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Coupling two unimodal maps with simple kneading sequences

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Abstract. The dynamics of a system of two diffusively coupled identical unimodal maps is investigated. The unimodal maps have simple kneading sequences which imply that their orbits are asymptotically periodic with period a power of two. Analogous results are obtained for the coupled system for all coupling parameters. We prove the eventual periodicity of the position of orbits with respect to the diagonal of the square phase space and the asymptotic periodicity for orbits whose coordinates have the same sign. Finally, we give a global condition for the existence of symmetric orbits that are examples of orbits with this asymptotic property.

1. *Introduction and definitions*

Mathematical studies of coupled map lattices (CMLs) have mainly considered either weakly coupled or strongly coupled systems. In the weakly coupled case, small perturbations of uncoupled (infinite) lattices of hyperbolic maps are analyzed using techniques from ergodic theory and statistical mechanics [**2, 6, 7, 11**]. In strongly coupled finite lattices, various regimes of synchronization take place depending on the normal Lyapunov spectrum [**1, 12**].

In this paper, we consider a CML consisting of two sites with unimodal local maps. We describe its global asymptotic dynamics, for *all* coupling parameters, independent of the explicit algebraic expression of the local map.

A (C^1-) *unimodal map* f is a C^1 map on $[-1, 1]$ satisfying $f'(x) \neq 0$ except at $x = 0$; $f(0) = 1$; f is increasing on $[-1, 0]$ and decreasing on $[0, 1]$; and $f([f(1), 1]) \subset [f(1), 1]$. An *S-unimodal map* is a C^3-unimodal map with negative Schwartzian derivative [**3**].

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Given a unimodal map f, the CML with two sites is the dynamical system $(\mathcal{M}, F_\epsilon)$ where the phase space is the square $\mathcal{M} = [f(1), 1]^2$ and

$$F_\epsilon(x, y) = \big((1-\epsilon)f(x) + \epsilon f(y), (1-\epsilon)f(y) + \epsilon f(x)\big), \quad (x, y) \in \mathcal{M},$$

and $\epsilon \in [0, \frac{1}{2}]$ is the coupling parameter. The map F_ϵ can be viewed as a composition of two maps $F_\epsilon = L_\epsilon \circ F_0$, where the coupling operator

$$L_\epsilon(x, y) = ((1-\epsilon)x + \epsilon y, (1-\epsilon)y + \epsilon x)$$

is a linear convex combination of coordinates. It follows from the properties of f that F_ϵ maps $\mathcal{M}$ into itself. The orbit with an initial point (x, y) is denoted by the sequence $\{(x^t, y^t)\}_{t\in\mathbb{N}}$ with $(x^0, y^0) = (x, y)$ and $(x^{t+1}, y^{t+1}) = F_\epsilon(x^t, y^t)$ for all $t \in \mathbb{N}$. For convenience, our index set $\mathbb{N}$ starts from zero.

The map F_ϵ commutes with the symmetry map $\theta : (x, y) \mapsto (y, x)$. An orbit with $x^t \neq y^t$, $t \in \mathbb{N}$, is said to be *symmetric* if there exists $t_0 \in \mathbb{N}$ such that $(x^{t_0}, y^{t_0}) = \theta(x^0, y^0)$. We take t_0 to be the smallest of such integers. Every symmetric orbit is periodic with a period $2t_0$. This symmetry implies that the *diagonal* of $\mathcal{M}$, $\{(x, x) : x \in [f(1), 1]\}$, is invariant under F_ϵ. Any orbit of F_ϵ lying in the diagonal is said to be *homogeneous*. Any orbit satisfying the condition $\lim_{t\to\infty}(x^t - y^t) = 0$ is said to be *asymptotically homogeneous*.

With respect to the coupling parameter ϵ, the map F_ϵ has another symmetry $F_{1-\epsilon}(x, y) = \theta(F_\epsilon(x, y))$, as observed in [**5**]. Therefore, when $\epsilon \in [\frac{1}{2}, 1]$, the orbit of F_ϵ starting from (x, y) is the sequence $\{\theta^t(x^t, y^t)\}$ where $\{(x^t, y^t)\}$ is the orbit of $F_{1-\epsilon}$ with $1 - \epsilon \in [0, \frac{1}{2}]$ starting from the same initial point. Hence, all our conclusions extend to the dynamics of F_ϵ when $\epsilon \in [\frac{1}{2}, 1]$.

Both analytic and numerical studies of systems of two coupled quadratic maps or systems of two coupled logistic maps can be found in the literature. These studies determined the parameter regions for the existence of some periodic orbits, investigated their bifurcations [**4, 9, 10**], and described the structure of attractors [**5**]. Even though the local map f has relatively simple dynamics, these studies have demonstrated that such systems possess rich and interesting dynamics such as *multistability*, i.e. the co-existence of a large number (as large as one wants) of distinct stable periodic orbits. Moreover, the symmetric periodic orbits were shown to undergo a Hopf bifurcation, creating an invariant closed curve [**4, 10**].

Independent of the explicit algebraic expression of f, two simple properties can be easily observed for F_ϵ. Firstly, every pair of hyperbolic periodic orbits of f is a hyperbolic periodic orbit of F_0. Thus, this orbit can be continued to obtain a periodic orbit for F_ϵ when ϵ is sufficiently small. Secondly, letting $K = \max_{x\in[f(1),1]} |f'(x)|$, the condition $K(1 - 2\epsilon) < 1$ implies that every orbit is asymptotically homogeneous, a property also known as *synchronization*. This is an immediate consequence of the relation

$$x^{t+1} - y^{t+1} = (1 - 2\epsilon)(f(x^t) - f(y^t)),$$

which follows directly from the definition of F_ϵ.

To learn more about the asymptotic behavior of orbits of F_ϵ, especially for intermediate values of ϵ, we consider unimodal maps f with kneading sequences which imply that every

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orbit of f is asymptotically periodic with period a power of two (see Proposition 2.1). Using the properties of kneading sequences, we obtain analogous results for the CML, namely, eventual periodicity of the position of orbits with respect to the diagonal (Theorem 2.2) and asymptotic periodicity for orbits with coordinates having the same sign (Theorem 2.3). Moreover, we give a global condition for the existence of symmetric orbits. These orbits are examples with the properties described in Theorems 2.2 and 2.3 and are believed to be the first non-homogeneous orbits to emerge when ϵ is relaxed from the synchronization region. To prove these results, we first review the dynamical properties of unimodal maps (with prescribed kneading sequences) in §3.1. Then a precise description of the iterates of any point under the CML is given. This description is the two-dimensional analogy of interval dynamics for unimodal maps. An immediate consequence is Theorem 2.2. Theorem 2.3 is proved in §2.3 using the additional consequences of the piecewise monotonicity of f and the positivity of the coupling operator. Finally, the proof of Proposition 2.1 and the existence of symmetric orbits are given in §3.4.

2. *Main results*

We start by recalling the definitions and results on unimodal maps with simple kneading sequences. The *itinerary* $I(x) = \{I_i(x)\}_{i\in\mathbb{N}}$ of a point x is either an infinite sequence of L's and R's, or a finite sequence of L's and R's, followed by C. The ith element of $I(x)$ is defined by

$$I_i(x) = \begin{cases} L & \text{if } f^i(x) < 0, \\ C & \text{if } f^i(x) = 0, \\ R & \text{if } f^i(x) > 0. \end{cases}$$

The *kneading sequence* of a unimodal map is the itinerary of the point 1, which largely determines the dynamics of the map. The simplest kneading sequences are $I^n := R^{*n} * R^\infty$ (A^∞ denotes the infinite concatenation $AAA\cdots$) which is 2^n-periodic and $J^n := R^{*n} * RC$, which is finite (see for instance [**3**] for their definitions using the $*$-product). They can also be directly defined by induction (see the inductive definition and simple properties in §3.1). When the kneading sequence is I^n or J^n, the asymptotic dynamics of f is simple, as claimed in Theorem C in [**8**]. We summarize the asymptotic behavior and provide additional properties in the following proposition.

In the rest of this paper, we assume that the kneading sequence of f is either I^n or J^n for some $n \in \mathbb{N}$.

PROPOSITION 2.1.

(1) *For every $x \in [f(1), 1]$, there exists $k \in \{0, \ldots, n+1\}$ such that*

$$\lim_{t\to\infty} (f^t(x) - f^{t+2^k}(x)) = 0.$$

(2) *For every $k \in \{0, \ldots, n\}$, there exists a unique 2^k-periodic point, x_k, between zero and $f^{2^k}(0)$.*

(3) *If f is S-unimodal and $n \geqslant 1$, then for each $k \in \{0, \ldots, n-1\}$ (and, possibly, for $k = n$ when $n \in \mathbb{N}$), this 2^k-periodic orbit is unstable with $(f^{2^k})'(x_k) < -1$.*

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For instance, for every μ smaller than the accumulation point of the period-doubling cascade, the kneading sequence of the quadratic map $f_\mu(x) = 1 - \mu x^2$ is either I^n or J^n for some $n \in \mathbb{N}$ (which depends on μ).

We now present analogous results for the CML. As we mentioned in the introduction, analytic and numerical studies of two coupled quadratic maps have revealed the presence of Hopf bifurcations and, thus, the possible existence of quasi-periodic orbits. Therefore, we cannot expect every orbit of F_ϵ to be asymptotically periodic. Nevertheless, for every orbit, we can prove the eventual periodicity of the position of the point (x^t, y^t) with respect to the diagonal, i.e. of the sign of $x^t - y^t$.

