POLAR MODULES

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This paper is concerned with what, for lack of a better name, the author proposes to call geometric modules over a field k. These modules are defined as follows. A geometric module M consists of a vector space, also denoted by M, over k, together with a ring of linear transformations A of M into itself subject to the following conditions:

(i) A is a homomorphic image of the ring $S_n = k[X_1, \ldots, X_n]$ of polynomials in n indeterminates $X_1, \ldots, X_n$ for some value of n;

(ii) M, considered as an A-module, is finitely generated.

It is clear that the value of n and the homomorphism of S_n onto A whose existence is postulated in (i) are not unique. However, given such a homomorphism, we can then consider M as an S-module, where $S = S_n$, and can then define the sequence of S-modules $\mathrm{Ext}^p_S(M, S)$. These modules can then in turn be considered as geometric modules, the associated ring of linear transformations being in each case a homomorphic image of S (and, in fact, of A). Now suppose we write $T^r(M) = \mathrm{Ext}^{n-r}_S(M, S)$. The basic theorem of the present paper then implies that $T^r(M)$ depends, to within isomorphism as a geometric module, only on M and r, and not on either S or the homomorphism of S onto A used in its definition. Further, $T^r(M) = 0$ if $r < 0$ or if $r > d$, where d is the transcendence degree over k of the ring A. The statement and proof of the basic theorem given in § 1 below is in rather more detail than is indicated above, since the possibility of M being a graded module is taken into account.

Apart from the basic theorem mentioned above, this paper is mainly concerned with the module $T^d(M)$ and its relation to M. We shall write M^* for $T^d(M)$ and call M^* the polar module of M. The most important property of M^* that we shall prove is the relation $M^{***} \simeq M^*$, a result closely related to the theorems of Macaulay and Grobner on reciprocal quotients. For other results relating to M^* the reader is referred to the body of the paper.

The present work owes its inspiration to the fundamental paper (5) by Serre on coherent faisceaux on an abstract algebraic variety, and is closely related to the unpublished Duality Theorem due to the same author. The present author would like to take this opportunity of expressing his gratitude to Prof. Serre for information concerning the latter theorem and some helpful comments on the earlier work (4) on which the present paper is based.

1. *The basic theorem.* We shall commence this section with the definition of a graded geometric module. A geometric module M is said to be a *graded geometric module* if M and A can be expressed as direct sums $M = \sum_{p=-\infty}^{\infty} M_p$ and $A = \sum_{q=-\infty}^{\infty} A_q$ of vector spaces M_p, A_q over k such that the following conditions are satisfied:

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(i) if $a_q, a_{q'}$ belong to $A_q, A_{q'}$ respectively, then $a_q a_{q'} \in A_{q+q'}$;

(ii) if $a_q \in A_q$ and $x_p \in M_p$, then $a_q x_p \in M_{p+q}$.

We term elements of M_p (or A_q) *homogeneous elements of* M (or A) *of degree* p (or q).

It is almost immediately obvious that if M is graded then we may choose the homomorphism of S_n onto A so that it satisfies the additional restriction that $X_i (i = 1, \ldots, n)$ has a homogeneous image in A. If we are considering graded modules we shall always assume that this restriction is in force.

Before proceeding to the proof of the basic theorem, we shall state three results, the first two classical and the third proved by the author in (4), on which this proof depends.

(*a*) Let $\mathfrak{a}$ be an ideal of S_n of rank r. Then the ring $A = S_n/\mathfrak{a}$ has transcendence degree $n-r$ over k.

(*b*) If $\mathfrak{a}$ is as in (*a*), then, if $p \leqslant r$, we can always find elements $g_1, \ldots, g_p$ in $\mathfrak{a}$ which satisfy the condition $(g_1, \ldots, g_{i-1}) : g_i = (g_1, \ldots, g_{i-1})$ for $i = 1, \ldots, p$, but this is not possible for $p > r$.

In (4), the author termed the greatest integer p for which elements $g_1, \ldots, g_p$ with the above property can be found in $\mathfrak{a}$, the grade of $\mathfrak{a}$; and proved that the grade of $\mathfrak{a}$ is always less than or equal to the rank of $\mathfrak{a}$. The above statement can then be expressed in the form that in S_n the grade of an ideal is always equal to its rank.

(*c*) Let A be a graded Noether ring and let $g_1, \ldots, g_r$ be homogeneous elements of A, g_i having degree m_i, subject to the condition that $(g_1, \ldots, g_{i-1}) : g_i = (g_1, \ldots, g_{i-1})$ for $i = 1, \ldots, r$. Let $\mathfrak{g}$ denote the ideal $(g_1, \ldots, g_r)$ and B the quotient ring $A/\mathfrak{g}$. Then, if M is a graded B-module (so that M can also be considered as an A-module),

$$\operatorname{Ext}_A^p(M, A) = 0 \quad (p < r),$$

$$\operatorname{Ext}_A^p(M, A) \simeq \operatorname{Ext}_B^{p-r}(M, B)\left(\sum_{i=1}^{r} m_i\right) \quad (p \geqslant r),$$

where if N is a graded module the symbol $N(m)$ denotes the same module with the degree of each homogeneous element decreased by m.

THEOREM 1·1. *Let M be a graded geometric module, the ring of linear transformations associated with M being denoted by A. Let h, h' be homomorphisms of $S = k[X_1, \ldots, X_m]$ and $S' = k[Y_1, \ldots, Y_n]$ respectively onto A, $h(X_i)$ being homogeneous of degree p_i and $h'(Y_j)$ being homogeneous of degree q_j. Then*

$$\operatorname{Ext}_S^{m-r}(M, S)\left(-\sum_{i=1}^{m} p_i\right) \simeq \operatorname{Ext}_{S'}^{n-r}(M, S')\left(-\sum_{j=1}^{n} q_j\right),$$

both sides being considered as geometric modules. Further, both sides are zero if $r < 0$ or $r > d$, where d is the transcendence degree of A over k.

Let S^* denote the ring $k[X_1, \ldots, X_m, Y_1, \ldots, Y_n]$ and let h^* denote the homomorphism of S^* onto A defined by the equations $h^*(X_i) = h(X_i), h^*(Y_j) = h'(Y_j)$. If we consider S and S' as sub-rings of S^*, h and h' can be considered as the restriction of h^* to S and S' respectively.

We now define S, S' and S^* as graded rings as follows. A monomial $X_1^{r_1} \ldots X_m^{r_m} Y_1^{s_1} \ldots Y_n^{s_n}$ is said to have weight $\sum_{i=1}^{m} r_i p_i + \sum_{j=1}^{n} s_j q_j$ in S^*. We then define S_p^* to be the vector space

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spanned by monomials of weight p. The resulting decomposition of S^* as a direct sum $S^* = \sum_{p=-\infty}^{\infty} S_p^*$ defines S^* as a graded ring. It is then clear that M is a graded S^*-module. S and S' can then, as sub-rings of S^*, be considered as graded rings, since $S = \Sigma S \cap S_p^*$ and $S' = \Sigma S' \cap S_p^*$, and then M is both a graded S-module and a graded S'-module.

