Ergod. Th. & Dynam. Sys. (2000), **20**, 953–962
Printed in the United Kingdom © *2000 Cambridge University Press*

Random iteration of Möbius transformations and Furstenberg's theorem

AMIRAN AMBROLADZE† and HANS WALLIN‡

† *Centre for Mathematics, Lund University/LTH, Box 118, S-221 00, Lund, Sweden*
(e-mail: amiran.ambroladze@math.lth.se)
‡ *Department of Mathematics, Umeå University, S-901 87 Umeå, Sweden*
(e-mail: Hans.Wallin@mathdept.umu.se)

(*Received* 9 *January* 1998 *and accepted in revised form* 2 *November* 1999)

Abstract. Let $Y_1, Y_2, \ldots$ be a sequence of independent random maps, identically distributed with respect to a probability measure μ on $SL(2, R)$. A (deep) theorem of Furstenberg gives abstract conditions under which for almost every such sequence the orbit of a non-zero initial point in R^2 tends to infinity exponentially fast. In the present paper we translate this statement into the set-up of Möbius transformations on the upper half-plane and provide a very explicit way to determine whether or not the required conditions are satisfied.

1. *Notation and definitions*

$\mathbb{C}, \overline{\mathbb{C}}, \overline{\mathbb{R}}$:	the complex plane, the extended complex plane, and the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, respectively.
$P(A)$:	the probability of the event A.
Möbius transformation:	$f(z) = (az + b)/(cz + d), ad - bc \neq 0$.
$\mathbb{H}$:	the upper half-plane, $\mathbb{H} = \{z = x + iy : y > 0\}$, with hyperbolic metric given by the distance function $ds = \lvert dz\rvert/\operatorname{Im} z$, where $\lvert dz\rvert$ denotes Euclidean distance.
$\overline{\mathbb{H}}$:	the closure of $\mathbb{H}$ as a subset of $\overline{\mathbb{C}}$.
Hyperbolic lines in $\mathbb{H}$:	Euclidean semi-circles in $\mathbb{H}$ with centers on the x-axis, and Euclidean half-lines $\{z = x + iy : x = \text{const.}, y > 0\}$ (see [**St**], p. 82).
$\mathcal{F} = \{f_j; p_j\}_{j=1}^N$:	iterated function system (IFS) with probabilities $p_j > 0$, $\sum_{j=1}^N p_j = 1$, of Möbius transformations f_j mapping $\mathbb{H}$ onto itself.
$\{F_n\}_{n=1}^\infty$:	independent identically distributed (iid) random variables with values in $\{f_j\}_{j=1}^N$ with distribution $P(F_n = f_j) = p_j$, $1 \leq j \leq N$.
Z_0:	an initial point in $\mathbb{H}$.

https://doi.org/10.1017/S0143385700000535 Published online by Cambridge University Press

$\{Z_n\}_{n=0}^{\infty}$:	a random orbit $Z_n = Z_n(Z_0) = F_n \circ F_{n-1} \circ \cdots \circ F_1(Z_0)$, $n \geq 1$.
$SL(2,\mathbb{R})$:	the set of 2×2 real matrices with determinant 1.
$\|A\|$:	the norm of the matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $\|A\| = \|a\|+\|b\|+\|c\|+\|d\|$.
μ:	a probability measure on $SL(2,\mathbb{R})$.
G_μ:	the smallest closed subgroup of $SL(2,\mathbb{R})$, containing the support of μ.
$\nu_n \to \nu$:	weak star convergence of measures, $\int \varphi \, d\nu_n \to \int \varphi \, d\nu$ for all continuous functions φ in $\overline{\mathbb{C}}$.

2. *Results*

The main result of this paper about iterations of Möbius transformations is the following theorem.

THEOREM 1. *Let μ be a probability measure on the set of Möbius transformations mapping the upper half-plane $\mathbb{H} = \{\operatorname{Im} z > 0\}$ onto itself. Assume that the transformations in the support of μ have no common fixed point in $\overline{\mathbb{H}}$ and no common invariant (hyperbolic) line in $\mathbb{H}$. Let $\{F_n\}$ be iid random variables with distribution μ. Then, for any initial point $Z_0 \in \mathbb{H}$, the orbit $Z_n = Z_n(Z_0) = F_n \circ F_{n-1} \circ \cdots \circ F_1(Z_0)$, $n \geq 1$, tends to $\overline{\mathbb{R}}$ almost surely. (That is, for arbitrarily fixed compact subset $K \subset \mathbb{H}$ and for almost all orbits $\{Z_n\}_{n\geq 1}$, only a finite number of points of the orbit belong to K.)*

As application to the theory of finite IFSs we give also the following discrete version of Theorem 1.

