \ell_2 \in L_t$). If $\ell_1^2 = \ell_2^2$, then $H_{\ell_1} = H_{\ell_2}$.*

Proof. If $\ell_1 - \ell_2 \in L_t$, then $\operatorname{div}(2t\ell_1) = \operatorname{div}(2t\ell_2)$ and $\widehat{\mathrm{SO}}^+(L_t)\,\ell_1 = \widehat{\mathrm{SO}}^+(L_t)\,\ell_2$. □

Definition. Let ℓ be a primitive vector of $\widehat{L}_t$. The integer $\Delta(\ell) = 2t\ell^2$ is called the *discriminant* of H_ℓ.

From the isomorphism $\widehat{L}_t/L_t \cong \mathbb{Z}/2t\mathbb{Z}$ one gets

COROLLARY 3·3. *The number of surfaces H_ℓ with fixed discriminant $\Delta = 2t\ell^2$, which are not Γ_t-equivalent, is equal to the number of solutions*

$$\#\{b \bmod 2t \mid b^2 \equiv \Delta \bmod 4t\}.$$

The standard definition of the Humbert surfaces (see [**vdG**], [**F**]) is given in terms of the moduli space of abelian surfaces with polarization $(1,t)$. Let us compare both definitions.

According to (1·8) and Proposition 1·2 we may rewrite the definition of $\mathscr{H}_\ell$ with $\ell = (e, a, -b/2t, c, f) \in \widehat{L}_t$ ($(e,a,b,c,f) = 1$) in coordinates (τ_i) of $\mathbb{H}_2$:

$$\mathscr{H}'_x = \psi_t^{-1}(\mathscr{H}_\ell) = \left\{ \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \in \mathbb{H}_2 \mid (\tau_2^2 - \tau_1\tau_3)f + c\tau_3 + b\tau_2 + ta\tau_1 + te = 0 \right\},$$

where $x = (te, ta, b, c, f)$. The number

$$2t\ell^2 = b^2 - 4f(te) - 4c(ta) = \Delta(\mathscr{H}'_x).$$

is by definition the discriminant of $\mathscr{H}'_x$. Let us introduce a lattice

$$N_t = \{(e,a,b,c,f) \in \mathbb{Z}^5 \mid e,\ a \equiv 0 \bmod t\}.$$

In accordance with Proposition 1·2 and Lemma 3·1 we have the following decomposition of the usual (in sense of [**vdG**], [**F**]) Humbert surface $H_\Delta \subset \mathscr{A}_t = \Gamma_t \setminus \mathbb{H}_2$:

$$H_\Delta = \pi_t \left(\bigcup_{\substack{x \in N_t,\ \text{primitive} \\ \Delta(x) = \Delta}} \mathscr{H}'_x \right) \cong \pi \left(\bigcup_{2t\ell^2 = \Delta} \bigcup_{g \in \widehat{SO}^+(L_t)} \mathscr{H}_{g\ell} \right),$$

where one takes the summation over representatives ℓ from the distinct orbits and π_t is the natural projection $\pi_t\colon \mathbb{H}_2 \to \mathscr{A}_t$. Thus the surface $\mathscr{H}_\ell$ defined above corresponds to an irreducible component of the surface H_Δ. Corollary 3·3 tells us that the number of the irreducible components of the H_Δ is equal to

$$\#\{b \bmod 2t \mid b^2 \equiv \Delta \bmod 4t\}.$$

This gives a new proof of Theorem 2·4 in [**vdG**] (see p. 212).

In Section 1 we fixed a basis of the lattice L_t such that

$$L_t = U(-1) \oplus U(-1) \oplus < 2t >,$$

where $U(-1)$ is the integral hyperbolic plane with the quadratic form $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

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and $< 2t >$ is the one dimensional $\mathbb{Z}$-lattice with even quadratic form $2t$. By $L_t^{(3)}$ we denote the orthogonal component in L_t of the first hyperbolic plane

$$L_t^{(3)} = \big(e_2 \wedge e_3,\ e_1 \wedge e_3 - te_2 \wedge e_4,\ e_4 \wedge e_1\big)\mathbb{Z}^3 \subset L_t.$$

It is easy to see that in any orbit $\widehat{\mathrm{SO}}(L_t)\ell$ there is a vector from $L_t^{(3)}$ and $\widehat{\mathrm{SO}}^+(L_t)\ell = \widehat{\mathrm{SO}}(L_t)\ell$. Thus any Humbert surface can be given in the form

$$\mathscr{H}_\ell = \{az_1 + bz_2 + cz_3 = 0\} \subset \mathbb{PH}_t^+, \quad \Delta(\mathscr{H}_\ell) = 2t\ell^2$$

or

$$\mathscr{H}'_x = \{ta\tau_1 + b\tau_2 + c\tau_3 = 0\} \subset \mathbb{H}_2, \quad \Delta(\mathscr{H}'_x) = b^2 - 4tac = 2t\ell^2$$

where $\ell = {}^t(0, a, -b/2t, c, 0) \in \widehat{L}_t$ and $x = (ta, b, c) \in N_t$.