THEOREM 2.2. *For every $(x, y) \in \mathcal{M}$ such that $x^t \neq y^t$ for all $t \in \mathbb{N}$, there exist $t_0 \in \mathbb{N}$ and $k \in \{0, \ldots, n\}$ so that for all $t \geqslant t_0$, we have*

$$(x^{t+2^k} - y^{t+2^k})(x^t - y^t) < 0. \tag{2.1}$$

For such an orbit, the integer k is clearly unique. However, for the coupled map all cases $k \in \{0, \ldots, n\}$ can occur. As a consequence, after a transient, every orbit satisfies

$$(x^{t+2^{n+1}} - y^{t+2^{n+1}})(x^t - y^t) \geqslant 0,$$

which may be useful for a stroboscopic plotting of the dynamics.

Not only is the position with respect to the diagonal eventually periodic, but the orbit itself can also be asymptotically periodic.

THEOREM 2.3. *Let $\{(x^t, y^t)\}_{t\in\mathbb{N}}$ be an orbit such that $x^t y^t \geqslant 0$ for all $t \in \mathbb{N}$. There exists $T \in \{0, \ldots, n+1\}$ such that*

$$\lim_{t\to\infty} (x^t - x^{t+2^T}) = \lim_{t\to\infty} (y^t - y^{t+2^T}) = 0.$$

Moreover, if the orbit is not asymptotically homogeneous, then $T = k + 1$, where k is the integer in Theorem 2.2 for which the relation (2.1) holds.

The condition $x^t y^t \geqslant 0$ for all t is not just a technical condition for our proof but also a natural one. Indeed, the Hopf bifurcations mentioned before, from which quasi-periodic orbits may be created, occur for orbits with coordinates of different signs, i.e. for which $x^t y^t < 0$ for some t [**4**].

In the introduction, we mentioned that every orbit is asymptotically homogeneous when the coupling parameter ϵ is sufficiently close to $\frac{1}{2}$. (As a consequence, the Hopf bifurcations must occur when ϵ is far from $\frac{1}{2}$.) If $n \geqslant 1$, the CML has non-homogeneous periodic orbits when ϵ is sufficiently small. Furthermore, a bifurcation analysis of the homogeneous 2^k-periodic orbit passing through the point (x_k, x_k) (see Proposition 2.1) shows that when its normal multiplier (i.e. the eigenvalue of the derivative $DF_\epsilon^{2^k}(x_k, x_k)$ in the direction perpendicular to the diagonal), $(1 - 2\epsilon)^{2^k}(f^{2^k})'(x_k)$, crosses -1, we have a period doubling bifurcation creating a symmetric 2^{k+1}-periodic orbit [**4**].

On the other hand, if $n \geqslant 1$, symmetric orbits also exist when $\epsilon = 0$. By the symmetry of F_ϵ, when these symmetric orbits are hyperbolic, they persist for small ϵ. The next statement tells us that such symmetric orbits exist for every coupling parameter between the period doubling bifurcation point and zero.

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THEOREM 2.4. *Let x_k where $k \in \{0, \dots, n\}$, be the 2^k-periodic point of f between zero and $f^{2^k}(0)$. If*

$$(1 - 2\epsilon)^{2^k} (f^{2^k})'(x_k) < -1, \tag{2.2}$$

then F_ϵ has a symmetric 2^{k+1}-periodic orbit.

In particular, if f is S-unimodal, then by Proposition 2.1(3), for each $k \in \{0, \dots, n-1\}$ (and possibly $k = n$), condition (2.2) holds if ϵ is chosen suitably close to zero.

3. *Proofs*

3.1. *Dynamical properties of unimodal maps.* In the analysis of the coupled system, we need certain properties of the iterates of some special triangles of the phase space. Given $a, b \in [-1, 1]$, $a < b$, let

$$\overline{T}_{a,b} = \{(x, y) \in [-1, 1]^2 \,:\, a \leqslant x \leqslant b \text{ and } x \leqslant y \leqslant b\}$$

denote a triangle above the diagonal and let $\underline{T}_{a,b} = \theta(\overline{T}_{a,b})$. Let $S_{a,b} = \overline{T}_{a,b} \cup \underline{T}_{a,b} = I^2_{a,b}$, where $I_{a,b} = [a, b]$. For convenience, let $U_{\{a,b\}} = U_{\min\{a,b\},\max\{a,b\}}$, where U is any of the symbols $\overline{T}$, $\underline{T}$, S, or I.

Given a unimodal map f, assume that, for some $a \in [-1, 1]$ and some $t \in \mathbb{N}$, the iterates $f^i(a)$ and $f^i(0)$, where $i \in \{1, \dots, t\}$, lie on the same side of zero. Then $f^{t+1}(I_{\{0,a\}}) = I_{\{f^{t+1}(0), f^{t+1}(a)\}}$ and the relative positions of $f^{t+1}(a)$ and $f^{t+1}(0)$ depend on the number of times when $f^i(a)$ is positive. In the next proposition, we present an alternative formulation of this property which will be used several times in later proofs. Recall that a finite sequence of L's and R's is said to be *odd* if it has an odd number of R's and *even* otherwise.

PROPOSITION 3.1. *Let $a \in [-1, 1] \setminus \{0\}$ and $t \in \mathbb{N}$ be such that $\{I_i(a)\}_{1 \leqslant i \leqslant t} = \{I_{i-1}(1)\}_{1 \leqslant i \leqslant t}$ if $t \geqslant 1$. If neither of these two sequences contains the symbol C, then we have:*

(i) $F_0^{t+1}(\overline{T}_{\{0,a\}}) = \theta^s(\overline{T}_{\{f^{t+1}(0), f^{t+1}(a)\}})$*; and*

(ii) $(-1)^s a(f^{t+1}(a) - f^{t+1}(0)) > 0$,

where $s = 0$ (respectively $s = 1$) if the sequence $\{I_i(a)\}_{0 \leqslant i \leqslant t}$ is even (respectively odd).

Proof. We use induction on t. For $t = 0$, the results follow from the fact that f is a bijection from $I_{\{0,a\}}$ to $I_{\{f(a),1\}}$ which preserves (respectively reverses) orientation if $a < 0$ (respectively $a > 0$).

Now, given $t \in \mathbb{N}$, assume that (i) and (ii) hold and suppose that

$$\{I_i(a)\}_{1 \leqslant i \leqslant t+1} = \{I_{i-1}(1)\}_{1 \leqslant i \leqslant t+1},$$

where all symbols are different from C. We first consider the case when the sequence $\{I_i(a)\}_{0 \leqslant i \leqslant t+1}$ is even. Then, either $\{I_i(a)\}_{0 \leqslant i \leqslant t}$ is even and $I_{t+1}(a) = L$, or $\{I_i(a)\}_{0 \leqslant i \leqslant t}$ is odd and $I_{t+1}(a) = R$. In the first situation, both $f^{t+1}(0) = f^t(1)$ and $f^{t+1}(a)$ are negative. Since f is continuous and strictly increasing on $[-1, 0]$, we have $(f^{t+1}(a) - f^{t+1}(0))(f^{t+2}(a) - f^{t+2}(0)) > 0$. Using the induction hypothesis, it then follows that

$$F_0^{t+2}(\overline{T}_{\{0,a\}}) = F_0(\overline{T}_{\{f^{t+1}(0), f^{t+1}(a)\}}) = \overline{T}_{\{f^{t+2}(0), f^{t+2}(a)\}},$$

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and $a(f^{t+2}(a) - f^{t+2}(0)) > 0$ because $a(f^{t+1}(a) - f^{t+1}(0)) > 0$. Similarly, in the second situation when $I_{t+1}(a) = R$, since both $f^{t+1}(0)$ and $f^{t+1}(a)$ are positive, we have

$$F_0^{t+2}(\overline{T}_{\{0,a\}}) = F_0(\underline{T}_{\{f^{t+1}(0),\, f^{t+1}(a)\}}) = \overline{T}_{\{f^{t+2}(0),\, f^{t+2}(a)\}},$$

and again $a(f^{t+2}(a) - f^{t+2}(0)) > 0$ because now $a(f^{t+1}(a) - f^{t+1}(0)) < 0$. Thus, both statements (i) and (ii) hold for $t+1$ in the case when the sequence $\{I_i(a)\}_{0 \leqslant i \leqslant t+1}$ is even. For the case when the sequence $\{I_i(a)\}_{0 \leqslant i \leqslant t+1}$ is odd, the proof follows analogously. □

The next proposition collects the properties of iterates $f^{2^k}(0)$ when the kneading sequence of f is either I^n or J^n. These properties are needed in proving the first two main results. Some of these properties can be found in [**3**].

PROPOSITION 3.2. *The following statements hold for every $k \in \{0, \ldots, n\}$.*

(1) $f^{2^k}(0) > 0$ *(respectively* $f^{2^k}(0) < 0$*) if k is even (respectively odd).*

(2) *If $k \geqslant 1$, then* $\{I_i(f^{2^k}(0))\}_{1 \leqslant i \leqslant 2^k - 1} = \{I_{i-1}(1)\}_{1 \leqslant i \leqslant 2^k - 1}$ *and neither of these two sequences contains the symbol C.*

(3) *The sequence* $\{I_i(f^{2^k}(0))\}_{0 \leqslant i \leqslant 2^k - 1}$ *is odd.*

(4) $S_{\{f^{2^n}(0),\, f^{2^{n+1}}(0)\}} \subset S_{\{0,\, f^{2^n}(0)\}}$.