THEOREM 1′. *Let $\mathcal{F} = \{f_1, \ldots, f_N; p_1, \ldots, p_N\}$, $p_j > 0$, $\sum_1^N p_j = 1$ be an IFS with probabilities of Möbius transformations $f_1, \ldots, f_N$ mapping the upper half-plane $\mathbb{H}$ onto itself, and having no common fixed point in $\overline{\mathbb{H}}$. Assume, in addition, that the Möbius transformations have no common invariant (hyperbolic) line in $\mathbb{H}$. Then, for any initial point $Z_0 \in \mathbb{H}$, the orbit $\{Z_n\}_0^{\infty}$ tends to $\overline{\mathbb{R}}$ almost surely.*

Remark 1. Theorem 1, in particular, implies that a Brownian motion on the hyperbolic plane is transient (that is a Brownian point goes to the boundary of $\mathbb{H}$ with probability 1).

Theorem 1 gives the most general geometrical conditions which guarantee almost sure convergence, in the sense that any further generalization of these conditions should contain some quantitative characterization of the functions (instead of their qualitative description in Theorem 1).

The assumption in Theorem 1 about a common invariant (hyperbolic) line in $\mathbb{H}$, is equivalent to the condition that there is no set $K = \{a, b\}$, $a, b \in \overline{\mathbb{R}}$, $(a \neq b)$ which is invariant for the system, that is such that $f_j(K) = K$ for all j, $1 \leq j \leq N$. (Actually, if such a set $K = \{a, b\}$ exists then the (hyperbolic) line ending at the points a and b, will also be invariant for the system.) From this, in particular, we get the following.

COROLLARY 1. *If the Möbius transformations in Theorem 1, mapping $\mathbb{H}$ onto $\mathbb{H}$, have no common fixed point in $\mathbb{H}$, and in addition, no common 2-periodic point on $\overline{\mathbb{R}}$, then for any initial point $Z_0 \in \mathbb{H}$, the orbit $\{Z_n\}_0^{\infty}$ tends to $\overline{\mathbb{R}}$ almost surely.*

https://doi.org/10.1017/S0143385700000535 Published online by Cambridge University Press

By a 2-periodic point for a function f we mean a point z for which $z = f(f(z))$.

We give one more corollary of Theorem 1, which is less general than Corollary 1, but instead very easy to check.

COROLLARY 2. *If the system in Theorem 1, of Möbius transformations of* $\mathbb{H}$ *onto* $\mathbb{H}$, *having no common fixed point in* $\mathbb{H}$, *contains at least one Möbius transformation* $f(z) = (az + b)/(cz + d)$, *whose matrix* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *has an eigenvalue* $\lambda = \alpha + i\beta$, *with* $\alpha\beta \neq 0$, *then the random orbit* $\{Z_n\}$ *converges to* $\overline{\mathbb{R}}$ *almost surely.*

Any Möbius transformation mapping $\mathbb{H}$ onto itself has a unique representation $f(z) = (ax + b)/(cz + d)$, with a, b, c, d real and $ad - bc = 1$. This allows us to establish a connection between iterations of Möbius transformations and iterations of matrices. This latter topic is very well studied and many interesting papers are devoted to this area (see for example [**Kai, HM**]). One of the most important and profound results about iterations of area-preserving matrices (det $= 1$) is the following theorem due to Furstenberg (we formulate this result for (2×2) matrices in the version given in [**BL**] (see also [**Fu**])).

THEOREM C. (Furstenberg) *Let* μ *be a probability measure on* $SL(2, \mathbb{R})$, *the set of* (2×2) *real matrices with determinant 1, such that the following holds, if* G_μ *is the smallest closed subgroup of* $SL(2, \mathbb{R})$ *which contains the support of* μ*:*

(i) G_μ *is not compact;*

(ii) *there does not exist a subset* L *of* $\mathbb{R}^2$ *which is a finite union of one-dimensional subspaces, such that* $M(L) = L$ *for any* M *in* G_μ.

Then, with probability 1, the norms $\|Y_n \dots Y_1 x\|$ *grow exponentially as* $n \to \infty$, *for all* $x \in \mathbb{R}^2 \setminus \{0\}$, *where* $\{Y_n\}$ *are iid random variables with values in* $SL(2, \mathbb{R})$ *and with distribution* μ.

