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A UNIFORM APPROACH TO AXIOMATIZING BUNDLED OPERATORS

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Abstract. Recent years have witnessed extensive logical studies of bundled operators. A known difficulty of such studies is how to axiomatize bundled operators. In this paper, we propose a uniform approach to axiomatizing such operators, which we call 'the method based on almost-definability schemas'. The approach is useful in finding axioms and inference rules, and in defining a suitable canonical relation, thus in the completeness proof of logical systems. This is, hopefully, good news for modal logicians who are interested in axiomatizing bundled operators. To explicate this approach, we choose four bundled operators—the operator N of purely physical necessity, the operator N' called 'All$_1$ and Only$_2$', the operator N'', and the operator N''', in the literature, where $N\varphi := \Box_1\varphi \wedge \neg\Box_2\varphi$, $N'\varphi := \Box_1\varphi \wedge \Box_2\neg\varphi$, $N''\varphi := \Box_1\varphi \vee \neg\Box_2\varphi$, $N'''\varphi := \Box_1\varphi \vee \Box_2\neg\varphi$. This approach can uniformly deal with axiomatizing these bundled operators. Among other contributions, we also answer several open questions, and obtain alternative axiomatizations which are deductively equivalent to the existing ones in the literature.

§1. Introduction. Recent years have witnessed extensive logical studies on bundled operators. Bundled operators are a kind of modal operators that pack several notions together. Where $\varphi(p_1, \dots, p_n)$ is any formula of a logical language $\mathcal{L}$ constructed using at most the propositional variables $p_1, \dots, p_n$, we write $\varphi(\psi_1, \dots, \psi_n)$ to denote the result of replacing every occurrence of each p_i in $\varphi(p_1, \dots, p_n)$ by ψ_i. Define an n-place operator $O_\varphi(\psi_1, \dots, \psi_n) = \varphi(\psi_1, \dots, \psi_n)$, for all $\psi_1, \dots, \psi_n \in \mathcal{L}$. We call O_φ the *bundled operator determined by* the formula φ (i.e., by $\varphi(p_1, \dots, p_n)$). For example, if $\varphi(p)$ is the formula $\Box p \vee \Box\neg p$, then O_φ is the non-contingency operator usually written as Δ (e.g., [33]); if $\varphi(p, q)$ is the formula $\Box(p \to q)$, then O_φ is the strict implication operator usually written as $\prec$; if $\varphi(p, q)$ is the formula $\Diamond p \wedge (\Box(p \to q) \vee \Box(p \to \neg q))$, then O_φ is the (possibility-based) dyadic non-contingency operator $\mathbf{\Delta}$ [16, 36]; if $\varphi(p)$ is the formula $\exists x \Box p$, then O_φ is the operator $\Box^x$ (e.g., [42]). A *bundled operator* is simply the bundled operator determined by some formula.[1]

There are a lot of bundled operators in the literature, for instance, the non-contingency operator $\Delta\varphi := \Box\varphi \vee \Box\neg\varphi$ studied in [19, 20, 26, 30, 33, 46], the accident

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[1] To our knowledge, the idea of 'bundled operators' dates back to [42], but the term 'bundled operators' is from [31], where the authors say "there are many ways to 'bundle' the quantifiers and modalities. We call these the 'bundled operators/modalities'." So they define bundled operators as the operators that bundle the quantifiers and modalities. Obviously, this definition is too narrow. Our definition here generalizes this definition, and is also a generalization of *modality* defined in [47], where only unary modal operators are included.

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operator $\bullet\varphi := \varphi \wedge \neg\Box\varphi$ studied in [9, 32, 39], the 'false belief' operator $W\varphi := \Box\varphi \wedge \neg\varphi$ studied in [15, 18, 24, 40, 43], the 'strong accident' operator $\odot\phi := \varphi \wedge \Box\neg\varphi$ studied in [34], the 'disjunctive ignorance' operator $\nabla\varphi \vee \bullet\varphi$ studied in [12], the 'strong non-contingency' operator $\blacktriangle\varphi := (\varphi \to \Box\varphi) \wedge (\neg\varphi \to \Box\neg\varphi)$ studied in [4, 11], the 'von Wright's deontic necessity' operator $\boxplus\varphi := \Box\varphi \wedge \Diamond\varphi$ studied in [5, 14], the 'true belief' operator $\boxdot\varphi := \Box\varphi \wedge \varphi$ studied in [7, lemma 3.86] (also see [45]), and the (possibility-based) dyadic non-contingency operator $\mathbf{\Delta}(\varphi, \psi) = \Diamond\varphi \wedge (\Box(\varphi \to \psi) \vee \Box(\varphi \to \neg\psi))$ studied in [16, 36], just to name a few. They characterize important notions in philosophy and philosophical logic. Despite being definable with other operators, studying them as first-class citizens has some advantages. First, the logical properties of the bundled operators can be seen more clearly. Second, the bundled operators are more succinctly expressed directly than by unpacking their occurrence into the conventional vocabulary by using the proposed definition (see, e.g., [8]).

However, given the fact that most of the bundled operators are non-normal, a known difficulty of such studies is how to axiomatize this kind of operators, when an operator of this kind is taken as primitive and even more particularly, is taken as the sole non-Boolean primitive. For instance, although the problem of axiomatizing non-contingency logic has been considered since 1960s [33], its axiomatization over symmetric frames has not been solved until [19]. One main reason for this is that the desired core axiom is hard to find due to its complexity, and the other one is that the completeness proof needs novel strategy. As another example, despite much work on the axiomatization of the logic of false belief, the problem of axiomatizing this logic over transitive frames has been thought of as a difficult issue (see [40, sec. 5] and [24, sec. 2.4]), where the author(s) spend a whole section on discussing the issue.

In this paper, we introduce a uniform approach to axiomatizing bundled operators.[2] The approach is called 'the method based on almost-definability schemas' (MBADS for short). The term *almost-definability schema* can be traced back to [19], where it is shown that although the necessity operator $\Box$ is not definable with the non-contingency operator Δ in general, $\Box$ is almost definable in terms of Δ, in that given a contingent proposition, $\Box$ is definable with Δ; in more detail, $\neg\Delta\psi \to (\Box\varphi \leftrightarrow (\Delta\varphi \wedge \Delta(\psi \to \varphi)))$ is a validity. With the help of the almost-definability schema, in addition to other contributions (e.g., the proposal of a suitable bisimulation notion for non-contingency logic), the desired core axiom for symmetric contingency logic is found, and a suitable definition of canonical model is proposed, which helps to uniformly handle completeness proofs of various axiomatizations for this logic.

The approach MBADS can be sketched as follows. Roughly speaking, given any tuples of operators $\langle O_1, \dots, O_m\rangle$ and $\langle O'_1, \dots, O'_n\rangle$, where $O_1, \dots, O_m$ constitute the primitive modalities of a familiar language L, $O'_1, \dots, O'_n$ constitute the primitive modalities of a relatively unfamiliar language L',[3] and every O_i differs from every O'_j, if given certain conditions, for every $i = 1, \dots, m$, O_i is definable with $O'_1, \dots, O'_n$, then we say O_i is almost definable in terms of $O'_1, \dots, O'_n$ (see Definition 1 for the formal definition). We call the so-obtained (frame) validity/validities 'almost-definability schema(s)'. Based on the schema(s), one can find suitable canonical relation and

[2] For the sake of simplicity, and also due to the fact that unary operators are mainly studied in the literature, in what follows, we only focus on the unary operators.

[3] The logical languages are usually finitary, which only contain finitely many primitive modalities.

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desired core axioms for the special systems. This is why we call the approach 'the MBADS'. Compared to other approaches, this approach has incomparable advantages in axiomatizing bundled operators, including seeking out the suitable canonical relation and the desired core axioms and inference rules.

The approach has been applied to many other logics. In addition to the above-mentioned non-contingency logic, examples include the bimodal logic with contingency and accident [10], the logic for disjunctive ignorance [12], the logic of von Wright's deontic necessity [14], the logic of temporal contingency [17], and the logic of false belief [18]. The almost-definability schemas involved in these logics are all valid. In this paper, we systematically propose this approach and extend it to the schemas that are valid on a class of frames, which is not necessarily the class of all frames. The core idea of this approach lies in the definability of familiar operators in terms of relatively unfamiliar ones given certain premises, thus one can 'reduce' the latter operators to the former ones in some way. To illustrate this approach, we choose four operators, among which one is the operator N of purely physical necessity introduced in [3], one is the operator N' called 'All_1 and Only_2' in [25], and another two are N'' and N''' proposed in [28]. The choice is mainly based on two reasons: firstly, the operators are interesting in their own right, where some open questions are raised; secondly, some existing bundled operators can be seen as their special instances, thus their axiomatizations can be obtained easily.

The remainder of the paper is structured as follows. In §2, we introduce the MBADS in a general way. §3 axiomatizes the operator N of purely physical necessity proposed by Bacon in [3] over various frames via MBADS. Among other results, we also axiomatize N over **S4**-frames, thereby partly answering an open question raised by Bacon. Through proper transformation, we also obtain the almost-definability schemas in terms of N'', and then we can also axiomatize N'' over various frame classes via MBADS. As the accident operator in the literature can be seen as a special instance of N, we also obtain some alternative axiomatizations of the logic of accident that are deductively equivalent to the existing ones in the literature. §4 axiomatizes the operator N' of 'All_1 and Only_2'. Among other results, we give a finite minimal axiomatization for N', which is absent in the literature concerning the logic of N'; moreover, we gain a lot of canonicity results, including the transitive axiomatization of the logic of false belief, which has been left open in the literature. §5 axiomatizes the operator N'' proposed in the literature via MBADS, whereas N'' amounts to $\neg N$, we only illustrate this with the minimal logic of N''. §6 axiomatizes the operator N''' proposed in the literature, which has connections with the operator of accident and the operator of non-contingency, and thus gives alternative axiomatizations of the logic concerning the two operators, which are deductively equivalent to the existing ones in the literature. §7 concludes with some remarks.

It is worth noting that the completeness results of §3–§6 immediately yield axiomatizations of various unimodal fragments of extensions of the bimodal logic $\mathbf{K}_2$ given by the operators N, N', N'', N''' defined as

$$\begin{array}{lcl} N\varphi & := & \Box_1\varphi \wedge \neg\Box_2\varphi, \\ N'\varphi & := & \Box_1\varphi \wedge \Box_2\neg\varphi, \\ N''\varphi & := & \Box_1\varphi \vee \neg\Box_2\varphi, \\ N'''\varphi & := & \Box_1\varphi \vee \Box_2\neg\varphi, \end{array}$$

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where $\Box_1, \Box_2$ are the two operators of $\mathbf{K}_2$ (see, e.g., [22, pp. 20–22]). Moreover, the finite model property and decidability are known for many developing systems of $\mathcal{L}(\Box_1, \Box_2)$, where $\mathcal{L}(\Box_1, \Box_2)$ extends the language of propositional logic with $\Box_1$ and $\Box_2$ as primitive modalities (see, e.g., [22]). These properties are inherited by the considered fragments which has N (N', N'', N''', resp.) as the sole primitive modality.

§2. The method based on almost-definability schemas. Given any unary operators $O_1, \dots, O_m$ and $O'_1, \dots, O'_n$ with some semantics, where $O_1, \dots, O_m$ constitute the primitive modalities of a familiar language L, $O'_1, \dots, O'_n$ constitute the primitive modalities of a relatively unfamiliar language L', and every O_i differs from every O'_j, if given certain conditions, for every $i = 1, \dots, m$, O_i is definable with $O'_1, \dots, O'_n$, then we say O_i is *almost definable* in terms of $O'_1, \dots, O'_n$. More specifically, we have the following.

DEFINITION 1. *Let $i \in \{1, \dots, m\}$. For a nonempty class $F = \langle W, R_1, \dots, R_m \rangle$ of L-frames and each L-modality O_i that is interpreted with R_i, we say that O_i is* almost definable in terms of $O'_1, \dots, O'_n$ over F, *if there are L'-formulas $FOR_i(\psi)$ and $FOR'_i(\varphi, \psi)$ as described below, for which we have:*[4] *for all $\varphi, \psi \in L'$,*

$$F \vDash FOR_i(\psi) \to (O_i\varphi \leftrightarrow FOR'_i(\varphi, \psi)).$$

Here $FOR_i(\psi)$ and $FOR'_i(\varphi, \psi)$ are L'-formulas which are constructed from ψ and from φ, ψ, respectively, using the Boolean connectives and modal operators $O'_1, \dots, O'_n$. If the relevant class F of frames is clear from the context (for example, as the class of all L-frames), we simply say that O_i is almost definable in terms of $O'_1, \dots, O'_n$, when O_i is almost definable in terms of $O_1, \dots, O_n$ over F.

As we will illustrate, the almost definability schemas are very useful in our axiomatizations. First, they help us find core axioms for special frames; second, they help us define suitable canonical relations in the canonical model, which can handle the completeness proofs of more proof systems than those in the literature.

In more detail, once we find the almost definability schemas of O_i in L', we can construct the canonical model for the minimal logic of L', where the canonical relations (denoted R^c_i for all $i = 1, \dots, m$) can be defined as follows, which is inspired by the almost definability schemas.

- If $FOR_i(\psi) \in w$ for *no* ψ, then $wR^c_i u$ iff $w = u$.
- If $FOR_i(\psi) \in w$ for some ψ, then $wR^c_i u$ is defined as in the canonical model for the minimal logic of L except that $O_i\varphi$ is replaced by $FOR'_i(\varphi, \psi)$.

Note that the choice of ψ in the above second case is inessential to R^c, and it matters that the ψ in $FOR'_i(\varphi, \psi)$ is required to be the same as that in $FOR_i(\psi)$. It is also worth noting that the second case of R^c is inspired by Almost Definability Schema in Definition 1. Specifically, given $FOR_i(\psi) \in w$ for some ψ, we replace $O_i\varphi$ with $FOR'_i(\varphi, \psi)$; otherwise, w can and only can access itself.

It is note remarking that instead of using the second item (denoted $(*)$) of the above definition of R^c_i, we can also use the following condition (denoted $(**)$):

[4] Here $F \vDash \varphi$ means that φ is valid over the class F. If F is the class of all frames, then we say that φ is valid. Note that we need to exclude the case that $FOR_i(\psi)$ is unsatisfiable in F, since it offers no help in defining the canonical relations.

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- Otherwise, for every ψ, if $FOR_i(\psi) \in w$, then $wR_i^c u$ is defined as in the canonical model for the minimal logic of L except that $O_i\varphi$ is replaced by $FOR'_i(\varphi, \psi)$.

This is because $(*)$ and $(**)$ are equivalent. The implication from $(**)$ to $(*)$ is immediate, and the converse also holds since the ψ in $FOR'_i(\varphi, \psi)$ is the same as that in $FOR_i(\psi)$. We prefer using $(*)$ due to its simplicity.

Then we can obtain the axioms and inference rules of the minimal system for L', according to the need of showing that the truth lemma holds and that the canonical model belongs to F in the completeness proof.[5]

§3. The operator of purely physical necessity. In [6], Arthur Burks establishes a bimodal system of logical necessity and physical necessity (or "causal necessity", as he calls it). According to this system, logical necessity implies physical necessity (in symbols, $\Box\varphi \rightarrow \boxed{\text{c}}\varphi$, where $\Box$ and $\boxed{\text{c}}$ are operators of logical necessity and physical necessity, respectively), but not vice versa, and both $\Box$ and $\boxed{\text{c}}$ are T necessity operators (namely, satisfy the conditions of K-axiom (namely, $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$ and $\boxed{\text{c}}(\varphi \rightarrow \psi) \rightarrow (\boxed{\text{c}}\varphi \rightarrow \boxed{\text{c}}\psi)$) and T-axiom (namely, $\Box\varphi \rightarrow \varphi$ and $\boxed{\text{c}}\varphi \rightarrow \varphi$) and necessitation rule). Kripke semantics for such bimodal systems have been given in the literature (e.g., [21, 37]).

Partly inspired by this, and partly inspired by Armstrong's idea that the laws of nature are logically contingent [2, p. 79], Bacon [3] proposed a notion of *purely physical necessity*, expressed by the operator N. According to his definition, a proposition is a law of nature in the sense of logical contingency, if it is physical but not logically necessary; in symbols, $N\varphi := \boxed{\text{c}}\varphi \wedge \neg\Box\varphi$. Then Bacon axiomatizes the logic which has purely physical necessity as the sole modality over the class of bi-relational frames with two accessibility relations R and S, R for logical necessity and S for physical necessity, such that $S \subseteq R$, and $\Box$ and $\boxed{\text{c}}$ are interpreted in terms of R and S in the usual way. Following Burks's idea, Bacon also assumes that R and S are reflexive, and axiomatizes this notion of necessity over such bi-relational frames. As Bacon noted, since N incorporates aspects of both logical and physical necessity, the axiomatization becomes tricky; in particular, he mentioned the difficulty of guaranteeing the transitivity of R in the completeness proof [3, p. 140]. As an open question, Bacon wonders what the **S4**- and **S5**-like extensions of the logic are.

The applications of $\boxed{\text{c}}\varphi \wedge \neg\Box\varphi$ to deontic logic are suggested in Jones and Pörn [29] and Åqvist [1]. In [29], to block various monotonicity-related paradoxes of deontic logic and implement the idea that genuine obligations are inherently susceptible of being violated, an Ought operator is introduced, defined as Ought $\varphi := O\varphi \wedge \neg O'\varphi$, where O and O' amount to the above $\boxed{\text{c}}$ and $\Box$, respectively, with different semantic restrictions to R and S.[6] In [1], a notion of supererogation is proposed, defined as Sup$\varphi :=$ Om-1$\varphi \wedge \neg$O1φ, where Om-1φ and O1φ, respectively, mean that φ is true

[5] In what follows, however, for readability, after we find the desired almost definability schemas, we first exhibit the axiomatizations and then show their soundness and completeness. But the reader can check that the axiomatizations can be indeed obtained via our approach MBADS.

[6] In details, both R and S are serial, $R \cap S = \emptyset$, and for every x in the model, either xRx or xSx.

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at all best worlds and that φ is true at all morally permissible worlds, so that "it is supererogatory that φ" means that φ is true at all best worlds but not at all morally permissible worlds.

3.1. Syntax and semantics. Throughout the paper, we assume **Prop** to be a nonempty set of propositional variables. We introduce the language of Bacon's logic of purely physical necessity. For the sake of references, we also introduce the language of the bimodal logic of logical necessity and physical necessity.

DEFINITION 2. *The language* $\mathcal{L}(\Box, \boxed{\mathrm{c}}, N)$ *is defined recursively as follows*:

$$\varphi ::= \quad p \mid \neg p \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid \boxed{\mathrm{c}}\varphi \mid N\varphi,$$

where $p \in$ ***Prop***.

Without the modal construct $N\varphi$, *we obtain the language* $\mathcal{L}(\Box, \boxed{\mathrm{c}})$ *of the bimodal logic of logical necessity and physical necessity, which is a fragment of Burks's logic of causal propositions without first-order quantifiers in* [6]; *with the modal construct* $N\varphi$ *only, we obtain* the language $\mathcal{L}(N)$ of Bacon's purely physical necessity logic. *In what follows, we will mainly focus on* $\mathcal{L}(N)$.

The language $\mathcal{L}(\Box, \boxed{\mathrm{c}}, N)$ is interpreted on models. A *model* is a tuple $\mathcal{M} = \langle W, S, R, V\rangle$, where W is a nonempty set of possible worlds, S and R are, respectively, physical and logical accessibility relations, and V : **Prop** $\to \mathcal{P}(W)$ is a valuation. Given $w \in W$, $S(w) = \{u \in W \mid wSu\}$ and $R(w) = \{u \in W \mid wRu\}$. A frame is a model without valuations. A pointed model is a pair consisting of a model and a world within it. We say that $\mathcal{M}$ is a P-model if both S and R have the property P. We say that a frame $\mathcal{F}$ is a P-frame if all models based on that frame are P-models. In particular, throughout the paper, we use **T**, **4**, and **S4** for the class of reflexive frames, the class of transitive frames, and the class of reflexive and transitive frames, respectively.

Note that, unlike Bacon, we do not assume that S and R are reflexive. This is because it is debatable which first-order properties the two accessibility relations should possess. For instance, despite following Burks's idea that both relations are reflexive (and $S \subseteq R$) without further restrictions, Bacon himself thinks that S and R are better to have **T** and **S5** structures, which means that S is reflexive and R is an equivalence relation. Moreover, as noted on page 228 of [28], the reflexivity of accessibility relation S for physical necessity (or 'nomic necessity', as Humberstone calls it), is disputable in the literature (see also Remark 4.2.1, p. 217 and Note 177, p. 219 and the references thereof). Rather, we will only assume that S is a subset of R. This is quite natural, given that logical necessity implies physical necessity; in other words, whatever is physically possible is also logically possible.

DEFINITION 3. *Given a model* $\mathcal{M} = \langle W, S, R, V\rangle$ *and a world* $w \in W$, *the semantics of* $\mathcal{L}(\Box, \boxed{\mathrm{c}}, N)$ *is defined recursively as follows*:

$$\begin{array}{lcl}
\mathcal{M}, w \vDash p & \iff & w \in V(p) \\
\mathcal{M}, w \vDash \neg\varphi & \iff & \mathcal{M}, w \nvDash \varphi \\
\mathcal{M}, w \vDash \varphi \wedge \psi & \iff & \mathcal{M}, w \vDash \varphi \text{ and } \mathcal{M}, w \vDash \psi \\
\mathcal{M}, w \vDash \Box\varphi & \iff & R(w) \vDash \varphi \\
\mathcal{M}, w \vDash \boxed{\mathrm{c}}\varphi & \iff & S(w) \vDash \varphi \\
\mathcal{M}, w \vDash N\varphi & \iff & S(w) \vDash \varphi \text{ and } R(w) \nvDash \varphi,
\end{array}$$

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where $R(w) \vDash \varphi$ *means that for all* $u \in R(w)$, $\mathcal{M}, u \vDash \varphi$, *whereas* $R(w) \nvDash \varphi$ *means that it is* not *the case that* $R(w) \vDash \varphi$; *similarly for* $S(w) \vDash \varphi$ *and* $S(w) \nvDash \varphi$.

If $\mathcal{M}, w \vDash \varphi$, *we say that* φ *is* true *in* $(\mathcal{M}, w)$; *if for all* $w \in W$ *we have* $\mathcal{M}, w \vDash \varphi$, *we say that* φ *is* valid on $\mathcal{M}$ *and write* $\mathcal{M} \vDash \varphi$; *if for all models* $\mathcal{M}$ *based on a frame* $\mathcal{F}$ *we have* $\mathcal{M} \vDash \varphi$, *we say that* φ *is* valid on $\mathcal{F}$ *and write* $\mathcal{F} \vDash \varphi$; *if for all frames* $\mathcal{F}$ *in a class of frames* F *we have* $\mathcal{F} \vDash \varphi$, *we say* φ *is* valid over F *and write* $F \vDash \varphi$, *when* F *is the class of all frames, we say* φ *is* valid *and write* $\vDash \varphi$.

Observe that there are two almost definability schemas—one for $\boxed{\mathrm{c}}$, and the other for $\Box$—in terms of N. Intuitively, they say that both physical necessity and logical necessity are definable with purely physical necessity, provided that there is a purely physically necessary proposition. They guide us to propose the desired canonical model later:

$$N\psi \to (\boxed{\mathrm{c}}\varphi \leftrightarrow N(\varphi \wedge \psi)) \qquad (\mathbf{NAD1})$$
$$N\psi \to (\Box\varphi \leftrightarrow \neg N\varphi \wedge N(\varphi \wedge \psi)) \qquad (\mathbf{NAD2}).$$

The following result shows that the two schemas are indeed valid (over the class of bi-relational frames). Note that the proof of (1) does not need any semantic restriction of accessibility relations, whereas (2) only uses $S \subseteq R$.

PROPOSITION 4. *Let* $\mathcal{F} = \langle W, S, R\rangle$ *such that* $S \subseteq R$.

(1) $\vDash N\psi \to (\boxed{\mathrm{c}}\varphi \leftrightarrow N(\varphi \wedge \psi))$.
(2) $\mathcal{F} \vDash N\psi \to (\Box\varphi \leftrightarrow \neg N\varphi \wedge N(\varphi \wedge \psi))$.

Proof. Let $\mathcal{M} = \langle W, S, R, V\rangle$ be a model and $w \in W$.

For (1), suppose that $\mathcal{M}, w \vDash N\psi$, to show that $\mathcal{M}, w \vDash \boxed{\mathrm{c}}\varphi \leftrightarrow N(\varphi \wedge \psi)$. By semantics, it is straightforward to see that $\vDash N(\varphi \wedge \psi) \to \boxed{\mathrm{c}}\varphi$. It remains only to show that $\mathcal{M}, w \vDash \boxed{\mathrm{c}}\varphi \to N(\varphi \wedge \psi)$. For this, assume that $\mathcal{M}, w \vDash \boxed{\mathrm{c}}\varphi$. Then $S(w) \vDash \varphi$. By supposition, we have $S(w) \vDash \psi$ and $R(w) \nvDash \psi$. This entails that $S(w) \vDash \varphi \wedge \psi$ and $R(w) \nvDash \varphi \wedge \psi$, and therefore $\mathcal{M}, w \vDash N(\varphi \wedge \psi)$.

For (2), assume that $\mathcal{M}, w \vDash N\psi$, to prove that $\mathcal{M}, w \vDash \Box\varphi \leftrightarrow \neg N\varphi \wedge N(\varphi \wedge \psi)$. First, suppose that $\mathcal{M}, w \vDash \neg N\varphi$ and $\mathcal{M}, w \vDash N(\varphi \wedge \psi)$. From the latter it follows that $S(w) \vDash \varphi \wedge \psi$, and then $S(w) \vDash \varphi$. This plus $\mathcal{M}, w \vDash \neg N\varphi$ implies that $R(w) \vDash \varphi$, and thus $\mathcal{M}, w \vDash \Box\varphi$. Conversely, suppose that $\mathcal{M}, w \vDash \Box\varphi$, then $R(w) \vDash \varphi$, thus $\mathcal{M}, w \vDash \neg N\varphi$. Since $S \subseteq R$, it follows that $S(w) \vDash \varphi$. Then similarly to the proof of (1), we can show that $\mathcal{M}, w \vDash N(\varphi \wedge \psi)$, as desired. □

REMARK 5. *In* [28, *p.* 229], *Humberstone also proposes several variations of* N, *namely* N', N'', N'''. *Among these variations,* $N''\varphi$ *is defined as* $\Box_1\varphi \vee \neg\Box_2\varphi$. *Semantically, given a model* $\mathcal{M} = \langle W, R_1, R_2, V\rangle$ *and* $w \in W$,

$$\mathcal{M}, w \vDash N''\varphi \iff R_1(w) \vDash \varphi \text{ or } R_2(w) \nvDash \varphi.$$

Thus $\neg N''\varphi := \Box_2\varphi \wedge \neg\Box_1\varphi$. *If we think as* $\Box_2$ *and* $\Box_1$ *as* $\boxed{\mathrm{c}}$ *and* $\Box$, *respectively, then* $\neg N''$ *amounts to* N. *Based on this observation, we can obtain two almost definability schemas in terms of* N'', *one for* $\Box_2$ *and the other for* $\Box_1$:

$$\neg N''\psi \to (\Box_1\varphi \leftrightarrow (N''\varphi \wedge \neg N''(\varphi \wedge \psi))) \qquad (\boldsymbol{N''AD1})$$
$$\neg N''\psi \to (\Box_2\varphi \leftrightarrow \neg N''(\varphi \wedge \psi)) \qquad (\boldsymbol{N''AD2}).$$

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And we can show the following:

(1) *Let* $\mathcal{F} = \langle W, R_1, R_2\rangle$ *such that* $R_2 \subseteq R_1$. *Then*

$$\mathcal{F} \vDash \neg N''\psi \to (\Box_1\varphi \leftrightarrow (N''\varphi \wedge \neg N''(\varphi \wedge \psi))).$$

(2) $\vDash \neg N''\psi \to (\Box_2\varphi \leftrightarrow \neg N''(\varphi \wedge \psi))$.