The definition of the $*$-product implies that the infinite sequences $I^n = \{I_i^n\}_{i \in \mathbb{N}} = R^{*n} * R^\infty$ satisfy the following inductive relations. We have $I^0 = R^\infty$ and for every $n \in \mathbb{N}$ and all $i \in \mathbb{N}$

$$I_{2i}^{n+1} = R \quad \text{and} \quad I_{2i+1}^{n+1} = \check{I}_i^n, \tag{3.1}$$

where $\check{R} = L$ and $\check{L} = R$. (For instance, $I^1 = (RL)^\infty$, $I^2 = (RLRR)^\infty$ and $I^3 = (RLRRRLRL)^\infty$.)

Similarly, the finite sequences $J^n = \{J_i^n\}_{0 \leqslant i \leqslant 2^{n+1} - 1} = R^{*n} * RC$ satisfy the induction $J^0 = RC$ and for every $n \in \mathbb{N}$,

$$J_{2i}^{n+1} = R \quad \text{and} \quad J_{2i+1}^{n+1} = \check{J}_i^n, \quad 0 \leqslant i \leqslant 2^{n+1} - 1,$$

where $\check{C} = C$.

These inductive relations allow us to prove the following properties which are also used to prove Lemma 3.9.

LEMMA 3.3. *For every $n \in \mathbb{N}$, we have:*

(A) $I_{2^n - 1}^n = R$ *(respectively* $I_{2^n - 1}^n = L$*) if n is even (respectively odd);*

(B) $\{I_i^{n+1}\}_{0 \leqslant i \leqslant 2^{n+1} - 2} = \{I_i^n\}_{0 \leqslant i \leqslant 2^{n+1} - 2}$*;*

(C) $\{J_i^n\}_{0 \leqslant i \leqslant 2^{n+1} - 2} = \{I_i^n\}_{0 \leqslant i \leqslant 2^{n+1} - 2}$ *and* $J_{2^{n+1} - 1}^n = C$*;*

(D) *the sequence I^n is 2^n-periodic;*

(E) *the sequence* $\{I_i^n\}_{0 \leqslant i \leqslant 2^n - 1}$ *is odd* [**3**].

Proof. (A) We have $I_0^1 = R$ and then the statement holds for $n = 0$. Let $n \geqslant 0$ be even and assume that $I_{2^n - 1}^n = R$. By relation (3.1), we have $I_{2^{n+1} - 1}^{n+1} = \check{I}_{2^n - 1}^n = L$ and then the statement holds for $n+1$, which is odd. The proof is similar when n is odd.

(B) For $n = 0$, we have $I_0^1 = R = I_0^0$ and the statement holds. Assume the statement holds for some $n \geqslant 0$. Then, by relation (3.1), we have

$$I_{2i+1}^{n+2} = \check{I}_i^{n+1} = \check{I}_i^n = I_{2i+1}^{n+1}, \quad 0 \leqslant i \leqslant 2^{n+1} - 2,$$

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and $I_{2i}^{n+1} = I_{2i}^{n}$ for every $i \in \{0, \ldots, 2^{n+1} - 1\}$, which implies that the statement holds for $n + 1$.

The proofs of (C) and (D) are similar and are left to the reader. □

Proof of Proposition 3.2. (1) Suppose first that the kneading sequence of f is I^k for some $k \in \mathbb{N}$. Then $I_0(f^{2^k}(0)) = I_{2^k-1}^k$ and the result follows from Lemma 3.3(A).

Now if the kneading sequence is I^n for some $n > k$, since we have $2^k - 1 \leqslant 2^{n+1} - 2$ by Lemma 3.3(B), it follows that $I_0(f^{2^k}(0)) = I_{2^k-1}^n = I_{2^k-1}^k$, so the result holds. The same argument and Lemma 3.3(C) show that the result is also true when the kneading sequence is J^n with $n \geqslant k$. Statement (1) is thus proved.

(2) The argument is similar to that above. First, suppose that the kneading sequence of f is I^k for some $k \in \mathbb{N}$. We have

$$\{I_i(f^{2^k}(0))\}_{1 \leqslant i \leqslant 2^k-1} = \{I_{i+2^k-1}(1)\}_{1 \leqslant i \leqslant 2^k-1} = \{I_{i-1}(1)\}_{1 \leqslant i \leqslant 2^k-1},$$

where the second equality follows from the periodicity of I^k (Lemma 3.3(D)). As before, this property extends to the case where the kneading sequence is either I^n or J^n with $n \geqslant k$.

(3) The argument is once again similar to that above. When the kneading sequence is I^k, the result is given by Lemma 3.3(E). The other cases follow from Lemmas 3.3(B) and 3.3(C).

(4) If the kneading sequence of f is J^n, then by Lemma 3.3(C), we have $f^{2^{n+1}}(0) = 0$ and the statement holds (we have equality of sets in this case).

Assume now that the kneading sequence of f is I^n for some $n \geqslant 1$. (The result in the case where $n = 0$ is obvious.) By statements (2) and (3), one can apply Proposition 3.1(ii) with $a = f^{2^n}(0)$ and $t = 2^n - 1$ to obtain the inequality $f^{2^n}(0)(f^{2^{n+1}}(0) - f^{2^n}(0)) < 0$. However, the periodicity of I^n implies the inequality $f^{2^n}(0) f^{2^{n+1}}(0) > 0$ and then the inclusion follows. □

3.2. *Proof of Theorem 2.2.* Using the results of the previous section together with the convexity of the coupling operator L_ϵ, we obtain a precise description of iterates of any point in $\mathcal{M}$ under the CML (Proposition 3.4 below). In particular, Theorem 2.2 follows immediately from this statement when we consider those $(x, y) \in \mathcal{M}$ for which $x^t \neq y^t$ for all $t \in \mathbb{N}$.

PROPOSITION 3.4. *For every $(x, y) \in \mathcal{M}$, there exist $k \in \{0, \ldots, n\}$ and $t_0 \in \mathbb{N}$ such that for all $t \in \mathbb{N}$ and $p \in \{0, \ldots, 2^k - 1\}$, there exists $\sigma(t, p) \in \mathbb{N}$ with $\sigma(t + 1, p) = \sigma(t, p) + 1$, so that*

$$(x^{t_0+2^k t+p}, y^{t_0+2^k t+p}) \in \theta^{\sigma(t,p)}(\overline{T}_{\{f^{2^k+p}(0), f^{2^{k+1}+p}(0)\}}),$$

if $k \in \{0, \ldots, n-1\}$ and

$$(x^{t_0+2^n t+p}, y^{t_0+2^n t+p}) \in \theta^{\sigma(t,p)}(\overline{T}_{\{f^{p}(0), f^{2^n+p}(0)\}}),$$

if $k = n$.

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In the proof, we use the following properties of the dynamics of certain triangles $\overline{T}_{\{a,b\}}$.

LEMMA 3.5.

(1) *For every* $t \in \{1, \dots, 2^n\}$, *there exists* $s(t, n) \in \{0, 1\}$ *such that*

$$F_\epsilon^t(\overline{T}_{\{0, f^{2^n}(0)\}}) \subset \theta^{s(t,n)}(\overline{T}_{\{f^t(0), f^{2^n+t}(0)\}}).$$

(2) *If* $n \geqslant 1$, *then for every* $k \in \{0, \dots, n-1\}$ *and every* $(x, y) \in \overline{T}_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}$, *at least one of the following relations holds:*

   (i) *For all* $t \in \{1, \dots, 2^k\}$, *there exists* $s(t, k) \in \{0, 1\}$ *such that* $F_\epsilon^t(x, y) \in \theta^{s(t,k)}(\overline{T}_{\{f^t(0), f^{2^k+t}(0)\}})$;

   (ii) *For all* $t \in \{1, \dots, 2^{k+1}\}$, *there exists* $s(t, k+1) \in \{0, 1\}$ *such that* $F_\epsilon^t(x, y) \in \theta^{s(t,k+1)}(\overline{T}_{\{f^t(0), f^{2^{k+1}+t}(0)\}})$.

(3) $s(2^k, k) = 1$ *for every* $k \in \{0, \dots, n\}$ *in all relations.*

Proof. (1) We will actually prove the following stronger statement. For every $k \in \{0, \dots, n\}$ and every $t \in \{1, \dots, 2^k\}$, there exists $s(t, k) \in \{0, 1\}$ such that

$$F_\epsilon^t(\overline{T}_{\{0, f^{2^k}(0)\}}) \subset \theta^{s(t,k)}(\overline{T}_{\{f^t(0), f^{2^k+t}(0)\}}). \tag{3.2}$$

Let $k \in \{0, \dots, n\}$. By Proposition 3.2(2), we can apply Proposition 3.1(i) with $a = f^{2^k}(0)$ to obtain the existence, for every $t \in \{1, \dots, 2^k\}$, of $s(t, k) \in \{0, 1\}$ such that

$$F_0^t(\overline{T}_{\{0, f^{2^k}(0)\}}) = \theta^{s(t,k)}(\overline{T}_{\{f^t(0), f^{2^k+t}(0)\}}). \tag{3.3}$$

(In the case where $k = 0$, this relation is obvious.)