For any $d||t$ we define the following subgroup of Γ_t^* and the corresponding quotient space of the moduli space $\mathscr{A}_t$

$$\Gamma_t^{(d)} = \Gamma_t \cup \Gamma_t V_d, \quad \mathscr{A}_t^{(d)} = \Gamma_t^{(d)} \setminus \mathbb{H}_2.$$

The ramification locus of the map $\mathscr{A}_t \to \mathscr{A}_t^{(d)}$ can consist of components of different dimension. In the next theorem we describe its *divisorial part* $D_t^{(d)}$.

THEOREM 3·4. *Let t be square free, $d > 1$ and $t_d = t/d$. Then*

$$D_t^{(d)} = \begin{cases} H_{4d} \cup H_d & \text{if} \quad \left(\dfrac{d}{4t_d}\right) = 1 \\ H_{4d} & \text{if} \quad \left(\dfrac{d}{4t_d}\right) \neq 1 \quad \text{and} \quad \left(\dfrac{d}{t_d}\right) = 1, \end{cases}$$

where (a/b) is the generalized Kronecker symbol of the quadratic residue.

Remarks. 1. For $d = t$

$$D_t^{(t)} = \begin{cases} H_{4t} \cup H_t & \text{if } t \equiv 1 \bmod 4 \\ H_{4t} & \text{otherwise.} \end{cases}$$

In particular $D_t^{(t)}$ is irreducible if $t \equiv 2$ or $3 \bmod 4$ (see Corollary 3·3).

2. For $d = 1$ Theorem 3·4 is still true if we denote by $D_t^{(1)}$ the divisorial part of the branch locus of the covering $\mathbb{H}_2 \to \mathscr{A}_t$. We note that $D_t^{(1)}$ was found in [**B**] by another method.

COROLLARY 3·5. *Let t be square free. The divisorial part D_t^* of the ramification locus of the map $\mathscr{A}_t \to \mathscr{A}_t^*$, where $\mathscr{A}_t^* = \Gamma_t^* \setminus \mathbb{H}_2$ is the 'minimal' Siegel modular threefold corresponding to polarization of type $(1, t)$, is the union of the following Humbert surfaces*

$$D_t^* = \bigcup_{d|t} \left(\varepsilon_1(d)H_{4d} \cup \varepsilon_2(d)H_d\right),$$

where $\varepsilon_1(d) = 1$ if $(d/t_d) = 1$, $\varepsilon_2(d) = 1$ if d is odd and $(d/4t_d) = 1$ and they equal 0 in all other cases. Moreover none of the above Humbert surfaces is Γ_t-equivalent.

Proof of corollary We have to prove only the last statement. If ℓ_1 and $\ell_2 \in \widehat{L}_t$ are two primitive vectors with norms $\ell_1^2 = 2/t_{d_1}$, $\ell_2^2 = 1/(2t_{d_2})$, then $\ell_1^2 \neq \ell_2^2$, since t is square free. □

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We break up the proof of Theorem 3·4 into several lemmas.

Let us consider a reflection with respect to a vector $v \in L_t \otimes \mathbb{R}$:

$$\sigma_v(x) = x - \frac{2(x,v)}{(v,v)}\, v.$$

It is known that $\sigma_v \in \mathrm{O}^+_{\mathbb{R}}(L_t)$ if and only if $v^2 > 0$. (This follows from the definition of the real spin norm.) If $\sigma_v \in \mathrm{O}^+_{\mathbb{R}}(L_t)$, then the set $\mathrm{Fix}_{\mathbb{PH}^+_t}(\sigma_v)$ of fix points of σ_v on $\mathbb{PH}^+_t$ is a complex surface $\mathscr{H}_v$. The opposite statement is also true.

LEMMA 3·6. *Let us suppose that the set of fix points of $\sigma \in \mathrm{SO}^+_{\mathbb{R}}(L_t)$ on $\mathbb{PH}^+_t$ is a complex surface. Then $-\sigma$ is a reflection with respect to a vector $v \in L_t \otimes \mathbb{R}$.*

Proof. Over $\mathbb{R}$ one can reduce the quadratic form S_t to $S = \mathrm{diag}\,(E_3, -E_2)$. The maximal compact subgroup $K_{\mathbb{R}}$ of the orthogonal group $\mathrm{SO}^+_{\mathbb{R}}(S)$ is isomorphic to $\mathrm{SO}(3) \times \mathrm{SO}(2)$ consisting of all elements which fix the point $Z_{\mathbf{i}} = {}^t(0,0,0,i,1) \in \mathbb{PH}^+_t$. Since the group $\mathrm{SO}^+_{\mathbb{R}}(S)$ acts transitively on the homogeneous domain we can suppose that $\sigma = \mathrm{diag}\,(A,B) \in K$ where $A \in \mathrm{SO}(3)$ and $B \in \mathrm{SO}(2)$. If $B \neq \pm E_2$, then B has only one fix point $\mathbf{i} = {}^t(i,1)$ on the projective line. If $\sigma = \mathrm{diag}\,(A,B)$ has at least three fixed points, then B has an eigenvalue λ of order two. A and B are orthogonal, thus all eigenvalues of σ are equal to ± 1.

