Canad. Math. Bull. Vol. **68** (4), 2025, pp. 1347–1358
http://dx.doi.org/10.4153/S0008439525100830
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Existence-uniqueness theory and small-data decay for a reaction-diffusion model of wildfire spread

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Abstract. I examine some analytical properties of a nonlinear reaction-diffusion system that has been used to model the propagation of a wildfire. I establish global-in-time existence and uniqueness of bounded mild solutions to the Cauchy problem for this system given bounded initial data. In particular, this shows that the model does not allow for thermal blow-up. If the initial temperature and fuel density also satisfy certain integrability conditions, the spatial L^2-norms of these global solutions are uniformly bounded in time. Additionally, I use a bootstrap argument to show that small initial temperatures give rise to solutions that decay to zero as time goes to infinity, proving the existence of initial states that do not develop into traveling combustion waves.

1 Introduction

In this article, I consider a reaction-diffusion model of wildfire propagation in a fuel layer with a very large spatial extent. Fix a spatial dimension $d \geq 1$. Consider a certain amount of solid fuel (trees, shrubberies, and so on) spread throughout $\mathbb{R}^d$ with mass density at spatial location x and time t given by $F(x,t)$. We then define the mass fraction of fuel $Y(x,t)$ by

$$Y(x,t) = \begin{cases} \frac{F(x,t)}{F(x,0)} & F(x,0) \neq 0 \\ 0 & F(x,0) = 0. \end{cases}$$

A fire with a dimensionless temperature field $T(x,t)$ is moving through space and consuming this fuel: temperature is normalized so that $T = 0$ is the ambient temperature of the system's environment. Also, we make the idealized assumption that the reaction rate governing combustion is given by an Arrhenius law activating at the ambient temperature $T = 0$:

$$r(T) = \begin{cases} e^{-\frac{1}{T}} & T > 0 \\ 0 & T \leq 0. \end{cases} \tag{1.1}$$

Received by the editors July 26, 2024; revised December 9, 2024; accepted December 9, 2024.
Published online on Cambridge Core July 8, 2025.
AMS subject classification: **35K57**, **80A25**.
Keywords: Wildfire spread models, solid-fuel combustion, reaction-diffusion equations in combustion theory, existence and uniqueness of solutions to reaction-diffusion equations, time decay of solutions to reaction-diffusion equations.

https://doi.org/10.4153/S0008439525100830 Published online by Cambridge University Press

Under these assumptions, Weber et al. [25] were the first to model the evolution of T and Y using the following Cauchy problem: given two parameters $\lambda, \beta \geq 0$,

$$\begin{cases} T_t = (\Delta - \lambda)\, T + Yr\,(T) \\ Y_t = -\beta Yr\,(T) \\ T|_{t=0} = T_0(x) \\ Y|_{t=0} = Y_0(x). \end{cases} \tag{1.2}$$

Up to a different choice of reaction rate, these are precisely the equations of first-order solid-fuel combustion at infinite Lewis number, with an added heat loss term [3, Section 1.3]. For an instructive derivation of (1.2) and an explanation of why it is useful for data assimilation, see Mandel et al. [13]. The discussion in Mandel et al. highlights that $r(0) = 0$ implies no combustion takes place at the ambient temperature $T = 0$. Consequently, (1.2) admits solutions that take the form of traveling combustion waves; this had already been confirmed using formal asymptotics and numerics in [25].

In light of the existence of traveling wave solutions, (1.2) has attracted attention from researchers in both mathematical modeling and dynamical systems theory. On the more concrete side, (1.2) has been applied to develop computationally efficient reduced-order models of wildfire dynamics [5, 11]. Also, Johnston [10] used (1.2) in a modeling pipeline where remote sensing data was utilized to estimate realistic parameter values. Additionally, variants of the model incorporating the effects of wind [2, 13] and radiative heat transfer [1, 15, 18, 24] have also been considered. When it comes to rigorous results on the behavior of solutions to (1.2), Billingham [4] proved the existence of traveling waves given suitable initial data and $d = 1$ (see also Babak et al. [2]). Linear stability of these traveling waves was examined numerically by Varas and Vega [22]. Existence and linear stability of traveling waves for related combustion models in $d = 1$ with *finite* Lewis number has also been considered, see for example [7, 17]. There is certainly no shortage of other interesting research on traveling waves in wildland fire propagation and other solid-fuel systems, but a complete literature review is beyond the scope of this brief article.