Then we can also axiomatize N'' *over various frame classes via MBADS. We will illustrate this by axiomatizing the minimal logic of* N'' *in* §5.

3.2. Bacon's proof system. First, we review Bacon's proof system in [3, p. 136] (also see [28, p. 228]), named **TN**,[7] which consists of the following axioms and inference rules:[8]

PC	All instances of propositional tautologies
N1	$N\varphi \to \varphi$
N2	$N\varphi \wedge N\psi \to N(\varphi \wedge \psi)$
N3	$N((\varphi \to \psi) \to \psi) \to N((\psi \to \varphi) \to \varphi)$
N4	$\neg N(\varphi \to \psi) \wedge N\psi \to (N\chi \to N((\varphi \to \psi) \to \chi))$
MP	From φ and $\varphi \to \psi$ infer ψ
N5	From φ infer $\neg N\varphi$
N6	From $\varphi \to \psi$ infer $N(\chi \to \psi) \to (N\varphi \to N\psi)$.

As mentioned before, axiom N1 is disputable. In a footnote appended to N3 [3, fn. 1, p. 136], Bacon says: "I suspect that N3 may be redundant." As Humberstone [28, p. 228] noted, this axiom would be redundant if one has a 'rule of congruentiality' for N, denoted as T3 by Bacon and as (NCong) by Humberstone, namely $\dfrac{\varphi \leftrightarrow \psi}{N\varphi \leftrightarrow N\psi}$, which is derivable without use of N3, since then N3 can be simplified to $N(\varphi \vee \psi) \to N(\psi \vee \varphi)$, which follows easily from the rule of congruentiality. However, as Bacon shows, the derivation of T3 does make use of N3.

Bacon then shows that **TN** is sound and strongly complete with respect to the class of all frames with the requirement that $S \subseteq R$ and S be reflexive (thus R be reflexive as well). The proof is via the Henkin-style canonical model method. However, it is not very clear what the canonical relations are. This was not done until Humberstone [28, p. 232], where purely physical necessity is also called 'purely nomic necessity', and $\boxed{\text{c}}$ and $\Box$ are replaced with $\Box_1$ and $\Box_2$, respectively. Based on Bacon's proof, Humberstone [28, p. 232] explicitly defines the canonical relations S^c and R^c as follows, where x, y are maximal consistent sets:

- wS^cu iff for all φ, if $N\varphi \in w$, then $\varphi \in u$;
- wR^cu iff for all φ, ψ, if $\neg N(\varphi \vee \psi) \wedge N\psi \in w$, then $\varphi \vee \psi \in u$.

Nevertheless, it is still unclear how S^c and R^c come about. Moreover, it is difficult to obtain the **S4** extension of **TN**, given that S^c is not transitive, which, in turn, is due

[7] In fact, Bacon [3] mixed the use of **TN** as a language (e.g., "Neither $\Box$ nor $\boxed{\text{c}}$ is definable in **TN**") and as a proof system (e.g., "**TN**-deducibility and -consistency are now defined in the usual way"). Here we use $\mathcal{L}(N)$ for the language and **TN** for the proof system.

[8] There may be some confusions in reading Bacon's original proof system **TN** due to the lack of necessary parentheses, but it should be clear from its use in the proof of the derivation of T1–T6 therein (see [3, pp. 136–137]).

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to the fact that $N\varphi \to NN\varphi$ is not valid over the class of **S4**-frames. Furthermore, to the best of our knowledge, the open question raised by Bacon, that is, what the **S4**- and **S5**-like extensions of **TN** are, has not been solved yet. In what follows, in addition to our new proof systems, we will also give different definitions for S^c and for R^c, which are inspired by the aforementioned two almost definability schemas (**NAD1**) and (**NAD2**). The new definitions can also handle the **S4**-extension of **TN**.

3.3. Our systems.

3.3.1. Minimal system and soundness.

DEFINITION 6. *The minimal system of* $\mathcal{L}(N)$, *denoted* $\mathbf{K^N}$, *consists of the following axioms and inference rules*:

PC	*All instances of propositional tautologies*
*A*1	$N\varphi \wedge N\psi \to N(\varphi \wedge \psi)$
*A*2	$N(\varphi \wedge \psi) \to N\varphi \vee N\psi$
MP	*From* φ *and* $\varphi \to \psi$ *infer* ψ
*R*1	*From* φ *infer* $\neg N\varphi$
*R*2	*From* $\varphi \to \psi$ *and* $\psi \to \chi$ *infer* $(N\varphi \wedge N\chi) \to N\psi$.

The above axioms and inference rules are obtained based on the need of showing that the truth lemma (Lemma 13) holds and that the canonical model is as desired. One may check that our $\mathbf{K^N}$ is simpler than Bacon's **TN**. Also, we do not have the debatable axiom N1. The rule R2 says that the operator N has an important property called 'Convexity' in [27, p. 548] and 'Interpolation' in [44, prop. 27].

Notions of derivation and theorems are defined as usual. The following proposition is important in the proof later.

PROPOSITION 7.

(1) *If* $\vdash \varphi \to \chi$, *then* $\vdash N(\varphi \wedge \psi) \wedge N\chi \to N\varphi$.
(2) *If* $\vdash \varphi \to \chi$, *then* $\vdash N(\varphi \wedge \psi) \wedge N\psi \to N(\chi \wedge \psi)$.
(3) *If* $\vdash \varphi \to \chi$, *then* $\vdash \neg N\varphi \wedge N(\varphi \wedge \psi) \to N(\chi \wedge \psi)$.
(4) *If* $\vdash \varphi \leftrightarrow \psi$, *then* $\vdash N\varphi \leftrightarrow N\psi$.

Proof. (1) Suppose that $\vdash \varphi \to \chi$. By axiom PC, $\vdash \varphi \wedge \psi \to \varphi$. Then using rule R2, we infer that $\vdash N(\varphi \wedge \psi) \wedge N\chi \to N\varphi$.

(2) Suppose that $\vdash \varphi \to \chi$. Then $\vdash \varphi \wedge \psi \to \chi \wedge \psi$. By axiom PC, $\vdash \chi \wedge \psi \to \psi$. Now using rule R2, we derive that $\vdash N(\varphi \wedge \psi) \wedge N\psi \to N(\chi \wedge \psi)$.

(3) Assume that $\vdash \varphi \to \chi$. By (2), $\vdash N(\varphi \wedge \psi) \wedge N\psi \to N(\chi \wedge \psi)$. Then by axiom A2, $\vdash N(\varphi \wedge \psi) \wedge \neg N\varphi \to N\psi$. Therefore, $\vdash \neg N\varphi \wedge N(\varphi \wedge \psi) \to N(\chi \wedge \psi)$.

(4) Suppose that $\vdash \varphi \leftrightarrow \psi$. Then $\vdash \varphi \to \psi$ and $\vdash \psi \to \varphi$. Then by R2, $\vdash N\varphi \wedge N\varphi \to N\psi$, that is, $\vdash N\varphi \to N\psi$. Similarly, we can show that $\vdash N\psi \to N\varphi$. Therefore, $\vdash N\varphi \leftrightarrow N\psi$. □

PROPOSITION 8. $\mathbf{K^N}$ *is sound with respect to the class of all frames.*

Proof. We only show the validity of axioms A1 and A2 and the validity-preserving of rules R1 and R2, since others are straightforward. Let $\mathcal{M} = \langle W, S, R, V \rangle$ be a model and $w \in W$.

For A1: suppose that $\mathcal{M}, w \vDash N\varphi$ and $\mathcal{M}, w \vDash N\psi$. Then $S(w) \vDash \varphi$ and $R(w) \nvDash \varphi$, and $S(w) \vDash \psi$. This implies that $S(w) \vDash \varphi \wedge \psi$ and $R(w) \nvDash \varphi \wedge \psi$. Therefore, $\mathcal{M}, w \vDash N(\varphi \wedge \psi)$.

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For A2: assume that $\mathcal{M}, w \vDash N(\varphi \wedge \psi)$. Then $S(w) \vDash \varphi \wedge \psi$ and $R(w) \nvDash \varphi \wedge \psi$. This entails that $S(w) \vDash \varphi$ and $S(w) \vDash \psi$, and either $R(w) \nvDash \varphi$ or $R(w) \nvDash \psi$. Therefore, $\mathcal{M}, w \vDash N\varphi \vee N\psi$.

For R1: suppose that $\vDash \varphi$, then for all models $\mathcal{M} = \langle W, S, R, V \rangle$ and $w \in W$, we have $R(w) \vDash \varphi$, which implies that $\mathcal{M}, w \vDash \neg N\varphi$. Thus $\vDash \neg N\varphi$.

For R2: assume that $\vDash \varphi \rightarrow \psi$ and $\vDash \psi \rightarrow \chi$. Suppose that $\mathcal{M}, w \vDash N\varphi \wedge N\chi$, to show that $\mathcal{M}, w \vDash N\psi$. From the supposition, it follows that $S(w) \vDash \varphi$ and $R(w) \nvDash \chi$. Then by assumption, we get $S(w) \vDash \psi$ and $R(w) \nvDash \psi$. Thus $\mathcal{M}, w \vDash N\psi$, as desired. □

3.3.2. Completeness. We proceed with the completeness of $\mathbf{K^N}$. This is shown via a construction of the canonical model, in which the crucial part is the definition of the canonical relations.

DEFINITION 9. *The* canonical model for $\mathbf{K^N}$ *is a tuple* $\mathcal{M}^c = \langle W^c, S^c, R^c, V^c \rangle$, *where*

- $W^c = \{w \mid w$ *is a maximal consistent set for* $\mathbf{K^N}\}$;
- *if* $N\psi \in w$ *for no* ψ, *then* wS^cu *iff* $w = u$, *and* wR^cu *iff* $w = u$;
- *if* $N\psi \in w$ *for some* ψ, *then* wS^cu *iff for all* φ, *if* $N(\varphi \wedge \psi) \in w$, *then* $\varphi \in u$, *and* wR^cu *iff for all* φ, *if* $\neg N\varphi \wedge N(\varphi \wedge \psi) \in w$, *then* $\varphi \in u$;
- $V^c(p) = \{w \in W^c \mid p \in w\}$.

It is required that the ψ in $N(\varphi \wedge \psi) \in w$ and $\neg N\varphi \wedge N(\varphi \wedge \psi) \in w$ be the same as that in $N\psi \in w$. Note that the choice of ψ in the above second case of the canonical relations is inessential to S^c and R^c (see Remark 10 below). It is also noteworthy that S^c and R^c are inspired by almost definability schemas (**NAD1**) and (**NAD2**), respectively, based on the analysis of §2. Recall that in the construction of the canonical model for the minimal logic of $\mathcal{L}(\Box, \boxed{c})$, the canonical relations S^c and R^c are defined as, respectively, wS^cu iff for all φ, if $\boxed{c}\varphi \in w$ then $\varphi \in u$, and wR^cu iff for all φ, if $\Box\varphi \in w$ then $\varphi \in u$.[9] Now according to (**NAD1**), given a precondition that $N\psi \in w$ for some ψ, $\boxed{c}\varphi \in w$ can be replaced with $N(\varphi \wedge \psi) \in w$, and according to (**NAD2**), $\Box\varphi \in w$ can be replaced with $\neg N\varphi \wedge N(\varphi \wedge \psi) \in w$; otherwise, we let w access and only access itself.

REMARK 10. *We noted that the choice of* ψ *in the above second case of the canonical relations is inessential to* S^c *and* R^c. *This means that, if* $N\psi_1 \in w$ *and* $N\psi_2 \in w$, *then it does not matter whether we choose* $\psi = \psi_1$ *or* $\psi = \psi_2$. *This is because* $N\psi \wedge N(\varphi \wedge \chi) \rightarrow N(\varphi \wedge \psi)$ *is derivable in* $\mathbf{K^N}$. *To see the derivability, note that by axiom A*1, $\vdash N\psi \wedge N(\varphi \wedge \chi) \rightarrow N(\varphi \wedge \psi \wedge \chi)$. *Since* $\vdash \varphi \wedge \psi \wedge \chi \rightarrow \varphi \wedge \psi$ *and* $\vdash \varphi \wedge \psi \rightarrow \psi$, *by rule R*2, $\vdash N(\varphi \wedge \psi \wedge \chi) \wedge N\psi \rightarrow N(\varphi \wedge \psi)$. *Therefore,* $\vdash N\psi \wedge N(\varphi \wedge \chi) \rightarrow N(\varphi \wedge \psi)$. *Given the precondition that* $N\psi \in w$ *for some* ψ, *the above second case for* wS^cu *is given by the following* (*denoted* (1)):

$$\textit{for all } \varphi, \textit{ if } N(\varphi \wedge \psi) \in w, \textit{ then } \varphi \in u.$$

[9] The minimal logic of $\mathcal{L}(\Box, \boxed{c})$ can be obtained from a fragment of Burks's logic of causal propositions without first-order quantifiers in [6] by deleting the axioms $\Box\varphi \rightarrow \boxed{c}\varphi$ and $\boxed{c}\varphi \rightarrow \varphi$, though no canonical relations are defined and no completeness proof is attempted therein (ibid., 381). These omissions are made good in [21, sec. 4], where $\Box$ and $\boxed{c}$ are replaced with L for the S5 operator and $\Box$ for the S4 operator, respectively.

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Now assume that for some ψ_1, ψ_2 *we have* $N\psi_1, N\psi_2 \in w$, *and* (1) *holds for the case* $\psi = \psi_1$. *Using the derivable formula in question, we can show that* (1) *also holds for the case* $\psi = \psi_2$. *The converse is similarly shown. This shows that* S^c *does not depend on the instance of* ψ.

To see that R^c *does not depend on the instance of* ψ *either, note that from the derivable formula in question, we can derive that* $\vdash \neg N\varphi \wedge N\psi \wedge N(\varphi \wedge \chi) \rightarrow \neg N\varphi \wedge N(\varphi \wedge \psi)$. *Then the proof goes as in the case of* S^c.

Propositions 11 and 12 list the logical properties of the functions λ_1 and λ_2, respectively. They are important in the proof of Lemma 13 below.

PROPOSITION 11. *Let* $w \in W^c$ *such that* $N\psi \in w$. *Define* $\lambda_1(w) = \{\chi \mid N(\chi \wedge \psi) \in w\}$. *Then*

(1) $\lambda_1(w) \neq \emptyset$.
(2) *If* $\varphi, \chi \in \lambda_1(w)$, *then* $\varphi \wedge \chi \in \lambda_1(w)$.
(3) *If* $\varphi \in \lambda_1(w)$ *and* $\vdash \varphi \rightarrow \chi$, *then* $\chi \in \lambda_1(w)$.

Proof. Suppose that $N\psi \in w$, where $w \in W^c$.

(1) By supposition and Proposition 7(4), we have $N(\top \wedge \psi) \in w$, thus $\top \in \lambda_1(w)$. Hence $\lambda_1(w) \neq \emptyset$.
(2) Assume that $\varphi \in \lambda_1(w)$ and $\chi \in \lambda_1(w)$, to show that $\varphi \wedge \chi \in \lambda_1(w)$. By assumption, $N(\varphi \wedge \psi) \in w$ and $N(\chi \wedge \psi) \in w$. Then using axiom A1 and Proposition 7(4), we infer that $N(\varphi \wedge \chi \wedge \psi) \in w$. Therefore, $\varphi \wedge \chi \in \lambda_1(w)$.
(3) Suppose that $\varphi \in \lambda_1(w)$ and $\vdash \varphi \rightarrow \chi$. From $\varphi \in \lambda_1(w)$, it follows that $N(\varphi \wedge \psi) \in w$. Then using supposition and Proposition 7(2), we derive that $N(\chi \wedge \psi) \in w$, and therefore $\chi \in \lambda_1(w)$. □

PROPOSITION 12. *Let* $w \in W^c$ *such that* $N\varphi \in w$. *Define* $\lambda_2(w) = \{\chi \mid \neg N\chi \wedge N(\chi \wedge \varphi) \in w\}$. *Then*

(1) $\lambda_2(w) \neq \emptyset$.
(2) *If* $\psi, \chi \in \lambda_2(w)$, *then* $\psi \wedge \chi \in \lambda_2(w)$.
(3) *If* $\psi \in \lambda_2(w)$ *and* $\vdash \psi \rightarrow \chi$, *then* $\chi \in \lambda_2(w)$.

Proof. Assume that $N\varphi \in w$, where $w \in W^c$.

(1) We claim that $\top \in \lambda_2(w)$. Since $\vdash \top$, by rule R1, $\vdash \neg N\top$, thus $\neg N\top \in w$; moreover, from the assumption, it follows that $N(\top \wedge \varphi) \in w$. Therefore, $\neg N\top \wedge N(\top \wedge \varphi) \in w$.
(2) Suppose that $\psi \in \lambda_2(w)$ and $\chi \in \lambda_2(w)$. Then $\neg N\psi \wedge N(\psi \wedge \varphi) \in w$ and $\neg N\chi \wedge N(\chi \wedge \varphi) \in w$. From $\neg N\psi \in w$ and $\neg N\chi \in w$ and axiom A2, it follows that $\neg N(\psi \wedge \chi) \in w$; from $N(\psi \wedge \varphi) \in w$ and $N(\chi \wedge \varphi) \in w$ and axiom A1, it follows that $N(\psi \wedge \chi \wedge \varphi) \in w$. Therefore, $\neg N(\psi \wedge \chi) \wedge N(\psi \wedge \chi \wedge \varphi) \in w$, thus $\psi \wedge \chi \in \lambda_2(w)$.
(3) Suppose that $\psi \in \lambda_2(w)$ and $\vdash \psi \rightarrow \chi$. Then $\neg N\psi \wedge N(\psi \wedge \varphi) \in w$. By Proposition 7(1), we infer that $\neg N\chi \in w$; by Proposition 7(3), we derive that $N(\chi \wedge \varphi) \in w$, and therefore $\chi \in \lambda_2(w)$. □

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LEMMA 13. *For all* $\varphi \in \mathcal{L}(N)$, *for all* $w \in W^c$, *we have*

$$\mathcal{M}^c, w \vDash \varphi \Longleftrightarrow \varphi \in w.$$

Proof. By induction on φ. The nontrivial case is $N\varphi$.

Suppose that $N\varphi \in w$, to show that $\mathcal{M}^c, w \vDash N\varphi$, that is, $S^c(w) \vDash \varphi$ and $R^c(w) \nvDash \varphi$. Let $u \in W^c$ such that wS^cu. By supposition and Proposition 7(4), $N(\varphi \wedge \varphi) \in w$. Then by definition of S^c, we obtain that $\varphi \in u$. By induction hypothesis, $\mathcal{M}^c, u \vDash \varphi$. This establishes that $S^c(w) \vDash \varphi$. We need to show that there exists $v \in W^c$ such that wR^cv and $\mathcal{M}^c, v \nvDash \varphi$. Define $\lambda_2(w) = \{\chi \mid \neg N\chi \wedge N(\chi \wedge \varphi) \in w\}$. By induction hypothesis and Lindenbaum's Lemma, it remains only to show $(\star)$: $\lambda_2(w) \cup \{\neg\varphi\}$ is consistent.

By Proposition 12(1), $\lambda_2(w) \neq \emptyset$. If $(\star)$ fails, then there exist $\chi_1, \ldots, \chi_n \in \lambda_2(w)$ such that $\vdash \chi_1 \wedge \cdots \wedge \chi_n \rightarrow \varphi$. By items (2) and (3) of Proposition 12, we can obtain that $\varphi \in \lambda_2(w)$, thus $\neg N\varphi \in w$, which is contrary to the supposition and consistency of w.

Conversely, assume that $N\varphi \notin w$, to prove that $\mathcal{M}^c, w \nvDash N\varphi$, that is, $S^c(w) \nvDash \varphi$ or $R^c(w) \vDash \varphi$. For this, we suppose that $R^c(w) \nvDash \varphi$, to demonstrate that $S^c(w) \nvDash \varphi$. By induction hypothesis, it suffices to find a $u \in W^c$ such that wS^cu and $\neg\varphi \in u$. By supposition, there exists $v \in W^c$ such that wR^cv and $\varphi \notin v$. We consider two cases.

- $N\psi \in w$ for no ψ. Then according to the definition of R^c, by wR^cv we have $w = v$. Now using the definition of S^c, wS^cv. As $\varphi \notin v$, $\neg\varphi \in v$.
- $N\psi \in w$ for some ψ. In this case, from wR^cv and $\varphi \notin v$, it follows that $\neg N\varphi \wedge N(\varphi \wedge \psi) \notin w$. By assumption, $\neg N\varphi \in w$. This gives us that $N(\varphi \wedge \psi) \notin w$. Define $\lambda_1(w) = \{\chi \mid N(\chi \wedge \psi) \in w\}$. In the sequel, we show that $\lambda_1(w) \cup \{\neg\varphi\}$ is consistent. The item (1) of Proposition 11 provides the nonemptiness of $\lambda_1(w)$. If $\lambda_1(w) \cup \{\neg\varphi\}$ is not consistent, then there are $\chi_1, \ldots, \chi_n \in \lambda_1(w)$ such that $\vdash \chi_1 \wedge \cdots \wedge \chi_n \rightarrow \varphi$. By items (2) and (3) of Proposition 11, we can get that $\varphi \in \lambda_1(w)$, that is, $N(\varphi \wedge \psi) \in w$: a contradiction. We have now shown that $\lambda_1(w) \cup \{\neg\varphi\}$ is consistent. By Lindenbaum's Lemma, there is a $u \in W^c$ such that wS^cu and $\neg\varphi \in u$, as desired.

□

By Proposition 8 and Lemma 13, we have the following result.

THEOREM 14. $\mathbf{K^N}$ *is sound and strongly complete with respect to the class of all frames.*

THEOREM 15. $\mathbf{K^N}$ *is sound and strongly complete with respect to the class of all frames* $\langle W, S, R\rangle$ *where* $S \subseteq R$.

Proof. By Theorem 14, it remains only to show that $S^c \subseteq R^c$. This is straightforward by definitions of S^c and R^c. □

We can show that all axioms and rules in **TN** (except for N1, which needs the reflexivity of S) are derivable in our $\mathbf{K^N}$, from a pure proof-theoretic perspective. According to the previous analysis, axiom N3 is immediate from Proposition 7(4). It remains only to consider axiom N4 and rule N6.

PROPOSITION 16. $(N4)$ *is provable in the proof system* $\mathbf{K^N}$. *That is,* $\vdash (\neg N(\varphi \rightarrow \psi) \wedge N\psi) \rightarrow (N\chi \rightarrow N((\varphi \rightarrow \psi) \rightarrow \chi))$.

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Proof. We have the following proof sequence in $\mathbf{K^N}$:

(1)	$(\psi \wedge \chi) \to ((\varphi \to \psi) \to \chi) \wedge (\varphi \to \psi)$	PC
(2)	$((\varphi \to \psi) \to \chi) \wedge (\varphi \to \psi) \to \chi$	PC
(3)	$N(\psi \wedge \chi) \wedge N\chi \to N(((\varphi \to \psi) \to \chi) \wedge (\varphi \to \psi))$	(1), (2), R2
(4)	$N\psi \wedge N\chi \to N(\psi \wedge \chi) \wedge N\chi$	A1
(5)	$N(((\varphi \to \psi) \to \chi) \wedge (\varphi \to \psi)) \to N((\varphi \to \psi) \to \chi) \vee N(\varphi \to \psi)$	A2
(6)	$N\psi \wedge N\chi \to N((\varphi \to \psi) \to \chi) \vee N(\varphi \to \psi)$	(3)–(5)
(7)	$(\neg N(\varphi \to \psi) \wedge N\psi) \to (N\chi \to N((\varphi \to \psi) \to \chi))$	(6).

□

PROPOSITION 17. *$(N6)$ is derivable in $\mathbf{K^N}$. That is, if $\vdash \varphi \to \psi$, then $\vdash N(\chi \to \psi) \to (N\varphi \to N\psi)$.*

Proof. Suppose that $\vdash \varphi \to \psi$. By axiom PC, $\vdash \psi \to (\chi \to \psi)$. Now using rule R2, we get $\vdash N\varphi \wedge N(\chi \to \psi) \to N\psi$, and thus $\vdash N(\chi \to \psi) \to (N\varphi \to N\psi)$. □

3.4. Extensions.

3.4.1. Serial logic. Define $\mathbf{D^N}$ to be the extension of $\mathbf{K^N}$ with the axiom $\neg N\bot$ (denoted ND). We recall that the relation S on W is serial, if for all $w \in W$, there is a $u \in W$ such that wSu.

THEOREM 18. *$\mathbf{D^N}$ is sound and strongly complete with respect to the class of all frames $\langle W, S, R\rangle$ where S is serial.*

Proof. For soundness, by Proposition 8, it suffices to show the validity of ND over serial frames. Suppose not, then there exists a serial model $\mathcal{M} = \langle W, S, R, V\rangle$ and $w \in W$ such that $\mathcal{M}, w \vDash N\bot$. Then $S(w) \vDash \bot$. Since S is serial, there exists $u \in W$ such that wSu and $\mathcal{M}, y \vDash \bot$: a contradiction.

For completeness, define $\mathcal{M}^c$ w.r.t. $\mathbf{D^N}$ as in Definition 9. By Theorem 14, it remains only to show that S^c is serial.

For this, let $w \in W^c$. If $N\psi \in w$ for no ψ, then as $w = w$, we have wS^cw. If $N\psi \in w$ for some ψ, then define $\lambda_1(w)$ as in Proposition 11. It suffices to show that $\lambda_1(w)$ is consistent. If not, then there are $\chi_1, \ldots, \chi_n \in \lambda_1(w)$ such that $\vdash \chi_1 \wedge \cdots \wedge \chi_n \to \bot$. Then by items (2) and (3) of Proposition 11, we can get $\bot \in \lambda_1(w)$, that is, $N(\bot \wedge \psi) \in w$. By Proposition 7(4), $N\bot \in w$. However, by axiom ND, $\neg N\bot \in w$, which is contrary to the consistency of w.

Thus $\lambda_1(x)$ is consistent. By Lindenbaum's Lemma, there exists $u \in W^c$ such that $\lambda_1(w) \subseteq u$. By definition of S^c, wS^cu. This establishes that S^c is serial. □

As $S^c \subseteq R^c$ (Theorem 15), we have more characterization results.

