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GEODESIC FLOWS MODELLED BY EXPANSIVE FLOWS

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Abstract Given a smooth compact surface without focal points and of higher genus, it is shown that its geodesic flow is semi-conjugate to a continuous expansive flow with a local product structure such that the semi-conjugation preserves time parametrization. It is concluded that the geodesic flow has a unique measure of maximal entropy.

Keywords: expansive flow; geodesic flow; conjugate points; focal points; time-preserving semi-conjugacy; measure of maximal entropy

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1. Introduction

The resemblance between the geodesic flow of a compact surface of genus greater than one and without conjugate points and the geodesic flow of a compact surface with constant negative curvature has been the focus of deep research in geometry and topological dynamics from the beginning of the 20th century. The first and perhaps most influential work on the subject was that by Morse [**33**] showing that every globally minimizing geodesic in the universal covering of a compact surface of genus greater than one is 'shadowed' by a geodesic in the hyperbolic plane. This beautiful result strongly suggested that the geodesic flow of a compact surface without conjugate points and of higher genus should be semi-conjugate to the geodesic flows of a constant negative curvature surface.

Recall that two continuous flows $\phi_t : Y \to Y$ and $\psi_t : X \to X$ acting on compact metric spaces Y and X are *semi-conjugate* if there exists a continuous and surjective map $\chi : Y \to X$ such that for each $p \in Y$ there exists a continuous and surjective reparametrization $\rho_p : \mathbb{R} \to \mathbb{R}$ such that

$$(\chi \circ \phi_t)(p) = (\psi_{\rho_p(t)} \circ \chi)(p).$$

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The map χ is called a *semi-conjugacy*; if χ is a homeomorphism then it is called a *conjugacy* and the flows are said to be *semi-conjugate*. Observe that χ maps orbits to orbits. We say that ϕ_t is *time-preserving semi-conjugate* to ψ_t if ρ_p is the identity for every p. The existence of conjugacies to nearby flows in a C^1 neighbourhood characterizes *structurally stable flows*, that is, Axiom A flows and, in particular, Anosov flows. Structural stability theory was developed in the 1960s and 70s, and its main ideas paved the way to the study of systems which show only weaker forms of stability like topological stability. A system is *C^k-topologically stable* if it is semi-conjugate to any nearby C^k system. Of course, C^k-structurally stable systems are C^k-topologically stable. But the converse is not true and there are many well-known counterexamples, many of them in the category of expansive, non-hyperbolic systems.

Definition 1.1. A continuous flow $\psi_t\colon X \to X$ without singular points on a metric space (X, d) is *expansive*[*] if there exists $\varepsilon > 0$ such that for every $x \in X$ and $y \in X$ for which there exists a continuous surjective function $\rho\colon \mathbb{R} \to \mathbb{R}$ with $\rho(0) = 0$ satisfying

$$d(\psi_t(x), \psi_{\rho(t)}(y)) \leq \varepsilon,$$

for every $t \in \mathbb{R}$ we have $\psi_{t(y)}(x) = y$ for some $|t(y)| < \varepsilon$. We call such ε an *expansivity constant*.

The definition of an expansive homeomorphism was introduced by Bowen prior to the above definition for flows (see also [8]), and Bowen's study of expansive homeomorphisms showed how to find weak stability properties in non-hyperbolic geodesic flows using a dynamical approach rather than the global geometry approach suggested by Morse's ideas.

Bowen [6] and Bowen and Walters [8] pointed out that expansive systems with a local product structure are C^0-topologically stable (see also [20, 44] in the case of flows); Lewowicz [31] and Hiraide [25] showed that expansive homeomorphisms of compact surfaces have an 'almost' local product structure. Paternain [37] and Inaba and Matsumoto [26] extended Lewowicz's work to show that expansive geodesic flows of compact surfaces have a local product structure and hence that they are C^0-topologically stable. Ruggiero [42] showed that expansive geodesic flows of compact manifolds without conjugate points have a local product structure, which also implies topological stability.

The global geometry point of view of weak stability theory of geodesic flows enjoyed a large development in the 1970s. One can mention the works of Eberlein [15] and Eberlein and O'Neill [18] about visibility manifolds extending Morse's [33] work: quasi-geodesics in visibility manifolds are close to true geodesics. The theory of visibility manifolds went further and introduced a whole body of tools to study coarse hyperbolic geometry of manifolds. Thurston and Gromov [24] introduced the notion of hyperbolic groups and not only extended Eberlein's theory of visibility manifolds but also created a rich theory to study coarse hyperbolic geometry in very general metric spaces. All the above results

[*] Observe that, in fact, our definition of expansivity made above is slightly stronger than in [8]; however, it appears naturally in the context of expansive geodesic flows (see, for example, [41, 42]). In the context of geodesic flows on compact manifolds without conjugate points, the two definitions are equivalent.