While there is plenty of mathematical work on the properties of traveling wave solutions to (1.2), to my knowledge there are no published results on the existence and uniqueness of global-in-time solutions to (1.2) for *generic* bounded initial data (see, however, the existence result for a related model in [3, Section 4.4]). In particular, with the exception of the lumped-parameter analysis in [18], I have not found any estimates that preclude finite-time blow-up of $|T(x,t)|$. Thermal blow-up phenomena indeed arise in certain solid-fuel combustion models [3, Section 3.2], so understanding whether or not blow-up occurs in solutions to (1.2) is a meaningful question. Also, in [13], the authors use a heuristic lumped-parameter argument to predict that a fire with a sufficiently low temperature eventually burns itself out: in the jargon of combustion theory, this means the model predicts a nonzero "auto-ignition temperature." I have not, however, found any rigorous proof of the claim that (1.2) has a nonzero auto-ignition temperature.

In this article, I prove three novel results to address the knowledge gaps discussed in the previous paragraph:

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(1) for any bounded initial state (T_0, Y_0) there exists a unique, bounded mild solution to (1.2) (to be defined precisely below) valid on the entire time interval $[0, \infty)$, provided $\lambda > 0$: in particular, thermal blow-up is impossible (Corollary 2.6);
(2) further, if Y_0 is also integrable and T_0 is also square-integrable, then the L^2_x norm of $T(x,t)$ is bounded uniformly in time (Corollary 2.6 as well);
(3) given an initial state T_0 that is small in a particular physically reasonable norm, the solution to (1.2) satisfies $\lim_{t\to 0} |T(x,t)| = 0$ uniformly in x for every $\lambda \geq 0$: morally, this confirms that (1.2) indeed has a nonzero auto-ignition temperature, and that not *every* initial state gives rise to a traveling wave solution (Theorem 2.9).

I hope these results prove useful to further investigations into the *nonlinear* stability of traveling waves and to the development of rigorous error estimates for numerical discretizations of (1.2).

Remark 1.1 In light of how the mass fraction is defined, we could consider the set of initial states $Y_0(x)$ to consist only of piecewise-constant functions taking the values 0 or 1. However, from an abstract point of view the initial-value problem (1.2) makes sense for a much wider variety of initial states. Consequently, in my opinion it is more elegant to allow $Y_0(x)$ to live in a "typical" function space rather than some purpose-built space of piecewise-constant functions. Additionally, I remark that due to the phenomenon of wildfire spotting (see for instance [14]), considering initial states where $Y = 0$ in some regions is physically relevant: although (1.2) does not include terms to describe spotting, adding such terms could make for significant future work.

1.1 Notation

For definitions of any of the objects appearing below, see [6].

- d always denotes a positive integer greater than or equal to 1.
- $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$.
- C denotes some non-negative constant that may change between lines. If α is some real parameter, then $C(\alpha)$ denotes a constant depending on α.
- If $a, b \in \mathbb{R}$, we say that $a \lesssim b$ if there exists C such that $a \leq bC$. If $a, b \in \mathbb{R}$, we say that $a \simeq b$ if $a \lesssim b$ and $b \lesssim a$.
- Given $k, \ell \in \mathbb{N}_0$ with $k \leq \ell$ and an ℓ-times continuously differentiable function $\varphi : \mathbb{R}^d \to \mathbb{R}$, $D^k_x \varphi$ denotes the set of all order k partial derivatives of φ.
- The symbol Δ denotes the Laplacian operator acting on functions with domain $\mathbb{R}^d$: $\Delta = \sum_{k=1}^d \left(\frac{\partial}{\partial x_k}\right)^2$.
- For $p \in [1, \infty]$, $L^p_x\left(\mathbb{R}^d\right)$ denotes the usual Lebesgue space of real-valued functions of $x \in \mathbb{R}^d$.
- Given a Banach space A and a time interval $[0, t_*]$, we let $C^0_t A$ denote the space of continuous curves $[0, t_*] \to A$. When the value of t_* has a substantial impact on the topic at hand, this value will be made clear. Likewise, $C^1_t A$ denotes the space of continuously differentiable curves in A, defined on some time interval.

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2 Results

2.1 Preliminaries

First, we recall some decay and smoothing estimates for the linear heat equation.

Lemma 2.1 *For any $p, q, r \in [1, \infty]$ with $p^{-1} = q^{-1} - r^{-1}$, any $\varphi \in L^q_x$, and any $k \in \mathbb{N}_0$, we have*

$$\left\| D_x^k e^{t\Delta} \varphi \right\|_{L^p_x} \lesssim t^{-\frac{1}{2}\left(\frac{d}{r}+k\right)} \left\| \varphi \right\|_{L^q_x} \quad \forall\ t > 0. \tag{2.1}$$

Proof This bound follows from the time-decay properties of the heat kernel on $\mathbb{R}^d$; for example, see [12, pp. 41–42] for an outline of the proof. ∎

In the important special case $p = q, r = \infty, k = 0$, we get

$$\left\| e^{t\Delta} \varphi \right\|_{L^p_x} \leq \left\| \varphi \right\|_{L^p_x} \quad \forall\ t > 0. \tag{2.2}$$

Additionally, we need a result about the function $r(T)$ defined in (1.1).