The proof of Theorem 1 above is based on the following result, which is a combination of Furstenberg's theorem, [**AW**, Lemma 1] and Propositions 2 and 3 below.

THEOREM 2. *Let* μ *be a probability measure on* $SL(2, \mathbb{R})$, *the set of* (2×2) *real matrices with determinant 1. Assume that the matrices in* supp μ *(the support of* μ*) have no common invariant ellipse, and no common invariant set of type* $\ell_1 \cup \ell_2$, *with lines* ℓ_1, ℓ_2 *(not necessarily different) passing through the origin. Then, with probability 1 the norms* $\|Y_n \dots Y_1 x\|$ *grow exponentially, as* $n \to \infty$, *for all* $x \in \mathbb{R}^2 \setminus \{0\}$, *where* $\{Y_n\}$ *are iid random variables with values in* $SL(2, \mathbb{R})$ *and distribution* μ.

COROLLARY 3. *If at least one matrix in* supp μ *has an eigenvalue* $\lambda = \alpha + i\beta$, *with* $\alpha\beta \neq 0$, *then the assertion of Theorem 2 is true.*

The conditions in Theorem 2, although much simpler to check, are equivalent to those in Furstenberg's theorem above (for (2×2) matrices).

The exceptional cases in Theorem 2 are actually three, when there is a basis in which all the matrices in supp μ have one of the following forms simultaneously:

(1) $\left(\begin{smallmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{smallmatrix}\right)$—simple rotation about the origin (invariant circles);

(2) $\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right)$—invariant real axis;

https://doi.org/10.1017/S0143385700000535 Published online by Cambridge University Press

(3) $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ or $\begin{pmatrix} 0 & -b \\ b^{-1} & 0 \end{pmatrix}$—invariant coordinate axes.

Note that in all three cases the eigenvalues of the matrices are real or purely imaginary.

3. *Discussion of the results*

In the case when $\operatorname{supp}\mu$ is a finite set, the following notation is often used:

$$\operatorname{supp}\mu = \mathcal{F} = \{f_j;\, p_j\}_{j=1}^{N}, \quad \sum_{1}^{N} p_j = 1, \quad p_j > 0, \tag{1}$$

where $p_j = \mu(f_j)$. $\mathcal{F}$ is said to be an IFS with probabilities.

Our original motivation to study the IFSs comes from its connection to continued fractions and rational approximation (see [**KWG, KW**]). Suppose for a moment that the transformations in $\mathcal{F}$ have the form $f_j(z) = a_j/(1+z)$, $i = 1, \dots, N$. Then the reversed iterates $\check{Z}_n = F_1 \circ F_2 \circ \cdots \circ F_n(Z_0)$, $n = 1, 2, \dots$ are the approximants of a continued fraction if $Z_0 = 0$, and of the so-called modified approximants if $Z_0 \neq 0$. There is an interplay between the iterates Z_n of an IFS and the reversed iterates $\check{Z}_n$, and because of that a study of IFSs of Möbius transformations may be used to get new insight into the theory of continued fractions.

Computer experiments described in [**KW**, Figure 6], indicated that the probabilistic attractor is the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ for the IFS $\mathcal{F} = \{a_1/(1+z), a_2/(1+z); \frac{1}{2}, \frac{1}{2}\}$ if $a_1, a_2 < -\frac{1}{4}$, $a_1 \neq a_2$. More specifically, computer experiments were made with real constants a_1 and a_2 less than $-\frac{1}{4}$, $a_1 \neq a_2$, and initial values $Z_0 \in \mathbb{C} \setminus \mathbb{R}$. The first 1000 points of the orbit $\{Z_n\}$ were hidden and then a large number of them were plotted. This gave a picture on the real axis for different choices of a_1, a_2, and Z_0.

Inspired by the computer experiments the following theorem was proved and conjecture stated in [**BWK**].

THEOREM A. [**BWK**] *Let $\mathcal{F} = \{-5/(1+z), -0.5/(1+z); \frac{1}{2}, \frac{1}{2}\}$. Then, for any initial point $Z_0 \in \overline{\mathbb{C}}$, the orbit $\{Z_n\}_0^\infty$ tends to the extended real line $\overline{\mathbb{R}}$ with probability 1, as n tends to infinity.*

CONJECTURE A. [**BWK**] *Theorem A remains true when -5 and -0.5 are replaced by a_1 and a_2, if $a_1, a_2 < -\frac{1}{4}$, $a_1 \neq a_2$.*

The proof of [**BWK**, Theorem A] depended on numerical computations for the functions $-5/(1+z)$ and $-0.5/(1+z)$. This means that Conjecture A could not be proved in the same way for general constants a_1 and a_2.