There are two possibilities for the set of eigenvalues of σ

$$\{\lambda(\sigma)\} = \{1,-1,-1,-1,-1\} \quad \text{or} \quad \{1,1,1,-1,-1\}.$$

In the first case $-\sigma$ is a reflection. In the second case σ can be written as a product of two reflections $\sigma_v\sigma_u$ with orthogonal u and v. Thus $\mathrm{Fix}_{\mathbb{PH}^+_t}(\sigma) = \mathscr{H}_u \bigcap \mathscr{H}_v$ and we have proved the lemma for non-trivial B.

If $B = \pm E_2$, then the same arguments show that σ is conjugate to

$$D = \begin{pmatrix} B_1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm E_2 \end{pmatrix} \qquad B_1 \in \mathrm{SO}(2) \tag{3·1}$$

if σ has at least two fixed points. If $B_1 \neq \pm E_2$, then $\mathrm{Fix}_{\mathbb{PH}^+_t}(D)$ is a subset of $\mathscr{H}_x \bigcap \mathscr{H}_y$, where x and y form an orthogonal basis of the plane of rotation of $B_1 \in \mathrm{SO}(2)$. □

LEMMA 3·7. *There is a one to one correspondence between the irreducible components H of the divisorial part $D_t^{(d)}$ and the surfaces H_ℓ defined by reflections $\sigma_\ell \in \Gamma_t V_d$.*

Proof. By $\mathscr{H}$ we denote an irreducible surface in $\mathbb{H}_2$ whose image is H. Let us suppose that $\mathscr{H} = \mathrm{Fix}_{\mathbb{H}_2}(G)$ with $G \in \Gamma_t V_d$. In accordance with Proposition 1·2 and Lemma 3·6 $\Psi(G) = \sigma_\ell$ is a reflection. Moreover $\psi_t(\mathscr{H}) = \mathscr{H}_\ell \subset \mathbb{PH}^+_t$ and σ_ℓ induces multiplication by ξ_d on the discriminant group A_t.

The reflection

$$\sigma_\ell(x) = x - \frac{2(x,\ell)}{(\ell,\ell)}\, \ell$$

depends only on the line $<\ell>$ defined by $\ell \in L_t$. It follows from the definition that σ_ℓ keeps the lattice L_t invariant if and only if $\ell^2 \mid 2D$ where $D = \mathrm{div}\,(\ell)$ (see the definition before Lemma 3·2). The surface H depends only on the class $\{\gamma G \gamma^{-1} \mid \gamma \in \Gamma_t\}$ and

$$\Psi(\gamma G \gamma^{-1}) = \beta \sigma_\ell \beta^{-1} = \sigma_{\beta\ell} \qquad \left(\gamma \in \Gamma_t,\ \beta = \Psi(\gamma) \in \widehat{\mathrm{SO}}^+(L_t)\right). \quad \square$$

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Therefore in order to find all Humbert surfaces in the divisorial part we have to classify the $\widehat{\mathrm{SO}}(L_t)$-orbits of vectors $\ell \in L_t$ with the additional condition $\ell^2 | 2\mathrm{div}\,(\ell)$.

LEMMA 3·8. *Let t be an arbitrary positive integer and $d||t$. There is a one to one correspondence between the $\widehat{\mathrm{SO}}(L_t)$-conjugacy classes of reflections σ_ℓ in the coset $(-\Gamma_t V_d)$ and the orbits of the primitive vectors in L_t, which satisfy the following conditions*:

$$\ell^2 = 2d \quad \text{and} \quad \begin{cases} \mathrm{div}\,(\ell) = 2t_d & \text{if } \left(\dfrac{d}{4t_d}\right) = 1 \\ \mathrm{div}\,(\ell) = t_d & \text{if } \left(\dfrac{d}{t_d}\right) = 1. \end{cases}$$