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suggest that the family of geodesic flows which are semi-conjugate to a geodesic flow of a manifold with negative curvature could be much larger than only the family of geodesic flows of compact surfaces without conjugate points.

In the 1980s, Ghys [**22**] proved the existence of a semi-conjugacy between the geodesic flow of a compact surface without conjugate points and of genus greater than one and a hyperbolic *geodesic* flow. However, in general, this semi-conjugacy is *not* time-preserving, that is, the reparametrization ρ_p is not the identity. While any semi-conjugacy between discrete systems is time preserving by definition, the existence of such a semi-conjugacy for flows is a much more delicate issue. Time-preserving semi-conjugacies are intimately related to spectral rigidity problems in continuous dynamical systems and in particular in Riemannian geometry: two geodesic flows which are time-preserving semi-conjugate have the same marked length spectrum of periodic geodesics. Indeed, the work of Otal [**36**], Croke [**13**], and Croke and Fathi [**14**] shows that if this is the case for the geodesic flows of two compact surfaces without conjugate points and of genus greater than one then both surfaces are in fact isometric.

In view of the above, the main result of the present article looks perhaps surprising.

Theorem A. *Let (M, g) be a C^∞ compact connected boundaryless surface without focal points and genus greater than one. Let $\phi_t \colon T_1M \to T_1M$ be its geodesic flow.*

Then there exists a compact 3-manifold X diffeomorphic to T_1M and a continuous flow $\psi_t \colon X \to X$ which is expansive and has a local product structure such that ϕ_t is time-preserving semi-conjugate to ψ_t.

So expansive flows arise as models of geodesic flows of compact surfaces without focal points and of genus greater than one up to time-preserving semi-conjugacies. By the above-mentioned spectral rigidity, we cannot expect that the the model flow ψ_t will be the geodesic flow of a compact surface if the original flow is non-expansive. This is why Ghys's method [**22**] does not provide a *time-preserving* semi-conjugacy. Indeed, if the model flow ψ_t was a geodesic flow, we would know that the corresponding surface would have no conjugate points by the result of Paternain [**37**]. Therefore, we could apply the spectral rigidity results mentioned above to show that both surfaces would have to be isometric. But the initial geodesic flow will not be expansive in general, owing to flat strips in the unit tangent bundle (see § 2).

The existence of a time-preserving semi-conjugacy to an expansive flow has some interesting applications in topological dynamics. In fact, this expansive flow inherits most of the features of the topological hyperbolic dynamics which are found in the theory of expansive geodesic flows in manifolds without conjugate points (see Theorem 5.1 and compare [**42**]). These results play a crucial part in the proof of the following result, which is related to the thermodynamical formalism and ergodic optimization and gives a first example of possible applications of Theorem A.

Theorem B. *The geodesic flow of a C^∞ compact connected boundaryless surface without focal points and of genus greater than one has a unique (hence ergodic) invariant probability measure of maximal entropy.*

Theorem B partially extends Knieper's result [**29**] about the uniqueness of maximal entropy measures for geodesic flows in rank one manifolds with non-positive curvature.

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Although his result is valid in any dimension, we would like to point out that in the case of surfaces our setting is more general (there exist surfaces without focal points which admit some regions with positive curvature).

Our approach to show Theorem B follows the classical thermodynamical formalism developed by Bowen [**7**, **20**] for expansive topological dynamical systems in compact metric spaces. Since the geodesic flow in Theorem B might not be expansive, we combine Theorem A with some recent results by Buzzi *et al.* [**11**] which reduce the study of maximal entropy measures of a so-called extension of an expansive system to the study of maximal entropy measures of expansive ones (see § 6).

This natural, more topological approach to studying ergodic invariant measures arising from topological dynamics seems promising regarding further generalizations under less (or no) restrictions on the curvature or Jacobi fields. We conjecture that Theorem A (and eventually Theorem B) will extend to surfaces without conjugate points where Green bundles are continuous (see [**15**] for the definition), in particular to so-called surfaces with bounded asymptote [**19**]. Such surfaces might admit focal points and in many respects be quite far from surfaces with non-positive curvature.

The main idea to prove Theorem A is in many ways a very natural one: the geodesic flow of a compact surface without conjugate points and of higher genus is not expansive in general. It may have regions of non-expansivity, which in the case of a surface without focal points consist precisely of flat strips. So we define an equivalence relation in the unit tangent bundle of the surface that identifies points in the same strip (see § 3, where further properties of such strips are studied). This relation naturally induces a quotient flow which preserves the classes and the time parametrization of the initial geodesic flow. The quotient space then carries a flow without 'non-expansive' orbits, and the hard part of the proof consists in showing that the quotient space has a good topological structure. We show that the quotient space is a topological 3-manifold, and hence carries a smooth manifold structure (see § 4). Finally, we show that the quotient flow is expansive with respect to any metric defined on the quotient space (see § 5).