We now choose elements $f_1, \ldots, f_m$ in S' (and therefore in S^*) homogeneous of degrees $p_1, \ldots, p_m$ and satisfying the conditions $h'(f_i) = h(X_i)\,(i = 1, \ldots, m)$. If we now write Z_i for $X_i - f_i$, it then follows that $h^*(Z_i) = 0\,(i = 1, \ldots, m)$. Let $\mathfrak{p}_h$ be the ideal of S^* generated by the elements $Z_1, \ldots, Z_h$. Then $\mathfrak{p}_h$ has rank at most h. On the other hand, every element of S^* is equivalent modulo $\mathfrak{p}_h$ to an element of the sub-ring $S'[X_{h+1}, \ldots, X_m]$ which is an integral domain and has transcendence degree $m+n-h$ over k. Hence $S^*/\mathfrak{p}_h$ has transcendence degree at most $m+n-h$ over k, and therefore the rank of $\mathfrak{p}_h$ is at least h. We have therefore proved that $\mathfrak{p}_h$ has rank exactly h. This also implies that $S^*/\mathfrak{p}_h$ is a homomorphic image of $S'[X_{h+1}, \ldots, X_m]$ of transcendence degree $m+n-h$ over k. Hence $S^*/\mathfrak{p}_h$ is isomorphic to the integral domain

$$S'[X_{h+1}, \ldots, X_m].$$

Finally, therefore, $\mathfrak{p}_h$ is a prime ideal of rank h.

Now put $h = m$. Then $S^*/\mathfrak{p}_m$ is isomorphic to S'. Hence we have a homomorphism k' of S^* onto S' satisfying the following conditions:

(i) k', restricted to S', induces the identity isomorphism of S' onto itself;

(ii) the kernel $\mathfrak{p}_m$ of k' is contained in the kernel of h^* and as a basis $Z_1, \ldots, Z_m$ which satisfies the condition $(Z_1, \ldots, Z_{i-1}) : Z_i = (Z_1, \ldots, Z_{i-1})$ for $i = 1, \ldots, m$.

Hence we are in a position to apply (c). This leads to the isomorphism

$$\mathrm{Ext}_{S'}^{n-r}(M, S')\left(-\sum_{j=1}^{n} q_j\right) \simeq \mathrm{Ext}_{S^*}^{m+n-r}(M, S^*)\left(-\sum_{i=1}^{m} p_i - \sum_{j=1}^{n} q_j\right).$$

In an exactly similar fashion we can show that

$$\mathrm{Ext}_{S}^{m-r}(M, S)\left(-\sum_{i=1}^{m} p_i\right) \simeq \mathrm{Ext}_{S^*}^{m+n-r}(M, S^*)\left(-\sum_{i=1}^{m} p_i - \sum_{j=1}^{n} q_j\right),$$

and combining these isomorphisms yields the isomorphism of the theorem.

Since $S = k[X_1, \ldots, X_m]$, S has global dimension m, whence $\mathrm{Ext}_S^p(M, N) = 0$ for $p > m$ and any pair of S-modules M and N. Hence $\mathrm{Ext}_S^{m-r}(M, S) = 0$ if $r < 0$. Further, since the kernel of the homomorphism of S onto A has rank $m-d$ by (a), an application of (b) and (c) implies that $\mathrm{Ext}_S^{m-r}(M, S) = 0$ if $r > d$.

Let M be a graded geometric module with associated ring of linear transformations A. If $S = k[X_1, \ldots, X_m]$ and if h is a homomorphism of S onto A, the image of X_i being homogeneous of degree p_i, then we define

$$T^r(M) = \mathrm{Ext}_S^{m-r}(M, S)\left(-\sum_{i=1}^{m} p_i\right),$$

where the right-hand side is calculated by considering M as an S-module.

Theorem 1·1 shows that $T^r(M)$ is independent of the choice of S and h involved in the above definition. We may also remark that the associated ring of linear transforma-

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tions of $T^r(M)$ is a homomorphic image of A. For if $\mathfrak{a}$ is the kernel of the homomorphism of S onto A, then multiplication by an element a of $\mathfrak{a}$ induces the zero transformation on M and hence on each of the satellite functor modules $\mathrm{Ext}_S^{m-r}(M, S)$.

If M is a geometric module with associated ring of linear transformations A, then by a sub-module of M we shall mean a sub-A-module. If N is a sub-module of M, then N, considered as a geometric module, has associated ring of linear transformations homomorphic to A. Similarly, we first consider the quotient module M/N as an A-module. Then, considering this module as a geometric module, the associated ring of linear transformations is again a homomorphic image of A.

THEOREM 1·2. *Let M be a (graded) geometric module, N a sub-module, and P the quotient module M/N. Then we have an exact sequence*

$$0 \to T^d(P) \to T^d(M) \to T^d(N) \to T^{d-1}(P) \to \ldots \to T^1(N) \to T^0(P) \to T^0(M) \to T^0(N) \to 0,$$

where d is the transcendence degree of the associated ring of linear transformations.

Suppose that we have a homomorphism of $S = k[X_1, \ldots, X_m]$ onto the associated ring of linear transformations of M. Then we can consider M, N, P as S-modules, and the above theorem is then an immediate consequence of the definition of $T^r(M)$ and the exact sequence

$$0 \to \mathrm{Ext}_S^0(P, S) \to \mathrm{Ext}_S^0(M, S) \to \mathrm{Ext}_S^0(N, S) \to \mathrm{Ext}_S^1(P, S) \to \ldots \to \mathrm{Ext}_S^{m-r}(P, S)$$
$$\to \mathrm{Ext}_S^{m-r}(M, S) \to \mathrm{Ext}_S^{m-r}(N, S) \to \mathrm{Ext}_S^{m-r+1}(P, S) \to \ldots,$$

if we observe that $\mathrm{Ext}_S^{m-d-1}(N, S) = \mathrm{Ext}_S^{m+1}(P, S) = 0$.

COROLLARY 1. *If the associated ring of linear transformations of N has transcendence degree $d' < d$, then*

$$T^{d-r}(M) \simeq T^{d-r}(P) \quad \text{if} \quad r < d - d'.$$

COROLLARY 2. *If the associated ring of linear transformations of P has transcendence degree $d' < d$, then*

$$T^{d-r}(M) \simeq T^{d-r}(N) \quad \text{if} \quad r < d - d' - 1.$$

2. *Ancillary results.* In this section we shall state, without proof, a number of known results which will be useful in the sequel.