Lemma 2.2 *For any $p \in \mathbb{N}_0$, there exists a constant $C(p)$ such that*

$$r(T) \leq C(p) T^p \quad \forall\ T \in [0, 1].$$

Proof This is obvious since $r(T)$ vanishes to all orders at $T = 0$: it is the canonical example of a smooth, non-analytic function. ∎

2.2 Mild solutions and their basic properties

We may use Duhamel's principle to re-formulate (1.2) in a way that lets us introduce solutions with limited regularity.

Definition 2.1 Let A, A_0, and B be Banach subspaces of L^∞_x, with $A \subseteq A_0$. Suppose we are given an initial datum $(T_0(x), Y_0(x)) \in A_0 \times B$. We say that $(T(x,t), Y(x,t)) \in C^0_t A \times C^0_t B$ is a *mild solution* of (1.2) if this pair satisfies

$$T(x,t) = e^{-\lambda t} e^{t\Delta} T_0 + \int_0^t e^{-\lambda(t-\tau)} e^{(t-\tau)\Delta} Y(x,\tau) r\left(T(x,\tau)\right)\, \mathrm{d}\tau \tag{2.3a}$$

$$Y(x,t) = Y_0(x) \exp\left(-\beta \int_0^t r\left(T(x,\sigma)\right)\, \mathrm{d}\sigma \right). \tag{2.3b}$$

Below, the time interval on which a mild solution is valid will be specified when it's vital to know. Immediately, one sees that mild solutions have some important properties.

(1) First, if $Y_0(x) \geq 0$ everywhere then $Y(x,t) \geq 0$ also holds. This makes perfect sense: an initially nonnegative fuel density should never become negative.
(2) Second, owing to the smoothing properties of the heat flow $e^{t\Delta}$, $T(x,t)$ is smooth in x for every $t > 0$. However, $Y(x,t)$ is only as smooth as $Y_0(x)$.

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(3) By construction, any mild solution (T, Y) is continuous in t: in fact, it's trivial to see that $Y(x,t)$ is also *continuously differentiable* in t. If we denote the right-hand side of (2.3a) by $\Phi[T, Y]$, then we have

$$\partial_t \Phi[T, Y] = (-\lambda + \Delta)\, e^{-\lambda t} e^{t\Delta} T_0 + Y(x,t) r(T(x,t)) + \int_0^t \cdots \mathrm{d}\tau,$$

where the unimportant particulars in the integral over τ have been omitted. Each term on the right-hand side of the above expression is continuous in t, so we conclude that $T(x,t)$ is continuously differentiable in t. Thus, in the notation of Definition 2.1, we could just as well *define* mild solutions as living in $C_t^1 A \times C_t^1 B$.

In light of the last two items above, I have *almost* shown that any mild solution is actually a classical solution to (1.2) (the converse statement is obvious). However, to avoid technicalities about the nature of the convergence of $(T(x,t), Y(x,t)) \to (T_0(x), Y_0(x))$ as $t \to 0^+$, I won't formalize such a statement here: in the sequel, the spatial and temporal regularity of mild solutions turns out to be sufficient to get the estimates I want.

2.3 *A priori* estimates and existence-uniqueness theory

In this subsection, I prove some basic *a priori* bounds on mild solutions to (1.2) with $\lambda > 0$. These bounds are then used to help construct global-in-time mild solutions of (1.2). We start with the easiest bound possible.

Proposition 2.3 *For any $p \in [1, \infty]$, if (T, Y) is a mild solution of (1.2), then*

$$\sup_{t>0} \|Y(x,t)\|_{L_x^p} \le \|Y_0(x)\|_{L_x^p} \tag{2.4}$$

provided the right-hand side is finite.

Proof Since $r(T)$ is nonnegative, $Y(x,t) \le Y_0(x)$ almost everywhere. ∎

According to Mandel et al. [13, Section 4.1], $Y(x,t)$ is expected to settle to a small but nonzero steady value for $t \gg 1$. I therefore do not think one can prove a variant of the above result that includes a time-dependent decay factor going to 0 as $t \to \infty$ on the right-hand side.