In [**AW**] we obtained a general convergence result for the orbit $\{Z_n\}$, but in a weaker sense than almost sure convergence. In fact, we established the following theorem.

THEOREM B. [**AW**, Corollary 1] *Let $\mathcal{F} = \{f_1, \dots, f_N;\, p_1, \dots, p_N\}$, $p_j > 0$, $\sum_1^N p_j = 1$ be an IFS with probabilities of Möbius transformations $f_1, \dots, f_N$ mapping the upper half-plane $\mathbb{H}$ onto itself, and having no common fixed point in $\mathbb{H}$. Then, for any initial point $Z_0 \in \mathbb{H}$, the orbit $\{Z_n\}_0^\infty$ tends to $\overline{\mathbb{R}}$ in probability in the sense that for any compact subset $K \subset \mathbb{H}$ we have*

$$P(Z_n \in K) \to 0, \quad \text{as} \quad n \to \infty.$$

https://doi.org/10.1017/S0143385700000535 Published online by Cambridge University Press

We might say that the probabilistic attractor of $\mathcal{F}$ lies on $\overline{\mathbb{R}}$ in Theorem B. It is easy to see that it is necessary in Theorem B to assume that $f_1, \ldots, f_N$ have no common fixed point in $\mathbb{H}$. In fact, otherwise we would get a contradiction to Theorem B by choosing Z_0 as this common fixed point. Concerning Theorem B we also note that the deterministic attractor for this case can be the whole $\overline{\mathbb{H}}$. (By the deterministic attractor for a given initial point Z_0 we mean the closure of the set of limit points of all orbits starting at Z_0.)

The last theorem explained the results of the computer experiments described above, but it did not answer the conjecture. It is Corollary 2 of the present paper (see above), which easily shows that Conjecture A, formulated above, is true. Actually, for a function of type $f(z) = a/(1+z)$ we have $\lambda_{1,2} = \frac{1}{2}(1 \pm \sqrt{1+4a})$, and the root $\sqrt{1+4a}$ is purely imaginary when $a < -\frac{1}{4}$. The conjecture is also true if we have more than two functions.

It is interesting to note that, as it follows from Corollary 1, the conjecture is true even for the values $a_1 < a_2 < 0$. Here we remark that the condition $a < 0$ is evidently necessary to guarantee that the function $f(z) = a/(1+z)$ maps $\mathbb{H}$ onto itself.

We conclude this section by referring to [**Am**], where the case of IFSs with general analytic functions is considered.

4. *The scheme of the proofs*

We start with the proof of Theorem 2. We divide it into the following two propositions, which have independent interest as well.

PROPOSITION 1. *Let μ be a probability measure on $SL(2,\mathbb{R})$ of (2×2) real matrices with determinant 1, and let G_μ be the smallest closed subgroup of $SL(2,\mathbb{R})$ which contains the support of μ. Then G_μ is compact if and only if all the matrices in G_μ (or, equivalently, in* supp μ*) have a common invariant ellipse.*

PROPOSITION 2. *Assume that the group G_μ in Proposition 1 is non-compact and that there exists a subset $L = \cup \ell_i$ of $\mathbb{R}^2$ of a finite union of n different one-dimensional subspaces ℓ_i, $i = 1, \ldots, n$, such that $A(L) = L$ for all A in G_μ. Then $n \leq 2$.*

These two propositions together with Furstenberg's theorem above give Theorem 2.

In the proof of Proposition 1 we use the following three lemmas.

LEMMA 1. *Let $\left\{A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right\}_A$ be a subset of $SL(2,\mathbb{R})$ and let $\{f_A(z) = (az+b)/(cz+d)\}_A$ be the associated Möbius transformations. Then, for any fixed $z \in \mathbb{H}$, $\{f_A(z)\}_A$ is an unbounded set in $\mathbb{H}$ (in the hyperbolic sense) if and only if the norms $\{\|A\|\}_A$ are unbounded (in $\mathbb{R}$ with the ordinary Euclidean metric).*

LEMMA 2. *In the notation of Lemma 1, the matrices $\{A\}_A$ have a common invariant ellipse if and only if the associated Möbius transformations $\{f_A\}_A$ have a common fixed point in $\mathbb{H}$.*

LEMMA 3. *Let μ be a probability measure on the set of Möbius transformations mapping the upper half-plane $\mathbb{H}$ onto itself. Assume that the transformations in the support of μ have no common fixed point in $\mathbb{H}$. Then for any initial fixed point $z \in \mathbb{H}$ the set $\{f(z)\}$, $f \in G_\mu$, is unbounded (in the hyperbolic sense), where G_μ denotes the smallest closed subgroup of Möbius transformations containing the support of μ.*

https://doi.org/10.1017/S0143385700000535 Published online by Cambridge University Press

Finally, from Theorem 2 we derive Theorem 1.