The first group of definitions and results that we shall consider refer to a fixed A-module M and its sub-modules, where A is a Noether ring and M is assumed to be finitely generated. For a fuller discussion of these results the reader is referred to Grundy (2), particularly pp. 251–5.

If M is as above and N is a sub-module of M, the radical N_ρ of N is the ideal of A consisting of all elements a of A such that $a^r M \in N$ for some integer r.

A sub-module N of M is termed a *primary sub-module* of M if whenever $ax \in N$, where $a \in A$ and $x \in M$, then either a belongs to N_ρ or $x \in N$. It then follows that N_ρ is a prime ideal of A.

A sub-module N of M is termed an *irreducible sub-module* of M if N is not the intersection of two sub-modules N_1, N_2 of M both of which properly contain N.

We may now state the following results:

(2·1) An irreducible sub-module of M is a primary sub-module.

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(2·2) Every sub-module N of M may be expressed as finite intersection of irreducible sub-modules of M.

(2·3) If $N = N_1 \cap \ldots \cap N_r = N_1' \cap \ldots \cap N_s'$ are two irredundant representations of N as intersections of irreducible sub-modules of M, then $r = s$ and the suffixes can always be so arranged that N_i and N_i' have the same radical.

(2·4) If N_1, N_2 are two primary sub-modules of M with the same radical $\mathfrak{p}$, then $N_1 \cap N_2$ is also primary, and has radical $\mathfrak{p}$.

(2·5) A sub-module N of M may always be expressed as an irredundant intersection $N = N_1 \cap \ldots \cap N_r$ of primary sub-modules N_i, where the radicals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ of $N_1, \ldots, N_r$ are distinct. The set of prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ is independent of the particular decomposition chosen.

(2·6) If in (2·5) the ideal $\mathfrak{p}_i$ is a minimal ideal in the set $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$, then the corresponding sub-module N_i of M is independent of the particular decomposition chosen.

Now suppose that M is a geometric module and A is its associated ring of linear transformations. We shall term M an equi-dimensional module if the prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ corresponding to a decomposition of the zero module of M as in (2·5) all have the same dimension (the dimension of $\mathfrak{p}_i$ being the transcendence degree of the integral domain $A/\mathfrak{p}_i$). From the choice of A it then follows that $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are the minimal prime ideals of A, which is therefore also equi-dimensional. Further, each of the ideals $\mathfrak{p}_i$ is minimal in the set $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ so that the decomposition of the zero module of M is in this case unique.

Our second group of definitions and results refer to the case where A is an Artin ring (i.e. satisfies the descending chain condition on ideals).

(2·7) An Artin ring A has a unique decomposition as a direct sum $A = A_1 \oplus \ldots \oplus A_p$, where A_i is a primary Artin ring, i.e. the radical $\mathfrak{m}_i$ of A_i is the only prime ideal of A_i.

(2·8) If M is an A-module then M has a unique decomposition $M = M_1 + \ldots + M_p$, where the A-module M_i can be considered as an A_i-module (that is, $ax = 0$ for all x in M_i if a belongs to the kernel of the homomorphism of A onto A_i).

(2·9) A finitely generated module M over an Artin ring possesses a composition series. The length of this composition series λ is the sum of the lengths λ^i of the sub-modules M_i of (2·8). If M has length 0, then M is the zero module, while if M has length 1, then M is module isomorphic to one of the (mutually non-isomorphic) modules $A_i/\mathfrak{m}_i$.

Our remaining definitions refer to the case where A is a primary Artin ring with radical $\mathfrak{m}$. If M is a finitely generated A-module, the upper Loewy series of M is the descending series of sub-modules

$$M \supset \mathfrak{m}M \supset \mathfrak{m}^2M \supset \ldots \supset \mathfrak{m}^\rho M = 0,$$

while the lower Loewy series is the ascending series of sub-modules

$$0 \subset (0:\mathfrak{m}) \subset (0:\mathfrak{m}^2) \subset \ldots \subset (0:\mathfrak{m}^\rho) = M,$$

where ρ is the exponent of the module M (that is, the least integer ρ such that $\mathfrak{m}^\rho M = 0$).

We now define $\overline{\lambda}_i$ to be the length of the module $\mathfrak{m}^{i-1}M/\mathfrak{m}^iM$ and λ_i to be the length of the module $(0:\mathfrak{m}^i)/(0:\mathfrak{m}^{i-1})$. These $\lambda_i, \overline{\lambda}_i$ are termed *Loewy invariants* of M.

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(2·10) (*a*) $\lambda_1 + \ldots + \lambda_\rho = \overline{\lambda}_1 + \ldots + \overline{\lambda}_\rho = \lambda$, where λ is the length of M;

(*b*) $\lambda_1 + \ldots + \lambda_i \geqslant \overline{\lambda}_p + \ldots + \overline{\lambda}_{p+1-i}$;

(*c*) $\overline{\lambda}_1$ is the number of elements in a minimal basis of M;

(*d*) λ_1 is the number of irreducible sub-modules of M in an irredundant representation of the zero module of M as an intersection of irreducible sub-modules.

Finally, suppose that M is a module over an Artin ring A. Then we shall denote the Loewy invariants of the module M_j of (2·9) by $\overline{\lambda}_{ji}$ and λ_{ji}.

3. *The polar module of a geometric module.* It will be convenient at this point to introduce the following notational convention. If M is a geometric module with associated ring of linear transformations A then the transcendence degree of A will be denoted by $d(M)$, or by d if no possibility of confusion arises. If M is a geometric module, the module $T^d(M)$ will be denoted by M^* and termed the *polar module* of M.

We shall be concerned in this section with the relationship between a module M and its polar module M^*. In order to avoid irrelevant notational complication in the proofs of the results that follow, we shall ignore the question of grading, and discuss the points that arise from considering M as a graded module at the end of the section.

For the reader's convenience, we reproduce the following theorem from (4).

THEOREM 3·1. *Let M be a geometric module with associated ring of linear transformations A. Let h be a homomorphism of $k[X_1, \ldots, X_m]$ onto A with kernel $\mathfrak{a}$. Let $g_1, \ldots, g_{m-d}$ be elements of $\mathfrak{a}$ satisfying the condition $(g_1, \ldots, g_{i-1}) : g_i = (g_1, \ldots, g_{i-1})$ for $i = 1, \ldots, m-d$. Finally, let $\mathfrak{g} = (g_1, \ldots, g_{m-d})$ and $B = k[X_1, \ldots, X_m]/\mathfrak{g}$. Then*

$$\text{(i)} \quad M^* \simeq \mathrm{Hom}_B(M, B),$$

$$\text{(ii)} \quad M^* \simeq \mathrm{Hom}_A(M, A^*),$$

where $A^* = T^d(A)$ is isomorphic as an A-module to $(\mathfrak{g} : \mathfrak{a})/\mathfrak{g}$.