Proof. One can suppose that $\ell = {}^t(0, a, b, c, 0) \in L_t$ and $(a, b, c) = 1$. For such ℓ the matrix of σ_ℓ has the following form

$$\sigma_\ell = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 + 2ca/\ell^2 & -4tba/\ell^2 & 2a^2/\ell^2 & 0 \\ 0 & 2cb/\ell^2 & 1 - 4tb^2/\ell^2 & 2ab/\ell^2 & 0 \\ 0 & 2c^2/\ell^2 & -4tbc/\ell^2 & 1 + 2ac/\ell^2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\ell^2 = 2tb^2 - 2ac$. $-\sigma_\ell \in \mathrm{SO}^+_{\mathbb{R}}$ if and only if $\ell^2 > 0$. On the discriminant group $-\sigma_\ell$ defines multiplication by

$$\xi(\ell) = \frac{4t}{\ell^2} b^2 - 1.$$

By definition $D = \mathrm{div}\,(\ell) = (a,\ 2tb,\ c)$, therefore $D \,|\, 2t$ and $D \,|\, \ell^2 \,|\, 2D$. We put

$$\ell = {}^t(0, Da_1,\ b,\ Dc_1, 0) \quad \text{with } \left(a_1,\ \frac{2tb}{D},\ c_1\right) = 1.$$

We have to consider four cases:

$$\ell^2 = D \quad \text{or} \quad \ell^2 = 2D \quad \text{and} \quad D \,|\, t \quad \text{or} \quad D \nmid t.$$

(1) Let us suppose that $D \,|\, t$ ($t_D = t/D$). Then we have

$$\ell^2 = 2D \Longleftrightarrow 1 = t_D b^2 - Da_1c_1 \Longleftrightarrow (D, t_D) = 1 \,\&\, \left(\frac{t_D}{D}\right) = 1,$$

$$\xi(-\sigma_\ell) = 2t_D b^2 - 1 = \begin{cases} -1 & \mod 2t_D \\ 1 & \mod 2D \end{cases} \Longrightarrow -\sigma_\ell \in V_{t_D}\Gamma_t$$

(see (1·5)). The case $\ell^2 = D$ leads trivially to a contradiction.

(2) Let us suppose that $D \,|\, 2t$, but $t/D \notin \mathbb{Z}$. In this case $D = 2D_1$, $D_1 | t$ and t_{D_1} is odd. For such D we have

$$\ell^2 = D \Longleftrightarrow 1 = t_{D_1} b^2 - 4D_1a_1c_1 \Longleftrightarrow (D_1, t_{D_1}) = 1 \,\&\, (t_{D_1} \text{ odd}) \,\&\, \left(\frac{t_{D_1}}{4D_1}\right) = 1,$$

$$\xi(-\sigma_\ell) = 2t_{D_1} b^2 - 1 = \begin{cases} -1 & \mod 2t_{D_1} \\ 1 & \mod 2D_1 \end{cases} \Longrightarrow -\sigma_\ell \in V_{t_{D_1}}\Gamma_t.$$

The case $\ell^2 = 2D$ leads to a contradiction to the primitivity of ℓ. □

For square free t the system of the surfaces $\{H_{\ell^*}\}$, where $\ell^* = \ell/(\mathrm{div}\,(\ell))$ and

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ℓ satisfies the condition of Lemma 3·8, contains all irreducible components of the Humbert surfaces from Theorem 3·4. This finishes the proof of Theorem 3·4.

The next corollary follows immediately from the proof of Lemma 3·8.

COROLLARY 3·9. *Let t be square free and $d||t$. If $(d/t_d) = 1$ and $(d/4t_d) \neq 1$ then there is, up to conjugation with respect to Γ_t, exactly one, and if $(d/4t_d) = 1$ then there are exactly two involutions in $\Gamma_t V_d$. They are $\Psi^{-1}(-\sigma_{\ell_1})$ (in the both cases), and $\Psi^{-1}(-\sigma_{\ell_2})$ (in the second case), where*

$$\ell_1 = {}^t\left(0, a_1, \frac{b_1}{t_d}, c_1, 0\right) \in \widehat{L}_t, \quad (a_1, b_1, c_1) = 1, \quad db_1^2 - t_d a_1 c_1 = 1$$

$$\ell_2 = {}^t\left(0, a_2, \frac{b_2}{2t_d}, c_2, 0\right) \in \widehat{L}_t, \quad (a_2, b_2, c_2) = 1, \quad db_2^2 - 4t_d a_2 c_2 = 1.$$

Remarks. 1. It is possible to apply Lemma 3·8, which has been proved for any integer t, to classify the divisorial part of the ramification locus of the covering $\mathscr{A}_t \to \mathscr{A}_t^*$ for any integer t.

2. Using the same method one can construct divisors on a homogeneous domain of type IV of any dimension n.

The ramification locus can also have components of smaller dimension. The proof of Lemma 3·6 shows us that the orthogonal group can contain a rotation in the positive definite subplane of the lattice L_t.