THEOREM 19. *$\mathbf{D^N}$ is sound and strongly complete with respect to the class of all frames $\langle W, S, R\rangle$ where S is serial and $S \subseteq R$ (thus R is serial).*

THEOREM 20. *$\mathbf{D^N}$ is sound and strongly complete with respect to the class of serial frames.*

3.4.2. Serial–reflexive logic. Define $\mathbf{DT^N}$ to be the extension of $\mathbf{K^N}$ with the axiom ND and the axiom $\neg N\varphi \wedge N(\varphi \wedge \psi) \to \varphi$ (denoted wNT).

THEOREM 21. *$\mathbf{DT^N}$ is sound and strongly complete with respect to the class of all frames $\langle W, S, R\rangle$ where S is serial and R is reflexive.*

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Proof. For soundness, by Theorem 18, it suffices to show the validity of axiom wNT. Let $\mathcal{M} = \langle W, S, R, V\rangle$ be a model such that S is serial and R is reflexive and $w \in W$. Suppose that $\mathcal{M}, w \vDash \neg N\varphi \wedge N(\varphi \wedge \psi)$, then $S(w) \nvDash \varphi$ or $R(w) \vDash \varphi$, and $S(w) \vDash \varphi \wedge \psi$. Thus $R(w) \vDash \varphi$. Since R is reflexive, $w \in R(w)$, and hence $\mathcal{M}, w \vDash \varphi$.

For completeness, define $\mathcal{M}^c$ w.r.t. $\mathbf{DT^N}$ as in Definition 9. By Theorem 18, it remains only to prove that R^c is reflexive. Let $w \in W^c$. If $N\psi \in w$ for no ψ, then as $w = w$, wR^cw; otherwise, for all φ, if $\neg N\varphi \wedge N(\varphi \wedge \psi) \in w$, then by axiom wNT, we infer that $\varphi \in w$, which implies that wR^cw again. □

Again, since $S^c \subseteq R^c$ (Theorem 15), we have the following.

THEOREM 22. $\mathbf{DT^N}$ *is sound and strongly complete with respect to the class of all frames* $\langle W, S, R\rangle$ *where* S *is serial and* R *is reflexive and* $S \subseteq R$.

3.4.3. Reflexive logic. Define $\mathbf{T^N}$ to be the extension of $\mathbf{K^N}$ with the axiom $N(\varphi \wedge \psi) \to \varphi$ (denoted NT).

THEOREM 23. $\mathbf{T^N}$ *is sound and strongly complete with respect to the class of all frames* $\langle W, S, R\rangle$ *where* S *is reflexive.*

Proof. For soundness, by Proposition 8, it remains only to show that axiom NT is valid over reflexive frames. Let $\mathcal{M} = \langle W, S, R, V\rangle$ be a reflexive model and $w \in W$. Suppose that $\mathcal{M}, w \vDash N(\varphi \wedge \psi)$, then $S(w) \vDash \varphi \wedge \psi$. Since S is reflexive, wSw. Thus $\mathcal{M}, w \vDash \varphi \wedge \psi$, and hence $\mathcal{M}, w \vDash \varphi$.

For completeness, define $\mathcal{M}^c$ w.r.t. $\mathbf{T^N}$ as in Definition 9. By Theorem 14, it suffices to show that S^c is reflexive.

For this, let $w \in W^c$. If $N\psi \in w$ for no ψ, then as $w = w$, we have wS^cw. If $N\psi \in w$ for some ψ, then for all φ, if $N(\varphi \wedge \psi) \in w$, by axiom NT, $\varphi \in w$. Thus again wS^cw. This completes the proof that S^c is reflexive, as desired. □

Again, as $S^c \subseteq R^c$ (Theorem 15), we have more characterization results.

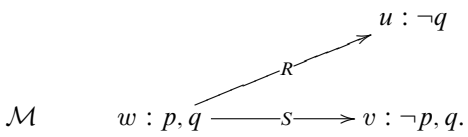
THEOREM 24. $\mathbf{T^N}$ *is sound and strongly complete with respect to the class of all frames* $\langle W, S, R\rangle$ *where* S *is reflexive and* $S \subseteq R$ *(thus* R *is reflexive).*

THEOREM 25. $\mathbf{T^N}$ *is sound and strongly complete with respect to the class of reflexive frames.*

We close this part with the following result, which will be used in §3.5.

PROPOSITION 26. $N\psi \to (\varphi \leftrightarrow N(\varphi \wedge \psi))$ *is not derivable from* $\mathbf{T^N}$.

Proof. We show that $\varphi \wedge N\psi \to N(\varphi \wedge \psi)$ is not derivable from $\mathbf{T^N}$. Consider the following reflexive model $\mathcal{M} = \langle W, S, R, V\rangle$, where for sake of simplicity we skip the reflexive arrows (the valuation of p at u is inessential):



Since $S(w) = \{w, v\}$ and q is true at both w and v, we have $S(w) \vDash q$; as wRu and $\mathcal{M}, u \nvDash q$, we infer that $R(w) \nvDash q$. This implies that $\mathcal{M}, w \vDash Nq$, and thus $\mathcal{M}, w \vDash p \wedge Nq$. However, since wSv and $\mathcal{M}, v \nvDash p \wedge q$, we derive that $S(w) \nvDash p \wedge q$, and thus

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$\mathcal{M}, w \not\models N(p \wedge q)$. Therefore, $p \wedge Nq \to N(p \wedge q)$ is not valid over reflexive frames. By the soundness of $\mathbf{T^N}$ w.r.t. the class of reflexive frames (Theorem 25), $p \wedge Nq \to N(p \wedge q)$ is not derivable in $\mathbf{T^N}$, and hence $\varphi \wedge N\psi \to N(\varphi \wedge \psi)$ is not derivable in $\mathbf{T^N}$. □

3.4.4. Transitive logic. Define $\mathbf{K4^N}$ to be the extension of $\mathbf{K^N}$ with the following axioms:[10]

A4-1 $N\psi \wedge N(\varphi \wedge \psi) \to N((N\psi' \to N(\varphi \wedge \psi')) \wedge \psi)$

A4-2 $N\psi \wedge \neg N\varphi \wedge N(\varphi \wedge \psi) \to \neg N(N\psi' \to \neg N\varphi \wedge N(\varphi \wedge \psi')) \wedge$
$\wedge N((N\psi' \to \neg N\varphi \wedge N(\varphi \wedge \psi')) \wedge \psi)$.

Although the two axioms are complicated, there is an explanation for how they get here. They are obtained from the transitive axioms for $\boxed{\mathrm{c}}$ and $\Box$, via a translation induced by almost definability schemas (**NAD1**) and (**NAD2**), respectively:

$$\begin{aligned}
& N\psi \to (\boxed{\mathrm{c}}\varphi \to \boxed{\mathrm{c}}(N\psi' \to \boxed{\mathrm{c}}\varphi)) && (1)\\
\iff\ & N\psi \to (\boxed{\mathrm{c}}\varphi \to \boxed{\mathrm{c}}(N\psi' \to N(\varphi \wedge \psi'))) && (2)\\
\iff\ & N\psi \to (N(\varphi \wedge \psi) \to N((N\psi' \to N(\varphi \wedge \psi')) \wedge \psi)) && (3)\\
\iff\ & N\psi \wedge N(\varphi \wedge \psi) \to N((N\psi' \to N(\varphi \wedge \psi')) \wedge \psi) && (4).
\end{aligned}$$

Here we write (1) rather than $\boxed{\mathrm{c}}\varphi \to \boxed{\mathrm{c}}\,\boxed{\mathrm{c}}\varphi$, since according to (**NAD1**), $\boxed{\mathrm{c}}$ is definable with N under the condition $N\psi$ for some ψ. The above transitions from (1) to (2) and then to (3) follow from Proposition 4(1). Then using simple propositional reasoning, we obtain the desired axiom (4), namely A4-1.

$$\begin{aligned}
& N\psi \to [\Box\varphi \to \Box(N\psi' \to \Box\varphi)] && (1')\\
\iff\ & N\psi \to [\Box\varphi \to \Box(N\psi' \to \neg N\varphi \wedge N(\varphi \wedge \psi'))] && (2')\\
\iff\ & N\psi \to [\neg N\varphi \wedge N(\varphi \wedge \psi) \to \neg N(N\psi' \to \neg N\varphi \wedge N(\varphi \wedge \psi')) \wedge \\
& \quad N((N\psi' \to \neg N\varphi \wedge N(\varphi \wedge \psi')) \wedge \psi)] && (3')\\
\iff\ & N\psi \wedge \neg N\varphi \wedge N(\varphi \wedge \psi) \to \neg N(N\psi' \to \neg N\varphi \wedge N(\varphi \wedge \psi')) \wedge \\
& \quad N((N\psi' \to \neg N\varphi \wedge N(\varphi \wedge \psi')) \wedge \psi) && (4').
\end{aligned}$$

Similarly, here we write (1′) rather than $\Box\varphi \to \Box\Box\varphi$, because according to (**NAD2**), $\Box$ is definable with N given $N\psi$ for some ψ. The above transitions from (1′) to (2′) and then to (3′) are due to Proposition 4(2). Then using simple propositional reasoning, we get the desired axiom (4′), viz. A4-2.

PROPOSITION 27. $\mathbf{K4^N}$ *is sound with respect to the class of transitive frames* $\langle W, S, R\rangle$ *where* $S \subseteq R$.

Proof. By Proposition 8, it remains only to show the validity of A4-1 and A4-2 over transitive frames $\langle W, S, R\rangle$ where $S \subseteq R$. Let $\mathcal{M} = \langle W, S, R, V\rangle$ be a model based on such a frame, and $w \in W$.

For A4-1: suppose that $\mathcal{M}, w \models N\psi$ and $\mathcal{M}, w \models N(\varphi \wedge \psi)$, to show that $\mathcal{M}, w \models N((N\psi' \to N(\varphi \wedge \psi')) \wedge \psi)$. Assume not, then by $\mathcal{M}, w \models N\psi$, we infer that $R(w) \not\models \psi$, thus $R(w) \not\models (N\psi' \to N(\varphi \wedge \psi')) \wedge \psi$. Then by assumption, $S(w) \not\models (N\psi' \to N(\varphi \wedge \psi')) \wedge \psi$. Moreover, $S(w) \models \psi$. This implies that $S(w) \not\models N\psi' \to N(\varphi \wedge \psi')$. Hence there exists $u \in W$ such that wSu and $\mathcal{M}, u \not\models N\psi' \to N(\varphi \wedge \psi')$, namely $\mathcal{M}, u \models N\psi'$ and $\mathcal{M}, u \not\models N(\varphi \wedge \psi')$. From $\mathcal{M}, u \models N\psi'$ it follows that $R(u) \not\models \psi'$, and

[10] Note that due to axiom A2, the conjunct $N\psi$ in axiom A4-2 can be left out.

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then $R(u) \not\models \varphi \wedge \psi'$. Then using $\mathcal{M}, u \not\models N(\varphi \wedge \psi')$, we derive that $S(u) \not\models \varphi \wedge \psi'$. However, we have also $S(u) \models \varphi \wedge \psi'$: firstly, as wSu and S is transitive, $S(u) \subseteq S(w)$. By $\mathcal{M}, w \models N(\varphi \wedge \psi)$, we have $S(w) \models \varphi \wedge \psi$, thus $S(w) \models \varphi$, and hence $S(u) \models \varphi$. Secondly, by $\mathcal{M}, u \models N\psi'$, we get $S(u) \models \psi'$. Thus $S(u) \models \varphi \wedge \psi'$. So far we arrive at a contradiction. This establishes that $\mathcal{M}, w \models N((N\psi' \rightarrow N(\varphi \wedge \psi')) \wedge \psi)$.

For A4-2: assume that $\mathcal{M}, w \models N\psi \wedge \neg N\varphi \wedge N(\varphi \wedge \psi)$, to show that $\mathcal{M}, w \models \neg N(N\psi' \rightarrow \neg N\varphi \wedge N(\varphi \wedge \psi'))$ and $\mathcal{M}, w \models N((N\psi' \rightarrow \neg N\varphi \wedge N(\varphi \wedge \psi')) \wedge \psi)$. By $\mathcal{M}, w \models \neg N\varphi$, we infer that either $S(x) \not\models \varphi$ or $R(w) \models \varphi$. By $\mathcal{M}, w \models N(\varphi \wedge \psi)$, we derive that $S(w) \models \varphi \wedge \psi$, thus $S(w) \models \varphi$. This entails that $R(w) \models \varphi$.

First, we show that $\mathcal{M}, w \models \neg N(N\psi' \rightarrow \neg N\varphi \wedge N(\varphi \wedge \psi'))$. Suppose not, that is, $\mathcal{M}, w \models N(N\psi' \rightarrow \neg N\varphi \wedge N(\varphi \wedge \psi'))$, then there exists a u such that wRu and $\mathcal{M}, u \not\models N\psi' \rightarrow \neg N\varphi \wedge N(\varphi \wedge \psi')$, thus $\mathcal{M}, u \models N\psi'$ and $\mathcal{M}, u \not\models \neg N\varphi \wedge N(\varphi \wedge \psi')$, and hence $\mathcal{M}, u \models N\varphi \vee \neg N(\varphi \wedge \psi')$. We consider two cases.

- $\mathcal{M}, u \models N\varphi$. Then there exists v such that uRv and $\mathcal{M}, v \not\models \varphi$. By wRu and uRv and the transitivity of R, we obtain that wRv. Now from $R(w) \models \varphi$, it follows that $\mathcal{M}, v \models \varphi$. A contradiction.
- $\mathcal{M}, u \models \neg N(\varphi \wedge \psi')$. Then either $S(u) \not\models \varphi \wedge \psi'$ or $R(u) \models \varphi \wedge \psi'$. If the former is the case, then there exists v such that uSv and $\mathcal{M}, v \not\models \varphi \wedge \psi'$. Since $\mathcal{M}, u \models N\psi'$, it follows that $S(u) \models \psi'$, thus $\mathcal{M}, v \models \psi'$, and hence $\mathcal{M}, v \not\models \varphi$. Since $S \subseteq R$, we have uRv, and then wRv due to wRu and the transitivity of R. Now using $R(w) \models \varphi$ again, we derive that $\mathcal{M}, v \models \varphi$. A contradiction again.

This completes the proof of $\mathcal{M}, w \models \neg N(N\psi' \rightarrow \neg N\varphi \wedge N(\varphi \wedge \psi'))$.

Second, we prove that $\mathcal{M}, w \models N((N\psi' \rightarrow \neg N\varphi \wedge N(\varphi \wedge \psi')) \wedge \psi)$. Suppose not, then either $S(w) \not\models (N\psi' \rightarrow \neg N\varphi \wedge N(\varphi \wedge \psi')) \wedge \psi$ or $R(w) \models (N\psi' \rightarrow \neg N\varphi \wedge N(\varphi \wedge \psi')) \wedge \psi$. By $\mathcal{M}, w \models N\psi$, it follows that $S(w) \models \psi$ and $R(w) \not\models \psi$. This would give us that $S(w) \not\models N\psi' \rightarrow \neg N\varphi \wedge N(\varphi \wedge \psi')$. Because $S \subseteq R$, this implies that $R(w) \not\models N\psi' \rightarrow \neg N\varphi \wedge N(\varphi \wedge \psi')$. Then similarly to the above proof, we can arrive a contradiction. This finishes the proof of $\mathcal{M}, w \models N((N\psi' \rightarrow \neg N\varphi \wedge N(\varphi \wedge \psi')) \wedge \psi)$, as desired. □

We proceed with the completeness proof of $\mathbf{K4^N}$. For this, define $\mathcal{M}^c$ w.r.t. $\mathbf{K4^N}$ as in Definition 9. By Theorem 15, it remains only to show that both S^c and R^c are transitive.

LEMMA 28. *S^c is transitive.*

Proof. Let $w, u, v \in W^c$. Suppose that wS^cu and uS^cv, to show that wS^cv. According to the definition of S^c, we consider the following cases:

- $N\psi \in w$ for no ψ. In this case, by wS^cu we have $w = u$. Then as uS^cv, we derive that wS^cv.
- $N\psi \in u$ for no ψ. In this case, from uS^cv we infer that $u = v$. Then since wS^cu, it follows that wS^cv.
- $N\psi \in w$ for some ψ, and $N\psi' \in u$ for some ψ'. In this case, assume for any φ that $N(\varphi \wedge \psi) \in w$, we hope to derive that $\varphi \in v$. By $N\psi \in w$, $N(\varphi \wedge \psi) \in w$ and axiom A4-1, we obtain that $N((N\psi' \rightarrow N(\varphi \wedge \psi')) \wedge \psi) \in w$. Then using wS^cu we infer that $N\psi' \rightarrow N(\varphi \wedge \psi') \in u$, thus $N(\varphi \wedge \psi') \in u$. Thanks to $N\psi' \in u$ and uS^cv, we conclude that $\varphi \in v$, as desired.

□

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LEMMA 29. *R^c is transitive.*

Proof. Let $w, u, v \in W^c$. Assume that wR^cu and uR^cv, to prove that wR^cv. Based on the definition of R^c, we consider the following cases:

- $N\psi \in w$ for no ψ. By wR^cu we get $w = u$. Then with uR^cv we derive that wR^cv.
- $N\psi \in u$ for no ψ. From uR^cv it follows that $u = v$. Then due to wR^cu, we infer that wR^cv.
- $N\psi \in w$ for some ψ, and $N\psi' \in u$ for some ψ'. Suppose for any φ that $\neg N\varphi \wedge N(\varphi \wedge \psi) \in w$, we hope to infer that $\varphi \in v$. By supposition and $N\psi \in w$ and axiom A4-2, we derive that $\neg N(N\psi' \rightarrow \neg N\varphi \wedge N(\varphi \wedge \psi')) \in w$ and $N((N\psi' \rightarrow \neg N\varphi \wedge N(\varphi \wedge \psi')) \wedge \psi) \in w$. Then as wR^cu, $N\psi' \rightarrow \neg N\varphi \wedge N(\varphi \wedge \psi') \in u$, thus $\neg N\varphi \wedge N(\varphi \wedge \psi') \in u$. Now using uR^cv, we conclude that $\varphi \in v$, as required.

□

Based on the previous analysis, we have the following result.

THEOREM 30. $\mathbf{K4^N}$ *is sound and strongly complete with respect to the class of transitive frames $\langle W, S, R\rangle$, where $S \subseteq R$.*

*3.4.5. **S4** logic.* Let $\mathbf{S4^N}$ be the extension of $\mathbf{K4^N}$ with the axiom NT.

THEOREM 31. $\mathbf{S4^N}$ *is sound and strongly complete with respect to the class of* **S4***-frames $\langle W, S, R\rangle$, where $S \subseteq R$.*

Proof. Direct by Theorems 23 and 30. □

This partly answers open questions raised in [3].

3.5. Connection with the logic of accidental truths. If S is the identity relation and R is reflexive, then N becomes the accident operator $\bullet$, and the language $\mathcal{L}(N)$ becomes $\mathcal{L}(\bullet)$. That is,

$$\mathcal{M}, w \vDash N\varphi \iff \mathcal{M}, w \vDash \varphi \text{ and } R(w) \nvDash \varphi.$$

Moreover, $\boxed{\mathrm{c}}\varphi$ becomes φ. That is,

$$\mathcal{M}, w \vDash \boxed{\mathrm{c}}\varphi \iff \mathcal{M}, w \vDash \varphi.$$

Then (**NAD1**) and (**NAD2**) become the following almost definability schemas:

$$N\psi \rightarrow (\varphi \leftrightarrow N(\varphi \wedge \psi)) \qquad (\mathbf{NAD1'})$$
$$N\psi \rightarrow (\Box\varphi \leftrightarrow \neg N\varphi \wedge N(\varphi \wedge \psi)) \qquad (\mathbf{NAD2}).$$

One may easily verify that according to the simplified semantics (where S is identity), (**NAD1′**) is valid over the class of all frames, whereas (**NAD2**) is valid over the class of (R-)reflexive frames. This contrasts with the fact that (**NAD1′**) is not valid under the original semantics, where S is not the identity relation (see the proofs of Proposition 26). To provide the completeness, we need to establish a proof system from which (**NAD1′**) is derivable. However, recall from Proposition 26 that the schema is not derivable from $\mathbf{T^N}$, so $\mathbf{T^N}$ is not a desired system. As we shall see, the extension

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of $\mathbf{T^N}$ with the formula $\varphi \wedge N\psi \to N(\varphi \wedge \psi)$ is a desired minimal logic of accidental truths. We denote the formula as A1′ and the extension as $\mathbf{T1^N}$.[11] We can see that the schema (**NAD1′**) is derivable from $\mathbf{T1^N}$ because of axioms A1′ and NT.

Based on the schemas (**NAD1′**) and **NAD2**, the canonical model for $\mathbf{T^N}$ can be simplified as follows.

DEFINITION 32. *The* canonical model for $\mathbf{T1^N}$ *is a tuple* $\mathcal{M}^c = \langle W^c, S^c, R^c, V^c\rangle$, *where the only change is* $N(\varphi \wedge \psi) \in w$ *in the definition of* S^c *is now becoming* $\varphi \in w$, *due to the derivable schema* (***NAD1′***). *In other words,*

- $W^c = \{w \mid w$ *is a maximal consistent set for* $\mathbf{T1^N}\}$;
- *if* $N\psi \in w$ *for no* ψ, *then* wS^cu *iff* $w = u$, *and* wR^cu *iff* $w = u$;
- *if* $N\psi \in w$ *for some* ψ, *then* wS^cu *iff for all* φ, *if* $\varphi \in w$, *then* $\varphi \in u$, *and* wR^cu *iff for all* φ, *if* $\neg N\varphi \wedge N(\varphi \wedge \psi) \in w$, *then* $\varphi \in u$;
- $V^c(p) = \{w \in W^c \mid p \in w\}$.

PROPOSITION 33. S^c *is the identity relation, and* R^c *is reflexive. Consequently,* $S^c \subseteq R^c$.

Proof. According to the definitions of S^c and R^c, we consider two cases. Let $w, u \in W^c$.

- $N\psi \in w$ for no ψ. Then wS^cu iff $w = u$, and wR^cu iff $w = u$.
- $N\psi \in w$ for some ψ. First, we show that wS^cu iff $w = u$. It follows from the fact that wS^cu iff $w \subseteq u$ (thus iff $w = u$).

  Second, we prove that wR^cw. Suppose for any φ that $\neg N\varphi \wedge N(\varphi \wedge \psi) \in w$, then since $\vdash N(\varphi \wedge \psi) \to \varphi$, we infer that $\varphi \in w$. Therefore, wR^cw.

Therefore, S^c is the identity relation, and R^c is reflexive. □

Recall that in §3.4.3, axiom NT is shown to be valid over reflexive frames, where the reflexivity of S is used. Since now S is the identity relation, S is automatically reflexive. So here NT is valid over the class of all frames, without need of the reflexivity of R. This gives us axiomatizations of $\mathcal{L}(\bullet)$ over various frames.

THEOREM 34.

(1) *Where* $\mathbf{T} \subseteq X$, $\mathbf{T1^N}$ *is sound and strongly complete with respect to the class of* X.
(2) *Where* $\mathbf{S4} \subseteq X \subseteq \mathbf{4}$ *and* $\mathbf{S41^N}$ *extends* $\mathbf{S4^N}$ *with axiom* A1′, $\mathbf{S41^N}$ *is sound and strongly complete with respect to* X.

Proof. It remains to show the truth lemma for $\mathbf{T1^N}$. That is to show, for all $\varphi \in \mathcal{L}(N)$, for all $w \in W^c$, we have

$$\mathcal{M}^c, w \vDash \varphi \text{ iff } \varphi \in w.$$

We proceed by induction on φ. The nontrivial case is $N\varphi$.

Suppose that $N\varphi \in w$, to show that $\mathcal{M}^c, w \vDash N\varphi$. By supposition and Proposition 7(4), we infer that $N(\varphi \wedge \varphi) \in w$. Then by axiom NT, we derive that $\varphi \in w$, which

[11] Actually, axiom A1 is derivable from A1′, due to the presence of axiom NT and Proposition 7(4). So we can leave axiom A1 out of $\mathbf{T1^N}$. The proof is as follows. By axiom NT, we have $\vdash N(\varphi \wedge \varphi) \to \varphi$. Then by Proposition 7(4) and TAUT, we can show that $\vdash N\varphi \to \varphi$. Now given axiom A1′, namely $\varphi \wedge N\psi \to N(\varphi \wedge \psi)$, we conclude that $\vdash N\varphi \wedge N\psi \to N(\varphi \wedge \psi)$, namely A1. (Note that the proof of Proposition 7(4) does not rely on axiom A1.)

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by induction hypothesis implies that $\mathcal{M}^c, w \vDash \varphi$. It suffices to show that $R^c(w) \nvDash \varphi$. For this, we show that $\lambda_2(w) \cup \{\neg\varphi\}$ is consistent, where $\lambda_2(w) = \{\chi \mid \neg N\chi \wedge N(\chi \wedge \varphi) \in w\}$. By Proposition 12(1), $\lambda_2(w) \neq \emptyset$.

If $\lambda_2(w) \cup \{\neg\varphi\}$ is not consistent, then there are $\chi_1, \ldots, \chi_n \in \lambda_2(w)$ such that

$$\vdash \chi_1 \wedge \cdots \wedge \chi_n \rightarrow \varphi.$$

By Proposition 12(2) and (3), we have $\varphi \in \lambda_2(w)$, thus $\neg N\varphi \in w$, which contradicts the supposition and the consistency of w.

We have shown that $\lambda_2(w) \cup \{\neg\varphi\}$ is consistent. Then by Lindenbaum's Lemma, there exists $u \in W^c$ such that wR^cu and $\neg\varphi \in u$, thus $\varphi \notin u$. By induction hypothesis, we conclude that $R^c(w) \nvDash \varphi$.

Conversely, assume that $N\varphi \notin w$, to prove that $\mathcal{M}^c, w \nvDash N\varphi$. If $\mathcal{M}^c, w \nvDash \varphi$, then $\mathcal{M}^c, w \nvDash N\varphi$. In what follows, we only consider the case that $\mathcal{M}^c, w \vDash \varphi$, in which case $\varphi \in w$. It suffices to show that $R^c(w) \vDash \varphi$. Let $u \in R^c(w)$. According to the definition of R^c, we consider two cases.

- $N\psi \in w$ for no ψ. Then $w = u$, and thus $\varphi \in u$.
- $N\psi \in w$ for some ψ. In this case, from $\varphi \in w$ and axiom A1′, it follows that $N(\varphi \wedge \psi) \in w$. Then by wR^cu, we obtain that $\varphi \in u$.

Both cases imply that $\varphi \in u$. By induction hypothesis, $\mathcal{M}^c, u \vDash \varphi$. Hence, $R^c(w) \vDash \varphi$, as desired. □

Recall that the minimal logic of accident is proposed in the literature (see, e.g., [9, 32, 38]). Here we choose the system $\mathbf{K}^\circ$ in [9] due to its simplicity, which consists of the following axioms and inference rules, where for the sake of reference, we write N for $\bullet$ and $\neg N$ for $\circ$ (namely, the negation of $\bullet$):

TAUT	All instances of propositional tautologies
$\circ\top$	$\neg N\top$
$\circ\neg$	$\neg\varphi \rightarrow \neg N\varphi$
$\circ\wedge$	$\neg N\varphi \wedge \neg N\psi \rightarrow \neg N(\varphi \wedge \psi)$
MP	From φ and $\varphi \rightarrow \psi$ infer ψ
R	From $\varphi \rightarrow \psi$ infer $\neg N\varphi \wedge \varphi \rightarrow \neg N\psi$.