(i) This is an immediate consequence of the result stated as (*c*) immediately preceding Theorem 1·1, if we observe that $\mathrm{Ext}^0_B(M, B) = \mathrm{Hom}_B(M, B)$ by definition.

(ii) Since M is an A-module, if h is any homomorphism of M into B, $a.h(x) = 0$ if a belongs to the kernel $\mathfrak{a}/\mathfrak{g}$ of the homomorphism of B onto A. Hence $h(x)$ must belong to $(\mathfrak{g} : \mathfrak{a})/\mathfrak{g}$ for all x in M. Write, temporarily, K for $(\mathfrak{g} : \mathfrak{a})/\mathfrak{g}$. Then $M^* = \mathrm{Hom}_B(M, K)$, and since both M and K are A-modules, we can replace the suffix B by A. Finally, putting $M = A$, we have $A^* = \mathrm{Hom}_A(A, K) = K$, which completes the proof.

LEMMA 3·2. *Let M, N be two modules over a commutative ring A and define $M^{(r)}$ inductively by $M^{(0)} = M$, $M^{(r+1)} = \mathrm{Hom}_A(M^{(r)}, N)$. Then*

(i) *there exists a natural homomorphism h_r of $M^{(r)}$ into $M^{(r+2)}$;*

(ii) *h_r is an isomorphism into if $r \geqslant 1$;*

(iii) *$h_r(M^{(r)})$ is a direct summand of $M^{(r+2)}$ if $r \geqslant 1$.*

Let x_r denote an element of $M^{(r)}$. Since x_{r+1} is, by definition, a homomorphism of $M^{(r)}$ into N we can define a 'product' $x_r x_{r+1}$ bilinear (over A) in x_r and x_{r+1} with values in N. Further, if $x_{r+1} \neq 0$, then $x_r x_{r+1}$ is not identically zero in x_r. If we fix x_r, the

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mapping $x_{r+1} \to x_r x_{r+1}$ of $M^{(r+1)}$ into N is clearly a homomorphism and therefore determines a unique element $h_r(x_r)$ of $M^{(r+2)}$ by the equation

$$x_r x_{r+1} = x_{r+1} . h_r(x_r).$$

The mapping h_r of $M^{(r)}$ into $M^{(r+2)}$ thus defined is clearly a homomorphism.

Now let M_1, M_2 be two A-modules and let θ be a homomorphism of M_1 into M_2. Then θ determines a sequence of homomorphisms θ_r, where θ_r maps $M_1^{(r)}$ into $M_2^{(r)}$ if r is even, and $M_2^{(r)}$ into $M_1^{(r)}$ if r is odd. This sequence of homomorphisms is defined inductively by taking θ_0 to be θ itself and using the equation

$$\theta_r(x_r) . x_{r+1} = x_r . \theta_{r+1}(x_{r+1}),$$

where if r is even x_r belongs to $M_1^{(r)}$ and x_{r+1} belongs to $M_2^{(r+1)}$, while if r is odd x_r belongs to $M_2^{(r)}$ and x_{r+1} to $M_1^{(r+1)}$. In either case

$$x_{r+1} . h_r(\theta_r(x_r)) = \theta_r(x_r) . x_{r+1} = x_r . \theta_{r+1}(x_{r+1}) = \theta_{r+1}(x_{r+1}) . h_r(x_r) = x_{r+1} . \theta_{r+2}(h_r(x_r)).$$

This can now be written

$$h_r \theta_r = \theta_{r+2} h_r,$$

and hence h_r is a natural homomorphism.

In particular, if we take for θ the homomorphism h_r of $M^{(r)}$ into $M^{(r+2)}$ we obtain a homomorphism of $M^{(r+3)}$ into $M^{(r+1)}$ which we shall denote by k_{r+1}. This is defined by

$$h_r(x_r) . x_{r+3} = x_r . k_{r+1}(x_{r+3}).$$

Let $r \geqslant 1$. Then

$$x_{r-1} . k_r(h_r(x_r)) = h_{r-1}(x_{r-1}) . h_r(x_r) = x_r . h_{r-1}(x_{r-1}) = x_{r-1} . x_r.$$

Hence $k_r h_r$ is the identity isomorphism of $M^{(r)}$ onto itself. This implies both that h_r is an isomorphism into and also that $M^{(r+2)}$ is the direct sum of $h_r(M^{(r)})$ and the kernel of the homomorphism k_r.

With these preliminaries, we shall now commence to study the relationship between the modules M and M^*. Using the notation of Theorem 3·1, what we have to study is the relationship between M and $\mathrm{Hom}_B(M, B)$. We take M^* in this form, as the ring B has several special properties which we shall use. The first step is to replace B by its ring of fractions F. Since, by Macaulay's theorem on unmixed ideals the ideal $\mathfrak{g}$ of Theorem 3·1 is unmixed, F is an Artin ring and is the direct sum of primary Artin rings $F_1, \ldots, F_q$ in 1-1 correspondence with the minimal prime ideals belonging to the ideal $\mathfrak{g}$ of $S = k[X_1, \ldots, X_m]$. We shall now consider the two F-modules $N = F \otimes_B M$ and $N' = F \otimes_B M^*$. Now $M^* = \mathrm{Hom}_B(M, B)$, whence we have

$$N' = F \otimes_B \mathrm{Hom}_B(M, B) = \mathrm{Hom}_F(N, F),$$

since F is a ring of fractions of B. Further, if we express N, N' as direct sums $N = \sum_{i=1}^{q} N_i$ and $N' = \sum_{i=1}^{q} N_i'$ where N_i and N_i' are, in effect, F_i-modules, it is easy to see that $N_i' = \mathrm{Hom}_{F_i}(N_i, F_i)$.

We shall use below a special property of the rings F_i derived above. This is, essentially, that the zero ideal of F is an irreducible ideal, or the equivalent statement that the

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primary components of the ideal $\mathfrak{g}$ of S are irreducible ideals. In the latter form, this is a classical result due to Macaulay (3) (also Grobner (1)). It will be more convenient to derive this result in the equivalent form stated in the following lemma.

LEMMA 3·3. *If P is an F-module of length* 1, *then $P' = \mathrm{Hom}_F(P, F)$ is also of length* 1, *and is isomorphic to P.*

If P has length 1, then P is isomorphic to the F-module $F_i/\mathfrak{m}_i$ for some i, where $\mathfrak{m}_i$ is the radical of F_i. Further P' is derived from the F_i-module $\mathrm{Hom}_{F_i}(P, F_i)$ by means of the homomorphism of F onto F_i.