Proposition 2.4 *Suppose $\lambda > 0$. If we are given an initial datum $(T_0, Y_0) \in L_x^\infty \times L_x^\infty$ and a corresponding mild solution $(T, Y) \in C_t^0 L_x^\infty \times C_t^0 L_x^\infty$ of (1.2), then*

$$\|T(x,t)\|_{L_x^\infty} \le e^{-\lambda t} \|T_0(x)\|_{L_x^\infty} + \lambda^{-1} \|Y_0(x)\|_{L_x^\infty}. \tag{2.5}$$

Proof Using Lemma 2.1 and $|r(T)| \le 1$, we have

$$\begin{aligned} \|T\|_{L_x^\infty} &\le e^{-\lambda t} \|T_0\|_{L_x^\infty} + \sup_{t>0} \|Yr(T)\|_{L_x^\infty} \int_0^t e^{-\lambda(t-\tau)}\, \mathrm{d}\tau \\ &\le e^{-\lambda t} \|T_0\|_{L_x^\infty} + \lambda^{-1} \left(1 - e^{-\lambda t}\right) \|Y_0\|_{L_x^\infty}. \end{aligned}$$

∎

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Proposition 2.5 *Suppose $\lambda > 0$. If we are given an initial datum $(T_0, Y_0) \in \left(L^2_x \cap L^\infty_x\right) \times L^\infty_x$ and a corresponding mild solution $(T, Y) \in C^0_t\left(L^2_x \cap L^\infty_x\right) \times C^0_t L^\infty_x$ of (1.2), then*

(2.6)
$$\|T\|^2_{L^2_x} \lesssim \min\left\{\|T_0\|^2_{L^2_x} + C\left(\|T_0\|_{L^\infty_x}, \|Y_0\|_{L^\infty_x}, \lambda, \beta\right)\|Y_0 - Y\|_{L^1_x},\ \|T_0\|^2_{L^2_x}\, e^{\|Y_0\|_{L^\infty_x} t}\right\}.$$

In particular, if $(T_0, Y_0) \in \left(L^2_x \cap L^\infty_x\right) \times \left(L^1_x \cap L^\infty_x\right)$ then

$$\sup_{t>0} \|T(x,t)\|_{L^2_x} \leq C\left(\|T_0\|_{L^2_x \cap L^\infty_x}, \|Y_0\|_{L^1_x \cap L^\infty_x}, \lambda, \beta\right). \tag{2.7}$$

Proof Recall that $T(x,t)$ is smooth in x and continuously differentiable in t, so we can write $T_t = (\Delta - \lambda)\, T + Yr(T)$ for $t > 0$. We then use a standard energy argument:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|T\|^2_{L^2_x} &= -\|T\|^2_{\dot{H}^1_x} - \frac{\lambda}{2}\|T\|^2_{L^2_x} + \int T(x,t)\, Y(x,t)\, r(T(x,t))\ \mathrm{d}x \\
&\leq \int T(x,t)\, Y(x,t)\, r(T(x,t))\ \mathrm{d}x \\
&= \beta^{-1}\int T(x,t)\ (-Y_t(x,t))\ \mathrm{d}x \\
&\leq \beta^{-1}\sup_{t>0}\|T(x,t)\|_{L^\infty_x}\int (-Y_t(x,t))\ \mathrm{d}x.
\end{aligned} \tag{2.8}$$

Integrating both sides of the above over $[0,t]$ and using Proposition 2.4 gives

$$\|T(x,t)\|^2_{L^2_x} \lesssim \|T_0(x)\|^2_{L^2_x} + C\left(\|T_0\|_{L^\infty_x}, \|Y_0\|_{L^\infty_x}, \lambda, \beta\right)\int_0^t \int (-Y_\tau(x,\tau))\ \mathrm{d}x\ \mathrm{d}\tau.$$

Then, we use the Fubini–Tonelli theorem (valid since $-Y_\tau$ is non-negative and measurable) and the fundamental theorem of calculus to obtain one part of (2.6). The special case (2.7) follows since $Y_0(x) - Y(x,t) \leq Y_0(x)$. For the other part of (2.6), we instead use $Tr(T) \lesssim T^2$ in (2.8), place Y in L^∞_x, and conclude with an application of Grönwall's inequality. ∎

Next, I demonstrate how these *a priori* estimates may be applied to obtain global-in-time mild solutions to (1.2).

Corollary 2.6 *Suppose we are given an initial datum $(T_0, Y_0) \in L^\infty_x \times L^\infty_x$. Then,*

- *(1.2) admits a unique mild solution $(T, Y) \in C^0_t L^\infty_x \times C^0_t L^\infty_x$ valid for at least some small time interval $[0, t^*]$;*
- *additionally, if $\lambda > 0$ then this solution is valid for $t \in [0, \infty)$;*
- *finally, if $\lambda > 0$ and we also have $T_0 \in L^2_x$, then T obeys the growth estimate (2.6).*

Proof Local-in-time existence and uniqueness follows from a routine fixed-point argument. Let $\Phi[T, Y]$ denote the right-hand side of (2.3a) and let $\Psi[T, Y]$ denote the right-hand side of (2.3b) (note that T_0, Y_0 are *fixed*). We want to show that there exist $R > 0$ and $t^* \in (0,1)$ such that $(T, Y) \mapsto (\Phi[T, Y], \Psi[T, Y])$ is a contraction mapping on the open ball $B(0,R) \subseteq C^0_t L^\infty_x \times C^0_t L^\infty_x$, where time takes values in