On the other hand, since $\epsilon \in [0, \frac{1}{2}]$, for any $a, b \in [-1, 1]$ and any $s \in \{0, 1\}$, we have

$$L_\epsilon(\theta^s(\overline{T}_{\{a,b\}})) \subset \theta^s(\overline{T}_{\{a,b\}}). \tag{3.4}$$

We now use induction on t. Using the decomposition $F_\epsilon = L_\epsilon \circ F_0$ and the relations (3.3) and (3.4), we obtain $F_\epsilon(\overline{T}_{\{0, f^{2^k}(0)\}}) \subset \theta^{s(1,k)}(\overline{T}_{\{f(0), f^{2^k+1}(0)\}})$. That is to say, the relation (3.2) holds for $t = 1$.

Assume now that this relation holds for some $t \in \{1, \dots, 2^k - 1\}$. Then by applying relations (3.3) and (3.4), we obtain

$$\begin{aligned} F_\epsilon^{t+1}(\overline{T}_{\{0, f^{2^k}(0)\}}) &\subset L_\epsilon \circ F_0(\theta^{s(t,k)}(\overline{T}_{\{f^t(0), f^{2^k+t}(0)\}})) \\ &= L_\epsilon(\theta^{s(t+1,k)}(\overline{T}_{\{f^{t+1}(0), f^{2^k+t+1}(0)\}})) \subset \theta^{s(t+1,k)}(\overline{T}_{\{f^{t+1}(0), f^{2^k+t+1}(0)\}}), \end{aligned}$$

showing that the relation (3.2) holds for $t + 1$ and hence holds for all $t \in \{1, \dots, 2^k\}$.

(2) We note that by the definition of f, for any $a, b \in [-1, 1]$ such that $a < 0 < b$, we have

$$\begin{aligned} F_0(\overline{T}_{a,b}) &\subset \{(x, y) \in \mathcal{M} : f(a) \leqslant x \leqslant 1 \text{ and } f(b) \leqslant y \leqslant 1\} \\ &\subset F_0(\overline{T}_{a,0}) \cup F_0(\overline{T}_{0,b}) = \overline{T}_{f(a),1} \cup \underline{T}_{f(b),1}. \end{aligned}$$

Using relations (3.3) and (3.4), we obtain the following inclusion for any $k \in \{0, \dots, n-1\}$:

$$F_\epsilon(\overline{T}_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}) \subset \theta^{s(1,k)}(\overline{T}_{f^{2^k+1}(0),1}) \cup \theta^{s(1,k+1)}(\overline{T}_{f^{2^{k+1}+1}(0),1}).$$

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Consider the case when

$$F_\epsilon(x, y) \in \theta^{s(1,k)}(\overline{T}_{f^{2^k+1}(0),1}) = F_0(\overline{T}_{\{0, f^{2^k}(0)\}}).$$

Using the same argument as that used in proving relation (3.2), we obtain by (3.3) that for every $t \in \{1, \ldots, 2^k\}$,

$$F_\epsilon^{t-1}(\theta^{s(1,k)}(\overline{T}_{f^{2^k+1}(0),1})) \subset F_0^{t-1}(\theta^{s(1,k)}(\overline{T}_{f^{2^k+1}(0),1})) = \theta^{s(t,k)}(\overline{T}_{\{f^t(0), f^{2^k+t}(0)\}}).$$

Therefore, relation (i) follows. In the case when

$$F_\epsilon(x, y) \in \theta^{s(1,k+1)}(\overline{T}_{f^{2^{k+1}+1}(0),1}),$$

since $k < n$ the proof is exactly the same, replacing k by $k + 1$ to obtain relation (ii).

(3) Finally, the fact that $s(2^k, k) = 1$ is a consequence of Propositions 3.1 and 3.2(3). □

Proof of Proposition 3.4. We consider two cases, namely $(x, y) \in S_{\{0, f^{2^n}(0)\}}$ and $(x, y) \notin S_{\{0, f^{2^n}(0)\}}$.

(a) Assume that $(x, y) \in S_{\{0, f^{2^n}(0)\}}$, i.e. $(x, y) \in \theta^s(\overline{T}_{\{0, f^{2^n}(0)\}})$ for some $s \in \{0, 1\}$. Applying Lemma 3.5(1) with $t = 2^n$ and using Proposition 3.2(4), it follows that

$$(x^{2^n}, y^{2^n}) \in \theta^{s+s(2^n,n)}(\overline{T}_{\{0, f^{2^n}(0)\}}).$$

Repeating this argument inductively, we obtain

$$(x^{2^n m}, y^{2^n m}) \in \theta^{s+ms(2^n,n)}(\overline{T}_{\{0, f^{2^n}(0)\}}),$$

for all $m \in \mathbb{N}$. For any given $p \in \{1, \ldots, 2^n - 1\}$, again applying Lemma 3.5(1) with $t = p$ to the point $(x^{2^n m}, y^{2^n m})$, we have

$$(x^{2^n m+p}, y^{2^n m+p}) \in \theta^{s+ms(2^n,n)+s(p,n)}(\overline{T}_{\{f^p(0), f^{2^n+p}(0)\}}),$$

for all $m \in \mathbb{N}$. The result follows with $k = n$, $t_0 = 0$ and $\sigma(t, p) = s + ts(2^n, n) + s(p, n)$ (letting $s(p, n) = 0$ if $p = 0$).

(b) Assume that $(x, y) \notin S_{\{0, f^{2^n}(0)\}}$. Since $\mathcal{M} = S_{f^2(0), f(0)}$, there exists $k' \in \{0, \ldots, n - 1\}$ such that $(x, y) \in S_{\{f^{2^{k'}}(0), f^{2^{k'+1}}(0)\}}$. Let j be the largest such k'. Then $(x, y) \in \theta^s(\overline{T}_{\{f^{2^j}(0), f^{2^{j+1}}(0)\}})$ for some $s \in \{0, 1\}$.

Applying Lemma 3.5(2) with $k = j$ to (x, y) and, repeatedly, to the subsequent iterates, either we have

$$(x^{2^j m}, y^{2^j m}) \in \theta^{s+ms(2^j,j)}(\overline{T}_{\{f^{2^j}(0), f^{2^{j+1}}(0)\}}),$$

for all $m \in \mathbb{N}$, or there exists $m' \in \mathbb{N}$ such that

$$(x^{2^j m'}, y^{2^j m'}) \notin \theta^{s+m's(2^j,j)}(\overline{T}_{\{f^{2^j}(0), f^{2^{j+1}}(0)\}}).$$

The latter situation occurs when $(x^{2^j(m'-1)}, y^{2^j(m'-1)})$ does not satisfy property (i) of Lemma 3.5(2) for $k = j$.

In the first case, using the same argument as that in case (a), we conclude that

$$(x^{2^j m+p}, y^{2^j m+p}) \in \theta^{s+ms(2^j,j)+s(p,j)}(\overline{T}_{\{f^p(0), f^{2^j+p}(0)\}}),$$

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for all $m \in \mathbb{N}$ and $p \in \{1, \dots, 2^j - 1\}$. Thus, the result holds with $k = j$, $t_0 = 0$ and $\sigma(t, p) = s + ts(2^j, j) + s(p, j)$.

In the second case, property (ii) of Lemma 3.5(2) with $k = j$ must apply to the point $(x^{2^j(m'-1)}, y^{2^j(m'-1)})$. Hence

$$(x^{2^j(m'+1)}, y^{2^j(m'+1)}) \in S_{\{f^{2^{j+1}}(0), f^{2^{j+2}}(0)\}}.$$

If $j = n - 1$, then by Proposition 3.2(4), we have $(x^{2^j(m'+1)}, y^{2^j(m'+1)}) \in S_{\{0, f^{2^n}(0)\}}$ and we are back in case (a) with $(x^{2^j(m'+1)}, y^{2^j(m'+1)})$ instead of (x, y). We conclude that the result holds with $k = n$ and $t_0 = 2^{n-1}(m' + 1)$.

Now, if $j \leqslant n - 2$, we apply Lemma 3.5(2) with $k = j + 1$ to the point $(x^{2^j(m'+1)}, y^{2^j(m'+1)})$ and, repeatedly, to the subsequent iterates. We conclude that either the result holds with $k = j + 1$ and $t_0 = 2^j(m' + 1)$ or there exists $m'' \in \mathbb{N}$ such that $(x^{2^j(m'+1)+2^{j+1}m''}, y^{2^j(m'+1)+2^{j+1}m''})$ does not belong to $\theta^{s+(m'+1)s(2^j,j)+m''s(2^{j+1},j+1)}(\overline{T}_{\{f^{2^{j+1}}(0), f^{2^{j+2}}(0)\}})$.