Now $F_i = Q/\mathfrak{g}Q$, where $Q = S\mathfrak{p}$, $\mathfrak{p}$ being one of the minimal prime ideals belonging to $\mathfrak{g}$. Q is then a regular local ring of dimension $m-d$ and the ideal $\mathfrak{g}Q$ has a basis $g_1, \ldots, g_{m-d}$ satisfying the condition $(g_1, \ldots, g_{j-1}):g_j = (g_1, \ldots, g_{j-1})$ for $j = 1, \ldots, m-d$. Hence if we consider P as a Q-module,

$$P' = \mathrm{Hom}_{F_i}(P, F_i) = \mathrm{Ext}_Q^{m-d}(P, Q).$$

Since Q is a regular local ring of dimension $m-d$, the maximal ideal $\mathfrak{m}$ of Q has a basis $x_1, \ldots, x_{m-d}$ satisfying the condition

$$(x_1, \ldots, x_{j-1}):x_j = (x_1, \ldots, x_{j-1}) \quad (j = 1, \ldots, m-d).$$

Hence $$\mathrm{Ext}_Q^{m-d}(P, Q) = \mathrm{Hom}_Q(P, Q/\mathfrak{m}) = \mathrm{Hom}_Q(Q/\mathfrak{m}, Q/\mathfrak{m}) = Q/\mathfrak{m} = P.$$

The theorem which follows is closely related to Grobner's theorem on reciprocal quotients.

THEOREM 3·4. *Let N be an F-module, and let $N' = \mathrm{Hom}_F(N, F)$. Then*

(i) *N and N' have the same length;*

(ii) *the homomorphism of N into $N'' = \mathrm{Hom}_F(N', F)$ is an isomorphism onto;*

(iii) *the Loewy invariants of N and N' are related as follows:*

$$\lambda_{ji}(N) = \overline{\lambda}_{ji}(N'), \quad \overline{\lambda}_{ji}(N) = \lambda_{ji}(N').$$

We shall first prove by induction on the length $\lambda(N)$ of N that $\lambda(N') \leqslant \lambda(N)$. The last lemma shows that this inequality holds if $\lambda(N) = 1$. Suppose that $\lambda(N) > 1$. Let P be a minimal sub-module of N, so that $\lambda(P) = 1$. Then $\lambda(N/P) = \lambda(N) - 1$. We have an exact sequence

$$0 \to \mathrm{Hom}_F(N/P, F) \to \mathrm{Hom}_F(N, F) \to \mathrm{Hom}_F(P, F).$$

Since $P' = \mathrm{Hom}_F(P, F)$ has length 1, the image of $N' = \mathrm{Hom}_F(N, F)$ in P' has length 0 or 1. Hence, using our inductive hypothesis,

$$\lambda(N') \leqslant \lambda(N/P) + 1 = \lambda(N).$$

Now N can be expressed as a quotient module of a free F-module L, that is we have an exact sequence

$$0 \to K \to L \to N \to 0.$$

From this we can deduce that $\lambda(N) = \lambda(L) - \lambda(K)$. We may deduce from this sequence the exact sequence

$$0 \to N' \to L' \to K',$$

from which we have

$$\lambda(N') \geqslant \lambda(L') - \lambda(K') \geqslant \lambda(L') - \lambda(K).$$

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Hence it will be sufficient to show that $\lambda(L') = \lambda(L)$. Since L' is isomorphic to L, this is immediate. Hence N and N' have equal length.

Let H be a sub-module of N. Then the homomorphisms of N into F which vanish on H form a sub-module of N' isomorphic to $(N/H)'$. Further, the quotient module $N'/(N/H)'$ is isomorphic to a sub-module of H' and has length equal to $\lambda(N') - \lambda((N/H)')$ which, using the result proved above, is equal to $\lambda(H)$ and therefore to $\lambda(H')$. Hence we may deduce that, if every homomorphism of N into F vanishes on H, then H must be of length zero, i.e. H must be the zero module. Since the kernel of the homomorphism of N into N'' is precisely the sub-module on which all homomorphisms of N into F vanish, it follows that N is mapped isomorphically into N''. Since, further, N and N'' have the same length, this isomorphism must be onto.

Consider the sub-module $\mathfrak{m}_j^i N_j$ of N_j where $\mathfrak{m}_j$ is the radical of F_j. A homomorphism of N_j into F_j which vanishes on this sub-module clearly belongs to the sub-module $(0:\mathfrak{m}_j^i)$ of N_j'. Conversely, any homomorphism which belongs to this latter sub-module of N_j' clearly vanishes on $\mathfrak{m}_j^i N_j$. Hence we can deduce immediately that $\bar{\lambda}_{ji}(N) = \lambda_{ji}(N')$ and by considering N' and $N'' \simeq N$, this also shows that $\bar{\lambda}_{ji}(N') = \lambda_{ji}(N)$.

We shall now apply the results obtained above to elucidate the relationship between M and M^*. In the theorem that follows, 0_d denotes the intersection of those primary components of the zero sub-module of M whose dimension is d.

THEOREM 3·5. (i) *The kernel of the homomorphism h_0 of M into M^{**} is 0_d where $d = d(M)$.*

(ii) *There exists an element a of A contained in no d-dimensional prime ideal of A such that $ha \,.\, M^{**} \in h_0(M)$.*

(iii) $M^{***} = M^*$.

Now h_0 induces a homomorphism of $N = M \otimes_B F$ into $N'' = M^{**} \otimes_B F$, which, by the last theorem, is an isomorphism onto. Hence the kernel of the homomorphism of M into M^{**} is contained in the kernel of the homomorphism of M into $M \otimes_B F$. This latter kernel consists of those elements of M annihilated by some non-zero-divisor of B. Now the elements of B act as operators on M through the homomorphism of B onto A. The elements of B which are not zero divisors in B have as images in A those elements which belong to no d-dimensional component of the zero ideal of A. Hence, if x in M is annihilated by an element of B which is not a zero divisor, x must belong to every primary component of the zero sub-module of M whose dimension is d. It is easy to see, conversely, that any such element of M is annihilated by an element of A which belongs to no d-dimensional component of the zero ideal of A, and consequently by an element of B which is not a zero-divisor. Hence every homomorphism of M into B must vanish on 0_d. This shows that 0_d is the kernel of the homomorphism of M into M^{**}.

Since the homomorphism of N into N'' is onto, every element of N'', and, a fortiori, every element of M^{**}, is of the form $b \otimes x$, where b belongs to F and x to $h_0(M)$. Since b belongs to F, we can find a' in B such that a' is not a zero-divisor in B and $a'b \in B$. Hence, if a is the image of a' in A, belongs to no d-dimensional component of the zero ideal of A, $a \,.\, b \otimes x$ belongs to $h_0(M)$. Since M^{**} has a finite set of generators, we can choose a such that $a \,.\, M^{**} \leqslant h_0(M)$.