Example (Brasch). The following example is due to Brasch. It shows that in general the ramification locus of the map $\mathscr{A}_t \to \mathscr{A}_t^*$ contains other components apart from the divisorial part described above. Let $t \equiv 1 \mod 4$. For an integer $f > 0$ put

$$c = -f^2 t - 1, \quad g = f^2.$$

Then the matrix

$$N = \begin{pmatrix} -f\sqrt{t} & 1/\sqrt{t} & 0 & f\sqrt{t} \\ c\sqrt{t} & 0 & f\sqrt{t} & f^2 t\sqrt{t} \\ c\sqrt{t} & 0 & f\sqrt{t} & -c\sqrt{t} \\ 0 & 1/\sqrt{t} & -1/\sqrt{t} & 0 \end{pmatrix}$$

is an element of Γ_t^*. One immediately checks that $N^2 = -E_4$. The fixed point set Fix N is a curve (cf. [**B**, Hilfssatz 2·5·3]). Moreover the curve Fix N is not contained in the fixed point set Fix I of an involution I in Γ_t^*. This would namely imply that $IN = -NI$. Now using the explicit form for N which follows from [**B**, Hilfssatz 2·8], a lengthy but straightforward calculation shows $c = -1$, a contradiction.

As a next step we want to interpret the surfaces H_t, resp. H_{4t} as moduli spaces of abelian surfaces with real multiplication. It is well known that there is a close connection between Hilbert modular surfaces and Humbert surfaces [**F**], [**vdG**, chapter IX]. Here we want to determine precisely which Hilbert modular surfaces correspond to H_t, resp. H_{4t}. Consider the ring $\mathfrak{o}$ of integers in the number field $\mathbb{Q}(\sqrt{t})$ and recall that

$$\mathfrak{o} = \mathbb{Z} + \mathbb{Z}\omega, \quad \omega = \tfrac{1}{2}(1 + \sqrt{t}) \quad \text{if } t \equiv 1 \bmod 4$$

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resp.

$$\mathfrak{o} = \mathbb{Z} + \mathbb{Z}\omega, \quad \omega = \sqrt{t} \quad \text{if } t \not\equiv 1 \bmod 4.$$

The *Hilbert modular group* $\mathrm{SL}_2(\mathfrak{o})$ acts on $\mathbb{H}_1 \times \mathbb{H}_1$ by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} (z_1, z_2) = \left(\frac{\alpha z_1 + \beta}{\gamma z_1 + \delta}, \frac{\alpha' z_2 + \beta'}{\gamma' z_2 + \delta'} \right)$$

where $'$ denotes the Galois automorphism $\sqrt{t} \mapsto -\sqrt{t}$. The quotient space $Y = \mathrm{SL}_2(\mathfrak{o})\backslash \mathbb{H}_1 \times \mathbb{H}_1$ is the standard *Hilbert modular surface* associated to $\mathbb{Q}(\sqrt{t})$. Let σ be the involution which interchanges the two factors of $\mathbb{H}_1 \times \mathbb{H}_1$, i.e.

$$\sigma(z_1, z_2) = (z_2, z_1).$$

Then the *symmetric* Hilbert modular group is

$$\mathrm{SL}_2^{\sigma}(\mathfrak{o}) = \mathrm{SL}_2(\mathfrak{o}) \bigcup \sigma \mathrm{SL}_2(\mathfrak{o})$$

and $Y^{\sigma} = \mathrm{SL}_2^{\sigma}(\mathfrak{o})\backslash \mathbb{H}_1 \times \mathbb{H}_1$ is the corresponding symmetric Hilbert modular surface. We shall first consider the Humbert surface H_t. In particular we assume that $t \equiv 1 \mod 4$. To every point $(z_1, z_2) \in \mathbb{H}_1 \times \mathbb{H}_1$ one can associate the lattice

$$\Lambda_{(z_1,z_2)} = \mathfrak{o} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \mathfrak{o} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \mathbb{Z} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \mathbb{Z} \begin{pmatrix} \eta z_1 \\ \eta' z_2 \end{pmatrix} + \mathbb{Z} \begin{pmatrix} -\eta'/\sqrt{t} \\ \eta/\sqrt{t} \end{pmatrix} + \mathbb{Z} \begin{pmatrix} \sqrt{t} \\ -\sqrt{t} \end{pmatrix}$$

where $\eta = \sqrt{(t)}\omega$. The form

$$E((x_1, x_2), (y_1, y_2)) = \mathrm{Im} \left(\frac{x_1 \bar{y}_1}{\mathrm{Im} z_1} + \frac{x_2 \bar{y}_2}{\mathrm{Im} z_2} \right)$$

defines a Riemann form for Λ_t. With respect to the basis given above this is just the alternating form W_t. The torus

$$A_{(z_1,z_2)} = \mathbb{C}^2 / \Lambda_{(z_1,z_2)}$$

is hence a $(1, t)$-polarized abelian surface with real multiplication in $\mathfrak{o}$. The Hilbert modular surface Y is the moduli space of these objects. The 2:1 cover $Y \to Y^{\sigma}$ identifies abelian surfaces whose real multiplication differs by the Galois conjugation. We have a 'forgetful' map

$$\Phi \colon Y^{\sigma} \to \mathscr{A}_t.$$

THEOREM 3·10. *Assume $t \equiv 1 \bmod 4$. The Humbert surface H_t is the image of the symmetric Hilbert modular surface Y^{σ} under the natural morphism $\Phi \colon Y^{\sigma} \to \mathscr{A}_t$ which is of degree 1 onto its image.*