Again, we see that if $j = n - 2$, then the point

$$(x^{2^j(m'+1)+2^{j+1}(m''+1)}, y^{2^j(m'+1)+2^{j+1}(m''+1)})$$

belongs to $S_{\{0, f^{2^n}(0)\}}$ and the result holds. If $j \leqslant n - 3$, we need to repeatedly apply Lemma 3.5(2) with $k = j + 2$ to this point, and so on. This process is finite and it ends within at most n steps. We conclude that the result holds for some $k \in \{j, \dots, n\}$ and $t_0 \in \mathbb{N}$. □

3.3. *Proof of Theorem 2.3.* Let $\{(x^t, y^t)\}_{t\in\mathbb{N}}$ be an orbit such that $x^t y^t \geqslant 0$ for all $t \in \mathbb{N}$. If $x^{t'} = y^{t'}$ for some $t' \in \mathbb{N}$, then $x^{t+1} = y^{t+1} = f(x^t)$ for all $t \geqslant t'$ and the result is given by Proposition 2.1(1). Therefore, in all of the proof, we also assume that $x^t \neq y^t$ for all t.

Let $k \in \{0, \dots, n\}$ and let $t_0 \in \mathbb{N}$ be integers for which Proposition 3.4 holds. We claim that for all $t \in \mathbb{N}$, we have

$$(x^{2^k t}, y^{2^k t}) \in S_{\{0, f^{2^k}(0)\}}. \tag{3.5}$$

This is immediate if $k = n$. If $k < n$, by assumption on the orbit, we have

$$(x^{t_0+2^k t}, y^{t_0+2^k t}) \in S_{\{0, f^{2^k}(0)\}} \cup S_{\{0, f^{2^{k+1}}(0)\}}.$$

(By Proposition 3.2(1), these squares only intersect at $(0, 0)$.) Suppose instead that $(x^{t_0+2^k t'}, y^{t_0+2^k t'}) \in S_{\{0, f^{2^{k+1}}(0)\}} \setminus S_{\{0, f^{2^k}(0)\}}$ for some $t' \in \mathbb{N}$. Then by the relation (3.2) with $t = 2^{k+1}$, we would have

$$(x^{t_0+2^k(t'+2)} - y^{t_0+2^k(t'+2)})(x^{t_0+2^k t'} - y^{t_0+2^k t'}) < 0.$$

However, we have assumed that Proposition 3.4 holds with k. So

$$(x^{t_0+2^k(t'+2)} - y^{t_0+2^k(t'+2)})(x^{t_0+2^k t'} - y^{t_0+2^k t'}) > 0,$$

and we have a contradiction.

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We now show that every orbit satisfying relation (3.5) is asymptotically periodic with period a factor of 2^{k+1}. Without loss of generality, we may assume that $t_0 = 0$. We may also assume that $(x^{2^{k+1}}, y^{2^{k+1}}) \neq (x, y)$, otherwise there is nothing to prove. For each point $(a, b) \in \mathcal{M}$, we introduce the notation $(a, b) \prec 0$ which means that both $\max\{a, b\} \leqslant 0$ and $\min\{a, b\} < 0$ and we divide all possible situations of $(x^{2^{k+1}}, y^{2^{k+1}}) \neq (x, y)$ into two cases:

(a) $(-1)^r(x^{2^{k+1}} - x, y^{2^{k+1}} - y) \prec 0$ for some $r \in \{0, 1\}$;

(b) $(-1)^r(x^{2^{k+1}} - x, y - y^{2^{k+1}}) \prec 0$ for some $r \in \{0, 1\}$.

In the first case, we use the following lemma, the proof of which is postponed to the end of this section.

LEMMA 3.6. *Let $a \in [-1, 1] \setminus \{0\}$ and $t \in \mathbb{N}$ be such that $\{I_i(a)\}_{1 \leqslant i \leqslant t} = \{I_{i-1}(1)\}_{1 \leqslant i \leqslant t}$ if $t \geqslant 1$. If neither of these sequences contains the symbol C, then for every pair of points $(x_1, y_1), (x_2, y_2) \in S_{\{0,a\}}$ with $(x_1 - x_2, y_1 - y_2) \prec 0$, we have*

$$(-1)^s(x_1^{t+1} - x_2^{t+1}, y_1^{t+1} - y_2^{t+1}) \prec 0$$

where $s = 0$ (respectively $s = 1$) if $\{I_i(a)\}_{0 \leqslant i \leqslant t}$ is even (respectively odd).

By relation (3.5) and Propositions 3.2(2) and 3.2(3), we apply Lemma 3.6 first to $(x^{2^{k+1}}, y^{2^{k+1}})$ and (x, y) with $a = f^{2^k}(0)$ and $t = 2^k - 1$ to obtain

$$(-1)^{r+1}(x^{2^{k+1}+2^k} - x^{2^k}, y^{2^{k+1}+2^k} - y^{2^k}) \prec 0.$$

Applying the lemma to the points $(x^{2^{k+1}+2^k}, y^{2^{k+1}+2^k})$ and (x^{2^k}, y^{2^k}), again with $a = f^{2^k}(0)$ and $t = 2^k - 1$, we obtain

$$(-1)^r(x^{2^{k+1}.2} - x^{2^{k+1}}, y^{2^{k+1}.2} - y^{2^{k+1}}) \prec 0.$$

Repeating this process, we have for all $t \in \mathbb{N}$

$$(-1)^r(x^{2^{k+1}(t+1)} - x^{2^{k+1}t}, y^{2^{k+1}(t+1)} - y^{2^{k+1}t}) \prec 0,$$

i.e. both sequences $\{x^{2^{k+1}t}\}$ and $\{y^{2^{k+1}t}\}$ are monotonic. Since they are also bounded, they converge. By the continuity of f, the limit is a periodic point of F_ϵ whose period must be a factor of 2^{k+1}.

In case (b), the proof uses another lemma whose proof is also given at the end of this section. Note that if $\epsilon = \frac{1}{2}$, then $x^1 = y^1$ for every $(x, y) \in \mathcal{M}$ and, thus, case (b) cannot occur.

LEMMA 3.7. *Let $a \in [-1, 1] \setminus \{0\}$ and $t \in \mathbb{N}$ be such that $\{I_i(a)\}_{1 \leqslant i \leqslant t} = \{I_{i-1}(1)\}_{1 \leqslant i \leqslant t}$ if $t \geqslant 1$. If neither of these sequences contains the symbol C, then for every $\epsilon \in [0, \frac{1}{2})$ and every pair of points $(x_1, y_1), (x_2, y_2) \in S_{\{0,a\}}$ with $(x_1 - x_2, y_2 - y_1) \prec 0$, at least one of the following statements hold:*

(i) *there exists $s \in \{0, 1\}$ (which depends on (x, y) and on a) such that*

$$(-1)^s(x_1^{t+1} - x_2^{t+1}, y_1^{t+1} - y_2^{t+1}) \prec 0;$$

(ii) *there exists $s \in \{0, 1\}$ depending only on a such that*

$$(-1)^s(x_1^{t+1} - x_2^{t+1}, y_2^{t+1} - y_1^{t+1}) \prec 0.$$

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In case (b), again using relation (3.5) and Propositions 3.2(2) and 3.2(3), we apply Lemma 3.7 with $a = f^{2^k}(0)$ and $t = 2^k - 1$. If statement (i) holds, then we are in the same situation as in case (a) with (x^{2^k}, y^{2^k}) instead of (x, y) and the orbit is asymptotically periodic with period a factor 2^{k+1}.

If statement (i) does not hold, then we once again apply Lemma 3.7 with $a = f^{2^k}(0)$ and $t = 2^k - 1$ to conclude that either we are in case (a) with $(x^{2^{k+1}}, y^{2^{k+1}})$ instead of (x, y), or

$$(-1)^r(x^{2^{k+1}.2} - x^{2^{k+1}}, y^{2^{k+1}} - y^{2^{k+1}.2}) \prec 0,$$

since s only depends on k and $(-1)^{2s} = 1$.

By induction on t, we are either in case (a) after a finite number of iterations and, therefore, the orbit is asymptotically periodic with period a factor 2^{k+1}, or else we have for all $t \in \mathbb{N}$

$$(-1)^r(x^{2^{k+1}(t+1)} - x^{2^{k+1}t}, y^{2^{k+1}t} - y^{2^{k+1}(t+1)}) \prec 0.$$

In the latter case, it also follows that both sequences $\{x^{2^{k+1}t}\}$ and $\{y^{2^{k+1}t}\}$ converge and the orbit is asymptotically periodic with period a factor of 2^{k+1}, which is the desired conclusion.

To finish the proof of Theorem 2.3, it remains to show that the period of every non-homogeneous periodic orbit passing $S_{\{0, f^{2^k}(0)\}}$ is also a multiple of 2^{k+1}.

To see this, recall that relation (3.2) imposes that every orbit such that $x^t \neq y^t$ and $(x^{2^k t}, y^{2^k t}) \in S_{\{0, f^{2^k}(0)\}}$ for all $t \in \mathbb{N}$ has the property

$$(x^{2^k(t+1)} - y^{2^k(t+1)})(x^{2^k t} - y^{2^k t}) \prec 0. \tag{3.6}$$

For a contradiction, assume now that there exists a non-homogeneous periodic orbit passing $S_{\{0, f^{2^k}(0)\}}$ with period $2^i j$, where $i \in \{0, \ldots, k\}$ and j is odd. By relation (3.6) and because j is odd, we obtain that $(x^{2^k j} - y^{2^k j})(x - y) \prec 0$. However, on the other hand, $2^k j$ is a multiple of $2^i j$ and then by periodicity we have $(x^{2^k j} - y^{2^k j})(x - y) = (x - y)^2 > 0$, which leads to a contradiction. □

Proof of Lemma 3.6. We use induction on t. First, note that if $(x_1 - x_2, y_1 - y_2) \prec 0$, then $L_\epsilon(x_1 - x_2, y_1 - y_2) \prec 0$. Now, if $(x_1, y_1), (x_2, y_2) \in S_{\{0,a\}}$ and $(x_1 - x_2, y_1 - y_2) \prec 0$, then by the piecewise monotonicity of f and the monotonicity of L_ϵ, we have

$$(-1)^s(x_1^1 - x_2^1, y_1^1 - y_2^1) \prec 0,$$

where $s = 0$ (respectively $s = 1$) if $I_0(a) = L$ (respectively $I_0(a) = R$). The statement holds for $t = 0$.