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By Lemma 3·2,

$$M^{***} = M^* + K,$$

where K is the kernel of the homomorphism of M^{***} onto M^*. Further, since $M^{***} \otimes_B F = M^* \otimes_B F$, it follows that $K \otimes_B F = 0$, i.e. if x belongs to K, then $bx = 0$ for some element b of B which is not a zero divisor. Now M^{***} is the module of homomorphisms of M^{**} into B, and hence $bx = 0$ implies that x is zero if b is not a zero divisor. Hence K must be the zero module, i.e. $M^{***} = M^*$.

COROLLARY 1. *A necessary and sufficient condition for the homomorphism of M into M^{**} to be an isomorphism into is that M is equi-dimensional.*

For M is equidimensional if and only if $0_d = 0$.

COROLLARY 2. *If $M_d = M/0_d$, then $M_d^* \simeq M^*$.*

The annihilator of 0_d, considered as an A-module, is an ideal of dimension less than d. Hence, by Corollary 1 to Theorem 1·2, it follows that $T^d(M) \simeq T^d(M_d)$, i.e. that $M_d^* \simeq M^*$.

Before stating the next theorem, we must consider the relationship of the Loewy invariants $\lambda_{ji}(N)$ and $\overline{\lambda}_{ji}(N)$ to the module M, where $N = M \otimes_B F$.

Our first step is to show that N essentially depends only on M and not on the choice of the ideal $\mathfrak{g}$ such that $B = S/\mathfrak{g}$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_p$ be the prime ideals of S of dimension d which belong to the kernel $\mathfrak{a}$ of the homomorphism of S onto A. Since $\mathfrak{g} \subset \mathfrak{a}$, and $\mathfrak{g}$ is unmixed of dimension d, $\mathfrak{p}_1, \ldots, \mathfrak{p}_p$ are among the prime ideals belonging to $\mathfrak{g}$. Let the latter be $\mathfrak{p}_1, \ldots, \mathfrak{p}_q$ where $q \geqslant p$. Let F' be the generalized ring of fractions of A, with the denominators elements of A belonging to no prime ideal of dimension d, i.e. not belonging to the images in A of $\mathfrak{p}_1, \ldots, \mathfrak{p}_p$. Then F' is a homomorphic image of F and, further, it is clear that any element of the kernel of the mapping of F onto F' annihilates the module N. Hence N can be considered as the F-module derived from the F'-module $M \otimes_A F'$. The Loewy invariants $\lambda_{ji}(N)$ and $\overline{\lambda}_{ji}(N)$ are then obtained as follows: $\lambda_{ji}(N) = \lambda_{ji}(M \otimes_A F')$, $\overline{\lambda}_{ji}(N) = \overline{\lambda}_{ji}(M \otimes_A F')$ if $j \leqslant p$ and

$$\lambda_{ji}(N) = \overline{\lambda}_{ji}(N) = 0$$

if $j > p$. We shall therefore term $\lambda_{ji}(N), \overline{\lambda}_{ji}(N)$ with $j \leqslant p$ the Loewy invariants of M and denote them by $\lambda_{ji}(M), \overline{\lambda}_{ji}(M)$.

THEOREM 3·6. *The Loewy invariants of M and M^* are related by the equations*

$$\lambda_{ji}(M) = \overline{\lambda}_{ji}(M^*), \quad \overline{\lambda}_{ji}(M) = \lambda_{ji}(M^*).$$

This is an immediate consequence of the above and the last part of Theorem 3·4.

THEOREM 3·7. *Let M be an equi-dimensional module. Then*

(i) *if $\overline{\lambda}_{j1} = 1$ for $j = 1, \ldots, p$, M is module isomorphic to an ideal of A;*

(ii) *if $\lambda_{j1} = 1$, then the primary component of the zero module of M corresponding to $\mathfrak{p}_j$ is an irreducible sub-module of M.*

Let $F' = F_1' + \ldots + F_p'$ be the ring of fractions of A. Since M is equi-dimensional, F' is the ring denoted by F' above. The assumption that $\overline{\lambda}_{ji} = 1$ implies that the F'-module $F_j' \otimes_A M$ is generated by a single element. For $\overline{\lambda}_{ji}$ is the dimension of the

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vector space $F'_j \otimes_A M/\mathfrak{m}_j \otimes_A M$ over the field $F'_j/\mathfrak{m}_j$ and it is well known that this is the minimum number of elements required to generate the module $F'_j \otimes_A M$. Let $u_1, \ldots, u_p$ generate, respectively, the modules $F'_1 \otimes_A M, \ldots, F'_j \otimes_A M$. Then $u = \sum_{j=1}^{p} u_j$ generates the module $F' \otimes_A M = F'_1 \otimes_A M + \ldots + F'_p \otimes_A M$. Since u is of the form v/a, where $v \in M$ and a is not a zero divisor of A, we may, by replacing u by v if necessary, assume that u belongs to M. Since M is a finitely generated module, we can find a in A such that aM is contained in the sub-module of M generated by u. Define, for all x in M, $h(x)$ by the equation

$$ax = h(x).u, \quad h(x) \in A.$$

Now $h(x)$ is uniquely determined. For if $bu = 0$ with b in A, then, for all x in M, $abx = 0$, whence $ab = 0$ and therefore as a is not a zero divisor of A, $b = 0$. Hence the mapping $x \to h(x)$ is a homomorphism of M into A and its image is an ideal of A. The kernel of this homomorphism is the sub-module $0:a$ of M. Since M is equi-dimensional and since a is not a zero divisor of A, this sub-module is the zero module, i.e. the homomorphism is an isomorphism into.

If $\lambda_{j1} = 1$, then the zero module of $F'_j \otimes_A M$ is irreducible (Grundy (2), p. 276). This implies that the component of the zero module of M corresponding to the prime ideal $\mathfrak{p}_j$ is an irreducible sub-module of M.

THEOREM 3·8. *Let A be an equi-dimensional geometric ring (i.e. a homomorphic image of $k[X_1, \ldots, X_m]$) such that the primary components of the zero ideal are irreducible. Let $\mathfrak{b}$ be an ideal of A not consisting entirely of zero divisors. Then $\mathfrak{b}^*$ is isomorphic to an ideal of A. This ideal is determined to within rational equivalence, and, if $\mathfrak{b}_1$ is any ideal of A isomorphic to $\mathfrak{b}^*$ as an A-module, $\mathfrak{b}_1$ is unmixed of rank 1 (or is A itself).*