Before giving the proof we turn to the Humbert surfaces H_{4t}. Consider the ring

$$\mathfrak{o}_2 = \mathbb{Z} + \mathbb{Z}\sqrt{t}.$$

Note that this is an order in $\mathfrak{o}$ if $t \equiv 1 \bmod 4$, whereas $\mathfrak{o} = \mathfrak{o}_2$ if $t \not\equiv 1 \bmod 4$. Let

$$\tilde{\mathfrak{o}}_2 = \tfrac{1}{2}\mathbb{Z} + \tfrac{1}{2}\sqrt{t}\mathbb{Z}, \quad \tilde{\mathfrak{o}}_2^{-1} = 2\mathbb{Z} + 2\sqrt{t}\mathbb{Z}.$$

The group

$$\mathrm{SL}_2(\mathfrak{o}_2, \tilde{\mathfrak{o}}_2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, d \in \mathfrak{o}_2,\ b \in \tilde{\mathfrak{o}}_2,\ c \in \tilde{\mathfrak{o}}_2^{-1},\ ad - bc = 1 \right\}$$

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acts on $\mathbb{H}_1 \times \mathbb{H}_1$ as well as its symmetric counterpart

$$\mathrm{SL}_2^{\sigma}(\mathfrak{o}_2, \tilde{\mathfrak{o}}_2) = \mathrm{SL}_2(\mathfrak{o}_2, \tilde{\mathfrak{o}}_2) \cup \sigma \mathrm{SL}_2(\mathfrak{o}_2, \tilde{\mathfrak{o}}_2).$$

Let

$$\widetilde{Y} = SL_2(\mathfrak{o}_2, \tilde{\mathfrak{o}}_2)\backslash \mathbb{H}_1 \times \mathbb{H}_1, \quad \widetilde{Y}^{\sigma} = \mathrm{SL}_2^{\sigma}(\mathfrak{o}_2, \tilde{\mathfrak{o}}_2)\backslash \mathbb{H}_1 \times \mathbb{H}_1$$

be the corresponding Hilbert modular surfaces. Again the Riemann form E induces a $(1, t)$-polarization on the tori

$$A_{(z_1,z_2)} = \mathbb{C}^2/\Lambda_{(z_1,z_2)}$$

where for $t \equiv 1 \bmod 4$

$$\Lambda_{(z_1,z_2)} = \mathbb{Z}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \mathbb{Z}\begin{pmatrix} 2\eta z_1 \\ 2\eta' z_2 \end{pmatrix} + \mathbb{Z}\begin{pmatrix} -\eta'/\sqrt{t} \\ \eta/\sqrt{t} \end{pmatrix} + \mathbb{Z}\begin{pmatrix} \sqrt{t}/2 \\ -\sqrt{t}/2 \end{pmatrix}$$

resp. $t \not\equiv 1 \bmod 4$

$$\Lambda_{(z_1,z_2)} = \mathbb{Z}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \mathbb{Z}\begin{pmatrix} \omega z_1 \\ \omega' z_2 \end{pmatrix} + \mathbb{Z}\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} + \mathbb{Z}\begin{pmatrix} \omega/2 \\ \omega'/2 \end{pmatrix}$$

Hence $\widetilde{Y}$, resp. $\widetilde{Y}^{\sigma}$ are moduli spaces of $(1, t)$-polarized abelian surfaces with real multiplication in $\mathfrak{o}_2$ and as before we have a canonical map

$$\widetilde{\Phi}\colon \widetilde{Y}^{\sigma} \to \mathscr{A}_t.$$

THEOREM 3·11. *The Humbert surface H_{4t} is the image of the symmetric Hilbert modular surface $\widetilde{Y}^{\sigma}$ under the natural map $\widetilde{\Phi}\colon \widetilde{Y}^{\sigma} \to \mathscr{A}_t$ which is of degree 1 onto its image.*

Proof of Theorems 3·10 *and* 3·11. We shall treat the case of H_{4t} and $t \equiv 1 \bmod 4$ in detail and then comment on the other cases. The proof is similar to the proofs in [**HL**, section 0], cf. also [**F**, Abschnitt 3]. Let

$$R = \begin{pmatrix} 1 & 2\eta \\ 1 & 2\eta' \end{pmatrix}$$

and consider the map

$$\begin{aligned} \widehat{\Phi}\colon \mathbb{H}_1 \times \mathbb{H}_1 &\to \mathbb{H}_2 \\ (z_1, z_2) &\mapsto {}^tR \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} R. \end{aligned}$$