Now, given $t \in \mathbb{N}$, assume that $\{I_i(a)\}_{1 \leqslant i \leqslant t+1} = \{I_{i-1}(1)\}_{1 \leqslant i \leqslant t+1}$ (where all symbols are different from C) and suppose that $\{I_i(a)\}_{0 \leqslant i \leqslant t+1}$ is even.

If $\{I_i(a)\}_{0 \leqslant i \leqslant t}$ is even and $I_{t+1}(a) = I_t(1) = L$, then by induction we have the hypothesis $(x_1^{t+1} - x_2^{t+1}, y_1^{t+1} - y_2^{t+1}) \prec 0$. Using the fact that f is strictly increasing on $[-1, 0]$ and the monotonicity of L_ϵ, we conclude that $(x_1^{t+2} - x_2^{t+2}, y_1^{t+2} - y_2^{t+2}) \prec 0$. If $\{I_i(a)\}_{0 \leqslant i \leqslant t}$ is odd and $I_{t+1}(a) = I_t(1) = R$, we have $(x_2^{t+1} - x_1^{t+1}, y_2^{t+1} - y_1^{t+1}) \prec 0$ and then $(x_1^{t+2} - x_2^{t+2}, y_1^{t+2} - y_2^{t+2}) \prec 0$, because now both $f^{t+1}(0)$ and $f^{t+1}(a)$ are positive. The statement then follows for $t + 1$ in the case where $\{I_i(a)\}_{0 \leqslant i \leqslant t+1}$ is even. The odd case follows analogously. □

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Proof of Lemma 3.7. We again use induction. If $I_0(a) = L$, then by the piecewise monotonicity of f, the assumption on (x_1, y_1) and (x_2, y_2) implies $(f(x_1)-f(x_2), f(y_2)-f(y_1)) \prec 0$ and then by the definition of F_ϵ,

$$x_1^1 - y_1^1 = (1-2\epsilon)(f(x_1) - f(y_1)) \prec (1-2\epsilon)(f(x_2) - f(y_2)) = x_2^1 - y_2^1.$$

This implies that either $(-1)^s(x_1^1 - x_2^1, y_1^1 - y_2^1) \prec 0$ for some $s \in \{0, 1\}$ or $(x_1^1 - x_2^1, y_2^1 - y_1^1) \prec 0$. Similarly, in the case where $I_0(a) = R$, we obtain that either $(-1)^s(x_1^1 - x_2^1, y_1^1 - y_2^1) \prec 0$ for some $s \in \{0, 1\}$ or $(x_2^1 - x_1^1, y_1^1 - y_2^1) \prec 0$. The statement follows for $t = 0$.

Assume now that the statement holds for some $t \in \mathbb{N}$. If property (i) holds, then, as in the proof of Lemma 3.6, we prove that the same property holds for $t+1$. In the case where property (ii) holds for t, then, as for $t = 0$, we prove that either (i) or (ii) holds (or both hold) for $t + 1$. □

3.4. *Existence of symmetric orbits.* In this section, we need the following additional property of the sequence $\{f^{2^k}(0)\}$ when the kneading sequence of f is either I^n and J^n.

PROPOSITION 3.8. *For every $k \in \{1, \dots, n\}$, we have $I_{\{f^{2^{k-1}}(0), f^{2^{k-1}+2^k}(0)\}} \subsetneq I_{\{0, f^{2^{k-1}}(0)\}}$.*

To prove this inclusion, we use the following lemma.

LEMMA 3.9. *Assume that $n \geqslant 1$ and let $k \in \{1, \dots, n\}$.*

(1) *We have $f^{2^{k-1}+2^k}(0) > 0$ (respectively $f^{2^{k-1}+2^k}(0) < 0$) if k is odd (respectively k is even).*

(2) *The sequence $\{I_i(f^{2^k}(0))\}_{0 \leqslant i \leqslant 2^{k-1}-1}$ is even.*

Proof. (1) We have $I_0(f^{2^{k-1}+2^k}(0)) = I_{2^{k-1}+2^k-1}(1)$ and $2^{k-1} + 2^k - 1 \leqslant 2^{n+1} - 2$ for all $n \geqslant k \geqslant 1$. Therefore, by Lemmas 3.3(B) and 3.3(C), we only need to show that $I^n_{2^{n-1}+2^n-1} = R$ when n is odd and $I^n_{2^{n-1}+2^n-1} = L$ when n is even. Since the sequence I^n is 2^n-periodic, we need to prove that $I^n_{2^{n-1}-1} = R$ when n is odd and $I^n_{2^{n-1}-1} = L$ when n is even. As in the proof of Lemma 3.3(A), these can easily be done by induction on n because we have $I^1_0 = R$ and $I^{n+1}_{2^n-1} = \check{I}^n_{2^{n-1}-1}$.

(2) Assume that the kneading sequence of f is I^k; the other cases follow from the fact that the proof only uses the first 2^{k-1} symbols of the kneading sequence and Lemmas 3.3(B) and 3.3(C).

By a change of the index, we have $\{I_i(f^{2^k}(0))\}_{1 \leqslant i \leqslant 2^{k-1}-1} = \{I^k_i\}_{0 \leqslant i \leqslant 2^{k-1}-2}$. Because $f^{2^k}(0) > 0$ when k is even and $f^{2^k}(0) < 0$ otherwise, to prove that $\{I_i(f^{2^k}(0))\}_{0 \leqslant i \leqslant 2^{k-1}-1}$ is even for every k, we only need to prove that the sequence $\{I^k_i\}_{0 \leqslant i \leqslant 2^{k-1}-2}$ is odd when k is even and even when k is odd.

We use induction on k. The sequence $\{I^2_i\}_{0 \leqslant i \leqslant 0} = R$ is odd and the statement holds for $k = 2$. Now we assume that the statement holds for some $k \geqslant 2$, k even. We have

$$\{I^{k+1}_i\}_{0 \leqslant i \leqslant 2^k-2} = \{I^{k+1}_{2i}\}_{0 \leqslant i \leqslant 2^{k-1}-1} \cup \{I^{k+1}_{2i+1}\}_{0 \leqslant i \leqslant 2^{k-1}-2}.$$

By the first equality in (3.1), the sequence $\{I^{k+1}_{2i}\}_{0 \leqslant i \leqslant 2^{k-1}-1}$ is even. Moreover $\{I^{k+1}_{2i+1}\}_{0 \leqslant i \leqslant 2^{k-1}-2} = \{\check{I}^k_i\}_{0 \leqslant i \leqslant 2^{k-1}-2}$, which is also an even sequence by the induction

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assumption. Therefore the statement holds for $k+1$, which is an odd number. A similar argument holds in the case where k is odd. This completes the proof. □

Proof of Proposition 3.8. By Lemma 3.9(1) and Proposition 3.2(1), for every $k \in \{1, \dots, n\}$, the points $f^{2^{k-1}}(0)$ and $f^{2^{k-1}+2^k}(0)$ lie on the same side of zero. Therefore, we only need to prove the inequality

$$f^{2^{k-1}}(0)(f^{2^{k-1}+2^k}(0) - f^{2^{k-1}}(0)) < 0. \tag{3.7}$$

By the same argument as in the proof of Lemma 3.9(2), we only have to prove this inequality when the kneading sequence is I^k. This sequence is 2^k-periodic. So we have

$$\{I_i(f^{2^k}(0))\}_{1 \leqslant i \leqslant 2^{k-1}-1} = \{I^k_{i-1}\}_{1 \leqslant i \leqslant 2^{k-1}-1} = \{I_{i-1}(1)\}_{1 \leqslant i \leqslant 2^{k-1}-1}.$$

We now apply Lemma 3.1(ii) with $a = f^{2^k}(0)$ and $t = 2^{k-1}-1$ and we use Lemma 3.9(2) to obtain the inequality $f^{2^k}(0)(f^{2^{k-1}+2^k}(0) - f^{2^{k-1}}(0)) > 0$. Thus, the fact that $f^{2^{k-1}}(0) f^{2^k}(0) < 0$ implies (3.7). □

3.4.1. *Proof of Proposition 2.1.* (1) This follows from [**8**]. We can also give a proof based on the same argument as that used in the proof of Theorem 2.3.

(2) Let us first prove the uniqueness of 2^k-periodic points in $I_{\{0, f^{2^k}(0)\}}$ by contradiction. Assume that there exist two such points, say x and x^*. This would mean that $(-1)^r(x-x^*, x-x^*) \prec 0$ for some $r \in \{0, 1\}$. Then, applying Lemma 3.6 with $a = f^{2^k}(0)$ and $t = 2^k - 1$, we obtain $(-1)^{r+1}(x-x^*, x-x^*) \prec 0$ by Proposition 3.2(3). However, this contradicts the definition of $\prec$.