Let the zero ideal of A have the decomposition $(0) = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_s$. Then the zero sub-module of $\mathfrak{b}$ (considered as an A-module) is the intersection of sub-modules $Q_1, \ldots, Q_s$ where $Q_i = \mathfrak{b} \cap \mathfrak{q}_i$. This representation is clearly irredundant, since, if $i_1, \ldots, i_r$ do not comprise the whole set $1, \ldots, s$, $Q_{i_1} \cap \ldots \cap Q_{i_r}$ contains the non-zero element ax where a is an element of $\mathfrak{b}$ which is not a zero divisor, and x is any non-zero element of $\mathfrak{q}_{i_1} \cap \ldots \cap \mathfrak{q}_{i_r}$. Next Q_i is an irreducible sub-module of $\mathfrak{b}$. For suppose that $Q_i = \mathfrak{c}_1 \cap \mathfrak{c}_2$ where $\mathfrak{c}_1, \mathfrak{c}_2$ are sub-modules of $\mathfrak{b}$, and therefore ideals of A contained in $\mathfrak{b}$. Since $\mathfrak{q}_i$ is irreducible, either $\mathfrak{c}_1$ or $\mathfrak{c}_2$ is contained in $\mathfrak{q}_i$ and hence in Q_i. Hence either $\mathfrak{c}_1$ or $\mathfrak{c}_2$ is equal to Q_i. Therefore $\lambda_{j1}(\mathfrak{b}) = 1$ for $j = 1, \ldots, s$ and hence, by Theorem 3·6, $\overline{\lambda}_{j1}(\mathfrak{b}^*) = 1$ for $j = 1, \ldots, s$. Hence, by Theorem 3·7 (i), $\mathfrak{b}^*$ is isomorphic to an ideal of A. This ideal cannot consist entirely of zero-divisors of A. For otherwise there would exist a non-zero element c of A such that $c\theta(x) = 0$ for every element θ of $\mathfrak{b}^*$ and every element x of $\mathfrak{b}$. If a is a non-zero-divisor contained in $\mathfrak{b}$, $\theta(ca) = 0$ for every element θ of $\mathfrak{b}^*$. This implies that ca is mapped into zero in the homomorphism of $\mathfrak{b}$ into $\mathfrak{b}^{**}$, which is impossible, since $\mathfrak{b}$ as an A-module, is equi-dimensional. Now let $\mathfrak{b}_1, \mathfrak{b}_2$ be two ideals of A isomorphic to $\mathfrak{b}^*$ as A-modules. Consider the A-module isomorphism between $\mathfrak{b}_1$ and $\mathfrak{b}_2$. Let $b_1 \in \mathfrak{b}_1$ be a non-zero-divisor and let the image of x in $\mathfrak{b}_1$ under the isomorphism be x'. Then $(b_1 x)' = b'_1 x = b_1 x'$. Hence $x' = b'_1 x/b_1$ and consequently $\mathfrak{b}_1$ and $\mathfrak{b}_2$ are rationally equivalent ideals.

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Let $\mathfrak{b}_1$ be an ideal of A which is isomorphic to $\mathfrak{b}^*$. Let $\mathfrak{b}_2$ be the intersection of the primary components of $\mathfrak{b}_1$ which are of rank 1 (for $\mathfrak{b}_2 = A$ if $\mathfrak{b}_1$ has no such components). Then the A-module $\mathfrak{b}_2/\mathfrak{b}_1$ has dimension at most $d-2$. Hence, by Corollary 2 to Theorem 1·2, it follows that the homomorphism of $\mathfrak{b}_2^*$ into $\mathfrak{b}_1^*$ is an isomorphism onto. This implies that $\mathfrak{b}_2^{**}$ is isomorphic to $\mathfrak{b}_1^{**} \simeq \mathfrak{b}^{***} \simeq \mathfrak{b}^* \simeq \mathfrak{b}_1$. Hence we have a homomorphism of $\mathfrak{b}_2$ onto $\mathfrak{b}_1 = \mathfrak{b}_2^{**}$ which induces the identity isomorphism on $\mathfrak{b}_1$. Consequently $\mathfrak{b}_2$ is the direct sum of $\mathfrak{b}_1$ and a module isomorphic to $\mathfrak{b}_2/\mathfrak{b}_1$. This is only possible if $\mathfrak{b}_2/\mathfrak{b}_1 = 0$, since otherwise every element of $\mathfrak{b}_1$ would be a zero-divisor, contrary to what has been proved above. Hence $\mathfrak{b}_1 \simeq \mathfrak{b}_2$ and therefore $\mathfrak{b}_1$ is unmixed of rank 1.

COROLLARY 1. *If M is a geometric module such that $M = M^{**}$, and if N is a module containing M such that N/M has dimension $\leqslant d-2$, then M is a direct summand of N.*

This follows by the argument used in the last paragraph above.

THEOREM 3·9. *Let A be as in the last theorem. Then A^{**} is a ring contained in the ring of fractions of A and is a finite A-module. If A is integrally closed in its ring of fractions, then the natural homomorphism of A into A^{**} is an isomorphism onto.*

By the second part of Theorem 3·1, $A^{**} = \mathrm{Hom}_A(A^*, A^*)$ and so has the structure of a ring. Further, the natural homomorphism of A into A^{**} is a ring homomorphism in this case, the homomorphism of A^* into itself induced by an element a of A maps an element θ of A^* into the element θ_a defined by $\theta_a(x) = \theta(ax)$. A^{**} is contained in the ring of fractions of A by Theorem 3·5 (ii) and is clearly a finite A-module. Since, by the restriction on A imposed in the last theorem, A is mapped isomorphically into A^{**}, we can consider A^{**} as an integral extension of A. Hence, if A is integrally closed in its ring of fractions, A^{**} must coincide with A, i.e. the natural homomorphism of A into A^{**} is an isomorphism onto.

We shall now consider what modifications are necessary in the theorems of this section if we assume that M is a graded module.

The theorems of this section that we shall consider in this connexion are Theorems 3·1, 3·5 and 3·8.

First consider Theorem 3·1. We shall suppose that M is a graded geometrical module with ring of linear transformations A, and that there exists a homomorphism of $S = k[X_1, \ldots, X_m]$ onto A, the homomorphism preserving degrees and homogeneity if we consider S as a graded ring in which X_i is homogeneous of degree p_i as described in the proof of Theorem 1·1. Then M^* is defined as a graded module by the formula

$$M^* = \mathrm{Ext}_S^{m-d}(M, S)(-p),$$

where $p = \Sigma p_i$. Now suppose that the elements $g_1, \ldots, g_{m-d}$ of Theorem 3·1 are homogeneous elements of degrees $m_1, \ldots, m_{m-d}$ and that B has the same meaning as in that theorem (B now being a graded ring). Then if we apply (*c*) of § 1 in the graded case we obtain

$$M^* = \mathrm{Hom}_B(M, B)(n),$$

where $n = \Sigma_{m_i} - p$. Finally,

$$M^* = \mathrm{Hom}_A(M, A^*),$$

where now $A^* = K(n)$, K being the graded module $\mathfrak{g}:\mathfrak{a}/\mathfrak{g}$.