In what follows, we will show that the systems $\mathbf{T1^N}$ and $\mathbf{K}^\circ$ are deductively equivalent, from a pure proof-theoretic perspective.

PROPOSITION 35. *All axioms and inference rules in* $\mathbf{K}^\circ$ *are derivable in* $\mathbf{T1^N}$.

Proof. Axiom $\circ\top$ is an immediate consequence of PC and R1, axiom $\circ\neg$ can be easily obtained from NT, and axiom $\circ\wedge$ is an equivalent transformation of A2. For the rule R, suppose that $\vdash \varphi \rightarrow \psi$, it remains only to show that $\vdash N\psi \wedge \varphi \rightarrow N\varphi$. By axiom A1′, we only need to show that $\vdash N\psi \wedge N(\varphi \wedge \psi) \rightarrow N\varphi$. This follows from Proposition 7(1). □

PROPOSITION 36. *All axioms and inference rules in* $\mathbf{T1^N}$ *are derivable in* $\mathbf{K}^\circ$.

Proof. The proof of axiom A1′. By TAUT, $\vdash \varphi \wedge \psi \rightarrow \psi$. Then using rule R, we infer that $\vdash \neg N(\varphi \wedge \psi) \wedge (\varphi \wedge \psi) \rightarrow \neg N\psi$. By propositional reasoning, we have $\vdash N\psi \wedge (\varphi \wedge \psi) \rightarrow N(\varphi \wedge \psi)$. Then by axiom $\circ\neg$, $\vdash N\psi \rightarrow \psi$. Therefore, $\vdash \varphi \wedge N\psi \rightarrow N(\varphi \wedge \psi)$.

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Axiom A2 is a direct consequence of $\circ\wedge$.

The proof of rule R1. Assume that $\vdash \varphi$. Then $\vdash \top \to \varphi$. Using R, we infer that $\vdash \neg N\top \wedge \top \to \neg N\varphi$. By $\circ\top$, it follows that $\vdash \neg N\top$. Hence, $\vdash \neg N\varphi$.

The proof of rule R2. Suppose that $\vdash \varphi \to \psi$ and $\vdash \psi \to \chi$, to prove that $\vdash (N\varphi \wedge N\chi) \to N\psi$. By $\vdash \psi \to \chi$ and R, we derive that $\vdash \neg N\psi \wedge \psi \to \neg N\chi$, that is, $\vdash N\chi \wedge \psi \to N\psi$. Since $\vdash \varphi \to \psi$, it follows that $\vdash N\chi \wedge \varphi \to N\psi$. Moreover, by $\circ\neg$, we get $\vdash N\varphi \to \varphi$. Hence, $\vdash (N\varphi \wedge N\chi) \to N\psi$.

The proof of axiom NT. By $\circ\neg$, it follows that $\vdash N(\varphi \wedge \psi) \to (\varphi \wedge \psi)$. Moreover, by TAUT, $\vdash (\varphi \wedge \psi) \to \varphi$. Therefore, $\vdash N(\varphi \wedge \psi) \to \varphi$.

The proof of axiom A1. By axiom $\circ\neg$, we have $\vdash N\varphi \to \varphi$. Then using A1′, we conclude that $\vdash N\varphi \wedge N\psi \to N(\varphi \wedge \psi)$. □

As a corollary, we give an alternative proof for the completeness of $\mathbf{K}^\circ$.

THEOREM 37. *Where* $\mathbf{T} \subseteq X$, $\mathbf{K}^\circ$ *is sound and strongly complete with respect to the class of* X.

Finally, since the operator $\boxdot$ of true belief is interdefinable with the accident operator of $\bullet$, since $\vDash \boxdot\varphi \leftrightarrow (\varphi \wedge \neg\bullet\varphi)$ and $\vDash \bullet\varphi \leftrightarrow \neg(\varphi \to \boxdot\varphi)$ [45, sec. 5.1], we can also obtain the axiomatizations of the logic of true belief over various frames.

§4. The operator of 'All$_1$ and Only$_2$'. In [28, p. 229], a variation of N, denoted N', is proposed, with the following semantics. Given an arbitrary model $\mathcal{M} = \langle W, R_1, R_2, V\rangle$ and $w \in W$, where R_1 and R_2 are arbitrary,

$$\mathcal{M}, w \vDash N'\varphi \iff R_1(w) \vDash \varphi \text{ and } R_2(w) \vDash \neg\varphi,$$

where for $i \in \{1, 2\}$, $R_i(w) \vDash \varphi$ means that for all $u \in R_i(w)$, $\mathcal{M}, u \vDash \varphi$.

Since R_2 is arbitrary, so is its complement (in symbols, $\overline{R_2} = (W \times W)\backslash R_2$). Thus we can also adopt the following equivalent semantics of N', called '[All$_1$-&-Only$_2$] semantics' in [25, p. 181]:

$$\mathcal{M}, w \vDash N'\varphi \iff R_1(w) \vDash \varphi \text{ and } \overline{R_2}(w) \vDash \neg\varphi.$$

In this sense, we call N' *the operator of* 'All$_1$ *and Only*$_2$', for want of a better name.[12] In what follows, we still adopt the first semantics of N'.

A proof system AO is then proposed and shown to be determined by the class of all frames (on the [All$_1$-&-Only$_2$] semantics). However, AO is an infinitary system which extends propositional calculus with the following schema for infinitely many finitary rules:[13] where $m, n \in \mathbb{N}$,

$$\text{[AO]} \quad \frac{(\psi_1 \wedge \cdots \wedge \psi_m) \to \chi \qquad \chi \to (\varphi_1 \vee \cdots \vee \varphi_n)}{(N'\varphi_1 \wedge \cdots \wedge N'\varphi_n \wedge N'\psi_1 \wedge \cdots \wedge N'\psi_m) \to N'\chi}.$$

To our knowledge, it is still unknown whether AO has an alternative *finite* axiomatization. In what follows, we will provide a finite minimal axiomatization for N' using our approach.

[12] N' also formalizes the notion of 'strong belief-disagreement' in [35].

[13] Note that in the conclusion of the rule [AO] on page 180 of [25], C should be $\Box C$.

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Define a language with N' and $\Box_1$ and $\Box_2$ as primitive modalities, and denote the language with N' as the sole primitive modality as $\mathcal{L}(N')$. The semantics of $\Box_1$ and $\Box_2$ are defined as $\mathcal{M}, w \vDash \Box_i\varphi$ iff $R_i(w) \vDash \varphi$, where $i \in \{1, 2\}$.

First, we observe two almost definability schemas—one for $\Box_1$ and the other for $\Box_2$—in terms of N'. Intuitively, they say that both necessity operators are definable with N', provided that there is a truth of the form $N'\psi$. They guide us to propose the desired canonical model later:

$$N'\psi \to (\Box_1\varphi \leftrightarrow N'(\psi \wedge \varphi)) \quad (\mathbf{N'AD1})$$
$$N'\psi \to (\Box_2\varphi \leftrightarrow N'(\psi \vee \neg\varphi)) \quad (\mathbf{N'AD2}).$$

Notions of truth, frame validity, and validity are defined as before. The following result states that the above two schemas are indeed valid.

PROPOSITION 38.

(1) $\vDash N'\psi \to (\Box_1\varphi \leftrightarrow N'(\psi \wedge \varphi))$.
(2) $\vDash N'\psi \to (\Box_2\varphi \leftrightarrow N'(\psi \vee \neg\varphi))$.

Proof. Let $\mathcal{M} = \langle W, R_1, R_2, V\rangle$ be a model and $w \in W$. Suppose that $\mathcal{M}, w \vDash N'\psi$, then $R_1(w) \vDash \psi$ and $R_2(w) \vDash \neg\psi$, thus $R_1(w) \vDash \psi \vee \neg\varphi$ and $R_2(w) \vDash \neg(\psi \wedge \varphi)$.

(1) Assume that $\mathcal{M}, w \vDash \Box_1\varphi$. Then $R_1(w) \vDash \varphi$. It then follows that $R_1(w) \vDash \psi \wedge \varphi$. Hence $\mathcal{M}, w \vDash N'(\psi \wedge \varphi)$.
   Conversely, assume that $\mathcal{M}, w \vDash N'(\psi \wedge \varphi)$. Then $R_1(w) \vDash \psi \wedge \varphi$, and thus $R_1(w) \vDash \varphi$. Hence $\mathcal{M}, w \vDash \Box_1\varphi$, as desired.
(2) Suppose that $\mathcal{M}, w \vDash \Box_2\varphi$. Then $R_2(w) \vDash \varphi$, and thus $R_2(w) \vDash \neg(\psi \vee \neg\varphi)$. Hence $\mathcal{M}, w \vDash N'(\psi \vee \neg\varphi)$.
   Conversely, assume that $\mathcal{M}, w \vDash N'(\psi \vee \neg\varphi)$, then $R_2(w) \vDash \neg(\psi \vee \neg\varphi)$, thus $R_2(w) \vDash \varphi$. Hence $\mathcal{M}, w \vDash \Box_2\varphi$, as desired.

□

Note that in the above result, there is no restriction to R_1 or R_2. One may impose certain conditions on these two accessibility relations. For instance, if $R_2 \subseteq R_1$, then we get the semantics of $\Box$ in [41], which has an equivalent semantics with non-normal possible worlds; if R_2 is the complement of R_1 (relative to $W \times W$), we get the operator of 'all and only' in [25]; if R_2 is the identity relation, then we obtain the operator W of false belief [18, 23, 24, 40].

4.1. Minimal system and soundness.

DEFINITION 39. *The minimal system of* $\mathcal{L}(N')$, *denoted* $\mathbf{K}^{\mathbf{N'}}$, *consists of the following axioms and inference rules*:

PC	*All instances of propositional tautologies*
*B*1	$N'\varphi \wedge N'\psi \to N'(\varphi \wedge \psi)$
*B*2	$N'\varphi \wedge N'\psi \to N'(\varphi \vee \psi)$
MP	*From* φ *and* $\varphi \to \psi$ *infer* ψ
$N'R$	*From* $\varphi \to \psi$ *and* $\psi \to \chi$ *infer* $(N'\varphi \wedge N'\chi) \to N'\psi$.

The above axioms and inference rules are obtained based on the need of showing the truth lemma (Lemma 44). Notions of derivation and theorems are defined as usual. The following proposition is important in the proof later.

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PROPOSITION 40.

(1) *If* $\vdash \varphi \leftrightarrow \psi$, *then* $\vdash N'\varphi \leftrightarrow N'\psi$.
(2) $\vdash N'(\psi \wedge \varphi) \wedge N'(\chi \vee \varphi) \rightarrow N'\varphi$.
(3) *If* $\vdash \varphi \rightarrow \chi$, *then* $\vdash (N'(\psi \wedge \varphi) \wedge N'\psi) \rightarrow N'(\psi \wedge \chi)$.
(4) *If* $\vdash \varphi \rightarrow \chi$, *then* $\vdash (N'\psi \wedge N'(\psi \vee \neg\varphi)) \rightarrow N'(\psi \vee \neg\chi)$.

Proof. Item (1) can be shown as Proposition 7(4), and (2) is immediate from the rule N′R. (3) Suppose that $\vdash \varphi \rightarrow \chi$, then $\vdash \psi \wedge \varphi \rightarrow \psi \wedge \chi$. Moreover, $\vdash \psi \wedge \chi \rightarrow \psi$. Now using the rule N′R, $\vdash (N'(\psi \wedge \varphi) \wedge N'\psi) \rightarrow N'(\psi \wedge \chi)$. (4) Assume that $\vdash \varphi \rightarrow \chi$, then $\vdash \neg\chi \rightarrow \neg\varphi$, thus $\vdash \psi \vee \neg\chi \rightarrow \psi \vee \neg\varphi$. Also, $\vdash \psi \rightarrow \psi \vee \neg\chi$. Using the rule N′R, we derive that $\vdash (N'\psi \wedge N'(\psi \vee \neg\varphi)) \rightarrow N'(\psi \vee \neg\chi)$. □

PROPOSITION 41. $\mathbf{K}^{\mathbf{N}'}$ *is sound with respect to the class of all frames.*

Proof. We only show the soundness of axioms B1, B2 and rule N′R. Let $\mathcal{M} = \langle W, R_1, R_2, V\rangle$ be a frame and $w \in W$.

For B1 and B2. Suppose that $\mathcal{M}, w \vDash N'\varphi$ and $\mathcal{M}, w \vDash N'\psi$. Then $R_1(w) \vDash \varphi$ and $R_2(w) \vDash \neg\varphi$, and $R_1(w) \vDash \psi$ and $R_2(w) \vDash \neg\psi$, which implies that $R_1(w) \vDash \varphi \wedge \psi$ and $R_2(w) \vDash \neg(\varphi \wedge \psi)$. Hence, $\mathcal{M}, w \vDash N'(\varphi \wedge \psi)$. Moreover, we have also $R_1(w) \vDash \varphi \vee \psi$ and $R_2(w) \vDash \neg(\varphi \vee \psi)$. Hence, $\mathcal{M}, w \vDash N'(\varphi \vee \psi)$.

For N′R. Assume that $\vDash \varphi \rightarrow \psi$ and $\vDash \psi \rightarrow \chi$, to show that $\vDash N'\varphi \wedge N'\chi \rightarrow N'\psi$. For this, suppose that $\mathcal{M}, w \vDash N'\varphi \wedge N'\chi$, then $R_1(w) \vDash \varphi$ and $R_2(w) \vDash \neg\chi$. By assumption, we infer that $R_1(w) \vDash \psi$ and $R_2(w) \vDash \neg\psi$. Therefore, $\mathcal{M}, w \vDash N'\psi$, as desired. □

4.2. Completeness. For completeness, we construct the canonical model for $\mathbf{K}^{\mathbf{N}'}$.

DEFINITION 42. *The canonical model for* $\mathbf{K}^{\mathbf{N}'}$ *is a tuple* $\mathcal{M} = \langle W^c, R_1^c, R_2^c, V^c\rangle$, *where*

- $W^c = \{w \mid w$ *is a maximal consistent set for* $\mathbf{K}^{\mathbf{N}'}\}$;
- *if* $N'\psi \in w$ *for no* ψ, *then* $wR_1^c u$ *iff* $w = u$, *and* $wR_2^c u$ *iff* $w = u$;
- *if* $N'\psi \in w$ *for some* ψ, *then* $wR_1^c u$ *iff for every* φ, *if* $N'(\psi \wedge \varphi) \in w$, *then* $\varphi \in u$, *and* $wR_2^c u$ *iff for every* φ, *if* $N'(\psi \vee \neg\varphi) \in w$, *then* $\varphi \in u$;
- $V^c(p) = \{w \in W^c \mid p \in w\}$.

It is required that the ψ in $N'(\psi \wedge \varphi) \in w$ and $N'(\psi \vee \neg\varphi) \in w$ be the same as that in $N'\psi \in w$. Note that the choice of ψ in the above second case of the canonical relations is inessential to R_1^c and R_2^c, as shown in Remark 10. It is also noteworthy that R_1^c and R_2^c in the above definition are inspired by almost definability schemas (**N′AD1**) and (**N′AD2**), respectively, based on the analysis of §2. Recall that in the construction of the canonical model for the minimal logic of $\mathcal{L}(\Box_1, \Box_2)$, the canonical relations R_1^c and R_2^c are defined as, respectively, $wR_1^c u$ iff for all φ, if $\Box_1\varphi \in w$ then $\varphi \in u$, and $wR_2^c u$ iff for all φ, if $\Box_2\varphi \in w$ then $\varphi \in u$. Now given a precondition that $N'\psi \in w$ for some ψ, according to (**N′AD1**), $\Box_1\varphi \in w$ can be replaced with $N'(\psi \wedge \varphi) \in w$, and according to (**N′AD2**), $\Box_2\varphi \in w$ can be replaced with $N'(\psi \vee \neg\varphi) \in w$; otherwise, we let w access and only access itself.

Proposition 43 lists some logical properties of Γ_1 and Γ_2. They are important in the proof of Lemma 44 below.

PROPOSITION 43. *Let* $w \in W^c$ *such that* $N'\psi \in w$. *Define* $\Gamma_1(w) = \{\chi \mid N'(\psi \wedge \chi) \in w\}$ *and* $\Gamma_2(w) = \{\chi \mid N'(\psi \vee \neg\chi) \in w\}$. *Then for any* $i \in \{1, 2\}$,

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(1) $\Gamma_i(w) \neq \emptyset$.
(2) *If* $\varphi, \chi \in \Gamma_i(w)$, *then* $\varphi \wedge \chi \in \Gamma_i(w)$.
(3) *If* $\varphi \in \Gamma_i(w)$ *and* $\vdash \varphi \to \chi$, *then* $\chi \in \Gamma_i(w)$.
(4) *If* $\varphi \in \Gamma_1(w)$ *and* $\neg\varphi \in \Gamma_2(w)$, *then* $N'\varphi \in w$.

Proof. Suppose that $N'\psi \in w$.

(1) By supposition and Proposition 40(1), we have $N'(\psi \wedge \top) \in w$ and $N'(\psi \vee \neg\top) \in w$, thus $\top \in \Gamma_1(w)$ and $\top \in \Gamma_2(w)$. Thus $\Gamma_1(w) \neq \emptyset$ and $\Gamma_2(w) \neq \emptyset$.
(2) Assume that $\varphi \in \Gamma_1(w)$ and $\chi \in \Gamma_1(w)$. Then $N'(\psi \wedge \varphi) \in w$ and $N'(\psi \wedge \chi) \in w$. By axiom B1 and Proposition 40(1), we infer that $N'(\psi \wedge (\varphi \wedge \chi)) \in w$, which implies that $\varphi \wedge \chi \in \Gamma_1(w)$.
Assume that $\varphi \in \Gamma_2(w)$ and $\chi \in \Gamma_2(w)$. Then $N'(\psi \vee \neg\varphi) \in w$ and $N'(\psi \vee \neg\chi) \in w$. By axiom B2 and Proposition 40(1), we derive that $N'(\psi \vee \neg(\varphi \wedge \chi)) \in w$, which entails that $\varphi \wedge \chi \in \Gamma_2(w)$.
(3) Assume that $\varphi \in \Gamma_1(w)$ and $\vdash \varphi \to \chi$. Then $N'(\psi \wedge \varphi) \in w$. Then using the supposition and Proposition 40(3), we get $N'(\psi \wedge \chi) \in w$, which implies that $\chi \in \Gamma_1(w)$.
Assume that $\varphi \in \Gamma_2(w)$ and $\vdash \varphi \to \chi$. Then $N'(\psi \vee \neg\varphi) \in w$. Now using the supposition and Proposition 40(4), we have $N'(\psi \vee \neg\chi) \in w$, which entails that $\chi \in \Gamma_2(w)$.
(4) Assume that $\varphi \in \Gamma_1(w)$ and $\neg\varphi \in \Gamma_2(w)$. Then $N'(\psi \wedge \varphi) \in w$ and $N'(\psi \vee \neg\neg\varphi) \in w$ (that is, $N'(\psi \vee \varphi) \in w$). By Proposition 40(2), $N'\varphi \in w$. □

LEMMA 44. *For all* $w \in W^c$, *for all* $\varphi \in \mathcal{L}(N')$, *we have*

$$\mathcal{M}^c, w \vDash \varphi \textit{ iff } \varphi \in w.$$

Proof. By induction on φ. The nontrivial case is $N'\varphi$.

Suppose that $N'\varphi \in w$, to show that $\mathcal{M}^c, w \vDash N'\varphi$, that is, $R_1^c(w) \vDash \varphi$ and $R_2^c(w) \vDash \neg\varphi$. By supposition and Proposition 40(1), $N'(\varphi \wedge \varphi) \in w$ and $N'(\varphi \vee \neg\neg\varphi) \in w$. Then given any $u \in W^c$ such that $wR_1^c u$, $\varphi \in u$; given any $u \in W^c$ such that $wR_2^c u$, $\neg\varphi \in u$. By induction hypothesis, $R_1^c(w) \vDash \varphi$ and $R_2^c(w) \vDash \neg\varphi$.

Conversely, assume that $N'\varphi \notin w$, to show that $\mathcal{M}^c, w \nvDash N'\varphi$, that is, $R_1^c(w) \nvDash \varphi$ or $R_2^c(w) \nvDash \neg\varphi$. If $N'\psi \in w$ for no ψ, then $R_1^c(w) = R_2^c(w) = \{w\}$, then we are done. In what follows, we consider the case that $N'\psi \in w$ for some ψ. Let $\Gamma_1(w) = \{\chi \mid N'(\psi \wedge \chi) \in w\}$ and $\Gamma_2(w) = \{\chi \mid N'(\psi \vee \neg\chi) \in w\}$. By induction hypothesis and Lindenbaum's Lemma, it remains only to show that $\Gamma_1(w) \cup \{\neg\varphi\}$ is consistent or $\Gamma_2(w) \cup \{\varphi\}$ is consistent. Note that by Proposition 43(1), $\Gamma_1(w)$ and $\Gamma_2(w)$ are nonempty.

If not, then there are $\chi_1, \ldots, \chi_n \in \Gamma_1(w)$ such that

$$\vdash \chi_1 \wedge \cdots \wedge \chi_n \to \varphi,$$

and there are $\delta_1, \ldots, \delta_m \in \Gamma_2(w)$ such that

$$\vdash \delta_1 \wedge \cdots \wedge \delta_n \to \neg\varphi.$$

By Proposition 43(2), we infer that $\chi_1 \wedge \cdots \wedge \chi_n \in \Gamma_1(w)$. Then by Proposition 43(3), we derive that $\varphi \in \Gamma_1(w)$. Similarly, we can show that $\neg\varphi \in \Gamma_2(w)$. Then by Proposition 43(4), $N'\varphi \in w$, contradicting the assumption. □

THEOREM 45. $\mathbf{K}^{\mathbf{N}'}$ *is strongly complete with respect to the class of all frames.*

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Based on the previous analysis, we obtain the completeness of $\mathbf{K}^{\mathbf{N}'}$ with respect to the class of all frames (on the [All_1-&-Only_2] semantics). Thus we have given a finite minimal axiomatization for the operator 'All_1 and Only_2'. This may also help find a finite axiomatization for the operator 'all and only', which is a special case of N' when R_2 is the complement of R_1 in the semantic structure.

4.3. Extensions. In this section, we consider extensions of $\mathbf{K}^{\mathbf{N}'}$ over various classes of frames. Although $\mathcal{L}(N')$ is not a normal modal logic, we can also define what it means by a formula in it being canonical for a frame property, just as the case for normal modal logics.

Let φ be a formula, and P be a frame property. We say that φ is *canonical for* P, if the canonical frame for any extension of $\mathbf{K}^{\mathbf{N}'}$ containing φ has property P, and φ is valid on any class of frames with property P. The following table summarizes that the formulas in the middle column are canonical for the frame properties in the right column.[14]

Notation	Axiom schemas	Frame properties
N′D1	$\neg N'\bot$	R_1 serial
N′D2	$\neg N'\top$	R_2 serial
N′T1	$N'(\varphi \wedge \psi) \rightarrow \psi$	R_1 reflexive
N′T2	$N'(\varphi \vee \neg\psi) \rightarrow \psi$	R_2 reflexive
N′41	$N'\psi \wedge N'(\psi \wedge \varphi) \rightarrow N'(\psi \wedge (N'\chi \rightarrow N'(\chi \wedge \varphi)))$	R_1 transitive
N′42	$N'\psi \wedge N'(\psi \vee \neg\varphi) \rightarrow N'(\psi \vee \neg(N'\chi \rightarrow N'(\chi \vee \neg\varphi)))$	R_2 transitive
MIX1	$N'\psi \rightarrow N'(\psi \wedge \varphi)$	$R_1 \subseteq R_2$
MIX2	$N'\psi \rightarrow N'(\psi \vee \neg\varphi)$	$R_2 \subseteq R_1$
N′ID1	$N'\psi \rightarrow (\varphi \leftrightarrow N'(\psi \wedge \varphi))$	R_1 identity
N′ID2	$N'\psi \rightarrow (\varphi \leftrightarrow N'(\psi \vee \neg\varphi))$	R_2 identity

THEOREM 46.

(1) *$N'D1$ is canonical for the seriality of R_1.*
(2) *$N'D2$ is canonical for the seriality of R_2.*
(3) *$N'T1$ is canonical for the reflexivity of R_1.*
(4) *$N'T2$ is canonical for the reflexivity of R_2.*
(5) *$N'41$ is canonical for the transitivity of R_1.*
(6) *$N'42$ is canonical for the transitivity of R_2.*
(7) *$MIX1$ is canonical for $R_1 \subseteq R_2$.*
(8) *$MIX2$ is canonical for $R_2 \subseteq R_1$.*
(9) *$N'ID1$ is canonical for R_1 being identity.*
(10) *$N'ID2$ is canonical for R_2 being identity.*

Proof.

(1) Suppose, for reductio, that there is a model $\mathcal{M} = \langle W, R_1, R_2, V\rangle$ with R_1 serial and a world $w \in W$ such that $\mathcal{M}, w \vDash N'\bot$. Then $R_1(w) \vDash \bot$. Since R_1 is serial, there is a $u \in W$ such that wR_1u and $\mathcal{M}, u \vDash \bot$: a contradiction.

Define the canonical frame $\mathcal{F}^c = \langle S^c, R_1^c, R_2^c\rangle$ as in Definition 42. Let $w \in W^c$. We consider two cases.

[14] Since we have congruentiality built in at the start (see Proposition 40(1)), it is worth mentioning that instead of N′T1, we could equivalently use the more T-like schema $N'\varphi \rightarrow \varphi$: we don't need to use two schematic letters here, for R_1-reflexivity.

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– $N'\psi \in w$ for no ψ. Then according to the definition of R_1^c, wR_1^cw.
– $N'\psi \in w$ for some ψ. Then define $\Gamma_1(w) = \{\chi \mid N'(\psi \wedge \chi) \in w\}$. According to the definition of R_1^c, we only need to show that $\Gamma_1(w)$ is consistent. If not, then there are $\chi_1, \ldots, \chi_n \in \Gamma_1(w)$ such that $\vdash \chi_1 \wedge \cdots \wedge \chi_n \to \bot$. By items (2) and (3) of Proposition 43, we infer that $\bot \in \Gamma_1(w)$, and thus $N'(\psi \wedge \bot) \in w$, that is, $N'\bot \in w$. However, by axiom N′D1, $\neg N'\bot \in w$: a contradiction.

This shows that R_1^c is serial.

(2) Assume, for a contradiction, that there is a model $\mathcal{M} = \langle W, R_1, R_2, V\rangle$ with R_2 serial and a world $w \in W$ such that $\mathcal{M}, w \vDash N'\top$. Then $R_2(w) \vDash \neg\top$. Since R_2 is serial, there exists $v \in W$ such that $wR2v$ and $\mathcal{M}, v \vDash \neg\top$: a contradiction.

Define the canonical frame $\mathcal{F}^c = \langle W^c, R_1^c, R_2^c\rangle$ as in Definition 42. Let $w \in W^c$. We consider two cases.