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$[0, t^*]$. Arguments similar to those applied in Propositions 2.3 and 2.4 show that this mapping takes $B(0,R)$ to itself if

$$\|T_0\|_{L_x^\infty} + \|Y_0\|_{L_x^\infty} + t^* R \le R. \tag{2.9a}$$

Analogous techniques, when combined with the mean value theorem in two variables, show that for any $(T_1, Y_1), (T_2, Y_2) \in B(0,R)$ we have the Lipschitz-type estimates

$$\begin{aligned}
\|\Phi[T_1, Y_1] - \Phi[T_2, Y_2]\|_{C_t^0 L_x^\infty} &\le CRt^* \|(T_1, Y_1) - (T_2, Y_2)\|_{C_t^0 L_x^\infty \times C_t^0 L_x^\infty}, \\
\|\Psi[T_1, Y_1] - \Psi[T_2, Y_2]\|_{C_t^0 L_x^\infty} &\le \beta Rt^* \|(T_1, Y_1) - (T_2, Y_2)\|_{C_t^0 L_x^\infty \times C_t^0 L_x^\infty},
\end{aligned}$$

where C is independent of R, t^*. Accordingly, $(T, Y) \mapsto (\Phi[T, Y], \Psi[T, Y])$ certainly defines a contraction on $B(0,R)$ if

$$t^* \le \frac{1}{2} \min\left\{1, (C+\beta)^{-1} R^{-1}\right\}. \tag{2.9b}$$

Choosing $R = 4\left(\|T_0\|_{L_x^\infty} + \|Y_0\|_{L_x^\infty}\right)$ and letting t^* saturate (2.9b), we find that (2.9a) is also satisfied. Applying Banach's contraction mapping theorem, the proof of existence and uniqueness of a mild solution within $B(0,R)$ is complete. Unconditional uniqueness then follows from the bootstrap principle (see for instance [19, pp. 131–132 for further details).

If $\lambda > 0$ then Propositions 2.3 and 2.4 allow us to extend this local solution to a global one. With this solution in hand, we apply Proposition 2.5 to conclude. ∎

Consequently, solutions to (1.2) do not exhibit thermal blow-up when $\lambda > 0$. In the case $\lambda = 0$ the techniques from Proposition 2.4 would yield

$$\|T(x,t)\|_{L_x^\infty} \le \|T_0(x)\|_{L_x^\infty} + t\|Y_0(x)\|_{L_x^\infty}.$$

While this does not give a nice uniform-in-time bound, it at least says that thermal blow-up does not occur in finite time irrespective of λ.

For the sake of completeness, I mention that we can obtain coarse estimates on ∇T without too much trouble.

Proposition 2.7 *Pick $p, q, \ell \in [1, \infty]$ with $p^{-1} = q^{-1} - \ell^{-1}$ and $\ell > d$. If $(T_0, Y_0) \in \left(L_x^p \cap L_x^\infty\right) \times \left(L_x^q \cap L_x^\infty\right)$ then the corresponding mild solution (T, Y) of (1.2) satisfies*

$$\|\nabla T(x,t)\|_{L_x^p} \lesssim t^{-\frac{1}{2}} \|T_0(x)\|_{L_x^p} + t^{\frac{1}{2}\left(1-\frac{d}{\ell}\right)} \|Y_0\|_{L_x^q} \quad \forall\ t > 0. \tag{2.10}$$

In particular,

$$\|\nabla T(x,t)\|_{L_x^\infty} \lesssim t^{-\frac{1}{2}} \|T_0(x)\|_{L_x^\infty} + t^{\frac{1}{2}} \|Y_0\|_{L_x^\infty} \quad \forall\ t > 0. \tag{2.11}$$

Proof Using (2.1), (2.4), and $|r(T)| \le 1$, we have

$$\|\nabla T(x,t)\|_{L_x^p} \lesssim t^{-\frac{1}{2}} \|T_0\|_{L_x^p} + \|Y_0\|_{L_x^q} \int_0^t (t-\tau)^{-\frac{1}{2}\left(\frac{d}{\ell}+1\right)} \, \mathrm{d}\tau$$

and we're all done upon computing the integral. ∎

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This re-affirms that solutions of (1.2) do not develop sharp edges in finite time (see the discussion in the second point under Definition 2.1). I expect that one can get rid of the temporally growing term in (2.10) using more sophisticated techniques.