5. *Proofs*

Proof of Lemma 1. Due to the preservation of hyperbolic distances under Möbius transformations, we may consider instead of a general point $z \in \mathbb{H}$, the point $z = i$. Then

$$f_A(i) = \frac{ai + b}{ci + d} = \frac{(bd + ac) + i(ad - bc)}{c^2 + d^2},$$

and since $1 = \det A = ad - bc$, we get

$$f_A(i) = \frac{bd + ac}{c^2 + d^2} + i\frac{1}{c^2 + d^2}. \tag{2}$$

Assume that $\{f_A(i)\}_A$ is an unbounded set in $\mathbb{H}$ (in the hyperbolic metric). This gives the following three alternatives: either the set $\{c^2 + d^2\}_A$ is unbounded (in $\mathbb{R}$), or $\{(bd + ac)/(c^2 + d^2)\}_A$ is unbounded, or the set $\{c^2 + d^2\}_A$ contains elements arbitrarily close to zero. All these alternatives together with the condition $\det A = ad - bc = 1$, give that $\{|a| + |b| + |c| + |d|\}_A$ is unbounded (in $\mathbb{R}$).

Now, conversely, assume that $\{\|A\|\}_A$ is unbounded, that is $\{|a| + |b| + |c| + |d|\}_A$ is unbounded. If we, in addition, have that $\{|c| + |d|\}_A$ is also unbounded, then (2) gives that $\{f_A(i)\}_A$ has elements arbitrarily close to the real line. If $\{|c| + |d|\}_A$ is bounded, then, by the assumption, $\{|a| + |b|\}_A$ is unbounded. In this case

$$|f_A(i)| = \left|\frac{ai + b}{ci + d}\right| = \frac{\sqrt{a^2 + b^2}}{\sqrt{c^2 + d^2}}$$

and $\{|f_A(i)|\}_A$ is unbounded in $\mathbb{R}$. In either case the set $\{f_A(i)\}_A$ is an unbounded subset of $\mathbb{H}$. Lemma 1 is proved. □

Proof of Lemma 2. Let the set of Möbius transformations $\{f_A\}_A$ have a common fixed point $w \in \mathbb{H}$. Choose $B \in SL(2, \mathbb{R})$, such that for the associated Möbius transformation f_B we have $f_B(i) = w$. (i is the imaginary unit.) Then the conjugate system $\{f_B^{-1} \circ f_A \circ f_B\}_A$ has i as a common fixed point. Note that $f_B^{-1} \circ f_A \circ f_B = f_{B^{-1}AB}$ and note also that a Möbius transformation $f(z) = (az + b)/(cz + d)$ has i as a fixed point ($f(i) = (ai + b)/(ci + d) = i$) if and only if $a = d$ and $b = -c$. If, in addition, the determinant $ad - bc = a^2 + b^2$ equals one, then the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ provides a rotation around the origin. This shows that the matrices $\{B^{-1}AB\}_A$ have a common invariant circle, for example the unit circle (centered at the origin) in $\mathbb{R}^2$. Then, evidently, the set $\{A\}_A$ has a common invariant ellipse which is the image of the unit circle under B. The converse assertion can be proved in the same way. Lemma 2 is proved. □

Proof of Lemma 3. This lemma was proved for the more general case of all isometries of the upper half-plane in [**AW**] (see Lemma 4.1 there). However, for its importance we include the proof (in the restrictive form of Lemma 3) here as well. This makes the present paper self-contained.

Before proceeding to the proofs of the lemma we recall that any Möbius transformation of the upper half-plane $\mathbb{H}$ is an orientation-preserving isometry of $\mathbb{H}$ (see [**St**, p. 89]).

https://doi.org/10.1017/S0143385700000535 Published online by Cambridge University Press

Geometrically any orientation-preserving isometry is either a rotation (conjugate to a function of the form $f(z) = e^{i\theta}z$ in D_1), a limit rotation (conjugate to $f(z) = z + \alpha$ in $\mathbb{H}$), or a translation (conjugate to $f(z) = \rho z$, $\rho > 0$, in $\mathbb{H}$, whose axis of translation is the imaginary half-axis).