To prove the existence, we observe that Propositions 3.2(1) and 3.2(4) imply that, for every $k \in \{0, \dots, n\}$, the product $(f^{2^{k+1}}(0) - f^{2^k}(0))(f^{2^k}(0) - 0) < 0$. By continuity, it follows that f^{2^k} has a fixed point in $I_{\{0, f^{2^k}(0)\}}$. The proof is complete when $k = 0$.

If $k \geqslant 1$, then Propositions 3.2 (statements (1) and (2)) and 3.8 show that

$$f^{2^{k-1}}(I_{\{0, f^{2^k}(0)\}}) \cap I_{\{0, f^{2^k}(0)\}} = I_{\{f^{2^{k-1}}(0), f^{2^{k-1}+2^k}(0)\}} \cap I_{\{0, f^{2^k}(0)\}} = \emptyset. \tag{3.8}$$

Therefore, every fixed point of f^{2^k} in $I_{\{0, f^{2^k}(0)\}}$ is actually a 2^k-periodic point of f.

(3) If f is S-unimodal, then it has at most one stable periodic orbit in $[f(1), 1]$ and this orbit attracts zero [**3**]. Let us prove that, when the kneading sequence is I^n or J^n, this orbit has period 2^n or 2^{n+1}.

We first use induction on t to show that $f^{2^n t}(0) \in I_{\{0, f^{2^n}(0)\}}$ and

$$f^{2^n(t+2)}(0) \in I_{\{f^{2^n(t+1)}(0), f^{2^n t}(0)\}}, \tag{3.9}$$

for all $t \in \mathbb{N}$. For $t = 0$, the result is given by Proposition 3.2(4). Assume now that the result holds for some $t \in \mathbb{N}$. Then, since both $f^{2^n t}(0)$ and $f^{2^n(t+2)}(0)$ belong to $I_{\{0, f^{2^n}(0)\}}$, we can apply Lemma 3.6 to $(f^{2^n(t+2)}(0), f^{2^n(t+2)}(0))$ and $(f^{2^n t}(0), f^{2^n t}(0))$ with $a = f^{2^n}(0)$ and $t = 2^n - 1$ using Proposition 3.2(2) to obtain

$$(f^{2^n(t+3)}(0) - f^{2^n(t+1)}(0))(f^{2^n(t+2)}(0) - f^{2^n t}(0)) < 0.$$

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Similarly, Lemma 3.6 applied to the points

$$(f^{2^n(t+2)}(0),\, f^{2^n(t+2)}(0)) \quad \text{and} \quad (f^{2^n(t+1)}(0),\, f^{2^n(t+1)}(0))$$

gives

$$(f^{2^n(t+3)}(0) - f^{2^n(t+2)}(0))(f^{2^n(t+2)}(0) - f^{2^n(t+1)}(0)) < 0.$$

These two inequalities, together with the assumption

$$f^{2^n(t+2)}(0) \in I_{\{f^{2^n(t+1)}(0),\, f^{2^n t}(0)\}},$$

imply

$$f^{2^n(t+3)}(0) \in I_{\{f^{2^n(t+2)}(0),\, f^{2^n(t+1)}(0)\}}.$$

In particular, we have $f^{2^n(t+3)}(0) \in I_{\{0,\, f^{2^n}(0)\}}$ and the induction follows.

Clearly, the property (3.9) implies that the sequence $\{f^{2^{n+1}t}(0)\}_{t\in\mathbb{N}}$ is monotonic and then converges to a limit in $I_{\{0,\, f^{2^n}(0)\}}$. The limit is a periodic point with a period factor of 2^{n+1}. However, $f^{2^{n-1}}(I_{\{0,\, f^{2^n}(0)\}}) \cap I_{\{0,\, f^{2^n}(0)\}} = \emptyset$ by (3.8). We conclude that the period of the stable periodic orbit is either 2^n or 2^{n+1} when the kneading sequence of a S-unimodal map is I^n or J^n.

Moreover, for every $k \in \{0, \ldots, n\}$, the itinerary $\{I_i(x_k)\}_{0\leqslant i\leqslant 2^k-1}$ is an odd sequence, by Proposition 3.2(3). This implies that the multiplier $(f^{2^k})'(x_k)$ is negative. Since, for each $k \in \{0, \ldots, n-1\}$, the 2^k-periodic orbit is unstable, we conclude that $(f^{2^k})'(x_k) < -1$ and the statement follows. (The same inequality may hold with $k = n$ if the stable periodic orbit has period 2^{n+1}, which is the case when the kneading sequence is J^n.)

3.4.2. *Proof of Theorem 2.4.* Given $(x, y) \in \mathcal{M}$, let $P_1(x, y) = x$ and $P_2(x, y) = y$ denote the projections on each coordinate. Given $k \in \mathbb{N}$, we define the functions

$$h_k(x, y) = P_1 F_\epsilon^{2^k}(x, y) - y \quad \text{and} \quad g_k(x, y) = P_2 F_\epsilon^{2^k}(x, y) - x.$$

The point $(x, y) \in \mathcal{M}$ with $x \neq y$ belongs to a symmetric 2^{k+1}-periodic orbit of F_ϵ if and only if

$$h_k(x, y) = g_k(x, y) = 0. \tag{3.10}$$

By Proposition 3.4, every symmetric 2^{k+1}-periodic orbit must pass the square $S_{\{f^{2^k}(0),\, f^{2^{k+1}}(0)\}}$ ($S_{\{0,\, f^{2^n}(0)\}}$ if $k = n$). Indeed, let $\{(x^t, y^t)\}_{t\in\mathbb{N}}$ be a symmetric 2^{k+1}-periodic orbit and let $j \in \{0, \ldots, n\}$ and $t_0 \in \mathbb{N}$ be integers for which Proposition 3.4 holds. By contradiction, we prove that $j = k$.

If $k < j \leqslant n$, then by Proposition 3.4, the points

$$(x^{2^j t_0},\, y^{2^j t_0}) \quad \text{and} \quad (x^{2^j(t_0+1)},\, y^{2^j(t_0+1)})$$

lie on different sides of the diagonal. However, the assumption $j > k$ implies that 2^j is a multiple of 2^{k+1} and, by periodicity, these points are the same. We have a contradiction.

If $0 \leqslant j < k$, then 2^k is a multiple of 2^{j+1}. Consequently, Proposition 3.4 implies that the points $(x^{2^j t_0},\, y^{2^j t_0})$ and $(x^{2^j t_0+2^k},\, y^{2^j t_0+2^k})$ lie on the same side of

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the diagonal. However, the orbit is supposed to be symmetric and 2^{k+1}-periodic. Therefore, $(x^{2^j t_0+2^k}, y^{2^j t_0+2^k}) = \theta(x^{2^j t_0}, y^{2^j t_0})$ and we also have a contradiction.

Hence, we look for a solution of (3.10) in $S_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}$ (in $S_{\{0, f^{2^n}(0)\}}$ if $k = n$). The idea of the proof is that the equations in (3.10) define C^1-curves with special relations that guarantee the existence of a common solution to both equations. Since $g_k(x, y) = h_k(y, x)$, we concentrate on the analysis of the function h_k.

Recall from Proposition 2.1 that x_k denotes the unique 2^k-periodic point of f between zero and $f^{2^k}(0)$.

PROPOSITION 3.10. *For every $k \in \{0, \dots, n\}$, there exists a connected C^1 curve H_k in $S_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}$ ($S_{\{0, f^{2^n}(0)\}}$ when $k = n$), going through (x_k, x_k) and joining the line segments $x = f^{2^k}(0)$ and $x = f^{2^{k+1}}(0)$ ($x = 0$ and $x = f^{2^n}(0)$ when $k = n$), on which $h_k(x, y) = 0$.*

The proof consists of three steps. We start by proving the existence of solutions of $h_k(x, y) = 0$ in the desired squares. Then we show that these solutions form smooth curves and finally that connected components of these curves join the desired line segments.

Proof. For the sake of clarity, we only provide a proof in the case $k < n$. The proof in the case $k = n$ is the same by replacing $f^{2^k}(0)$ by $f^{2^n}(0)$ and $f^{2^{k+1}}(0)$ by zero.

Existence of solutions. Lemma 3.5(2) with $t = 2^k$ implies that for every $(x, y) \in S_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}$, the iterate $F_\epsilon^{2^k}(x, y) \in S_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}$. (In the case where $k = n$, we use Lemma 3.5(1) to get $F_\epsilon^{2^n}(x, y) \in S_{\{0, f^{2^n}(0)\}}$ for every $(x, y) \in S_{0, \{f^{2^n}(0)\}}$.) Indeed, this is immediate in case (i) and it follows from Proposition 3.8 in case (ii). In other words for every $(x, y) \in S_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}$, we have $P_1 F_\epsilon^{2^k}(x, y) \in I_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}$. By continuity, we conclude that for every $x \in I_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}$, there exists $y \in I_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}$ such that $h_k(x, y) = 0$.

Existence of the C^1-curves. We prove that the partial derivatives of h_k are not simultaneously zero when $h_k(x, y) = 0$ in $S_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}$.