– $N'\psi \in w$ for no ψ. Then according to the definition of R_2^c, we have wR_2^cw.
– $N'\psi \in w$ for some ψ. Then define $\Gamma_2(w) = \{\chi \mid N'(\psi \vee \neg\chi) \in w\}$. According to the definition of R_2^c, we only need to show that $\Gamma_2(w)$ is consistent. If not, then there are $\chi_1, \ldots, \chi_n \in \Gamma_2(w)$ such that $\vdash \chi_1 \wedge \cdots \wedge \chi_n \to \bot$. By items (2) and (3) of Proposition 43, we infer that $\bot \in \Gamma_2(w)$, and thus $N'(\psi \vee \neg\bot) \in w$, that is, $N'\top \in w$. However, by axiom N′D2, $\neg N'\top \in w$: a contradiction.

This shows that R_2^c is serial.

(3) Let $\mathcal{M} = \langle W, R_1, R_2, V\rangle$ be a model with R_1 reflexive, and $w \in W$. Assume that $\mathcal{M}, w \vDash N'(\varphi \wedge \psi)$. Then $R_1(w) \vDash \varphi \wedge \psi$. By reflexivity of R_1, wR_1w. Then $\mathcal{M}, w \vDash \psi$.

Define the canonical frame $\mathcal{F}^c = \langle W^c, R_1^c, R_2^c\rangle$ as in Definition 42. Let $w \in W^c$. We consider two cases.

– $N'\psi \in w$ for no ψ. Then according to the definition of R_1^c, wR_1^cw.
– $N'\psi \in w$ for some ψ. Then define $\Gamma_1(w) = \{\chi \mid N'(\psi \wedge \chi) \in w\}$. According to the definition of R_1^c, we only need to show that $\Gamma_1(w) \subseteq w$. For this, suppose for any χ that $\chi \in \Gamma_1(w)$, that is, $N'(\psi \wedge \chi) \in w$. By axiom N′T1, we obtain that $\chi \in w$.

This shows that R_1^c is reflexive.

(4) Let $\mathcal{M} = \langle W, R_1, R_2, V\rangle$ be a model with R_2 reflexive, and $w \in W$. Suppose that $\mathcal{M}, w \vDash N'(\varphi \vee \neg\psi)$. Then $R_2(w) \vDash \neg(\varphi \vee \neg\psi)$. By reflexivity of R_2, wR_2w. Then $\mathcal{M}, w \vDash \psi$. Define the canonical frame $\mathcal{F}^c = \langle W^c, R_1^c, R_2^c\rangle$ as in Definition 42. Let $w \in W^c$. We consider two cases.

– $N'\psi \in w$ for no ψ. Then according to the definition of R_2^c, wR_2^cw.
– $N'\psi \in w$ for some ψ. Then define $\Gamma_2(w) = \{\chi \mid N'(\psi \vee \neg\chi) \in w\}$. According to the definition of R_2^c, we only need to show that $\Gamma_2(w) \subseteq w$. For this, suppose for any χ that $\chi \in \Gamma_2(w)$, that is, $N'(\psi \vee \neg\chi) \in w$. By axiom N′T2, we obtain that $\chi \in w$.

This shows that R_2^c is reflexive.

(5) Let $\mathcal{M} = \langle W, R_1, R_2, V\rangle$ be a model with R_1 transitive, and $w \in W$. Suppose, for reductio, that $\mathcal{M}, w \vDash N'\psi \wedge N'(\psi \wedge \varphi)$, and $\mathcal{M}, w \nvDash N'(\psi \wedge (N'\chi \to N'(\chi \wedge \varphi)))$. From $\mathcal{M}, w \vDash N'\psi$, it follows that $R_2(w) \vDash \neg\psi$. Thus there exists $u \in W$ such that wR_1u and $\mathcal{M}, u \nvDash \psi \wedge (N'\chi \to N'(\chi \wedge \varphi))$. We have

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also $R_1(w) \vDash \psi$. This implies that $\mathcal{M}, u \nvDash N'\chi \to N'(\chi \wedge \varphi)$, thus $\mathcal{M}, u \vDash N'\chi$ and $\mathcal{M}, u \nvDash N'(\chi \wedge \varphi)$. From the former, it follows that $R_1(u) \vDash \chi$ and $R_2(u) \vDash \neg\chi$; from the latter, it follows that either $R_1(u) \nvDash \chi \wedge \varphi$ or $R_2(u) \nvDash \neg(\chi \wedge \varphi)$. Thus $R_1(u) \nvDash \chi \wedge \varphi$. However, by wR_1u and transitivity of R_1, $R_1(u) \subseteq R_1(w)$. Now using $\mathcal{M}, w \vDash N'(\psi \wedge \varphi)$, we obtain that $R_1(w) \vDash \varphi$ and thus $R_1(u) \vDash \varphi$, and hence $R_1(u) \vDash \chi \wedge \varphi$: a contradiction.

Define the canonical frame $\mathcal{F}^c = \langle W^c, R_1^c, R_2^c \rangle$ as in Definition 42. Let $w, u, v \in W^c$. Assume that $wR_1^c u$ and $uR_1^c v$, to show that $wR_1^c v$. We consider the following cases.

- $N'\psi \in w$ for no ψ. Then by $wR_1^c u$ we infer that $w = u$. As $uR_1^c v$, we derive that $wR_1^c v$.
- $N'\psi \in u$ for no ψ. From $uR_1^c v$ it follows that $u = v$. This plus $wR_1^c u$ entails that $wR_1^c v$.
- $N'\psi \in w$ and $N'\chi \in u$ for some ψ and χ. Suppose for any φ that $N'(\psi \wedge \varphi) \in w$, it remains only to show that $\varphi \in v$. By axiom N′41, $N'(\psi \wedge (N'\chi \to N'(\chi \wedge \varphi))) \in w$. Due to $wR_1^c u$, we have $N'\chi \to N'(\chi \wedge \varphi) \in u$, thus $N'(\chi \wedge \varphi) \in u$. Then using $uR_1^c v$, we get $\varphi \in v$, as desired.

This shows that R_1^c is transitive.

(6) Let $\mathcal{M} = \langle W, R_1, R_2 \rangle$ be a model with R_2 transitive, and $w \in W$. Assume, for a contradiction, that $\mathcal{M}, w \vDash N'\psi \wedge N'(\psi \vee \neg\varphi)$ and $\mathcal{M}, w \nvDash N'(\psi \vee \neg(N'\chi \to N'(\chi \vee \neg\varphi)))$. By $\mathcal{M}, w \vDash N'\psi$, we have $R_1(w) \vDash \psi$. Then by $\mathcal{M}, w \nvDash N'(\psi \vee \neg(N'\chi \to N'(\chi \vee \neg\varphi)))$, we infer that $R_2(w) \nvDash \neg(\psi \vee \neg(N'\chi \to N'(\chi \vee \neg\varphi)))$; in other words, there is a $u \in W$ such that wR_2u and $\mathcal{M}, u \vDash \psi \vee \neg(N'\chi \to N'(\chi \vee \neg\varphi))$. We have also $R_2(w) \vDash \neg\psi$. Thus $\mathcal{M}, u \vDash \neg(N'\chi \to N'(\chi \vee \neg\varphi))$, that is, $\mathcal{M}, u \vDash N'\chi$ and $\mathcal{M}, u \nvDash N'(\chi \vee \neg\varphi)$. From the former, it follows that $R_1(u) \vDash \chi$ and $R_2(u) \vDash \neg\chi$; from the latter, it follows that either $R_1(u) \nvDash \chi \vee \neg\varphi$ or $R_2(u) \nvDash \neg(\chi \vee \neg\varphi)$. This implies that $R_2(u) \nvDash \neg(\chi \vee \neg\varphi)$. However, as $\mathcal{M}, w \vDash N'(\psi \vee \neg\varphi)$, we derive that $R_2(w) \vDash \varphi$. By wR_2u and transitivity of R_2, $R_2(u) \subseteq R_2(w)$, thus $R_2(u) \vDash \varphi$, and hence $R_2(u) \vDash \neg(\chi \vee \neg\varphi)$. A contradiction.

Define the canonical frame $\mathcal{F}^c = \langle W^c, R_1^c, R_2^c \rangle$ as in Definition 42. Let $w, u, v \in W^c$. Suppose that $wR_2^c u$ and $uR_2^c v$, to prove that $wR_2^c v$. We consider the following cases:

- $N'\psi \in w$ for no ψ. Then by $wR_2^c u$ we obtain that $w = u$. This plus $uR_2^c v$ implies that $wR_2^c v$.
- $N'\psi \in u$ for no ψ. By $uR_2^c v$ we infer that $u = v$. Then using $wR_2^c u$ we derive that $wR_2^c v$.
- $N'\psi \in w$ and $N'\chi \in u$ for some ψ and χ. Assume for any φ that $N'(\psi \vee \neg\varphi) \in w$, to show that $\varphi \in v$. By axiom N′42, $N'(\psi \vee \neg(N'\chi \to N'(\chi \vee \neg\varphi))) \in w$. Then using $wR_2^c u$, we get $N'\chi \to N'(\chi \vee \neg\varphi) \in u$, thus $N'(\chi \vee \neg\varphi) \in u$. Now due to $uR_2^c v$, we conclude that $\varphi \in v$, as desired.

This shows that R_2^c is transitive.

(7) Let $\mathcal{M} = \langle W, R_1, R_2, V \rangle$ be a model with $R_1 \subseteq R_2$, and $w \in W$. Suppose, for reductio, that $\mathcal{M}, w \vDash N'\psi$ and $\mathcal{M}, w \nvDash N'(\psi \wedge \varphi)$. Then either $R_1(w) \nvDash \psi \wedge \varphi$ or $R_2(w) \nvDash \neg(\psi \wedge \varphi)$. By $\mathcal{M}, w \vDash N'\psi$, $R_2(w) \vDash \neg\psi$, thus $R_2(w) \vDash \neg(\psi \wedge \varphi)$, and hence $R_1(w) \nvDash \psi \wedge \varphi$. So there exists $u \in W$ such that wR_1u and $\mathcal{M}, u \nvDash \psi \wedge \varphi$. Using $\mathcal{M}, w \vDash N'\psi$ again, we derive that $R_1(w) \vDash \psi$,

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thus $\mathcal{M}, u \vDash \psi$. Moreover, $R_2(w) \vDash \neg\psi$ and $R_1(w) \subseteq R_2(w)$ implies that $R_1(w) \vDash \neg\psi$, and thus $\mathcal{M}, u \vDash \neg\psi$. A contradiction.

Define the canonical frame $\mathcal{F}^c = \langle W^c, R_1^c, R_2^c \rangle$ as in Definition 42. Let $w, u \in W^c$. Assume that $wR_1^c u$, to show that $wR_2^c u$. We consider two cases.

- $N'\psi \in w$ for no ψ. Then from $wR_1^c u$ it follows that $w = u$. Then $wR_2^c u$.
- $N'\psi \in w$ for some ψ. Suppose for any φ that $N'(\psi \vee \neg\varphi) \in w$, to prove that $\varphi \in u$. By $N'\psi \in w$ and axiom MIX1, we have $N'(\psi \wedge \varphi) \in w$. Now using $wR_1^c u$, we derive that $\varphi \in u$, as desired.

This shows that $R_1^c \subseteq R_2^c$.

(8) Let $\mathcal{M} = \langle W, R_1, R_2, V \rangle$ be a model with $R_2 \subseteq R_1$, and $w \in W$. Assume, for a contradiction, that $\mathcal{M}, w \vDash N'\psi$ and $\mathcal{M}, w \nvDash N'(\psi \vee \neg\varphi)$. Then either $R_1(w) \nvDash \psi \vee \neg\varphi$ or $R_2(w) \nvDash \neg(\psi \vee \neg\varphi)$. By $\mathcal{M}, w \vDash N'\psi$, it follows that $R_1(w) \vDash \psi$, thus $R_1(w) \vDash \psi \vee \neg\varphi$. Hence $R_2(w) \nvDash \neg(\psi \vee \neg\varphi)$. Thus there exists $u \in W$ such that wR_2u and $\mathcal{M}, u \vDash \psi \vee \neg\varphi$. Using $\mathcal{M}, w \vDash N'\psi$ again, we derive that $R_2(w) \vDash \neg\psi$, thus $\mathcal{M}, u \vDash \neg\psi$. Further, $R_1(w) \vDash \psi$ and $R_2(w) \subseteq R_1(w)$ entails that $R_2(w) \vDash \psi$, thus $\mathcal{M}, u \vDash \psi$. A contradiction.

Define the canonical frame $\mathcal{F}^c = \langle W^c, R_1^c, R_2^c \rangle$ as in Definition 42. Let $w, u \in W^c$. Suppose that $wR_2^c u$, to prove that $wR_1^c u$. We consider two cases.

- $N'\psi \in w$ for no ψ. Then from $wR_2^c u$ it follows that $w = u$. Then $wR_1^c u$.
- $N'\psi \in w$ for some ψ. Assume for any φ that $N'(\psi \wedge \varphi) \in w$, to show that $\varphi \in u$. By $N'\psi \in w$ and axiom MIX2, we infer that $N'(\psi \vee \neg\varphi) \in w$. Now using $wR_2^c u$ we conclude that $\varphi \in u$, as desired.

This shows that $R_2^c \subseteq R_1^c$.

(9) Let $\mathcal{M} = \langle W, R_1, R_2, V \rangle$ be a model with the identity relation R_1, and $w \in W$. Assume that $\mathcal{M}, w \vDash N'\psi$, to show that $\mathcal{M}, w \vDash \varphi \leftrightarrow N'(\psi \wedge \varphi)$. By assumption, $R_1(w) \vDash \psi$ and $R_2(w) \vDash \neg\psi$. Thus $R_2(w) \vDash \neg(\psi \wedge \varphi)$. Moreover, due to $R_1(w) = \{w\}$, $\mathcal{M}, w \vDash \psi$.

Suppose that $\mathcal{M}, w \vDash \varphi$. Then $\mathcal{M}, w \vDash \psi \wedge \varphi$. Since $R_1(w) = \{w\}$, $R_1(w) \vDash \psi \wedge \varphi$, and thus $\mathcal{M}, w \vDash N'(\psi \wedge \varphi)$.

Conversely, suppose that $\mathcal{M}, w \vDash N'(\psi \wedge \varphi)$, then $R_1(w) \vDash \psi \wedge \varphi$. Since $R_1(w) = \{w\}$, we derive that $\mathcal{M}, w \vDash \psi \wedge \varphi$, and thus $\mathcal{M}, w \vDash \varphi$.

Define the canonical frame $\mathcal{F}^c = \langle W^c, R_1^c, R_2^c \rangle$ as in Definition 42. In what follows, we show that $R_1^c = \{(w, w) \mid w \in W^c\}$; equivalently, for all $w, u \in W^c$, $wR_1^c u$ iff $w = u$.

Let $w, u \in W^c$. Assume that $wR_1^c u$, to show that $w = u$, it suffices to show that $w \subseteq u$. If $N'\psi \in w$ for no ψ, then clearly, from the assumption it follows that $w = u$. If $N'\psi \in w$ for some ψ. If $\varphi \in w$, then by axiom N′ID1, we infer that $N'(\psi \wedge \varphi) \in w$. This plus the assumption implies that $\varphi \in u$.

Conversely, suppose that $w = u$, to prove that $wR_1^c u$. If $N'\psi \in w$ for no ψ, then by supposition, it is obvious that $wR_1^c u$. If $N'\psi \in w$ for some ψ, then for any φ, if $N'(\psi \wedge \varphi) \in w$, then by axiom N′ID1, we derive that $\varphi \in w$, thus $\varphi \in u$, and therefore $wR_1^c u$, as desired.

(10) Let $\mathcal{M} = \langle W, R_1, R_2, V \rangle$ be a model with the identity relation R_2, and $w \in W$. Suppose that $\mathcal{M}, w \vDash N'\psi$, to show that $\mathcal{M}, w \vDash \varphi \leftrightarrow N'(\psi \vee \neg\varphi)$. By supposition, $R_1(w) \vDash \psi$ and $R_2(w) \vDash \neg\psi$. Thus $R_1(w) \vDash \psi \vee \neg\varphi$. Moreover, due to $R_2(w) = \{w\}$, $\mathcal{M}, w \vDash \neg\psi$.

Assume that $\mathcal{M}, w \vDash \varphi$. If $\mathcal{M}, w \nvDash N'(\psi \vee \neg\varphi)$, then either $R_1(w) \nvDash \psi \vee \neg\varphi$ or $R_2(w) \nvDash \neg(\psi \vee \neg\varphi)$. The first case contradicts $R_1(w) \vDash \psi \vee \neg\varphi$.

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If the second case holds, due to $R_2(w) = \{w\}$, then $\mathcal{M}, w \not\models \neg(\psi \vee \neg\varphi)$, equivalently, $\mathcal{M}, w \not\models \neg\psi \wedge \varphi$. This contradicts $\mathcal{M}, w \models \neg\psi$ and the assumption. Hence, $\mathcal{M}, w \models N'(\psi \vee \neg\varphi)$.

Conversely, assume that $\mathcal{M}, w \models N'(\psi \vee \neg\varphi)$, to show that $\mathcal{M}, w \models \varphi$. By assumption, $R_2(w) \models \neg(\psi \vee \neg\varphi)$. As $R_2(w) = \{w\}$, we infer that $\mathcal{M}, w \models \neg(\psi \vee \neg\varphi)$, and thus $\mathcal{M}, w \models \varphi$.

Define the canonical frame $\mathcal{F}^c = \langle W^c, R_1^c, R_2^c\rangle$ as in Definition 42. In what follows, we show that $R_2^c = \{(w, w) \mid w \in W^c\}$; equivalently, for all $w, u \in W^c$, $wR_2^c u$ iff $w = u$.

Let $w, u \in W^c$. Suppose that $wR_2^c u$, to show that $w = u$, it suffices to show that $w \subseteq u$. If $N'\psi \in w$ for no ψ, then clearly, from supposition it follows that $w = u$. If $N'\psi \in w$ for some ψ. If $\varphi \in w$, then by axiom N′ID2, we infer that $N'(\psi \vee \neg\varphi) \in w$. This plus the supposition implies that $\varphi \in u$.

Conversely, assume that $w = u$, to prove that $wR_2^c u$. If $N'\psi \in w$ for no ψ, then by assumption, it is obvious that $wR_2^c u$. If $N'\psi \in w$ for some ψ, then for any φ, if $N'(\psi \vee \neg\varphi) \in w$, then by axiom N′ID2, we derive that $\varphi \in w$, thus $\varphi \in u$, and therefore $wR_2^c u$, as desired. □

Note that the above theorem gives us a lot of axiomatizations. In particular, from (10), we know that $\mathbf{K}^{\mathbf{N}'} + \mathrm{N}'\mathrm{ID2}$ is a minimal axiomatization of the logic $\mathcal{L}(W)$ of false beliefs which has W as the sole modality. With (5) as well, we get that $\mathbf{K}^{\mathbf{N}'} + \mathrm{N}'\mathrm{ID2} + \mathrm{N}'41$ is the transitive axiomatization of $\mathcal{L}(W)$, which solves an open question raised in the literature of the logic of false beliefs (see, for instance, [24, 40]). From (9), we obtain the minimal axiomatization of the logic of weak essence and strong accident [34]. Moreover, from (7) and (8), it follows that $\mathbf{K}^{\mathbf{N}'} + \mathrm{MIX1} + \mathrm{MIX2}$ is the minimal axiomatization of the logic of belief inconsistency.

§5. The operator N''. In [28, p. 229], Humberstone also proposes another variation of N, denoted N'', which is defined as $N''\varphi := \Box_1\varphi \vee \neg\Box_2\varphi$. Syntactically, by extending the language of propositional logic with the sole modality N'', we obtain the language $\mathcal{L}(N'')$. Semantically, given a model $\mathcal{M} = \langle W, R_1, R_2, V\rangle$ and $w \in W$, $\Box_1$ and $\Box_2$ are interpreted as before, and

$$\mathcal{M}, w \models N''\varphi \iff R_1(w) \models \varphi \text{ or } R_2(w) \not\models \varphi.$$

We observe the following almost-definability schemas, which guides us to propose the desired canonical model later:

$$\neg N''\psi \to (\Box_1\varphi \leftrightarrow (N''\varphi \wedge \neg N''(\varphi \wedge \psi))) \quad (\mathbf{N''AD1})$$
$$\neg N''\psi \to (\Box_2\varphi \leftrightarrow \neg N''(\varphi \wedge \psi)) \quad (\mathbf{N''AD2}).$$

PROPOSITION 47. *Let $\mathcal{F} = \langle W, R_1, R_2\rangle$ such that $R_2 \subseteq R_1$. Then*

(1) $\mathcal{F} \models \neg N''\psi \to (\Box_1\varphi \leftrightarrow (N''\varphi \wedge \neg N''(\varphi \wedge \psi)))$.
(2) $\models \neg N''\psi \to (\Box_2\varphi \leftrightarrow \neg N''(\varphi \wedge \psi))$.

Proof. Let $\mathcal{M} = \langle W, R_1, R_2, V\rangle$ be a model such that $R_2 \subseteq R_1$ and $w \in W$. Suppose that $\mathcal{M}, w \models \neg N''\psi$, then $R_1(w) \not\models \psi$ and $R_2(w) \models \psi$.

(1) It remains only to show that $\mathcal{M}, w \models \Box_1\varphi \leftrightarrow (N''\varphi \wedge \neg N''(\varphi \wedge \psi))$.
Assume that $\mathcal{M}, w \models \Box_1\varphi$. Then $R_1(w) \models \varphi$, and thus $\mathcal{M}, w \models N''\varphi$. From $R_1(w) \not\models \psi$, it follows that $R_1(w) \not\models \varphi \wedge \psi$; moreover, since $R_1(w) \models \varphi$ and

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$R_2 \subseteq R_1$, we obtain $R_2(w) \vDash \varphi$, and thus $R_2(w) \vDash \varphi \wedge \psi$. This implies that $\mathcal{M}, w \vDash \neg N''(\varphi \wedge \psi)$. Therefore, $\mathcal{M}, w \vDash N''\varphi \wedge \neg N''(\varphi \wedge \psi)$.

Conversely, assume that $\mathcal{M}, w \vDash N''\varphi \wedge \neg N''(\varphi \wedge \psi)$. Then from $\mathcal{M}, w \vDash N''\varphi$, it follows that $R_1(w) \vDash \varphi$ or $R_2(w) \nvDash \varphi$; from $\mathcal{M}, w \vDash \neg N''(\varphi \wedge \psi)$, it follows that $R_2(w) \vDash \varphi \wedge \psi$, thus $R_2(w) \vDash \varphi$. Therefore, $R_1(w) \vDash \varphi$, which entails that $\mathcal{M}, w \vDash \Box_1\varphi$.

(2) It suffices to prove that $\mathcal{M}, w \vDash \Box_2\varphi \leftrightarrow \neg N''(\varphi \wedge \psi)$.

Assume that $\mathcal{M}, w \vDash \Box_2\varphi$. Then $R_2(w) \vDash \varphi$. Since $R_2(w) \vDash \psi$, it follows that $R_2(w) \vDash \varphi \wedge \psi$. Moreover, as $R_1(w) \nvDash \psi$, we infer that $R_1(w) \nvDash \varphi \wedge \psi$. Hence, $\mathcal{M}, w \vDash \neg N''(\varphi \wedge \psi)$.

Conversely, assume that $\mathcal{M}, w \vDash \neg N''(\varphi \wedge \psi)$. Then $R_2(w) \vDash \varphi \wedge \psi$, thus $R_2(w) \vDash \varphi$, and hence $\mathcal{M}, w \vDash \Box_2\varphi$.

□

Note that the proof of (2) does not need any semantic restriction of accessibility relations, whereas (1) only uses $R_2 \subseteq R_1$.

Unlike N and N', the operator N'' does not have the property of convexity. In other words, the transition from $\varphi \to \psi$ and $\psi \to \chi$ to $(N''\varphi \wedge N''\chi) \to N''\psi$ does not preserve validity.

PROPOSITION 48. $\dfrac{\varphi \to \psi \quad \psi \to \chi}{(N''\varphi \wedge N''\chi) \to N''\psi}$ *is* not *validity-preserving.*

Proof. Since $\vDash p \wedge q \to q$ and $\vDash q \to q \vee r$, it remains only to show that $\nvDash N''(p \wedge q) \wedge N''(q \vee r) \to N''q$. Consider the following model $\mathcal{M}$ (other valuations are inessential):

$$u : \neg q, r \xleftarrow{\quad 1 \quad} w \xrightarrow{\quad 2 \quad} v : \neg p, q.$$

Since wR_2v and $\mathcal{M}, v \nvDash p \wedge q$, we have $R_2(w) \nvDash p \wedge q$, thus $\mathcal{M}, w \vDash N''(p \wedge q)$. Moreover, as $R_1(w) = \{u\}$ and $\mathcal{M}, u \vDash q \vee r$, we infer that $R_1(w) \vDash q \vee r$, thus $\mathcal{M}, w \vDash N''(q \vee r)$. However, $R_1(w) \nvDash q$ and $R_2(w) \vDash q$, hence $\mathcal{M}, w \nvDash N''q$. Therefore, $\nvDash N''(p \wedge q) \wedge N''(q \vee r) \to N''q$. □

Despite this, N'' has a property called 'coconvexity'.[15] That is, the transition from $\varphi \to \psi$ and $\psi \to \chi$ to $N''\psi \to (N''\varphi \vee N''\chi)$ preserves validity.

PROPOSITION 49. $\dfrac{\varphi \to \psi \quad \psi \to \chi}{N''\psi \to (N''\varphi \vee N''\chi)}$ *is validity-preserving.*

Proof. Suppose that $\vDash \varphi \to \psi$ and $\vDash \psi \to \chi$, to show that $\vDash N''\psi \to (N''\varphi \vee N''\chi)$. Let $\mathcal{M} = \langle W, R_1, R_2, V \rangle$ be a model and $w \in W$. Assume that $\mathcal{M}, w \vDash N''\psi$. Then $R_1(w) \vDash \psi$ or $R_2(w) \nvDash \psi$. If the former holds, then from $\vDash \psi \to \chi$, it follows that $R_1(w) \vDash \chi$, and thus $\mathcal{M}, w \vDash N''\chi$; if the latter holds, then from $\vDash \varphi \to \psi$, it follows that $R_2(w) \nvDash \varphi$, and thus $\mathcal{M}, w \vDash N''\varphi$. Therefore, $\mathcal{M}, w \vDash N''\varphi \vee N''\chi$, as desired. □

5.1. Axiomatization. As commented in Remark 5, we can axiomatize $\mathcal{L}(N'')$ over various frames. Due to space limitation, we illustrate this with the minimal logic.

[15] The term 'coconvexity' comes from [27, p. 584].

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5.1.1. Minimal system and soundness.

DEFINITION 50. *The minimal system of* $\mathcal{L}(N'')$, *denoted* $\mathbf{K}^{\mathbf{N}''}$, *consists of the following axioms and inference rules*:

PC	*All instances of propositional tautologies*
*C*1	$N''\varphi \wedge N''\psi \to N''(\varphi \wedge \psi)$
*C*2	$N''(\varphi \wedge \psi) \to N''\varphi \vee N''\psi$
MP	*From* φ *and* $\varphi \to \psi$ *infer* ψ
N''*R*1	*From* φ *infer* $N''\varphi$
N''*R*2	*From* $\varphi \to \psi$ *and* $\psi \to \chi$ *infer* $N''\psi \to (N''\varphi \vee N''\chi)$.

Notions of derivation and theorems are defined as usual. The following proposition is important in the proof later.

PROPOSITION 51.