We divide the proof of the lemma into two cases: in case 1 below we assume that among the transformations in the support of μ we have at least one Möbius transformation (not the identity) with a fixed point on the boundary of $\mathbb{H}$ (limit rotation or translation according to the classification above). Consequently, in case 2 we can assume that all the transformations in the support of μ are rotations. Then, by the assumptions of Lemma 3 we have at least two rotations with different rotation centers (fixed points).

Case 1. Suppose that among the functions in the support of μ there is a Möbius transformation, say f, which has a fixed point on the extended real line $\overline{\mathbb{R}}$. We may assume, without loss of generality, that this fixed point is infinity. Then $f(z) = az + b$, $a > 0$, $b \in \mathbb{R}$. If infinity is an attractive fixed point of f (the case $a > 1$), Lemma 3 is evidently true. In fact, applying only f each time we get $|Z_n| \to \infty$. The same argument applies when infinity is a repelling fixed point of f (the case $a < 1$). Then there exists an attractive fixed point of f on $\mathbb{R}$ as well. If $a = 1$ and $b \neq 0$, Lemma 3 is again evident. If $a = 1$ and $b = 0$, f is the identity map (which we may ignore).

Case 2. In this case we use the unit disk model for the hyperbolic plane and assume that the functions in the support of μ are Möbius transformations mapping the unit disk D_1 onto itself. We can also assume that these transformations contain at least two rotations, say f_1 and f_2, with fixed points a_1 and a_2, respectively, in D_1, $a_1 \neq a_2$.

Let $A = A(Z_0)$ be the closure in the Euclidean metric of the set consisting of the orbits with initial point Z_0 obtained by iterating with f_1 and f_2 only. Suppose now that Lemma 3 is not true. Then we may choose an initial point $Z_0 \in D_1$ such that A is a compact subset of D_1.

Consider the following function defined for $z \in D_1 : z \mapsto h(z, a_1) + h(z, a_2)$, where $h(\cdot, \cdot)$ denotes the hyperbolic distance. Denote by w_0 the point where this continuous function has its maximum on the compact set A. If we start with w_0 and iterate with f_1 only (or with f_2 only), the point w_0 moves on some circle C_1 (or some circle C_2) with hyperbolic center at a_1 (or at a_2). Furthermore, under every iteration the point moves on C_1 (or C_2) along an arc of the same hyperbolic length. In fact, this is obvious if a_1 (or a_2) is the origin. Otherwise, we map C_1 (or C_2) by a Möbius transformation onto a circle with center at the origin and use the invariance of hyperbolic distance.

Assume that the hyperbolic length C_1 is not less than that of C_2, i.e. that $h(C_1) \geq h(C_2)$. Denote by γ_1 the open arc of C_1 in the exterior of C_2 and by $\tilde{\gamma}_1$ the complementary arc, $\tilde{\gamma}_1 = C_1 \setminus \gamma_1$. Below we shall prove that $h(\gamma_1) > h(\tilde{\gamma}_1)$. Now, assuming this inequality, we finish the proof of Lemma 3. Since under the iteration of the function f_1 the point w_0 and its images move a fixed hyperbolic distance along C_1, there is at least one point among these images which belongs to γ_1 (here we use that $h(\gamma_1) > h(\tilde{\gamma}_1)$). Denote this point by w_1. Then, evidently, $w_1 \in A$, $h(w_1, a_1) = h(w_0, a_1)$ and $h(w_1, a_2) > h(w_0, a_2)$. Consequently, $h(w_1, a_1) + h(w_1, a_2) > h(w_0, a_1) + h(w_0, a_2)$, which contradicts the extremal choice of the point w_0.

We now go back and prove that $h(\gamma_1) > h(\tilde{\gamma}_1)$. We may assume that $a_1 = 0$ since

https://doi.org/10.1017/S0143385700000535 Published online by Cambridge University Press

otherwise we could consider a conjugate IFS with this property, obtained by means of a Möbius transformation and preserving, consequently, hyperbolic distances. Then the hyperbolic length $h(\cdot)$ along C_1 is proportional to the ordinary Euclidean length $\ell(\cdot)$, and so it remains to prove that $\ell(\gamma_1) > \ell(\tilde{\gamma}_1)$. A simple geometric argument shows that this latter inequality follows on proving

$$\ell(C_1) > \ell(C_2). \tag{3}$$

Denote by C_2^* the circle centered at the origin with $h(C_2^*) = h(C_2)$. Then by the assumption $h(C_2) \leq h(C_1)$ we have $h(C_2^*) \leq h(C_1)$. Consequently, since both C_2^* and C_1 have centers at the origin, we get $\ell(C_2^*) \leq \ell(C_1)$.