2.4 Decay of small-data solutions

Next I show that, if the initial temperature T_0 is sufficiently small, then $\lim_{t\to\infty} |T(x,t)| = 0$ uniformly. This confirms that (1.2) has a nonzero auto-ignition temperature up to some mild assumptions on T_0. From our work in the last section, we know that $\lambda = 0$ gives the worst L^∞_x temperature bounds possible, so in this section we choose $\lambda = 0$: if we can establish decay with $\lambda = 0$ then it is even simpler to establish decay with $\lambda > 0$. While this means we don't have access to the existence-uniqueness results from the previous section, we can still build a temporally-decaying global solution using the assumption of small initial data and the bootstrap principle. This strategy for establishing time decay is well-known in the theory of dispersive PDEs, see for instance [16].

Throughout this section, we suppose that $\varepsilon_0 \in (0,1)$ is a small parameter (to be constrained later) and

$$\|T_0\|_{L^1_x \cap L^\infty_x} < \varepsilon_0 \quad \text{and} \quad \|Y_0\|_{L^\infty_x} < \infty. \tag{2.12}$$

The requirement that T_0 is small in L^1_x expresses that the initial temperature is approximately localized in space. Strictly speaking, smallness in any L^q_x-norm with $q < \infty$ would also capture localization, but the choice $q = 1$ is a minor technical one giving us access to a good decay estimate. Now, given $t_* \in (0,\infty]$ and a sufficiently nice function $T(x,t)$ defined for $t \in [0,t_*]$, we set

$$\|T(x,t)\|_{X_{t_*}} \doteq \sup_{t\in[0,t_*]} \left[(1+t)^{\frac{d}{2}} \|T(x,t)\|_{L^\infty_x} + \|T(x,t)\|_{L^1_x} \right].$$

This norm naturally gives rise to a Banach space X_{t_*} defined by

$$X_{t_*} \doteq \left\{ T\colon [0,t_*] \to L^1_x \cap L^\infty_x \text{ continuous} \;\middle|\; \|T(x,t)\|_{X_{t_*}} < \infty \right\}.$$

The first order of business is to prove a new *a priori* estimate for solutions that are small in X_{t_*}.

Proposition 2.8 *Take $\varepsilon_1 \in [\varepsilon_0, 1)$. Assume that (2.12) holds and that there is a corresponding mild solution $(T,Y) \in X_{t_*} \times C^0_t L^\infty_x$ to (1.2) valid on $[0,t_*]$. Further, assume that $\|T\|_{X_{t_*}} \le \varepsilon_1$. Then, there exists an absolute constant $C > 0$ such that*

$$\|T(x,t)\|_{X_{t_*}} \le 2^{1+d/2}\varepsilon_0 + C\|Y_0\|_{L^\infty_x}\varepsilon_1^{11}. \tag{2.13}$$

Proof Let's look at the L^∞_x part of the X_{t_*} norm first. Together with the hypothesis of our claim, Lemma 2.2 (with $p = 11$) implies

$$\|r(T)\|_{L^\infty_x} \le C\|T\|^{11}_{L^\infty_x} \le C\left(\varepsilon_1 (1+t)^{-\frac{d}{2}}\right)^{11} \le C\varepsilon_1^{11}(1+t)^{-\frac{11}{2}d}$$

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and

$$\|r(T)\|_{L^1_x} \le C\left\|T^{10}\right\|_{L^\infty_x}\|T\|_{L^1_x} \le C\left[\varepsilon_1(1+t)^{-\frac{d}{2}}\right]^{10}\varepsilon_1 \le C\varepsilon_1^{11}(1+t)^{-5d}.$$

There is nothing special about the choice $p = 11$: any large enough natural number will do, and the rapid vanishing of $r(T)$ at $T = 0$ guarantees we can pick such a large p. We first focus on the $t > 1$ case. Combining the estimates on $r(T)$ with the definition of a mild solution, (2.1), the easy bound $2t \ge 1 + t$ for $t > 1$, (2.2), and (2.12), we have

$$\begin{aligned}
\|T(x,t)\|_{L^\infty_x} &\le 2^{d/2}(1+t)^{-\frac{d}{2}}\varepsilon_0 + \int_0^t \left\|e^{(t-\tau)\Delta}Y(x,\tau)r(T(x,\tau))\right\|_{L^\infty_x}\,\mathrm{d}\tau\\
&\le 2^{d/2}(1+t)^{-\frac{d}{2}}\varepsilon_0 + \int_0^{t/2}(t-\tau)^{-\frac{d}{2}}\|Y(x,\tau)\,r(T(x,\tau))\|_{L^1_x}\,\mathrm{d}\tau\\
&\quad + \int_{t/2}^t \|Y(x,\tau)\,r(T(x,\tau))\|_{L^\infty_x}\,\mathrm{d}\tau\\
&\le 2^{d/2}(1+t)^{-\frac{d}{2}}\varepsilon_0 + C\|Y_0\|_{L^\infty_x}\varepsilon_1^{11}\left[\int_0^{t/2}(1+t-\tau)^{-\frac{d}{2}}(1+\tau)^{-5d}\,\mathrm{d}\tau\right.\\
&\quad \left. + \int_{t/2}^t (1+\tau)^{-\frac{11}{2}d}\right]\\
&\le 2^{d/2}(1+t)^{-\frac{d}{2}}\left[\varepsilon_0 + C\|Y_0\|_{L^\infty_x}\varepsilon_1^{11}\right].
\end{aligned}$$