(1) *If* $\vdash \varphi \to \chi$, *then* $\vdash N''\varphi \to N''(\varphi \wedge \psi) \vee N''\chi$.
(2) *If* $\vdash \varphi \to \chi$, *then* $\vdash N''(\chi \wedge \psi) \to N''(\varphi \wedge \psi) \vee N''\psi$.
(3) *If* $\vdash \varphi \to \chi$, *then* $\vdash N''(\chi \wedge \psi) \wedge N''\varphi \to N''(\varphi \wedge \psi)$.
(4) *If* $\vdash \varphi \leftrightarrow \psi$, *then* $\vdash N''\varphi \leftrightarrow N''\psi$.

Proof.

(1) Suppose that $\vdash \varphi \to \chi$. Since $\vdash \varphi \wedge \psi \to \varphi$, by rule N$''$R2, we infer that $\vdash N''\varphi \to N''(\varphi \wedge \psi) \vee N''\chi$.
(2) Assume that $\vdash \varphi \to \chi$. Then $\vdash \varphi \wedge \psi \to \chi \wedge \psi$. Moreover, $\vdash \chi \wedge \psi \to \psi$. Now using N$''$R2, we derive that $\vdash N''(\chi \wedge \psi) \to N''(\varphi \wedge \psi) \vee N''\psi$.
(3) Suppose that $\vdash \varphi \to \chi$. By (2), $\vdash N''(\chi \wedge \psi) \to N''(\varphi \wedge \psi) \vee N''\psi$. Then by axiom C1, we obtain that $\vdash N''\varphi \wedge N''\psi \to N''(\varphi \wedge \psi)$. Therefore, $\vdash N''(\chi \wedge \psi) \wedge N''\varphi \to N''(\varphi \wedge \psi)$.
(4) Assume that $\vdash \varphi \leftrightarrow \psi$. Then $\vdash \varphi \to \psi$ and $\vdash \psi \to \varphi$. Using rule N$''$R2, we get that $\vdash N''\psi \to (N''\varphi \vee N''\varphi)$, and thus $\vdash N''\psi \to N''\varphi$. Similarly, we can show that $\vdash N''\varphi \to N''\psi$. Therefore, $\vdash N''\varphi \leftrightarrow N''\psi$.

□

PROPOSITION 52. $\mathbf{K}^{\mathbf{N}''}$ *is sound with respect to the class of all frames.*

Proof. We only need to show the validity of axioms C1 and C2, since the validity-preserving of N$''$R2 has been shown in Proposition 49, and other axiom and inference rules are straightforward. Let $\mathcal{M} = \langle W, R_1, R_2, V\rangle$ be a model and $w \in W$.

For C1, suppose that $\mathcal{M}, w \vDash N''\varphi$ and $\mathcal{M}, w \vDash N''\psi$. Then $R_1(w) \vDash \varphi$ or $R_2(w) \nvDash \varphi$, and $R_1(w) \vDash \psi$ or $R_2(w) \nvDash \psi$. If $R_2(w) \nvDash \varphi$ or $R_2(w) \nvDash \psi$, then $R_2(w) \nvDash \varphi \wedge \psi$, and thus $\mathcal{M}, w \vDash N''(\varphi \wedge \psi)$. Otherwise, that is, if $R_1(w) \vDash \varphi$ and $R_1(w) \vDash \psi$, then $R_1(w) \vDash \varphi \wedge \psi$, and thus again $\mathcal{M}, w \vDash N''(\varphi \wedge \psi)$. Therefore, $\mathcal{M}, w \vDash N''(\varphi \wedge \psi)$.

For C2, assume that $\mathcal{M}, w \vDash N''(\varphi \wedge \psi)$, then $R_1(w) \vDash \varphi \wedge \psi$ or $R_2(w) \nvDash \varphi \wedge \psi$. If the former is the case, then $R_1(w) \vDash \varphi$ and $R_1(w) \vDash \psi$, which implies that $\mathcal{M}, w \vDash N''\varphi$ and $\mathcal{M}, w \vDash N''\psi$; if the latter is the case, then $R_2(w) \nvDash \varphi$ or $R_2(w) \nvDash \psi$, which implies that $\mathcal{M}, w \vDash N''\varphi$ or $\mathcal{M}, w \vDash N''\psi$. Therefore, $\mathcal{M}, w \vDash N''\varphi \vee N''\psi$, as desired. □

5.1.2. Completeness. We continue with the completeness of $\mathbf{K}^{\mathbf{N}''}$. This is shown via a construction of the canonical model, where again, the crucial part is the definition of canonical relations.

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DEFINITION 53. *The canonical model for* $\mathbf{K}^{\mathbf{N}''}$ *is the tuple* $\mathcal{M}^c = \langle W^c, R_1^c, R_2^c, V^c \rangle$, *where*

- $W^c = \{w \mid w \text{ is a maximal consistent set for } \mathbf{K}^{\mathbf{N}''}\}$;
- *if* $\neg N''\psi \in w$ *for no* ψ, *then* $wR_1^c u$ *iff* $w = u$, *and* $wR_2^c u$ *iff* $w = u$;
- *if* $\neg N''\psi \in w$ *for some* ψ, *then* $wR_1^c u$ *iff for all* φ, *if* $N''\varphi \wedge \neg N''(\varphi \wedge \psi) \in w$, *then* $\varphi \in u$, *and* $wR_2^c u$ *iff for all* φ, *if* $\neg N''(\varphi \wedge \psi) \in w$, *then* $\varphi \in u$;
- $V^c(p) = \{w \in W^c \mid p \in w\}$.

It is required that the ψ in $N''\varphi \wedge \neg N''(\varphi \wedge \psi) \in w$ and $\neg N''(\varphi \wedge \psi) \in w$ be the same as that in $\neg N''\psi \in w$. Note that the choice of ψ in the above second case of the canonical relations is inessential to R_1^c and R_2^c, as shown in Remark 10. It is also worth noting that R_1^c and R_2^c are inspired by the almost-definability schemas ($\mathbf{N}''\mathbf{AD1}$) and ($\mathbf{N}''\mathbf{AD2}$), respectively, based on the analysis of §2. Recall that in the construction of the canonical model for the minimal logic of $\mathcal{L}(\Box_1, \Box_2)$, the canonical relations R_1^c and R_2^c are defined as, respectively, $wR_1^c u$ iff for all φ, if $\Box_1\varphi \in w$ then $\varphi \in u$, and $wR_2^c u$ iff for all φ, if $\Box_2\varphi \in w$ then $\varphi \in u$. Now given a precondition that $\neg N''\psi \in w$ for some ψ, according to ($\mathbf{N}''\mathbf{AD1}$), $\Box_1\varphi \in w$ can be replaced with $N''\varphi \wedge \neg N''(\varphi \wedge \psi) \in w$, and according to ($\mathbf{N}''\mathbf{AD2}$), $\Box_2\varphi \in w$ can be replaced with $\neg N''(\varphi \wedge \psi) \in w$; otherwise, we let w access and only access itself.

Propositions 54 and 55 list some logical properties of μ_1 and μ_2, respectively. They are important in the proof of Lemma 56 below.

PROPOSITION 54. *Let* $w \in W^c$ *such that* $\neg N''\varphi \in w$. *Define* $\mu_1(w) = \{\chi \mid N''\chi \wedge \neg N''(\chi \wedge \varphi) \in w\}$. *Then*

(1) $\mu_1(w) \neq \emptyset$.
(2) *If* $\psi, \chi \in \mu_1(w)$, *then* $\psi \wedge \chi \in \mu_1(w)$.
(3) *If* $\psi \in \mu_1(w)$ *and* $\vdash \psi \to \chi$, *then* $\chi \in \mu_1(w)$.

Proof. Suppose $w \in W^c$ such that $\neg N''\varphi \in w$.

(1) By axiom PC and rule N″R1, $N''\top \in w$; by supposition and Proposition 51(4), $\neg N''(\top \wedge \varphi) \in w$. Thus $N''\top \wedge \neg N''(\top \wedge \varphi) \in w$, and hence $\top \in \mu_1(w)$. This entails that $\mu_1(w) \neq \emptyset$.
(2) Assume that $\psi, \chi \in \mu_1(w)$. Then $N''\psi \wedge \neg N''(\psi \wedge \varphi) \in w$ and $N''\chi \wedge \neg N''(\chi \wedge \varphi) \in w$. By axiom C1, $N''(\psi \wedge \chi) \in w$; by axioms C2, PC and Proposition 51(4), we can show that $\neg N''((\psi \wedge \chi) \wedge \varphi) \in w$. Thus $N''(\psi \wedge \chi) \wedge \neg N''((\psi \wedge \chi) \wedge \varphi) \in w$, and hence $\psi \wedge \chi \in \mu_1(w)$.
(3) Assume that $\psi \in \mu_1(w)$ and $\vdash \psi \to \chi$. Then $N''\psi \wedge \neg N''(\psi \wedge \varphi) \in w$. By Proposition 51(1), $N''\chi \in w$; by Proposition 51(3), $\neg N''(\chi \wedge \varphi) \in w$. Thus $N''\chi \wedge \neg N''(\chi \wedge \varphi) \in w$, and hence $\chi \in \mu_1(w)$.

□

PROPOSITION 55. *Let* $w \in W^c$ *such that* $\neg N''\psi \in w$. *Define* $\mu_2(w) = \{\chi \mid \neg N''(\chi \wedge \psi) \in w\}$. *Then*

(1) $\mu_2(w) \neq \emptyset$.
(2) *If* $\varphi, \chi \in \mu_2(w)$, *then* $\varphi \wedge \chi \in \mu_2(w)$.
(3) *If* $\varphi \in \mu_2(w)$ *and* $\vdash \varphi \to \chi$, *then* $\chi \in \mu_2(w)$.

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Proof. Suppose that $w \in W^c$ such that $\neg N''\psi \in w$.

(1) By supposition and axiom PC and Proposition 51(4), we have $\neg N''(\top \wedge \psi) \in w$, and thus $\top \in \mu_2(w)$.
(2) Assume that $\varphi, \chi \in \mu_2(w)$. Then $\neg N''(\varphi \wedge \psi) \in w$ and $\neg N''(\chi \wedge \psi) \in w$. Then by axioms C2 and PC and Proposition 51(4), it follows that $\neg N''((\varphi \wedge \chi) \wedge \psi) \in w$, and thus $\varphi \wedge \chi \in \mu_2(w)$.
(3) Assume that $\varphi \in \mu_2(w)$ and $\vdash \varphi \to \chi$. Then $\neg N''(\varphi \wedge \psi) \in w$. By $\vdash \varphi \to \chi$ and Proposition 51(2), it follows that $\neg N''(\chi \wedge \psi) \in w$, and thus $\chi \in \mu_2(w)$.

□

LEMMA 56. *For all $w \in W^c$, for all $\varphi \in \mathcal{L}(N'')$, we have*

$$\mathcal{M}^c, w \vDash \varphi \text{ iff } \varphi \in w.$$

Proof. By induction on φ, where the nontrivial case is $N''\varphi$.

Suppose that $N''\varphi \in w$, to show that $\mathcal{M}^c, w \vDash N''\varphi$. If $R_1^c(w) \vDash \varphi$, then we immediately have $\mathcal{M}^c, w \vDash N''\varphi$. Otherwise, that is, $R_1^c(w) \nvDash \varphi$, it remains only to show that $R_2^c(w) \nvDash \varphi$. It follows by induction hypothesis that there exists $u \in W^c$ such that $wR_1^c u$ and $\varphi \notin u$. According to the definition of R_1^c and R_2^c, we distinguish two cases.

- $\neg N''\psi \in w$ for no ψ. Then as $wR_1^c u$, we have $w = u$, and thus $\varphi \notin w$, which by induction hypothesis implies that $\mathcal{M}^c, w \nvDash \varphi$. Since $w = w$, we infer that $wR_2^c w$, and thus $R_2^c(w) \nvDash \varphi$.
- $\neg N''\psi \in w$ for some ψ. In this case, from $wR_1^c u$ and $\varphi \notin u$ and $N''\varphi \in w$, it follows that $\neg N''(\varphi \wedge \psi) \notin w$, that is, $N''(\varphi \wedge \psi) \in w$. To show that $R_2^c(w) \nvDash \varphi$, by Lindenbaum's Lemma and induction hypothesis, it suffices to show that $\mu_2(w) \cup \{\neg\varphi\}$ is consistent, where $\mu_2(w) = \{\chi \mid \neg N''(\chi \wedge \psi) \in w\}$. By Lemma 55(1), $\mu_2(w) \neq \emptyset$.

  If $\mu_2(w) \cup \{\neg\varphi\}$ is inconsistent, then there are $\chi_1, \ldots, \chi_n \in \mu_2(w)$ such that

  $$\vdash \chi_1 \wedge \cdots \wedge \chi_n \to \varphi.$$

  By Lemma 55(2) and (3), we obtain that $\varphi \in \mu_2(w)$, thus $\neg N''(\varphi \wedge \psi) \in w$: a contradiction. Thus $\mu_2(w) \cup \{\neg\varphi\}$ is consistent, as desired.

Conversely, assume that $N''\varphi \notin w$, to prove that $\mathcal{M}^c, w \nvDash N''\varphi$, that is, $R_1^c(w) \nvDash \varphi$ and $R_2^c(w) \vDash \varphi$. By assumption, $\neg N''\varphi \in w$. Using axiom PC and Proposition 51(4), we derive that $\neg N''(\varphi \wedge \varphi) \in w$. Then for all $u \in W^c$ such that $wR_2^c u$, we have $\varphi \in u$, which by induction hypothesis entails that $\mathcal{M}^c, u \vDash \varphi$. This means that $R_2^c(w) \vDash \varphi$. By Lindenbaum's Lemma and induction hypothesis, it remains only to show that $\mu_1(w) \cup \{\neg\varphi\}$ is consistent, where $\mu_1(w) = \{\chi \mid N''\chi \wedge \neg N''(\chi \wedge \varphi) \in w\}$. By Lemma 54(1), $\mu_1(w) \neq \emptyset$.

If $\mu_1(w) \cup \{\neg\varphi\}$ is inconsistent, then there are $\chi_1, \ldots, \chi_n \in \mu_1(w)$ such that

$$\vdash \chi_1 \wedge \cdots \wedge \chi_n \to \varphi.$$

By Lemma 54(2) and (3), we derive that $\varphi \in \mu_1(w)$, and thus $N''\varphi \in w$, which contradicts the assumption. □

THEOREM 57. $\mathbf{K}^{\mathbf{N}''}$ *is sound and strongly complete with respect to the class of all frames.*

THEOREM 58. $\mathbf{K}^{\mathbf{N}''}$ *is sound and strongly complete with respect to the class of all frames $\langle W, R_1, R_2\rangle$, where $R_2 \subseteq R_1$.*

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Proof. By Theorem 57, it remains only to show that $R_2^c \subseteq R_1^c$. This is straightforward by definitions of R_1^c and R_2^c. □

We can also axiomatize N'' over special frames, like the case for the operator N, with the only change that all occurrences of N are replaced with $\neg N''$. The reason is that, as observed in Remark 5, $\neg N''\varphi$ is equivalent to $\Box_2\varphi \wedge \neg\Box_1\varphi$, and if we think of $\Box_2$ and $\Box_1$ as $\boxed{\mathrm{c}}$ and $\Box$, respectively, then $\neg N''$ amounts to N.

§6. The operator N'''. Another variation of N, namely N''', is also proposed in [28, p. 229], where $N'''\varphi := \Box_1\varphi \vee \Box_2\neg\varphi$. Syntactically, by enriching the language of propositional logic with the sole modality N''', we obtain the language $\mathcal{L}(N''')$. Semantically, as before, given a model $\mathcal{M} = \langle W, R_1, R_2, V\rangle$ and $w \in W$, $\Box_1$ and $\Box_2$ are interpreted as before, and

$$\mathcal{M}, w \vDash N'''\varphi \iff R_1(w) \vDash \varphi \text{ or } R_2(w) \vDash \neg\varphi.$$

This is exactly the semantics of the operator $\boxplus$ in [13]. First, we observe the following almost-definability schemas, which guide us to propose the desired canonical model later:

$$\neg N'''\psi \to (\Box_1\varphi \leftrightarrow N'''\varphi \wedge N'''(\varphi \vee \psi)) \qquad (\mathbf{N'''AD1})$$
$$\neg N'''\psi \to (\Box_2\varphi \leftrightarrow N'''\neg\varphi \wedge N'''(\varphi \vee \psi)) \qquad (\mathbf{N'''AD2}).$$

PROPOSITION 59. *Let $\mathcal{F} = \langle W, R_1, R_2\rangle$ such that $R_1 \subseteq R_2$. Then*

(1) $\mathcal{F} \vDash \neg N'''\psi \to (\Box_1\varphi \leftrightarrow N'''\varphi \wedge N'''(\varphi \vee \psi))$.
(2) $\mathcal{F} \vDash \neg N'''\psi \to (\Box_2\varphi \leftrightarrow N'''\neg\varphi \wedge N'''(\varphi \vee \psi))$.

Proof. Let $\mathcal{M} = \langle W, R_1, R_2, V\rangle$ be a model such that $R_1 \subseteq R_2$ and $w \in W$. Suppose that $\mathcal{M}, w \vDash \neg N'''\psi$, then $R_1(w) \nvDash \psi$ and $R_2(w) \nvDash \neg\psi$ (thus $R_2(w) \nvDash \neg(\varphi \vee \psi)$).

(1) It suffices to show that $\mathcal{M}, w \vDash \Box_1\varphi \leftrightarrow N'''\varphi \wedge N'''(\varphi \vee \psi)$.
Firstly, assume that $\mathcal{M}, w \vDash \Box_1\varphi$, then $R_1(w) \vDash \varphi$. Thus $R_1(w) \vDash \varphi \vee \psi$. It then follows that $\mathcal{M}, w \vDash N'''\varphi$ and $\mathcal{M}, w \vDash N'''(\varphi \vee \psi)$.
Conversely, assume, for a contradiction, that $\mathcal{M}, w \vDash N'''\varphi$ and $\mathcal{M}, w \vDash N'''(\varphi \vee \psi)$ but $\mathcal{M}, w \nvDash \Box_1\varphi$. By $\mathcal{M}, w \nvDash \Box_1\varphi$, we have $R_1(w) \nvDash \varphi$, and then due to $\mathcal{M}, w \vDash N'''\varphi$, we infer that $R_2(w) \vDash \neg\varphi$. Moreover, by $\mathcal{M}, w \vDash N'''(\varphi \vee \psi)$, we derive that $R_1(w) \vDash \varphi \vee \psi$ or $R_2(w) \vDash \neg(\varphi \vee \psi)$, and thus $R_1(w) \vDash \varphi \vee \psi$. This entails that $R_1(w) \nvDash \neg\varphi$: otherwise, we would have $R_1(w) \vDash \psi$, a contradiction. Since $R_1 \subseteq R_2$, it follows that $R_2(w) \nvDash \neg\varphi$: a contradiction.

(2) It remains only to prove that $\mathcal{M}, w \vDash \Box_2\varphi \leftrightarrow N'''\neg\varphi \wedge N'''(\varphi \vee \psi)$.
Firstly, assume that $\mathcal{M}, w \vDash \Box_2\varphi$, then $R_2(w) \vDash \varphi$. It follows immediately that $\mathcal{M}, w \vDash N'''\neg\varphi$. If $\mathcal{M}, w \nvDash N'''(\varphi \vee \psi)$, then $R_1(w) \nvDash \varphi \vee \psi$. As $R_1 \subseteq R_2$, $R_2(w) \nvDash \varphi \vee \psi$, and thus $R_2(w) \nvDash \varphi$: a contradiction. Hence $\mathcal{M}, w \vDash N'''(\varphi \vee \psi)$.
Conversely, assume towards contradiction that $\mathcal{M}, w \vDash N'''\neg\varphi$ and $\mathcal{M}, w \vDash N'''(\varphi \vee \psi)$, but $\mathcal{M}, w \nvDash \Box_2\varphi$. From $\mathcal{M}, w \vDash N'''\neg\varphi$, it follows that $R_1(w) \vDash \neg\varphi$ or $R_2(w) \vDash \varphi$; from $\mathcal{M}, w \nvDash \Box_2\varphi$, it follows that $R_2(w) \nvDash \varphi$. This entails that $R_1(w) \vDash \neg\varphi$. By $\mathcal{M}, w \vDash N'''(\varphi \vee \psi)$, we obtain $R_1(w) \vDash \varphi \vee \psi$ or $R_2(w) \vDash \neg(\varphi \vee \psi)$. If $R_1(w) \vDash \varphi \vee \psi$, then since $R_1(w) \vDash \neg\varphi$, it follows

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that $R_1(w) \vDash \psi$: a contradiction; if $R_2(w) \vDash \neg(\varphi \vee \psi)$, then $R_2(w) \vDash \neg\psi$: a contradiction again.

□

Unlike N and N' but like N'', the operator N''' does not have the property of convexity. In other words, the transition from $\varphi \to \psi$ and $\psi \to \chi$ to $(N'''\varphi \wedge N'''\chi) \to N'''\psi$ does not preserve validity. Here we introduce some notation. We say that a frame (model) with accessibility relations R_1 and R_2 is a $\subseteq$-*frame* ($\subseteq$-model, resp.), if $R_1 \subseteq R_2$.

PROPOSITION 60. $\dfrac{\varphi \to \psi \quad \psi \to \chi}{(N'''\varphi \wedge N'''\chi) \to N'''\psi}$ *is* not *validity-preserving* (*even on* $\subseteq$-*frames*).

Proof. Since $\vDash p \wedge q \to q$ and $\vDash q \to q \vee r$, it remains only to show that $\nvDash N'''(p \wedge q) \wedge N'''(q \vee r) \to N'''q$. Consider the following model $\mathcal{M}$:

$$u : \neg q, r \xleftarrow{1,2} w \xrightarrow{2} v : \neg p, q.$$

One may check that $\mathcal{M}$ is a $\subseteq$-model. Moreover, $R_2(w) \vDash \neg(p \wedge q)$, thus $\mathcal{M}, w \vDash N'''(p \wedge q)$; $R_1(w) \vDash q \vee r$, thus $\mathcal{M}, w \vDash N'''(q \vee r)$. However, $R_1(w) \nvDash q$ and $R_2(w) \nvDash \neg q$, thus $\mathcal{M}, w \nvDash N'''q$. Therefore, $\mathcal{M}, w \nvDash N'''(p \wedge q) \wedge N'''(q \vee r) \to N'''q$, as desired. □

Despite this, like N'', N''' has the property of coconvexity. That is, the transition from $\varphi \to \psi$ and $\psi \to \chi$ to $N'''\psi \to (N'''\varphi \vee N'''\chi)$ preserves validity.

PROPOSITION 61. $\dfrac{\varphi \to \psi \quad \psi \to \chi}{N'''\psi \to (N'''\varphi \vee N'''\chi)}$ *is validity-preserving.*

Proof. Suppose that $\vDash \varphi \to \psi$ and $\vDash \psi \to \chi$, to show that $\vDash N'''\psi \to (N'''\varphi \vee N'''\chi)$. For this, assume that there is a model $\mathcal{M} = \langle W, R_1, R_2, V\rangle$ and $w \in W$ such that $\mathcal{M}, w \vDash N'''\psi$. By $\mathcal{M}, w \vDash N'''\psi$, we have $R_1(w) \vDash \psi$ or $R_2(w) \vDash \neg\psi$. If the former is the case, then by $\vDash \psi \to \chi$, we infer that $R_1(w) \vDash \chi$, which implies that $\mathcal{M}, w \vDash N'''\chi$; if the latter is the case, then by $\vDash \varphi \to \psi$, we derive that $R_2(w) \vDash \neg\varphi$, which entails that $\mathcal{M}, w \vDash N'''\varphi$. Either case gives us $\mathcal{M}, w \vDash N'''\varphi \vee N'''\chi$, as desired. □

6.1. Axiomatizations. In this part, we axiomatize $\mathcal{L}(N''')$ over various frames. We start with the minimal logic.

6.1.1. Minimal logic and soundness.

DEFINITION 62. *The minimal system of* $\mathcal{L}(N''')$, *denoted* $\mathbf{K}^{\mathbf{N}'''}$, *consists of the following axioms and inference rules*:

PC	*All instances of propositional tautologies*
*D*1	$N'''\top$
*D*2	$N'''\bot$
*D*3	$N'''\varphi \wedge N'''\psi \to N'''(\varphi \wedge \psi)$
*D*4	$N'''\varphi \wedge N'''\psi \to N'''(\varphi \vee \psi)$
*D*5	$N'''\varphi \to N'''(\varphi \vee \psi) \vee N'''(\neg\varphi \vee \chi)$
*D*6	$\neg N'''\psi \wedge N'''\neg\varphi \wedge N'''(\varphi \vee \psi) \to N'''\varphi$

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MP *From* φ *and* $\varphi \to \psi$ *infer* ψ
$N'''R1$ *From* $\varphi \to \psi$ *and* $\psi \to \chi$ *infer* $N'''\psi \to (N'''\varphi \vee N'''\chi)$
$N'''R2$ *From* $\psi \to \chi$ *infer* $\neg N'''\varphi \wedge N'''\psi \wedge N'''(\psi \vee \varphi) \to N'''\chi$
$N'''R3$ *From* $\psi \to \neg\chi$ *infer* $\neg N'''\varphi \wedge N'''\neg\psi \wedge N'''(\psi \vee \varphi) \to N'''\chi$.

As before, all these axioms and inference rules can be obtained according to the need of showing that the truth lemma (Lemma 68) holds and that the canonical model is as desired. Axiom D6 plays the core role in guaranteeing the canonical model below to be a $\subseteq$-model. The following result is derivable from N'''R1 and N'''R2.

PROPOSITION 63.

(1) *If* $\vdash \varphi \leftrightarrow \psi$, *then* $\vdash N'''\varphi \leftrightarrow N'''\psi$.
(2) *If* $\vdash \psi \to \chi$, *then* $\vdash \neg N'''\varphi \wedge N'''(\psi \vee \varphi) \to N'''(\chi \vee \varphi)$.

PROPOSITION 64. $\mathbf{K}^{\mathrm{N}'''}$ *is sound with respect to the class of all* $\subseteq$*-frames.*

Proof. The soundness of N'''R1 has been shown in Proposition 61. The verifications for D1 and D2 are straightforward. Let $\mathcal{M} = \langle W, R_1, R_2, V\rangle$ be a model such that $R_1 \subseteq R_2$ and $w \in W$.

For D3. Assume that $\mathcal{M}, w \vDash N'''\varphi$ and $\mathcal{M}, w \vDash N'''\psi$, to show that $\mathcal{M}, w \vDash N'''(\varphi \wedge \psi)$. By assumption, $R_1(w) \vDash \varphi$ or $R_2(w) \vDash \neg\varphi$, and $R_1(w) \vDash \psi$ or $R_2(w) \vDash \neg\psi$. If $R_2(w) \vDash \neg\varphi$ or $R_2(w) \vDash \neg\psi$, then $R_2(w) \vDash \neg(\varphi \wedge \psi)$, and thus $\mathcal{M}, w \vDash N'''(\varphi \wedge \psi)$; otherwise, then $R_1(w) \vDash \varphi$ and $R_1(w) \vDash \psi$, and thus $R_1(w) \vDash \varphi \wedge \psi$, and we also have $\mathcal{M}, w \vDash N'''(\varphi \wedge \psi)$.