Now we shall prove that

$$\ell(C_2^*) > \ell(C_2) \tag{4}$$

which, together with the previous inequality, gives (3). The inequality (4) means that among all circles with equal hyperbolic length the one with center at the origin has largest Euclidean length. We shall prove this fact by proving the following equivalent statement: among all circles with equal Euclidean length, the one with center at the origin has smallest hyperbolic length. To prove this we consider two circles S_1 and S_2 with the same Euclidean radius r, S_1 centered at the origin and S_2, without loss of generality, at a point $x_0 \in \mathbb{R}$ with $x_0 \neq 0$. Their diameters are $\Delta_1 = (-r, r)$ and $\Delta_2 = (x_0 - r, x_0 + r)$, respectively. We want to prove that $h(S_1) < h(S_2)$. This inequality is equivalent to the corresponding inequality for the diameters, $h(\Delta_1) < h(\Delta_2)$. However, by the definition of the hyperbolic metric, $h(\Delta_j) = \int_{\Delta_j} dx/(1 - x^2)$, $j = 1, 2$, and by simple calculations we get that $h(\Delta_1) < h(\Delta_2)$. Lemma 3 is proved. □

Proof of Proposition 1. In one direction the assertion is evident: if all matrices in G_μ have a common invariant ellipse, then the group $G_\mu$is compact. Now assume that the matrices in G_μ have no common invariant ellipse. Due to Lemma 2 this is equivalent to the fact that the associated Möbius transformations f_A, $A \in G_\mu$, have no common fixed point in $\mathbb{H}$. Next we use Lemma 3, which says that in this case the set $\{f_A(z)\}_A$ is unbounded (in the hyperbolic sense) in $\mathbb{H}$, for any fixed $z \in \mathbb{H}$. Finally, Lemma 1 shows that $\{\|A\|\}_A$ is unbounded and, hence, that G_μ is non-compact, which finishes the proof. Proposition 1 is proved. □

Proof of Proposition 2. Consider the unit disk Δ in $\mathbb{R}^2$. The lines ℓ_i, $i = 1, \ldots, n$ intersect each other at the origin and divide Δ into $2n$ sectors; denote them by $D_1, D_2, \ldots, D_{2n}$. Let $A \in G_\mu$. Then $A(\Delta)$ is an ellipse centered at the origin. We also have that $A(L) = L$. The set L divides $A(\Delta)$ into $2n$ parts; denote them by $S_1, S_2, \ldots, S_{2n}$. The matrix $A \in G_\mu$ gives a one-to-one area-preserving transformation in $\mathbb{R}^2$. This means that for any S_i, $1 \leq i \leq 2n$, there exists a D_j, $1 \leq j \leq 2n$, such that the areas of S_i and D_j are equal. Now, Proposition 2 will be proved by showing that if $n > 2$, then at least one of the parts S_i, $1 \leq i \leq 2n$, will have an arbitrarily small area, provided that the norm of A is sufficiently large. We first assume that $n = 3$, i.e. that $L = \ell_1 \cup \ell_2 \cup \ell_3$, where ℓ_1, ℓ_2, ℓ_3 are (different) lines passing through the origin. If the norm of A is large, then $A(\Delta)$ is a long and thin ellipse, with area equal to one and centered at the origin. Denote by P_1 and P_2 the

https://doi.org/10.1017/S0143385700000535 Published online by Cambridge University Press

points of the ellipse having maximal distance from each other, and denote by ℓ the straight line passing through these points. Then ℓ passes through the origin as well. Let ℓ_1 form the minimal angle with ℓ among the three lines ℓ_1, ℓ_2, ℓ_3. Then the parts S_i, $1 \le i \le 6$, of $A(\Delta)$ lying between the lines ℓ_2 and ℓ_3 and not containing the points P_1 and P_2 will have arbitrarily small area if $A(\Delta)$ is sufficiently long and thin. This follows from simple geometrical considerations, which we omit. The proof proceeds in the same way if $n > 3$. Proposition 2 is proved. □

Now, as we have already remarked, Proposition 1, Proposition 2 and Furstenberg's theorem above give Theorem 2.