When $k = 0$, we have $(\partial h_0/\partial x)(x, y) = (1 - \epsilon) f'(x)$. When $k \geqslant 1$, by the chain rule, we have

$$\frac{\partial h_k}{\partial x}(x, y) = ((1 - \epsilon) f'(x^{2^k - 1}), \epsilon f'(y^{2^k - 1})) \prod_{i=1}^{2^k - 2} DF_\epsilon(x^i, y^i) \begin{pmatrix} (1 - \epsilon) f'(x) \\ \epsilon f'(x) \end{pmatrix}, \quad (3.11)$$

where $DF_\epsilon(x, y)$ denotes the Jacobian matrix computed at (x, y). By Lemma 3.5(2) and Proposition 3.2(2), the coordinates x^i and y^i, $i \in \{1, \dots, 2^k - 1\}$, are either both positive or both negative for every $(x, y) \in S_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}$. Therefore, all products

$$f'(x) \prod_{i=1}^{2^k - 1} f'(z^i), \quad (3.12)$$

where z^i can be either x^i or y^i, have the same sign. Since the expression of $(\partial h_k/\partial x)(x, y)$ is a linear combination of terms in the form of (3.12) with non-negative coefficients, we

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conclude that $(\partial h_k/\partial x)(x, y) = 0$ in $S_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}$ only if $f'(x) = 0$, i.e. if $x = 0$, by the properties of f.

Now we consider $(\partial h_k/\partial y)(x, y)$ when $x = 0$ and $y \in I_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}$. We have $(\partial h_0/\partial y)(x, y) = \epsilon f'(y) - 1 < 0$ and, when $k \geqslant 1$,

$$\frac{\partial h_k}{\partial y}(x, y) = ((1-\epsilon) f'(x^{2^k-1}), \epsilon f'(y^{2^k-1})) \prod_{i=1}^{2^k-2} DF_\epsilon(x^i, y^i) \begin{pmatrix} \epsilon f'(y) \\ (1-\epsilon) f'(y) \end{pmatrix} - 1. \tag{3.13}$$

If $x = 0$ and $y \in I_{\{0, f^{2^k}(0)\}}$, then the relation (3.2), the piecewise monotonicity of f, and Proposition 3.2(3) imply that all products in the linear combination in (3.13) are non-positive. Consequently, we have $(\partial h_k/\partial y)(0, y) \leqslant -1$ and the two partial derivatives of h_k are not simultaneously zero.

Assume now that $x = 0$ and $y \in I_{\{0, f^{2^{k+1}}(0)\}}$. (Note that this case does not occur when $k = n$.) Since $(x, y) \in S_{\{0, f^{2^{k+1}}(0)\}}$, the relation (3.2) implies that $F_\epsilon^{2^k}(x, y) \in S_{\{f^{2^k}(0), f^{2^k+2^{k+1}}(0)\}}$. However, Propositions 3.8 and 3.2(1) imply that $S_{\{f^{2^k}(0), f^{2^k+2^{k+1}}(0)\}} \cap S_{\{0, f^{2^{k+1}}(0)\}} = \emptyset$. Consequently, there is no solution for $h_k(x, y) = 0$ in $S_{\{0, f^{2^{k+1}}(0)\}}$. Thus, we conclude that the partial derivatives of h_k are not simultaneously zero when $h_k(x, y) = 0$.

By the previous existence of solutions and the implicit function theorem, we conclude that for any point (x_0, y_0) not on the boundary of $S_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}$ and satisfying $h_k(x_0, y_0) = 0$, there is an open neighborhood of (x_0, y_0) such that its intersection with the set $\{(x, y) : h_k(x, y) = 0\}$ is diffeomorphic to an open interval. If (x_0, y_0) is on the boundary of this square, the intersection is diffeomorphic to a semi-closed interval. This proves the existence of a C^1-curve of solutions $h_k(x, y) = 0$ in $S_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}$. Let H_k be the connected component of this curve going through (x_k, x_k).

Existence of curves with desired properties. We may assume that $\epsilon > 0$, the case when $\epsilon = 0$ being trivial. If $\epsilon \in (0, \frac{1}{2}]$, then for any $a, b \in [-1, 1]$ and any $(x, y) \in I_{\{a,b\}}$, $P_1 L_\epsilon(x, y)$, $P_2 L_\epsilon(x, y)$ are interior points of the interval $I_{\{a,b\}}$ unless $x = y = a$ or $x = y = b$.

By the implicit function theorem, the curve H_k extends to the boundary of the square $S_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}$. Since $F_\epsilon^{2^k}(S_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}) \subset S_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}$, we have that $P_1 F_\epsilon^{2^k}(x, y)$ is an interior point of the interval $I_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}$ for every $(x, y) \in S_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}$ unless $x = y = f^{2^k}(0)$ or $x = y = f^{2^{k+1}}(0)$. This implies that the curve H_k can only meet the boundary on the line segments $x = f^{2^k}(0)$ and $x = f^{2^{k+1}}(0)$. By Proposition 2.1(2), the curve crosses the diagonal of $S_{\{f^{2^k}(0), f^{2^{k+1}}(0)\}}$ once at the point (x_k, x_k). Therefore, H_k joins the two line segments. □

By symmetry, we have G_k, the connected smooth curve on which $g_k(x, y) = 0$. G_k passes the point (x_k, x_k) and joins the line segments $y = f^{2^k}(0)$ and $y = f^{2^{k+1}}(0)$ if $k \in \{0, \ldots, n-1\}$ and the line segments $y = 0$ and $y = f^{2^n}(0)$ if $k = n$.

To complete the proof, we now show that, under condition (2.2), the curves H_k and G_k intersect at the point (x_k, x_k) in a way that guarantees an off-diagonal intersection (see the example shown in Figure 1).

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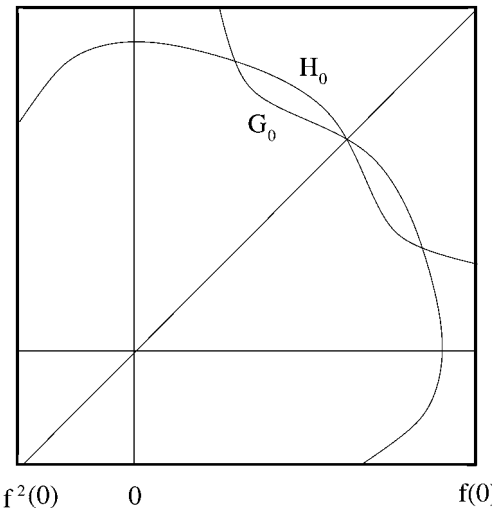


FIGURE 1. The curves H_0 and G_0 when $n \geqslant 1$ and the condition (2.2) is satisfied.

LEMMA 3.11. *If* $(1-2\epsilon)^{2^k}(f^{2^k})'(x_k) < -1$, *then, at the point* (x_k, x_k), *the slope of the tangent line to* H_k *is less than* -1 *and the slope of the tangent line to* G_k *is greater than* -1.

Proof. Formulas (3.11) and (3.13) evaluated at (x_k, x_k) give us

$$\frac{\partial h_k}{\partial x}(x_k, x_k) = (f^{2^k})'(x_k)(1-\epsilon, \epsilon)\begin{pmatrix} 1-\epsilon & \epsilon \\ \epsilon & 1-\epsilon \end{pmatrix}^{2^k-2}\begin{pmatrix} 1-\epsilon \\ \epsilon \end{pmatrix},$$

and

$$\frac{\partial h_k}{\partial y}(x_k, x_k) = (f^{2^k})'(x_k)(1-\epsilon, \epsilon)\begin{pmatrix} 1-\epsilon & \epsilon \\ \epsilon & 1-\epsilon \end{pmatrix}^{2^k-2}\begin{pmatrix} \epsilon \\ 1-\epsilon \end{pmatrix} - 1.$$

We have

$$\begin{pmatrix} 1-\epsilon & \epsilon \\ \epsilon & 1-\epsilon \end{pmatrix} = (1-\epsilon)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \epsilon\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and then

$$\begin{pmatrix} 1-\epsilon & \epsilon \\ \epsilon & 1-\epsilon \end{pmatrix}^{2^k} = \sum_{i=0}^{2^{k-1}} C_{2^k}^{2i}(1-\epsilon)^{2i}\epsilon^{2^k-2i}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{i=1}^{2^{k-1}} C_{2^k}^{2i-1}(1-\epsilon)^{2i-1}\epsilon^{2^k-2i+1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

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The slopes of the tangent lines follow directly from this expression. The slope for H_k is

$$\begin{aligned} S_k :&= -\frac{\partial h_k}{\partial x}(x_k, x_k) \Big/ \frac{\partial h_k}{\partial y}(x_k, x_k) \\ &= \left[-(f^{2^k})'(x_k) \sum_{i=0}^{2^{k-1}} C_{2^k}^{2i} (1-\epsilon)^{2i} \epsilon^{2^k-2i} \right] \\ &\quad \times \left[(f^{2^k})'(x_k) \sum_{i=1}^{2^{k-1}} C_{2^k}^{2i-1} (1-\epsilon)^{2i-1} \epsilon^{2^k-2i+1} - 1 \right]^{-1}. \end{aligned}$$

The slope for G_k is the inverse $1/S_k$. With the denominator of S_k being negative, the condition $S_k < -1$, which leads to the desired conclusion, is equivalent to $(1-2\epsilon)^{2^k} (f^{2^k})'(x_k) < -1$ and the proof is complete. □

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