For $t \le 1$ we may apply an almost identical approach, but we don't split the Duhamel term and we only use (2.2). This takes care of the L^∞_x piece. The L^1_x estimate is obtained by a similar but even easier argument using (2.2) to control $\left\|e^{(t-\tau)\Delta}Yr(T)\right\|_{L^1_x}$. ∎

Theorem 2.9 *There exists* $\varepsilon_0 = \varepsilon_0\left(\|Y_0\|_{L^\infty_x}, d\right) \in (0,1)$ *such that, under the hypothesis (2.12), (1.2) admits a unique global-in-time mild solution* $(T,Y) \in X_\infty \times C^0_t L^\infty_x$ *obeying the decay estimate*

$$\|T(x,t)\|_{L^\infty_x} \lesssim_d \varepsilon_0(1+t)^{-\frac{d}{2}}. \tag{2.14}$$

In particular, $\lim_{t\to\infty}|T(x,t)| = 0$ *uniformly in x.*

Proof Using fixed-point iteration as in Corollary 2.6, one finds a $t_* > 0$ such that (1.2) has a unique local-in-time mild solution valid on $[0,t_*]$ with $T(x,t) \in X_{t_*}$. Further, the map $\tau \mapsto \|T\|_{X_\tau}$ is easily shown to be continuous for $\tau \in [0,t_*]$. Accordingly, for sufficiently small t_* and $\alpha > 2 + \frac{d}{2}$ we know $\|T\|_{X_{t_*}} \le 2^\alpha\varepsilon_0$. Now, apply Proposition 2.8 with $\varepsilon_1 = 2^\alpha\varepsilon_0$ to discover

$$\|T\|_{X_{t_*}} \le 2^{1+d/2}\varepsilon_0\left(1 + 2^{11\alpha-1-\frac{d}{2}}C\|Y_0\|_{L^\infty_x}\varepsilon_0^{10}\right).$$

By choosing $\varepsilon_0 < \frac{1}{2}\left(2^{11\alpha-1-\frac{d}{2}}C\|Y_0\|_{L^\infty_x}\right)^{-\frac{1}{10}}$, the above becomes $\|T\|_{X_{t_*}} \le 2^{2+d/2}\varepsilon_0$. Since $\alpha > 2 + \frac{d}{2}$, this bound is nontrivial. The bootstrap principle (see for example [19]) allows us to iterate this argument, giving a global solution satisfying $\|T\|_{X_\infty} \lesssim_d \varepsilon_0$. ∎

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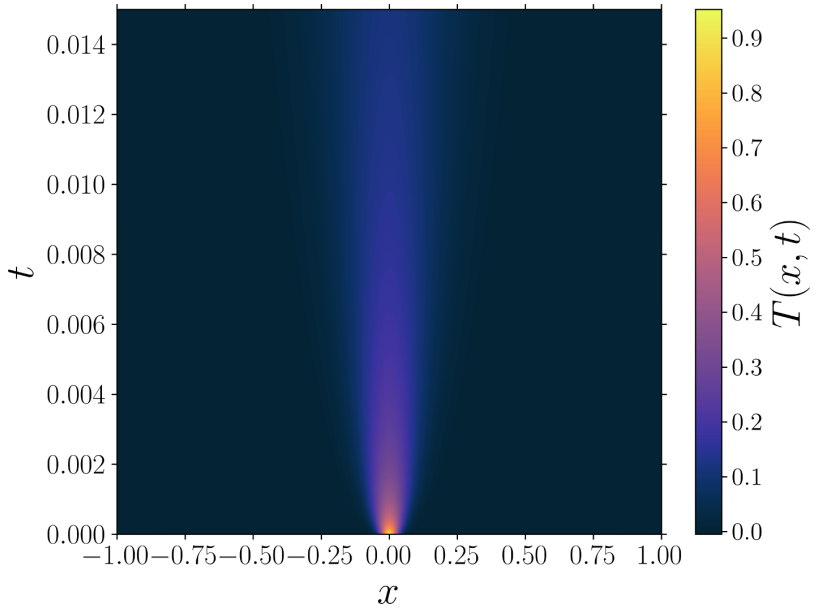


Figure 1: Evolution of the initial state (2.15) with $\beta = 1, \lambda = 0$. Note the rapid decay in time despite the initial amplitude being much larger than ε_0.