Proof of Theorem 1. Denote by A_f, $f \in \operatorname{supp} \mu$, the matrices corresponding to the Möbius transformations in Theorem 1. We may assume that $A_f \in SL(2, \mathbb{R})$. We check now that these matrices satisfy all the assumptions of Theorem 2, that is that they have no common invariant ellipse, no common invariant line ℓ passing through the origin and no common invariant set $\ell_1 \cup \ell_2$, where $\ell_1, \ell_2, \ell_1 \ne \ell_2$ are lines in $\mathbb{R}^2$ passing through the origin. The existence of a common invariant ellipse is excluded by Lemma 2, since otherwise the functions $f \in \operatorname{supp} \mu$ would have a common invariant point in $\mathbb{H}$.

If now the matrices A_f have a common invariant line ℓ passing through the origin, then we may assume that ℓ is the x-axis (otherwise we could consider a conjugate system). In this case (easy to check) any matrix A in $\{A_f\}$ has the form $A = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$. This means that the corresponding Möbius transformations are linear functions, and consequently they have a common fixed point $z = \infty$ on $\overline{\mathbb{H}}$; but the assumptions in Theorem 1 exclude this possibility. Finally, if we assume that the matrices A_f have a common invariant set $\ell_1 \cup \ell_2$ ($\ell_1 \ne \ell_2$), then we may assume (again due to possibility of conjugation) that ℓ_1 and ℓ_2 are the x- and y-axis. Then any matrix A in $\{A_f\}$ has one of the forms: either $A = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ (when each line ℓ_1 and ℓ_2 separately is invariant for A), or $A = \begin{pmatrix} 0 & -b \\ b^{-1} & 0 \end{pmatrix}$ (the case when ℓ_1 and ℓ_2 change places under the transformation A). In either case the corresponding Möbius transformations ($f(z) = a^2 z$ or $f(z) = -b^2/z$) have a common invariant (hyperbolic) line in $\mathbb{H}$. This line is the positive imaginary axis: $\{\operatorname{Re} z = 0, \operatorname{Im} z > 0\}$. This possibility is also excluded by the assumptions in Theorem 1. So we have checked that the matrices A_f satisfy all the conditions of Theorem 2. Then Theorem 2 in particular asserts that $\|Y_n \cdots Y_1\| \to \infty$, as $n \to \infty$, almost surely. Then Lemma 1 gives that $Z_n(Z_0) = F_n \circ \cdots \circ F_1(Z_0)$ tends to $\overline{\mathbb{R}}$, as $n \to \infty$, almost surely. Theorem 1 is proved. □

References

[Am] A. Ambroladze. Ergodic properties of random iterations of analytic functions. *Ergod. Th. & Dynam. Sys.* **19** (1999), 1379–1388.

[AW] A. Ambroladze and H. Wallin. Random iteration of isometries of the hyperbolic plane. *Markov Processes and Related Fields* **5**(1) (1999), 69–88.

[BWK] A. Barrlund, H. Wallin and J. Karlsson. Iteration of Möbius transformations and attractors on the real line. *Comp. Math. Appl.* **33** (1997), 1–12.

https://doi.org/10.1017/S0143385700000535 Published online by Cambridge University Press

[BL] P. Bougerol and J. Lacroix. *Products of Random Matrices with Applications to Schrödinger Operators*. Birkhäuser, Boston, Basel, Stuttgart, 1985.

[Fu] H. Furstenberg. Noncommuting random products. *Trans. Amer. Math. Soc.* **108** (1963), 377–428.

[HM] G. Högnäs and A. Mukherjea. *Probability Measures on Semigroups*. Plenum, New York, 1995.

[KW] J. Karlsson and H. Wallin. Continued fractions and iterated function systems. *Proc. Conf. Loen 1992, Continued Fractions and Orthogonal Functions*. Marcel Dekker, New York, 1994, pp. 191–210.

[KWG] J. Karlsson, H. Wallin and J. Gelfgren. Iteration of Möbius transformations and continued fractions. *Rocky Mountain J. Math.* **21** (1991), 451–472.

[Kai] T. Kaijser. On invariant measures associated to products of random matrices. *Festschrift in Honor of L. Carleson and Y. Domar (Proc. Conf., Uppsala 1993). Acta Univ. Upsaliensis*, Uppsala, 1995, pp. 191–203.

[St] J. Stillwell. *Geometry of Surfaces*. Springer, New York, 1992.

https://doi.org/10.1017/S0143385700000535 Published online by Cambridge University Press