Then

$$\mathrm{Im}\widehat{\Phi} = \{-(t^2 - t)\tau_1 + 2t\tau_2 - \tau_3 = 0\} = \mathscr{H}'_{4t}$$

and modulo

$$X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t & 1 & 0 & 0 \\ 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma_t$$

this is equivalent to $\mathscr{H}_{4t} = \{t\tau_1 - \tau_3 = 0\}$. Let $A_{\widehat{\Phi}(z_1,z_2)}$ be the abelian surface associ-

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ated to the period matrix

$$\begin{pmatrix} {}^tR\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} R \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \end{pmatrix}.$$

Then $A_{(z_1,z_2)}$ and $A_{\widehat{\Phi}(z_1,z_2)}$ are isomorphic as polarized abelian surfaces since

$${}^tR\begin{pmatrix} z_1 & 2\eta z_1 & -\eta'/\sqrt{t} & \sqrt{t}/2 \\ z_2 & 2\eta' z_2 & \eta/\sqrt{t} & -\sqrt{t}/2 \end{pmatrix} = \begin{pmatrix} {}^tR\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} R \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \end{pmatrix}.$$

Hence $\widehat{\Phi}$ is a lift of the map $\widetilde{\Phi}$. Next we consider the homomorphism

$$\Psi\colon \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}) \longrightarrow \mathrm{Sp}_4(\mathbb{R})$$

$$\Psi\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) = \begin{pmatrix} {}^tR & 0 \\ 0 & R^{-1} \end{pmatrix} \begin{pmatrix} d(a_1,a_2) & d(b_1,b_2) \\ d(c_1,c_2) & d(d_1,d_2) \end{pmatrix} \begin{pmatrix} {}^tR & 0 \\ 0 & R \end{pmatrix}$$

where $d(a_1,a_2) = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$, etc. Via the embedding

$$\begin{aligned} \mathrm{SL}_2(\mathbb{Q}(\sqrt{t})) &\to \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}) \\ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} &\mapsto \left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}\right) \end{aligned}$$

this also defines a homomorphism

$$\widehat{\Psi}\colon \mathrm{SL}_2(\mathbb{Q}(\sqrt{t})) \to \mathrm{Sp}_4(\mathbb{R}).$$

For $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ we find that

$$S = \begin{pmatrix} {}^tRJ\,{}^tR^{-1} & 0 \\ 0 & R^{-1}JR \end{pmatrix} \in \Gamma_t.$$

Clearly S is an involution. Setting $\widehat{\Psi}(\sigma) = S$ we can extend $\widehat{\Psi}$ to a homomorphism

$$\widehat{\Psi}_\sigma\colon \mathrm{SL}_2^\sigma(\mathbb{Q}(\sqrt{t})) \to \mathrm{Sp}_4(\mathbb{R})$$

and one checks easily that $\widehat{\Phi}$ is $\widehat{\Psi}_\sigma$-equivariant. Let G, resp. $G_\mathbb{R}$ be the stabilizer of $\mathscr{H}'_{4t}$ in Γ_t, resp. $\mathrm{Sp}_4(\mathbb{R})$. As in [**HL**, lemma 0·8], cf. also [**F**, Korollar 3·2·8] one shows that $G_\mathbb{R}$ is the group generated by the image of Ψ and S. The result follows if we can prove that $G = \widehat{\Psi}_\sigma(\mathrm{SL}_2^\sigma(\mathfrak{o}_2, \tilde{\mathfrak{o}}_2))$. For this it is now enough to prove the following.

LEMMA 3·12. *Let* $\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$; $i = 1, 2$ *and assume that*

$$M = \begin{pmatrix} {}^tR & 0 \\ 0 & R^{-1} \end{pmatrix} \begin{pmatrix} d(a_1,a_2) & d(b_1,b_2) \\ d(c_1,c_2) & d(d_1,d_2) \end{pmatrix} \begin{pmatrix} {}^tR^{-1} & 0 \\ 0 & R \end{pmatrix} \in \Gamma_t.$$

Then $a_1, d_1 \in \mathfrak{o}_2$, $b_1 \in \tilde{\mathfrak{o}}_2$, $c_1 \in \tilde{\mathfrak{o}}_2^{-1}$ *and* $a_2 = a_1'$, $b_2 = b_1'$, $c_2 = c_1'$, $d_2 = d_2'$.