For D4. Suppose that $\mathcal{M}, w \vDash N'''\varphi$ and $\mathcal{M}, w \vDash N'''\psi$, to prove that $\mathcal{M}, w \vDash N'''(\varphi \vee \psi)$. By supposition, $R_1(w) \vDash \varphi$ or $R_2(w) \vDash \neg\varphi$, and $R_1(w) \vDash \psi$ or $R_2(w) \vDash \neg\psi$. If $R_1(w) \vDash \varphi$ or $R_1(w) \vDash \psi$, then $R_1(w) \vDash \varphi \vee \psi$, and thus $\mathcal{M}, w \vDash N'''(\varphi \vee \psi)$; otherwise, we would have $R_2(w) \vDash \neg\varphi$ and $R_2(w) \vDash \neg\psi$, which implies that $R_2(w) \vDash \neg(\varphi \vee \psi)$, and hence $\mathcal{M}, w \vDash N'''(\varphi \vee \psi)$ again. Therefore $\mathcal{M}, w \vDash N'''(\varphi \vee \psi)$.

For D5. Assume that $\mathcal{M}, w \vDash N'''\varphi$. Then $R_1(w) \vDash \varphi$ or $R_2(w) \vDash \neg\varphi$. The first case implies that $R_1(w) \vDash \varphi \vee \psi$, thus $\mathcal{M}, w \vDash N'''(\varphi \vee \psi)$. The second case implies that $R_2(w) \vDash \neg\varphi \vee \chi$. Since $R_1 \subseteq R_2$, it follows that $R_1(w) \vDash \neg\varphi \vee \chi$, hence $\mathcal{M}, w \vDash N'''(\neg\varphi \vee \chi)$. Therefore, $\mathcal{M}, w \vDash N'''(\varphi \vee \psi) \vee N'''(\neg\varphi \vee \chi)$.

For D6. Suppose that $\mathcal{M}, w \vDash \neg N'''\psi$ and $\mathcal{M}, w \vDash N'''\neg\varphi$ and $\mathcal{M}, w \vDash N'''(\varphi \vee \psi)$, to prove that $\mathcal{M}, w \vDash N'''\varphi$. By $\mathcal{M}, w \vDash \neg N'''\psi$, we infer that $R_1(w) \nvDash \psi$ and $R_2(w) \nvDash \neg\psi$. By $\mathcal{M}, w \vDash N'''\neg\varphi$, we derive that $R_1(w) \vDash \neg\varphi$ or $R_2(w) \vDash \varphi$. By $\mathcal{M}, w \vDash N'''(\varphi \vee \psi)$, we obtain that $R_1(w) \vDash \varphi \vee \psi$ or $R_2(w) \vDash \neg(\varphi \vee \psi)$. If $R_2(w) \vDash \neg(\varphi \vee \psi)$, then $R_2(w) \vDash \neg\psi$: a contradiction, and thus $R_1(w) \vDash \varphi \vee \psi$. From $R_1(w) \nvDash \psi$, it follows that $R_1(w) \nvDash \neg\varphi$, thus $R_2(w) \vDash \varphi$. Since $R_1 \subseteq R_2$, we have $R_1(w) \vDash \varphi$, and therefore $\mathcal{M}, w \vDash N'''\varphi$.

For N'''R2. Assume that $\vDash \psi \to \chi$ and $\mathcal{M}, w \vDash \neg N'''\varphi \wedge N'''\psi \wedge N'''(\psi \vee \varphi)$, to show that $\mathcal{M}, w \vDash N'''\chi$. By $\mathcal{M}, w \vDash \neg N'''\varphi$, $R_1(w) \nvDash \varphi$ and $R_2(w) \nvDash \neg\varphi$. By $\mathcal{M}, w \vDash N'''\psi$, it follows that $R_1(w) \vDash \psi$ or $R_2(w) \vDash \neg\psi$. By $\mathcal{M}, w \vDash N'''(\psi \vee \varphi)$, $R_1(w) \vDash \psi \vee \varphi$ or $R_2(w) \vDash \neg(\psi \vee \varphi)$. Since $R_2(w) \nvDash \neg\varphi$, we have $R_2(w) \nvDash \neg(\psi \vee \varphi)$, and thus $R_1(w) \vDash \psi \vee \varphi$. Moreover, it is impossible that $R_2(w) \vDash \neg\psi$: otherwise, as $R_1 \subseteq R_2$, we get $R_1(w) \vDash \neg\psi$, and then $R_1(w) \vDash \varphi$, a contradiction. This shows that $R_1(w) \vDash \psi$. Then by assumption, we infer that $R_1(w) \vDash \chi$, and thus $\mathcal{M}, w \vDash N'''\chi$.

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For $\mathbf{N}'''$R3. Suppose that $\vDash \psi \to \neg\chi$ and $\mathcal{M}, w \vDash \neg N'''\varphi \wedge N'''\neg\psi \wedge N'''(\psi \vee \varphi)$, to prove that $\mathcal{M}, w \vDash N'''\chi$. Since $\mathcal{M}, w \vDash \neg N'''\varphi$, it follows that $R_1(w) \nvDash \varphi$ and $R_2(w) \nvDash \neg\varphi$. By $\mathcal{M}, w \vDash N'''(\psi \vee \varphi)$, we obtain that $R_1(w) \vDash \psi \vee \varphi$ or $R_2(w) \vDash \neg(\psi \vee \varphi)$. If the latter is the case, $R_2(w) \vDash \neg\varphi$, a contradiction, and thus $R_1(w) \vDash \psi \vee \varphi$. By $\mathcal{M}, w \vDash N'''\neg\psi$, it follows that $R_1(w) \vDash \neg\psi$ or $R_2(w) \vDash \psi$. If $R_1(w) \vDash \neg\psi$, then $R_1(w) \vDash \varphi$, a contradiction. Thus $R_2(w) \vDash \psi$, and hence using the supposition, we derive that $R_2(w) \vDash \neg\chi$, which implies that $\mathcal{M}, w \vDash N'''\chi$. □

6.1.2. Completeness. For completeness, we construct the canonical model for $\mathbf{K}^{\mathbf{N}'''}$.

DEFINITION 65. *The canonical model for* $\mathbf{K}^{\mathbf{N}'''}$ *is a tuple* $\mathcal{M}^c = \langle W^c, R_1^c, R_2^c, V^c\rangle$, *where*

- $W^c = \{w \mid w$ *is a maximal consistent set for* $\mathbf{K}^{\mathbf{N}'''}\}$;
- *if* $\neg N'''\psi \in w$ *for no* ψ, *then* wR_1^cu *iff* $w = u$, *and* wR_2^cu *iff* $w = u$;
- *if* $\neg N'''\psi \in s$ *for some* ψ, *then* wR_1^cu *iff for every* φ, *if* $N'''\varphi \wedge N'''(\varphi \vee \psi) \in w$, *then* $\varphi \in u$, *and* wR_2^cu *iff for every* φ, *if* $N'''\neg\varphi \wedge N'''(\varphi \vee \psi) \in w$, *then* $\varphi \in u$;
- $V^c(p) = \{w \in W^c \mid p \in w\}$.

As before, it is required that the ψ in $N'''\varphi \wedge N'''(\varphi \vee \psi) \in w$ and $N'''\neg\varphi \wedge N'''(\varphi \vee \psi) \in w$ be the same as that in $\neg N'''\psi \in w$. Note that the choice of ψ in the above second case of the canonical relations is inessential to R_1^c and R_2^c, as shown in Remark 10. It is also worth noting that R_1^c and R_2^c in the above definition are inspired by almost definability schemas ($\mathbf{N}'''\mathbf{AD1}$) and ($\mathbf{N}'''\mathbf{AD2}$), respectively, based on the analysis of §2. Recall that in the construction of the canonical model for the minimal logic of $\mathcal{L}(\Box_1, \Box_2)$, the canonical relations R_1^c and R_2^c are defined as, respectively, wR_1^cu iff for all φ, if $\Box_1\varphi \in w$ then $\varphi \in u$, and wR_2^cu iff for all φ, if $\Box_2\varphi \in w$ then $\varphi \in u$. Now given a precondition that $\neg N'''\psi \in w$ for some ψ, according to ($\mathbf{N}'''\mathbf{AD1}$), $\Box_1\varphi \in w$ can be replaced with $N'''\varphi \wedge N'''(\varphi \vee \psi) \in w$, and according to ($\mathbf{N}'''\mathbf{AD2}$), $\Box_2\varphi \in w$ can be replaced with $N'''\neg\varphi \wedge N'''(\varphi \vee \psi) \in w$; otherwise, we let w access and only access itself.

Proposition 66 lists some logical properties of Σ_1 and Σ_2, respectively. They are important in the proof of Lemma 68 below.

PROPOSITION 66. *Let* $w \in W^c$ *and* $\neg N'''\varphi \in w$. *Define* $\Sigma_1(w) = \{\chi \mid N'''\chi \wedge N'''(\chi \vee \varphi) \in w\}$ *and* $\Sigma_2(w) = \{\chi \mid N'''\neg\chi \wedge N'''(\chi \vee \varphi) \in w\}$. *Then for* $i \in \{1, 2\}$,

(1) $\Sigma_i(w) \neq \emptyset$.
(2) *If* $\psi, \chi \in \Sigma_i(w)$, *then* $\psi \wedge \chi \in \Sigma_i(w)$.
(3) *If* $\psi \in \Sigma_1(w)$ *and* $\vdash \psi \to \chi$, *then* $\chi \in \Sigma_1(w)$.
(4) *If* $\psi \in \Sigma_2(w)$ *and* $\vdash \psi \to \neg\chi$, *then* $N'''\chi \in w$.

Proof. Let $\neg N'''\varphi \in w$, where $w \in W^c$.

(1) By axioms D1 and D2.
(2) Suppose that $\psi \in \Sigma_1(w)$ and $\chi \in \Sigma_1(w)$. Then $N'''\psi \wedge N'''(\psi \vee \varphi) \in w$ and $N'''\chi \wedge N'''(\chi \vee \varphi) \in w$. By axiom D3 and Proposition 63(1), we obtain that $N'''(\psi \wedge \chi) \in w$ and $N'''((\psi \wedge \chi) \vee \varphi) \in w$. Therefore, $\psi \wedge \chi \in \Sigma_1(w)$.

Assume that $\psi \in \Sigma_2(w)$ and $\chi \in \Sigma_2(w)$. Then $N'''\neg\psi \wedge N'''(\psi \vee \varphi) \in w$, and $N'''\neg\chi \wedge N'''(\chi \vee \varphi) \in w$. Then by axiom D4, $N'''(\neg\psi \vee \neg\chi) \in w$, equivalently, $N'''\neg(\psi \wedge \chi) \in w$; by axiom D3 and Proposition 63(1), $N'''((\psi \wedge \chi) \vee \varphi) \in w$. Therefore, $\psi \wedge \chi \in \Sigma_2(w)$.

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(3) Suppose that $\psi \in \Sigma_1(w)$ and $\vdash \psi \to \chi$. Then $N'''\psi \wedge N'''(\psi \vee \varphi) \in w$. Since $\neg N'''\varphi \in w$, by rule $\mathrm{N}'''\mathrm{R2}$ and Proposition 63(2), it follows that $N'''\chi \wedge N'''(\chi \vee \varphi) \in w$, and therefore, $\chi \in \Sigma_1(w)$.
(4) Assume that $\psi \in \Sigma_2(w)$ and $\vdash \psi \to \neg\chi$. Then $N'''\neg\psi \wedge N'''(\psi \vee \varphi) \in w$. Since $\neg N'''\varphi \in w$, by rule $\mathrm{N}'''\mathrm{R3}$, it follows that $N'''\chi \in w$.

□

PROPOSITION 67. *$\mathcal{M}^c$ is indeed a $\subseteq$-model. That is, $R_1^c \subseteq R_2^c$.*

Proof. This is provided by axiom D6. □

LEMMA 68. *For all $w \in W^c$, for all $\varphi \in \mathcal{L}(N''')$, we have*

$$\mathcal{M}^c, w \vDash \varphi \textit{ iff } \varphi \in w.$$

Proof. By induction on φ. The nontrivial case is $N'''\varphi$.

Suppose that $N'''\varphi \in w$ (equivalently, $N'''\neg\neg\varphi \in w$), to show that $\mathcal{M}^c, s \vDash N'''\varphi$. Assume not, then $R_1^c(w) \nvDash \varphi$ and $R_2^c(w) \nvDash \neg\varphi$. By induction hypothesis, this means that (1) there exists $u \in W^c$ such that $wR_1^c u$ and $\varphi \notin u$ and (2) there exists $v \in W^c$ such that $wR_2^c v$ and $\neg\varphi \notin v$ (thus $\varphi \in v$). According to the definition of R_1^c and R_2^c, we consider two cases.

- $\neg N'''\psi \in w$ for no ψ. Then $w = u$ and $w = v$, thus $u = v$: a contradiction.
- $\neg N'''\psi \in w$ for some ψ. In this case, by (1), we infer that $N'''\varphi \wedge N'''(\varphi \vee \psi) \notin w$, thus $N'''(\varphi \vee \psi) \notin w$; by (2), we derive that $N'''\neg\neg\varphi \wedge N'''(\neg\varphi \vee \psi) \notin w$, thus $N'''(\neg\varphi \vee \psi) \notin w$. By axiom D5, $N'''\varphi \notin w$, which contradicts the supposition.

  Conversely, suppose that $N'''\varphi \notin w$, then $\neg N'''\varphi \in w$. To prove that $\mathcal{M}, w \nvDash N'''\varphi$, by induction hypothesis, it remains only to show (1) and (2). Define Σ_1 and Σ_2 as in Proposition 66. According to the definition of R_1^c and R_2^c and Lindenbaum's Lemma, it suffices to show that both $\Sigma_1(w) \cup \{\neg\varphi\}$ and $\Sigma_2(w) \cup \{\varphi\}$ are consistent. By Proposition 66(1), $\Sigma_1(w)$ and $\Sigma_2(w)$ are both nonempty.

  If $\Sigma_1(w) \cup \{\neg\varphi\}$ is inconsistent, then there are $\varphi_1, \ldots, \varphi_n \in \Sigma_1(w)$ and

$$\vdash \varphi_1 \wedge \cdots \wedge \varphi_n \to \varphi.$$

  Then by items (2) and (3) of Proposition 66, we derive that $N'''\varphi \in w$: a contradiction. Therefore, $\Sigma_1(w) \cup \{\neg\varphi\}$ is consistent.

  If $\Sigma_2(w) \cup \{\varphi\}$ is inconsistent, then there exist $\chi_1, \ldots, \chi_m \in \Sigma_2(w)$ and

$$\vdash \chi_1 \wedge \cdots \wedge \chi_m \to \neg\varphi.$$

  Now using items (2) and (4) of Proposition 66, we derive that $N'''\varphi \in w$: a contradiction. Therefore, $\Sigma_2(w) \cup \{\varphi\}$ is consistent.

□

THEOREM 69. *$\mathbf{K}^{\mathbf{N}'''}$ is sound and strongly complete with respect to the class of $\subseteq$-frames.*

6.2. Extensions.

6.2.1. Serial logic.

THEOREM 70. *$\mathbf{K}^{\mathbf{N}'''}$ is sound and strongly complete with respect to the class of serial $\subseteq$-frames.*

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Proof. By Theorem 69, we only need to show the completeness. Define $\mathcal{M}^c$ as in Definition 65. It remains only to show the seriality of R_1^c. Let $w \in W^c$.

If $\neg N'''\psi \in w$ for no ψ, then as $w = w$, we have $wR_1^c w$. If $\neg N'''\psi \in w$ for some ψ, define $\Sigma_1(w)$ as in Proposition 66, it suffices to prove that $\Sigma_1(w)$ is consistent. The item (1) of Proposition 66 provides the nonemptiness of $\Sigma_1(w)$. Suppose that $\Sigma_1(w)$ is not consistent, then there are $\chi_1, \ldots, \chi_n \in \Sigma_1(w)$ such that $\vdash \chi_1 \wedge \cdots \wedge \chi_n \to \bot$. By items (2) and (3) of Proposition 66, we infer that $\bot \in \Sigma_1(w)$, and thus $N'''(\bot \vee \psi) \in w$, equivalently, $N'''\psi \in w$: a contradicition. □

6.2.2. Reflexive logic. Extending $\mathbf{K}^{\mathbf{N}'''}$ with the following axiom (denoted N'''T), we obtain the system $\mathbf{T}^{\mathbf{N}'''}$:

$$\neg N'''\psi \wedge N'''\varphi \wedge N'''(\varphi \vee \psi) \to \varphi.$$

THEOREM 71. *$\mathbf{T}^{\mathbf{N}'''}$ is sound and strongly complete with respect to the class of reflexive $\subseteq$-frames.*

Proof. For soundness, by Theorem 69, it remains only to show the validity of N'''T. Let $\mathcal{M} = \langle W, R_1, R_2, V\rangle$ be a reflexive $\subseteq$-model and $w \in W$. Suppose that $\mathcal{M}, w \vDash \neg N'''\psi \wedge N'''\varphi \wedge N'''(\varphi \vee \psi)$, to show that $\mathcal{M}, w \vDash \varphi$. Since $\mathcal{M}, w \vDash \neg N'''\psi$, we have $R_1(w) \nvDash \psi$ and $R_2(w) \nvDash \neg\psi$, thus $R_2(w) \nvDash \neg(\varphi \vee \psi)$. Due to $\mathcal{M}, w \vDash N'''(\varphi \vee \psi)$, we get $R_1(w) \vDash \varphi \vee \psi$ or $R_2(w) \vDash \neg(\varphi \vee \psi)$, and then $R_1(w) \vDash \varphi \vee \psi$. Plus $R_1(w) \nvDash \psi$, this implies that $R_1(w) \nvDash \neg\varphi$. Since $R_1 \subseteq R_2$, it follows that $R_2(w) \nvDash \neg\varphi$. Now using $\mathcal{M}, w \vDash N'''\varphi$, we derive that $R_1(w) \vDash \varphi$, from which and the reflexivity of R_1, it follows that $\mathcal{M}, w \vDash \varphi$.

For completeness, define $\mathcal{M}^c$ w.r.t. $\mathbf{T}^{\mathbf{N}'''}$ as in Definition 65. By Theorem 69, it suffices to prove that R_1^c is reflexive.

Let $w \in W^c$. If $\neg N'''\psi \in w$ for no ψ, then as $w = w$, we immediately have $wR_1^c w$. If $\neg N'''\psi \in w$ for some ψ, assume for any φ that $N'''\varphi \wedge N'''(\varphi \vee \psi) \in w$. By axiom N'''T, $\varphi \in w$. Thus $wR_1^c w$ again. □

We conclude this part with two unprovable results which will be used in §6.3.

PROPOSITION 72. *$\neg N'''\psi \to (\varphi \leftrightarrow N'''\varphi \wedge N'''(\varphi \vee \psi))$ is not derivable from $\mathbf{T}^{\mathbf{N}'''}$.*

Proof. We only show that $\neg N'''\psi \wedge \varphi \to N'''(\varphi \vee \psi)$ is not derivable from $\mathbf{T}^{\mathbf{N}'''}$. For this, consider the following reflexive $\subseteq$-model $\mathcal{M} = \langle W, R_1, R_2, V\rangle$, where we omit all reflexive arrows for sake of simplicity:

$$u : \neg p, \neg q \xleftarrow{1,2} w : p \xrightarrow{2} v : q.$$

One may verify that $\mathcal{M}, w \vDash \neg N'''q$ (since wR_1u and $u \nvDash q$ and wR_2v and $v \vDash q$) and $\mathcal{M}, w \vDash p$, whereas $\mathcal{M}, w \nvDash N'''(p \vee q)$ (because $u \nvDash p \vee q$ and $v \vDash p \vee q$). Thus, $\mathcal{M}, w \nvDash \neg N'''q \wedge p \to N'''(p \vee q)$, and hence $\mathcal{M}, w \nvDash \neg N'''\psi \wedge \varphi \to N'''(\varphi \vee \psi)$ and therefore $\mathcal{M} \nvDash \neg N'''\psi \wedge \varphi \to N'''(\varphi \vee \psi)$. By soundness of $\mathbf{T}^{\mathbf{N}'''}$ (Theorem 71), $\neg N'''\psi \wedge \varphi \to N'''(\varphi \vee \psi)$ is not derivable from $\mathbf{T}^{\mathbf{N}'''}$. □

PROPOSITION 73. *$\neg N'''\psi \to (\varphi \leftrightarrow N'''(\varphi \vee \psi))$ is not derivable from $\mathbf{T}^{\mathbf{N}'''}$.*

Proof. We only show that $\neg N'''\psi \wedge N'''(\varphi \vee \psi) \to \varphi$ is not derivable from $\mathbf{T}^{\mathbf{N}'''}$. For this, consider the following reflexive $\subseteq$-model $\mathcal{M} = \langle W, R_1, R_2, V\rangle$, where we omit

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all reflexive arrows for sake of simplicity:

$$u : p, \neg q \xleftarrow{\quad 1,2 \quad} w : \neg p, q \xrightarrow{\quad 2 \quad} v : q.$$

One may verify that $\mathcal{M}, w \vDash \neg N'''q$ (since wR_1u and $u \nvDash q$ and wR_2v and $v \vDash q$) and $\mathcal{M}, w \vDash N'''(p \vee q)$ (because $R_1(w) = \{w, u\} \vDash p \vee q$), whereas $\mathcal{M}, w \nvDash p$. Thus, $\mathcal{M}, w \nvDash \neg N'''q \wedge N'''(p \vee q) \to p$, and hence $\mathcal{M}, w \nvDash \neg N'''\psi \wedge N'''(\varphi \vee \psi) \to \varphi$ and therefore $\mathcal{M} \nvDash \neg N'''\psi \wedge N'''(\varphi \vee \psi) \to \varphi$. By soundness of $\mathbf{T}^{\mathbf{N}'''}$ (Theorem 71), $\neg N'''\psi \wedge N'''(\varphi \vee \psi) \to \varphi$ is not derivable from $\mathbf{T}^{\mathbf{N}'''}$. □

6.2.3. Transitive logic. Extending $\mathbf{K}^{\mathbf{N}'''}$ with the following axioms, we obtain the system $\mathbf{K4}^{\mathbf{N}'''}$:

$\mathrm{N}'''4\text{-}1$ $\neg N'''\psi \wedge N'''\varphi \wedge N'''(\varphi \vee \psi) \to N'''(\neg N'''\psi' \to N'''\varphi \wedge N'''(\varphi \vee \psi')) \wedge$
$N'''((\neg N'''\psi' \to N'''\varphi \wedge N'''(\varphi \vee \psi')) \vee \psi)$

$\mathrm{N}'''4\text{-}2$ $\neg N'''\psi \wedge N'''\neg\varphi \wedge N'''(\varphi \vee \psi) \to N'''\neg(\neg N'''\psi' \to N'''\neg\varphi \wedge$
$N'''(\varphi \vee \psi')) \wedge N'''((\neg N'''\psi' \to N'''\neg\varphi \wedge N'''(\varphi \vee \psi')) \vee \psi)$.

Similar to axioms A4-1 and A4-2 in §3.4.4, axioms $\mathrm{N}'''4\text{-}1$ and $\mathrm{N}'''4\text{-}2$ are obtained from the transitive axioms for $\Box_1$ and $\Box_2$ via a translation induced by almost definability schemas ($\mathbf{N}'''\mathbf{AD1}$) and ($\mathbf{N}'''\mathbf{AD2}$), respectively.

THEOREM 74. $\mathbf{K4}^{\mathbf{N}'''}$ *is sound and strongly complete with respect to the class of transitive* $\subseteq$*-frames.*

Proof. For soundness, by Theorem 69, it suffices to show the validity of axioms $\mathrm{N}'''4\text{-}1$ and $\mathrm{N}'''4\text{-}2$. To see this, just notice that in the transformation of the two axioms starting from the schemas ($\mathbf{N}'''\mathbf{AD1}$) and ($\mathbf{N}'''\mathbf{AD2}$), every step is equivalent, and that the proof of the validity of the two schemas only uses $R_1 \subseteq R_2$ (see Proposition 59).

For completeness, define $\mathcal{M}^c$ w.r.t. $\mathbf{K4}^{\mathbf{N}'''}$ as in Definition 65. By Theorem 69, it remains only to show that both R_1^c and R_2^c are transitive. For this, let $w, u, v \in W^c$.

Suppose that wR_1^cu and uR_1^cv, to show that wR_1^cv. We consider the following cases:

- $\neg N'''\psi \in w$ for no ψ. In this case, $w = u$, and thus wR_1^cv.
- $\neg N'''\psi' \in u$ for no ψ'. In this case, $u = v$, and hence wR_1^cv.
- $\neg N'''\psi \in w$ for some ψ and $\neg N'''\psi' \in u$ for some ψ'. Assume for any φ that $N'''\varphi \wedge N'''(\varphi \vee \psi) \in w$, to prove that $\varphi \in v$. By axiom $\mathrm{N}'''4\text{-}1$, $N'''(\neg N'''\psi' \to N'''\varphi \wedge N'''(\varphi \vee \psi')) \wedge N'''((\neg N'''\psi' \to N'''\varphi \wedge N'''(\varphi \vee \psi')) \vee \psi) \in w$. From wR_1^cu, it follows that $\neg N'''\psi' \to N'''\varphi \wedge N'''(\varphi \vee \psi') \in u$, thus $N'''\varphi \wedge N'''(\varphi \vee \psi') \in u$. Now using uR_1^cv, we derive that $\varphi \in v$, as desired.

Now suppose that wR_2^cu and uR_2^cv, to prove that wR_2^cv. Again, we consider the following cases:

- $\neg N'''\psi \in w$ for no ψ. In this case, $w = u$, and thus wR_2^cv.
- $\neg N'''\psi' \in u$ for no ψ'. In this case, $u = v$, and hence wR_2^cv.
- $\neg N'''\psi \in w$ for some ψ and $\neg N'''\psi' \in u$ for some ψ'. Assume for any φ that $N'''\neg\varphi \wedge N'''(\varphi \vee \psi) \in w$, to show that $\varphi \in v$. By axiom $\mathrm{N}'''4\text{-}2$, $N'''\neg(\neg N'''\psi' \to N'''\neg\varphi \wedge N'''(\varphi \vee \psi')) \wedge N'''((\neg N'''\psi' \to N'''\neg\varphi \wedge N'''(\varphi \vee \psi')) \vee \psi) \in w$. From wR_2^cu, it follows that $\neg N'''\psi' \to N'''\neg\varphi \wedge$

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$N'''(\varphi \vee \psi') \in u$, thus $N'''\neg\varphi \wedge N'''(\varphi \vee \psi') \in u$. Then using $uR_2^c v$, we infer that $\varphi \in v$, as desired. □

6.2.4. **S4**-*logic.* Define $\mathbf{S4}^{\mathbf{N}'''}$ to be the extension of $\mathbf{K4}^{\mathbf{N}'''}$ with axiom N'''T. By Theorems 71 and 74, we have the following result.