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Now we turn to Theorem 3·5. The essential part of this theorem is that the natural homomorphism of M^* into M^{***} is an isomorphism onto. If M, and hence M^* and M^{***} are graded modules, it is important to verify that this isomorphism is an isomorphism between graded modules, that is, it preserves homogeneity and degrees. This will follow if we can show that the natural homomorphism of M into M^{**} preserves homogeneity and degrees, since we then need only replace M by M^*. This, however, follows immediately from the formula

$$M^* = \mathrm{Hom}_A(M, A^*),$$

since, if θ is a homomorphism of M into A^* which is homogeneous of degree r and if x is an element of M which is homogeneous of degree s, then $\theta(x)$ is an element of A^* which is homogeneous of degree $r+s$. Now the image of x in M^{**} is the homomorphism $\theta \to \theta(x)$ of M^* into A^*. This maps the homogeneous element θ of degree r onto the element $\theta(x)$ of degree $r+s$, and is therefore an element of M^{**} of degree s. Since this is also the degree of x, it follows that the homomorphism of M into M^{**} preserves homogeneity and degrees.

Finally we turn to Theorem 3·8. Here we shall make the assumption that the ideal $\mathfrak{b}$ entering in this theorem is a graded ideal, and therefore, qua module, a graded module. Theorem 3·8 states that there is an isomorphism of $\mathfrak{b}^*$ onto an ideal $\mathfrak{b}_1$ of A. However, we have no guarantee that this isomorphism will preserve homogeneity and degrees or even that the ideal $\mathfrak{b}_1$ is a graded module. In order to ensure the preservation of homogeneity and degrees it will first be necessary to replace the statement that $\mathfrak{b}^*$ is isomorphic to an ideal $\mathfrak{b}_1$ by the statement that $\mathfrak{b}^*$ is isomorphic to a fractional ideal $\mathfrak{b}_1$. Even with this assumption the problem of proving a graded analogue of Theorem 3·8 appears to the author to present some difficulty and we shall therefore make a further restriction on the ring A. We shall assume that A is generated over k by elements of degree 1. This effectively implies that S is graded so that $X_1, \ldots, X_m$ all have degrees 1. This assumption is in force in the following theorem:

THEOREM 3·10. *Let A be a graded geometric ring which is generated by elements of degree* 1 *(over k) and which is equi-dimensional of dimension $d > 0$, the primary components of the zero ideal being irreducible. Let $\mathfrak{b}$ be a homogeneous ideal of A which contains a homogeneous element which is not a zero-divisor. Then $\mathfrak{b}^*$, considered as a graded module is isomorphic to a homogeneous fractional ideal $\mathfrak{b}_1$ of A.*

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_q$ be the minimal prime ideals of A. By Theorem 3·8 there is an isomorphism of $\mathfrak{b}^*$ onto an ideal $\mathfrak{b}_1$ of A, $\mathfrak{b}_1$ not necessarily being a homogeneous ideal, and the isomorphism therefore not necessarily being an isomorphism of graded modules. Let $P_1, \ldots, P_q$ be the sub-modules of $\mathfrak{b}^*$ whose images under this isomorphism are $\mathfrak{p}_1 \cap \mathfrak{b}_1, \ldots, \mathfrak{p}_q \cap \mathfrak{b}_1$. Then, if an element x of $\mathfrak{b}^*$ has the property that $ax = 0$ for some non-zero element a of A, x must belong to one of the sub-modules P_i. Further, if $ax \in P_i$, then either a belongs to $\mathfrak{p}_i$ or x belongs to P_i, that is, P_i is a primary sub-module of $\mathfrak{b}^*$ with associated prime ideal $\mathfrak{p}_i$. It was also shown in the course of the proof of Theorem 3·8 that $\mathfrak{b}_1$ does not consist entirely of zero-divisors. This implies that $\mathfrak{b}^*$ must contain an element not in any of the sub-modules P_i and a fortiori, that P_i does not contain every element of $\mathfrak{b}^*$. Hence, for each i, we can find an element x_i in $\mathfrak{b}^*$ not in P_i which is

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homogeneous. Next $\mathfrak{p}_i$ cannot contain every element of A of degree 1, and we can therefore find a_i of degree 1 in A but not in $\mathfrak{p}_i$. It then follows that $a_i^r x_i$ does not belong to P_i. Hence we can choose x_i in $\mathfrak{b}^*$ but not in P_i of any sufficiently high degree. Finally, therefore, we may assume that $x_1, \ldots, x_q$ are all homogeneous elements of the same degree.

We can find homogeneous elements $b_1, \ldots, b_q$ in A such that b_i does not belong to $\mathfrak{p}_i$ but belongs to $\mathfrak{p}_j$ for all $j \neq i$. Since every power of b_i also has this property, we may assume that $b_1, \ldots, b_q$ are all of the same degree. Hence the element $x = \sum_{i=1}^{q} b_i x_i$ is a homogeneous element of $\mathfrak{b}^*$ and does not belong to any of the sub-modules $P_1, \ldots, P_q$. Hence if $ax = 0$, where a belongs to A, $a = 0$.

It follows from Theorem 3·8 that we can find an element u of $\mathfrak{b}^*$, not necessarily homogeneous, and an element a of A, also not necessarily homogeneous but not a zero divisor, such that, for all y in $\mathfrak{b}^*$, ay belongs to the sub-module (u) of $\mathfrak{b}^*$. Take y to be the element x defined above. Then if $ax = cu$, $c \in A$, c cannot be a zero-divisor of A, since otherwise there would exist a non-zero element d of A such that $dx = 0$. Hence, for every element y of $\mathfrak{b}^*$, acy belongs to the sub-module generated by x. Hence, to sum up, we may assume that there is a homogeneous element u of $\mathfrak{b}^*$ and an element a of A such that $a\mathfrak{b}^* \subset (u)$, a not being a zero divisor.

Now consider the set of elements a of A such that $a\mathfrak{b}^* \subset (u)$. Since u is homogeneous, this set is a homogeneous ideal of A, and does not consist entirely of zero divisors. Then it is well known that the set must contain a homogeneous element that is not a zero divisor. (This can be proved by a method similar to the method used to construct $x = \Sigma b_i x_i$ above.) Hence we may also assume that a is a homogeneous element of A. We now consider the isomorphism of $\mathfrak{b}^*$ into A $y \to y^*$ where $ay = y^*u$. This preserves homogeneity but not degrees. In fact, the degree of y^* exceeds that of y by $\deg(a) - \deg(u)$. However, A contains homogeneous elements which are not zero divisors for all sufficiently high degrees and hence we can find homogeneous elements f and g in A which are not zero divisors and which satisfy

$$\deg(g) - \deg(f) = \deg(a) - \deg(u).$$

Then the mapping $y \to fy^*/g$ is an isomorphism of $\mathfrak{b}^*$ on a homogeneous fractional ideal of A.

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