The ε_0 from Theorem 2.9 should be regarded as a lower bound on the auto-ignition temperature for the model (1.2), but my methods cannot determine how sharp this bound is. However, a simple numerical experiment shows that much more work is needed to obtain a precise bound on the auto-ignition temperature. Fix $d = 1$. Using a symbolic computation software like WolframAlpha, one can verify that the constant $C(p)$ from Lemma 2.2 with $p = 11$ is on the order of 10^6 (simply ask the software to find the maximum of $z^{-11}r(z)$). Consequently, a bit of arithmetic and the physically meaningful bound $\|Y_0\|_{L^\infty_x} \leq 1$ show that ε_0 is on the order of about 10^{-2}. Figure 1 displays a numerical simulation of (1.2) with $\beta = 1, \lambda = 0$, and initial state

$$T_0(x) = \exp\left(-\left(\frac{x}{0.03}\right)^2\right) \tag{2.15a}$$

$$Y_0(x) = \begin{cases} 1 & x \in [-0.75, 0.75] \\ 0 & \text{else} \end{cases} \tag{2.15b}$$

generated using a Chebyshev differentiation matrix in space [21], exponential Euler integration in time for the temperature [9], and regular Euler integration in time for the mass ratio. Even though this initial state has an amplitude $\|T_0(x)\|_{L^\infty_x}$ that is about one hundred times larger than ε_0, it still exhibits rapid decay: by $t = 0.015$, $\|T(x,t)\|_{L^\infty_x}$ has already dropped to approximately 10% of its initial value. Accordingly, the ε_0 obtained in Theorem 2.9 is far too small to be a sharp lower bound on the auto-ignition temperature. Determining a sharp estimate for the auto-ignition temperature, whether by numerical or analytical means, would be an excellent topic

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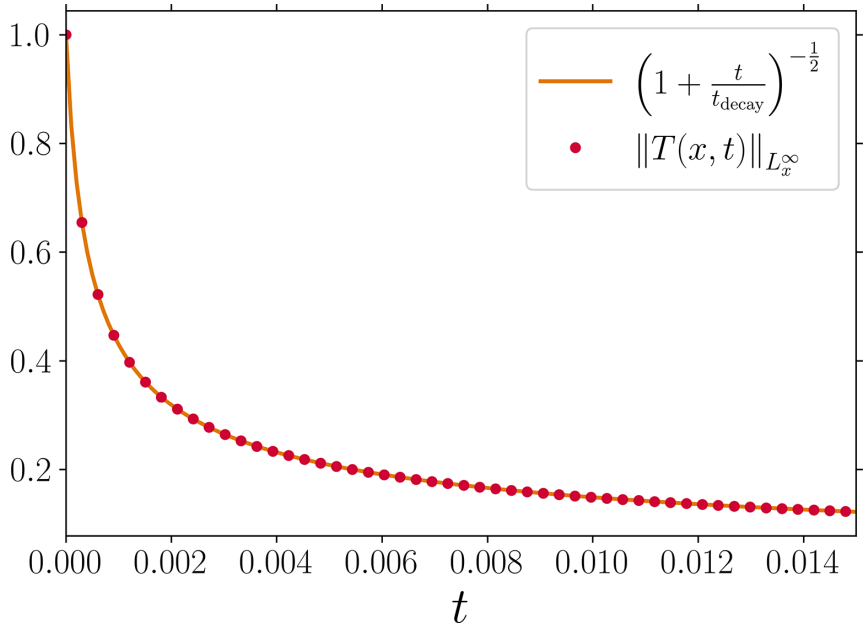


Figure 2: Empirical fit for $\|T(x,t)\|_{L^\infty_x}$ inspired by (2.14). The value of the decay timescale is $t_{\text{decay}} = 2.267 \times 10^{-4}$.

for future investigations. Finally, a curve of the form $t \mapsto \left(1 + t/t_{\text{decay}}\right)^{-\frac{1}{2}}$ (where t_{decay} is a decay timescale) can be fit to the numerically-determined amplitude $t \mapsto \|T(x,t)\|_{L^\infty_x}$ using nonlinear least squares; see Figure 2. The similarity between this fit function and the right-hand side of (2.14) suggests that a variant of (2.14) may still hold for initial data near the *actual* auto-ignition temperature, rather than just for data that is smaller than ε_0.

Acknowledgements I'd like to thank Carrie Clark for helpful comments on the draft manuscript, and Dominic Shillingford for invaluable discussions. I am also grateful to the anonymous referees, whose comments led to strong improvements in the article. Figure 1 uses a cmocean colourmap [20].

Data availability statement The code for reproducing the figures (which makes use of NumPy [8] and SciPy [23]) is available on GitHub at github.com/ageorgemorgan/wildfire_rd/blob/main/mandel_model_autoignition.ipynb.

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