THEOREM 75. $\mathbf{S4}^{\mathbf{N}'''}$ *is sound and strongly complete with respect to the class of* $\mathbf{S4}$ $\subseteq$*-frames.*

6.3. Connection with the logic of accidental truths. Let R_1 be the identity relation and R_2 be reflexive. Of course, $R_1 \subseteq R_2$. Then $N'''\varphi$ becomes $\varphi \vee \Box_2\neg\varphi$, equivalently, $\neg\varphi \to \Box_2\neg\varphi$. This is exactly $\circ\neg\varphi$, where the essential operator $\circ$ is the negation of the accident operator $\bullet$. That is,

$$\mathcal{M}, w \vDash N'''\varphi \iff \mathcal{M}, w \vDash \varphi \text{ or } R_2(w) \vDash \neg\varphi.$$

Moreover, $\Box_1\varphi$ becomes φ. That is,

$$\mathcal{M}, w \vDash \Box_1\varphi \iff \mathcal{M}, w \vDash \varphi.$$

Then ($\mathbf{N}'''\mathbf{AD1}$) and ($\mathbf{N}'''\mathbf{AD2}$) become the following almost-definability schemas:

$$\neg N'''\psi \to (\varphi \leftrightarrow N'''\varphi \wedge N'''(\varphi \vee \psi)) \qquad (\mathbf{N}'''\mathbf{AD1}')$$
$$\neg N'''\psi \to (\Box_2\varphi \leftrightarrow N'''\neg\varphi \wedge N'''(\varphi \vee \psi)) \qquad (\mathbf{N}'''\mathbf{AD2}).$$

As a matter of fact, ($\mathbf{N}'''\mathbf{AD1}'$) can be simplified as

$$\neg N'''\psi \to (\varphi \leftrightarrow N'''(\varphi \vee \psi)) \qquad (\mathbf{N}'''\mathbf{AD1}'').$$

One may easily verify that according to this simplified semantics (that is, R_1 being identity), ($\mathbf{N}'''\mathbf{AD1}'$) and ($\mathbf{N}'''\mathbf{AD1}''$) are valid over the class of all frames, whereas ($\mathbf{N}'''\mathbf{AD2}$) is valid over the class of (R_2-)reflexive frames. This contrasts with the fact that ($\mathbf{N}'''\mathbf{AD1}'$) and ($\mathbf{N}'''\mathbf{AD1}''$) are not valid under the original semantics, where R_1 is not the identity relation (see the proofs of Propositions 72 and 73). To provide the completeness, we need to establish a proof system from which ($\mathbf{N}'''\mathbf{AD1}'$) and ($\mathbf{N}'''\mathbf{AD1}''$) are derivable. However, recall from Propositions 72 and 73 that the two schemas are not derivable from $\mathbf{T}^{\mathbf{N}'''}$, so $\mathbf{T}^{\mathbf{N}'''}$ is not a desired system. We will show that the extension of $\mathbf{T}^{\mathbf{N}'''}$ with $\varphi \to N'''\varphi$ and $\neg N'''\psi \wedge N'''(\varphi \vee \psi) \to \varphi$ is a desired minimal logic of accidental truths. We denote the formulas as A1''' and A2''' and the extension as $\mathbf{T1}^{\mathbf{N}'''}$. We can see that the schemas ($\mathbf{N}'''\mathbf{AD1}'$) and ($\mathbf{N}'''\mathbf{AD1}''$) are derivable from $\mathbf{T1}^{\mathbf{N}'''}$ because of axioms A1''' and A2'''.

Based on the schemas ($\mathbf{N}'''\mathbf{AD1}'$) and ($\mathbf{N}'''\mathbf{AD2}'$), the canonical model for $\mathbf{T}^{\mathbf{N}'''}$ can be simplified as follows.

DEFINITION 76. *The* canonical model for $\mathbf{T1}^{\mathbf{N}'''}$ *is a tuple* $\mathcal{M}^c = \langle W^c, R_1^c, R_2^c, V^c\rangle$, *where the only change is that* $N'''\varphi \wedge N'''(\varphi \vee \psi) \in w$ *in the definition of* R_1^c *in Definition* 65 *now becomes* $\phi \in w$, *due to the derivable schema* ($\boldsymbol{N'''AD1'}$). *In other words,*

- $W^c = \{w \mid w$ *is a maximal consistent set for* $\mathbf{T1}^{\mathbf{N}'''}\}$;
- *if* $\neg N'''\psi \in w$ *for no* ψ, *then* $wR_1^c u$ *iff* $w = u$, *and* $wR_2^c u$ *iff* $w = u$;
- *if* $\neg N'''\psi \in w$ *for some* ψ, *then* $wR_1^c u$ *iff for every* φ, *if* $\varphi \in w$, *then* $\varphi \in u$, *and* $wR_2^c u$ *iff for every* φ, *if* $N'''\neg\varphi \wedge N'''(\varphi \vee \psi) \in w$, *then* $\varphi \in u$;
- $V^c(p) = \{w \in W^c \mid p \in w\}$.

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PROPOSITION 77. *R_1^c is the identity relation, and R_2^c is reflexive. Consequently, $R_1^c \subseteq R_2^c$.*

Proof. The proof of the case R_1^c is the same as the case of S^c in Proposition 33.

For the case R_2^c. Let $w \in W^c$. If $\neg N'''\psi \in w$ for no ψ, then as $w = w$, $wR_2^c w$. If $\neg N'''\psi \in w$ for some ψ, if $N'''\neg\varphi \wedge N'''(\varphi \vee \psi) \in w$, then by axiom D6 and axiom N'''T, we derive that $\varphi \in w$, and thus $wR_2^c w$. □

Recall that in Theorem 71, axiom N'''T is shown to be valid over reflexive $\subseteq$-frames, where the reflexivity of R_1 and $R_1 \subseteq R_2$ are used. Since now R_1 is the identity relation and R_2 is reflexive, the reflexivity of R_1 and $R_1 \subseteq R_2$ are automatic. In fact, one may show that NT is valid over the class of all frames (due to the validity of the schema $\mathbf{N'''AD1'}$), without need of the reflexivity of R_2. This gives us axiomatizations of $\mathcal{L}(\bullet)$ over various frames.

THEOREM 78.

(1) *Where $\mathbf{T} \subseteq X$, $\mathbf{T1}^{\mathbf{N'''}}$ is sound and strongly complete with respect to the class of X.*
(2) *Where $\mathbf{S4} \subseteq X \subseteq \mathbf{4}$ and $\mathbf{S41}^{\mathbf{N'''}}$ extends $\mathbf{S4}^{\mathbf{N'''}}$ with $A1'''$ and $A2'''$, $\mathbf{S41}^{\mathbf{N'''}}$ is sound and strongly complete with respect to X.*

Proof. It remains to show the truth lemma for $\mathbf{T1}^{\mathbf{N'''}}$. That is, for all $\varphi \in \mathcal{L}(N''')$, for all $w \in W^c$, we have

$$\mathcal{M}^c, w \vDash \varphi \text{ iff } \varphi \in w.$$

The proof proceeds by induction on φ. The nontrivial case is $N'''\varphi$.

Suppose, for reductio, that $N'''\varphi \in w$ but $\mathcal{M}^c, w \nvDash N'''\varphi$. Then $\mathcal{M}^c, w \nvDash \varphi$ and $R_2^c(w) \nvDash \neg\varphi$. By induction hypothesis, $\varphi \notin w$ (thus $\neg\varphi \in w$) and there exists $u \in W^c$ such that $wR_2^c u$ and $\neg\varphi \notin u$. Since $w \neq u$, by definition of R_2^c, it must be the case that $\neg N'''\psi \in w$ for some ψ. Thus $N'''\neg\neg\varphi \wedge N'''(\neg\varphi \vee \psi) \notin w$. By supposition and Proposition 63(1), we infer that $N'''\neg\neg\varphi \in w$, and thus $N'''(\neg\varphi \vee \psi) \notin w$. From $\neg\varphi \in w$, it follows that $\neg\varphi \vee \psi \in w$, then by axiom A1''', we derive that $N'''(\neg\varphi \vee \psi) \in w$: a contradiction.

Conversely, assume that $N'''\varphi \notin w$ (thus $\neg N'''\varphi \in w$), to show that $\mathcal{M}^c, w \nvDash N'''\varphi$. By induction hypothesis, it remains only to prove that $\varphi \notin w$ and there exists $u \in W^c$ such that $wR_2^c u$ and $\neg\varphi \notin u$. By assumption and axiom A1''', $\varphi \notin w$. Moreover, we show that $\Sigma_2(w) \cup \{\varphi\}$ is consistent, where $\Sigma_2(w) = \{\chi \mid N'''\neg\chi \wedge N'''(\chi \vee \varphi) \in w\}$. By Lemma 66(1), $\Sigma_2(w) \neq \emptyset$.

If not, then there are $\psi_1, \ldots, \psi_n \in \Sigma_2(w)$ such that

$$\vdash \psi_1 \wedge \cdots \wedge \psi_n \rightarrow \neg\varphi.$$

Using Lemma 66(2) and (4), we derive that $N'''\varphi \in w$: a contradiction.

We have shown that $\Sigma_2(w) \cup \{\varphi\}$ is consistent. By Lindenbaum's Lemma, there exists $u \in W^c$ such that $\Sigma_2(w) \cup \{\varphi\} \subseteq u$. By definition of R_2^c, $wR_2^c u$ and $\varphi \in u$, thus $\neg\varphi \notin u$, as desired. □

6.4. Connection with the logic of contingency. Let $R_1 = R_2 = R$. Then N''' becomes the non-contingency operator Δ. That is,

$$\mathcal{M}, w \vDash N'''\varphi \iff R(w) \vDash \varphi \text{ or } R(w) \vDash \neg\varphi.$$

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It is easy to see that $\vDash N'''\varphi \leftrightarrow N'''\neg\varphi$. Then ($\mathbf{N'''AD1}$) and ($\mathbf{N'''AD2}$) become the following:

$$\neg N'''\psi \rightarrow (\Box_1\varphi \leftrightarrow N'''\varphi \wedge N'''(\varphi \vee \psi)) \qquad (\mathbf{N'''AD1'})$$
$$\neg N'''\psi \rightarrow (\Box_2\varphi \leftrightarrow N'''\varphi \wedge N'''(\varphi \vee \psi)) \qquad (\mathbf{N'''AD2'}).$$

In this case, as $\vDash N'''\top \leftrightarrow N'''\bot$ and $\vDash N'''\varphi \leftrightarrow N'''\neg\varphi$, $\mathbf{K^{N'''}}$ can be replaced by the following simpler proof system (denoted $\mathbf{K1^{N'''}}$), where we omit axiom PC and the inference rule MP:

D1	$N'''\top$
D2′	$N'''\varphi \leftrightarrow N'''\neg\varphi$
D3	$N'''\varphi \wedge N'''\psi \rightarrow N'''(\varphi \wedge \psi)$
D5	$N'''\varphi \rightarrow N'''(\varphi \vee \psi) \vee N'''(\neg\varphi \vee \chi)$
N‴R2	From $\psi \rightarrow \chi$ infer $\neg N'''\varphi \wedge N'''\psi \wedge N'''(\psi \vee \varphi) \rightarrow N'''\chi$.

Notice that $\mathbf{K1^{N'''}}$ extends $\mathbf{K^{N'''}}$ with axiom D2′. To see that $\mathbf{K1^{N'''}}$ extends $\mathbf{K^{N'''}}$, we only need to consider the rule N‴R1, since the other axioms and rules are easy to show.

PROPOSITION 79. *N‴R1, namely* $\dfrac{\varphi \rightarrow \psi \quad \psi \rightarrow \chi}{N'''\psi \rightarrow (N'''\varphi \vee N'''\chi)}$, *is derivable in* $\mathbf{K1^{N'''}}$.

Proof. We have the proof sequence in $\mathbf{K^{N'''}}$:

(1)	$\psi \rightarrow \chi$	premise
(2)	$\neg N'''\chi \wedge N'''\psi \wedge N'''(\psi \vee \chi) \rightarrow N'''\chi$	(1), N‴R2
(3)	$N'''\psi \wedge N'''(\psi \vee \chi) \rightarrow N'''\chi$	(2)
(4)	$\varphi \rightarrow \psi$	premise
(5)	$\neg\psi \rightarrow \neg\varphi$	(4)
(6)	$N'''\neg\psi \wedge N'''(\neg\psi \vee \neg\varphi) \rightarrow N'''\neg\varphi$	(5), similar to (1)–(3)
(7)	$N'''\psi \wedge N'''(\neg\psi \vee \neg\varphi) \rightarrow N'''\varphi$	(6), D2′
(8)	$N'''\psi \rightarrow N'''(\psi \vee \chi) \vee N'''(\neg\psi \vee \neg\varphi)$	D5
(9)	$N'''\psi \rightarrow (N'''\varphi \vee N'''\chi)$	(3), (7), (8).

□

Moreover, the canonical model for $\mathbf{K^{N'''}}$ becomes the following.

DEFINITION 80. *The canonical model for* $\mathbf{K1^{N'''}}$ *is a tuple* $\mathcal{M}^c = \langle W^c, R_1^c, R_2^c, V^c\rangle$, *where*

- $W^c = \{w \mid w$ *is a maximal consistent set for* $\mathbf{K1^{N'''}}\}$;
- *if* $\neg N'''\psi \in w$ *for no* ψ, *then* wR_1^cu *iff* $w = u$, *and* wR_2^cu *iff* $w = u$;
- *if* $\neg N'''\psi \in s$ *for some* ψ, *then* sR_1^ct *iff for every* φ, *if* $N'''\varphi \wedge N'''(\varphi \vee \psi) \in w$, *then* $\varphi \in u$, *and* wR_2^cu *iff for every* φ, *if* $N'''\varphi \wedge N'''(\varphi \vee \psi) \in w$, *then* $\varphi \in u$;
- $V^c(p) = \{w \in W^c \mid p \in w\}$.

One may easily verify that $R_1^c = R_2^c$.

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Then along with the completeness proof of $\mathbf{K}^{\mathbf{N}'''}$ in §6.1.2, we can finally show the following.

THEOREM 81. $\mathbf{K1}^{\mathbf{N}'''}$ *is sound and strongly complete with respect to the class of all frames.*

This gives an alternative axiomatization of contingency logic $\mathcal{L}(\Delta)$ over the class of all frames. Then we can also axiomatize N''' $(= \Delta)$ over various classes of special frames, as in [19, 20].

Recall that various axiomatizations of $\mathcal{L}(\Delta)$ are proposed in the literature (see, for instance, [19, 20, 26, 30, 33, 46]). We may show that our systems are equivalent to the corresponding ones from a proof-theoretic perspective. For the sake of simplicity, we illustrate with the minimal logics. We choose the minimal logic given in [30], denoted $\boldsymbol{K\Delta}$ here.[16] The system $\boldsymbol{K\Delta}$ consists of the following axioms and inference rules, where we write N''' for Δ for convenience:

PL	All substitution instances of tautologies
A1	$N'''\neg\varphi \to N'''\varphi$
A2	$N'''\varphi \wedge \neg N'''(\varphi \wedge \psi) \to \neg N'''\psi$
A3	$N'''\varphi \wedge \neg N'''(\varphi \vee \psi) \to N'''(\neg\varphi \vee \chi)$
RΔ	From φ infer $N'''\varphi$
REΔ	From $\varphi \leftrightarrow \psi$ infer $N'''\varphi \leftrightarrow N'''\psi$
MP	From φ and $\varphi \to \psi$ infer ψ.

In what follows, we show that our system $\mathbf{K1}^{\mathbf{N}'''}$ is equivalent to $\boldsymbol{K\Delta}$.

PROPOSITION 82. *All axioms and inference rules in* $\boldsymbol{K\Delta}$ *are derivable in* $\mathbf{K1}^{\mathbf{N}'''}$.

Proof. Axiom **A1** is direct from D2′. Axioms **A2** and **A3** are equivalent transformations of D3 and D5, respectively. Rule **REΔ** can be shown from N‴R1, which is derivable in $\mathbf{K1}^{\mathbf{N}'''}$ (Proposition 79). In details, the proof of **REΔ** is as follows. Suppose that $\vdash \varphi \leftrightarrow \psi$, then $\vdash \varphi \to \psi$ and $\vdash \psi \to \varphi$. By N‴R1, we infer that $\vdash N'''\psi \to N'''\varphi \vee N'''\varphi$, equivalently, $\vdash N'''\psi \to N'''\varphi$. Similarly, we can show that $\vdash N'''\varphi \to N'''\psi$. Therefore, $\vdash N'''\varphi \leftrightarrow N'''\psi$. The proof of **R$\Delta$** is the following. Assume that $\vdash \varphi$, then $\vdash \varphi \leftrightarrow \top$. By **RE$\Delta$**, $\vdash N'''\varphi \leftrightarrow N'''\top$. Then using D1 (namely $\vdash N'''\top$), we derive that $\vdash N'''\varphi$. □

PROPOSITION 83. *All axioms and inference rules in* $\mathbf{K1}^{\mathbf{N}'''}$ *are derivable in* $\boldsymbol{K\Delta}$.

Proof. Axiom D1 is immediate by **PL** and **RΔ**. Axiom D2′ can be easily shown from **A1** and **REΔ**. Axioms D3 and D5 are equivalent transformations of **A2** and **A3**, respectively.

The proof of N‴R2 is as follows. Suppose that $\vdash \psi \to \chi$. By **RΔ**, $\vdash N'''(\psi \to \chi)$. By **A2**, $\vdash N'''(\psi \to \chi) \wedge N'''(\psi \vee \chi) \to N'''((\psi \to \chi) \wedge (\psi \vee \chi))$. By **PL** and **RE$\Delta$**, we can show that $\vdash N'''((\psi \to \chi) \wedge (\psi \vee \chi)) \leftrightarrow N'''\chi$. Thus $\vdash N'''(\psi \to \chi) \wedge N'''(\psi \vee \chi) \to N'''\chi$, thus $\vdash N'''(\psi \vee \chi) \to N'''\chi$. By axiom **A2** again, $\vdash \neg N'''\varphi \wedge N'''(\psi \vee \varphi) \to \neg N'''(\neg\psi \vee \varphi)$. Moreover, by axiom **A3**, $\vdash N'''\psi \wedge \neg N'''(\neg\psi \vee \varphi) \to N'''(\psi \vee \chi)$. Therefore, $\vdash \neg N'''\varphi \wedge N'''\psi \wedge N'''(\psi \vee \varphi) \to N'''\chi$, as desired. □

[16] On p. 231 of [30], $\boldsymbol{K\Delta}$ is miswritten as $\boldsymbol{K4\Delta}$, and **REΔ** is miswritten as **RE**.

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§7. Conclusion. In this paper, we proposed a uniform approach to axiomatizing bundled operators, called the 'MBADS'. The core idea lies in the definability of familiar operators in terms of relatively unfamiliar ones given some certain premises, thus one can 'reduce' the latter operators to the former ones in some way. This approach extends the cases where the schemas are valid to those that are valid on a class of frames, which is not necessarily the class of all frames. We illustrated this approach with several bundled operators such as the operator of purely physical necessity N, the operator N' called 'All$_1$ and Only$_2$', the operator N'', and the operator N''' in the literature. With the aid of this approach, among other contributions, we answered several open questions, including **S4**-like logic of $\mathcal{L}(N)$, a finite minimal axiomatization for the operator 'All$_1$ and Only$_2$', the transitive axiomatization of the logic of false belief; also, we obtained various alternative axiomatizations, which are shown to be deductively equivalent to the existing ones in the literature.

For us, the most difficult part of this approach is finding the desired almost-definability schema(s). Once we find the desired almost-definability schemas, the axiomatization can be done step by step: first, according to the almost-definability schema(s), define the canonical relation(s), and thus the canonical model. Second, in the completeness proof, show that the truth lemma holds and that the canonical model is the desired one. Third, in the process of the second step, find out which axioms and inference rules are needed.

In comparison with other approaches, this approach is useful in finding axioms and inference rules, and in defining suitable canonical relation(s), thus in the completeness proof of logical systems. In these aspects, our approach MBADS seems to be incomparable with other approaches. This is, hopefully, good news for modal logicians who are interested in axiomatizing bundled operators. When we deal with the axiomatizations of the bundled operators, we can consider to use this approach.

Although this paper focuses solely on relational semantics, we believe that this approach can be generalized to other semantics as well. We have already extended this approach to neighborhood semantics, whose details will be presented in a separate paper.

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BIBLIOGRAPHY

[1] Åqvist, L. (2000). Three characterizability problems in deontic logic. *Nordic Journal of Philosophical Logic*, **5**, 65–82.

[2] Armstrong, D. M. (1979). Laws of nature: Why they cannot be mere uniformities. *Proceedings of the Russellian Society*, **4**, 46–61.

[3] Bacon, J. (1981). Purely physical necessity. *Theoria*, **47**, 135–141.

https://doi.org/10.1017/S1755020325100981 Published online by Cambridge University Press

[4] Balbiani, P., & Fan, J. (2017). A complete axiomatization of Euclidean strong non-contingency logic. In *Proceedings of 12th Tbilisi Symposium of Language, Logic and Computation (TbiLLC)*. Lagodekhi, Georgia: Springer, pp. 43–48.

[5] Bezerra, E., & Venturi, G. (2021). A non-standard Kripke semantics for the minimal deontic logic. *Logic and Logical Philosophy*, **30**(1), 97–107.

[6] Burks, A. (1951). The logic of causal propositions. *Mind*, **60**(239), 363–382.

[7] Chagrov, A., & Zakharyaschev, M. (1997). *Modal Logic*. Oxford: Clarendon Press.

[8] van Ditmarsch, H., Fan, J., van der Hoek, W., & Iliev, P. (2014). Some exponential lower bounds on formula-size in modal logic. In R. Goré, B. Kooi, and A. Kurucz, editors. *Advances in Modal Logic*. UK: College Publications, Vol. 10. pp. 139–157.

[9] Fan, J. (2015). Logics of essence and accident. Preprint, arXiv:1506.01872, unpublished manuscript.

[10] ———. (2019). Bimodal logics with contingency and accident. *Journal of Philosophical Logic*, **48**, 425–445.

[11] ———. (2019). Strong non-contingency: On the modal logics of an operator expressively weaker than necessity. *Notre Dame Journal of Formal Logic*, **60**(3), 407–435.

[12] ———. (2021). A logic for disjunctive ignorance. *Journal of Philosophical Logic*, **50**(6), 1293–1312.

[13] ———. (2021). Two variants of noncontingency operator. *Notre Dame Journal of Formal Logic*, **62**(3), 491–525.

[14] ———. (2022). A logic of von Wright's deontic necessity. *Studies in Logic*, **15**(3), 1–17.

[15] ———. (2022). Unknown truths and false beliefs: Completeness and expressivity results for the neighborhood semantics. *Studia Logica*, **110**, 1–45.

[16] ———. (2023). Some notes on dyadic contingency. *Journal of Logic, Language and Information*, **32**, 209–217.

[17] ———. (2024). A logic of temporal contingency. *Erkenntnis*, **89**, 2611–2640. https://link.springer.com/article/10.1007/s10670-022-00644-5.

[18] ———. (2025). Logics of false belief and reliable belief. *Notre Dame Journal of Formal Logic*, **66**(4), 521–542. https://doi.org/10.1215/00294527-2025-0015.

[19] Fan, J., Wang, Y., & van Ditmarsch, H. (2014). Almost necessary. *Advances in Modal Logic*, **10**, 178–196.

[20] ———. (2015). Contingency and knowing whether. *The Review of Symbolic Logic*, **8**(1), 75–107.

[21] Fitting, M. (1969). Logics with several modal operators. *Theoria*, **35**(3), 259–266.

[22] Gabbay, D. M., Kurucz, A., Wolter, F., & Zakharyaschev, M. (2003). *Many-Dimensional Modal Logics: Theory and Applications* (first edition). North Holland: Elsevier.

[23] Gilbert, D., Kubyshkina, E., Petrolo, M., & Venturi, G. (2022). Logics of ignorance and being wrong. *Logic Journal of the IGPL*, **30**(5), 870–885.

[24] Gilbert, D., & Venturi, G. (2017). Neighborhood semantics for logics of unknown truths and false beliefs. *The Australasian Journal of Logic*, **14**(1), 246–267.

[25] Humberstone, L. (1987). The modal logic of 'all and only'. *Notre Dame Journal of Formal Logic*, **28**(2), 177–188.

https://doi.org/10.1017/S1755020325100981 Published online by Cambridge University Press

[26] ———. (1995). The logic of non-contingency. *Notre Dame Journal of Formal Logic*, **36**(2), 214–229.

[27] ———. (2012). Minimally congruential contexts: Observations and questions on embedding E in K. *Notre Dame Journal of Formal Logic*, **54**(3), 581–598.

[28] ———. (2016). *Philosophical Applications of Modal Logic*. London: College Publications.

[29] Jones, A. J. I., & Pörn, I. (1985). Ideality, sub-ideality and denotic logic. *Synthese*, **65**(2), 275–290.

[30] Kuhn, S. (1995). Minimal non-contingency logic. *Notre Dame Journal of Formal Logic*, **36**(2), 230–234.

[31] Liu, M., Padmanabha, A., Ramanujam, R., & Wang, Y. (2023). Are bundles good deals for first-order modal logic? *Information and Computation*, **293**, 105062.

[32] Marcos, J. (2005). Logics of essence and accident. *Bulletin of the Section of Logic*, **34**(1), 43–56.

[33] Montgomery, H., & Routley, R. (1966). Contingency and non-contingency bases for normal modal logics. *Logique et Analyse*, **9**, 318–328.

[34] Pan, T., & Yang, C. (2017). A logic for weak essence and strong accident. *Logique et Analyse*, **238**, 179–190.

[35] Pan, T. Q. (2011). On logic of belief-disagreement among agents. In H. van Ditmarsch, J. Lang, S. Ju, editors. *Logic, Rationality, and Interaction: Third International Workshop, LORI 2011, Guangzhou, China, October 10–13, 2011 Proceedings 3*. Berlin: Springer, pp. 392–393.

[36] Pizzi, C. (2022). Possibility and dyadic contingency. *Journal of Logic, Language and Information*, **31**(3), 451–463.

[37] Rennie, M. K. (1970). Models for multiply modal systems. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, **16**(2), 175–186.

[38] Steinsvold, C. (2008). Completeness for various logics of essence and accident. *Bulletin of the Section of Logic*, **37**(2), 93–101.

[39] ———. (2008). A note on logics of ignorance and borders. *Notre Dame Journal of Formal Logic*, **49**(4), 385–392.

[40] ———. (2011). Being wrong: Logics for false belief. *Notre Dame Journal of Formal Logic*, **52**(3), 245–253.

[41] Vander Nat, A. (1979). Beyond non-normal possible worlds. *Notre Dame Journal of Formal Logic*, **20**(3), 631–635.

[42] Wang, Y. (2017). A new modal framework for epistemic logic. In Lang, J., editor. Tark, Liverpool, UK. pp. 515–534.

[43] Witczak, T. (2022). A note on the intuitionistic logic of false belief. *Bulletin of the Section of Logic*, **51**(1), 57–71.

[44] Xiong, Z., & Agotnes, T. (2023). The logic of secrets and the interpolation rule. *Annals of Mathematics and Artificial Intelligence*, **91**(4), 375–407.

[45] Yang, Y. (2024). Logics of true belief. *Notre Dame Journal of Formal Logic*, **65**(1), 55–80.

[46] Zolin, E. (1999). Completeness and definability in the logic of noncontingency. *Notre Dame Journal of Formal Logic*, **40**(4), 533–547.

[47] ———. (2000). Embeddings of propositional monomodal logics. *Logic Journal of the IGPL*, **8**(6), 861–882.

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