Proof of the lemma. We write $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Straightforward calculation gives

$$A = \frac{1}{2(\eta' - \eta)} \begin{pmatrix} 2\eta' a_1 - 2\eta a_2 & -a_1 + a_2 \\ 4\eta\eta'(a_1 - a_2) & -2\eta a_1 + 2\eta' a_2 \end{pmatrix}.$$

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From $A_{12} \in \mathbb{Z}$ we find $(a_1 - a_2)/(2\sqrt{t}) \in \mathbb{Z}$, i.e.

$$a_2 = a_1 + 2n\sqrt{t}, \quad n \in \mathbb{Z}.$$

$A_{11} \in \mathbb{Z}$ gives $a_1\omega' + a_2\omega \in \mathbb{Z}$. Hence

$$a_1 + 2n\sqrt{t}\omega' = a_1(\omega + \omega') + 2n\sqrt{t}\omega' = a_1\omega + a_2\omega' \in \mathbb{Z}. \tag{3·2}$$

I.e. $a_1 \in \mathbb{Z} + \mathbb{Z}\sqrt{t} = \mathfrak{o}_2$. We can write $a_1 = \alpha + \beta\sqrt{t}$ with $\alpha, \beta \in \mathbb{Z}$. By 3·2 we have $a_1 + n(\sqrt{t} + t) \in \mathbb{Z}$ and hence $\beta = -n$. But then $a_2 = a_1 - 2\beta\sqrt{t} = \alpha - \beta\sqrt{t} = a_1'$. Note also that for $a_1 \in \mathfrak{o}_2$ and $a_2 = a_1'$ one has $A \in \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ t\mathbb{Z} & \mathbb{Z} \end{pmatrix}$. Similarly we obtain

$$B = \begin{pmatrix} b_1 + b_2 & 2\eta b_1 + 2\eta' b_2 \\ 2\eta b_1 + 2\eta' b_2 & 4\eta^2 b_1 + 4\eta'^2 b_2 \end{pmatrix}.$$

Since $B_{11} \in \mathbb{Z}$ we find that $b_2 = n - b_1$ for some $n \in \mathbb{Z}$. Using this and $B_{12} \in t\mathbb{Z}$ one concludes that

$$(b_1\omega - b_2\omega') \in \frac{\sqrt{t}}{2}\mathbb{Z}.$$

Hence

$$b_1 - n\omega' = b_1(\omega + \omega') - n\omega' = b_1\omega - b_2\omega' \in \frac{\sqrt{t}}{2}\mathbb{Z}. \tag{3·3}$$

This shows $b_1, b_2 \in \mathfrak{d}_2$. Writing $b_1 = (\alpha + \beta\sqrt{t})/2$ with $\alpha, \beta \in \mathbb{Z}$ one has from (3.3)

$$\frac{\alpha}{2} + \frac{\beta}{2}\sqrt{t} - n\omega' \in \frac{\sqrt{t}}{2}\mathbb{Z}$$

i.e. $\alpha = n$. Hence $b_2 = n - b_1 = (\alpha - \beta\sqrt{t})/2 = b_1'$. Conversely if $b_1 \in \mathfrak{d}_2$ and $b_2 = b_1'$, then $B \in \begin{pmatrix} \mathbb{Z} & t\mathbb{Z} \\ t\mathbb{Z} & t\mathbb{Z} \end{pmatrix}$. For C one computes

$$C = \frac{1}{4t}\begin{pmatrix} 4\eta'^2 c_1 + 4\eta^2 c_2 & -2\eta' c_1 - 2\eta c_2 \\ -2\eta' c_1 - 2\eta c_2 & c_1 + c_2 \end{pmatrix}.$$

Comparing this with the situation for B one finds that $c_1 \in \mathfrak{d}_2^{-1}, c_2 = c_1'$. Finally

$$D = \frac{1}{2(\eta' - \eta)}\begin{pmatrix} 2\eta' d_1 - 2\eta d_2 & 4\eta\eta'(d_1 - d_2) \\ -d_1 + d_2 & -2\eta d_1 + 2\eta' d_2 \end{pmatrix}.$$

This case is analogous to A and one obtains $d_1 \in \mathfrak{o}_2,\ d_2 = d_1'$. This proves Lemma 3·12.

The case of the Humbert surface H_t, $t \equiv 1 \bmod 4$ can be treated in the same manner if we choose $R = \begin{pmatrix} 1 & \eta \\ 1 & \eta' \end{pmatrix}$. This is analogous to [**HL**, section 0] where the case $t = 5$ was done. Finally we have to deal with H_{4t} in case $t \not\equiv 1 \bmod 4$. Here we can choose $R = \begin{pmatrix} 1 & \omega \\ 1 & \omega' \end{pmatrix}, \omega = \sqrt{t}$. In this case

$$\operatorname{Im}\widehat{\Phi} = \{t\tau_1 - \tau_3 = 0\} = \mathscr{H}_{4t}$$

and the above arguments go through essentially unchanged.

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Added in proof. 1. The assumption that t is square free in Proposition 1·2 and Corollary 1·3 is unnecessary. Using arguments similar to those in [**Be**] one can show directly that the paramodular group Γ_t is isomorphic to the corresponding orthogonal group for arbitrary t.

2. Theorem 1·5 gives a negative answer to a question raised by Shioda [**S**, question 5]. If A is a general $(1,t)$-polarized abelian surface, then A and its dual A' are not isomorphic (as projective surfaces), but their Kummer surfaces are isomorphic.

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