2. Preliminaries

Standing assumption

Throughout the paper (M, g) will be a C^∞ compact connected Riemannian manifold without boundary. We shall always assume that M has *no conjugate points*, that is, at every point the exponential map is non-singular. In particular, we will study the particular subclass of manifolds *without focal points*, that is, if $J(t)$ is a Jacobi field along a geodesic in M with $J(0) = 0$ then $\|J(t)\|$ is strictly increasing in t.

Each vector $\theta \in TM$ determines a unique geodesic $\gamma_\theta(\cdot)$ such that $\gamma'_\theta(0) = \theta$. The geodesic flow $(\phi_t)_{t\in\mathbb{R}}$ acts on TM by $\phi_t(\theta) = \gamma'_\theta(t)$. We shall study its restriction to the unit tangent bundle T_1M, which is invariant. All geodesics will be parametrized by arc length.

We shall denote by $\tilde{M}$ the universal covering of M and endow it with the pullback $\tilde{g}$ of the metric g by the covering map $\pi\colon \tilde{M} \to M$ which gives the Riemannian manifold $(\tilde{M}, \tilde{g})$. For this manifold we consider also geodesics and the geodesic flow which acts on $T_1\tilde{M}$, and we will also denote them by $\gamma_{\bar\theta}(\cdot)$ for given vector $\bar\theta \in T_1\tilde{M}$ and $(\phi_t)_{t\in\mathbb{R}}$ (the

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domain of the flow ϕ_t is enough to specify the dynamical system under consideration), respectively. We denote by $\bar{\pi}\colon T_1\tilde{M} \to T_1M$ the natural projection. The distance associated with the Riemannian metric g will be denoted by d_g and the one associated with $\tilde{g}$ by $d_{\tilde{g}}$. We will omit the metric and simply write d if there is no danger of confusion. The Riemannian metric on M lifts to the Sasaki metric on TM, which we denote by d_S. We shall use the same notation for the Sasaki distances in $T_1\tilde{M}$ and $\widetilde{T_1M}$. For any $\theta \in TM$ we will consider the orthogonal decomposition of $T_\theta TM$ into horizontal and vertical parts $T_\theta TM = H_\theta \oplus V_\theta$, similar for $\bar{\theta} \in T\tilde{M}$.

Two geodesics γ_1 and γ_2 in $\tilde{M}$ are *asymptotic* (as $t \to \infty$) if $d(\gamma_1(t), \gamma_2(t))$ is bounded as $t \to \infty$, that is, there exists $C > 0$ such that $d(\gamma_1(t), \gamma_2(t)) \le C$ for all $t \ge 0$, and *bi-asymptotic* if $d(\gamma_1(t), \gamma_2(t))$ is bounded as $t \to \pm\infty$, that is, the previous inequality holds for all $t \in \mathbb{R}$.

Given a metric space (X, d) and two subsets $Z_1, Z_2 \subset X$, let us denote by $d_H(Z_1, Z_2)$ the Hausdorff distance between Z_1 and Z_2.

2.1. Horospheres and invariant submanifolds

A very special property of manifolds with no conjugate points is the existence of the so-called Busemann functions: given $\bar{\theta} = (p, v) \in T_1\tilde{M}$, the *forward* and *backward Busemann functions* $b^{\pm}_{\bar{\theta}}\colon \tilde{M} \to \mathbb{R}$ associated with $\bar{\theta}$ are defined by

$$b^+_{\bar{\theta}}(x) := \lim_{t\to+\infty} d(x, \gamma_{\bar{\theta}}(t)) - t \quad \text{and} \quad b^-_{\bar{\theta}}(x) := \lim_{t\to+\infty} d(x, \gamma_{\bar{\theta}}(-t)) - t,$$

respectively. The level sets of the Busemann functions are the *horospheres*. We define the (level 0) *positive* and *negative horosphere* of $\bar{\theta} \in T_1\tilde{M}$ by

$$H^+(\bar{\theta}) := (b^+_{\bar{\theta}})^{-1}(0) \quad \text{and} \quad H^-(\bar{\theta}) := (b^-_{\bar{\theta}})^{-1}(0),$$

respectively.

Horospheres in $\tilde{M}$ lift naturally to horospheres in $T_1\tilde{M}$ as follows. Consider the gradient vector fields $\nabla b^{\pm}_{\bar{\theta}}$ and define the *positive* and *negative horocycles* $\tilde{\mathscr{F}}^{\mathrm{s/u}}(\bar{\theta})$ in $T_1\tilde{M}$ through $\bar{\theta}$ to be the restriction of $\nabla b^{\pm}_{\bar{\theta}}$ to $H^{\pm}(\bar{\theta})$

$$\tilde{\mathscr{F}}^{\mathrm{s}}(\bar{\theta}) := \{(q, -\nabla_q b^+_{\bar{\theta}})\colon q \in H^+(\bar{\theta})\} \quad \text{and} \quad \tilde{\mathscr{F}}^{\mathrm{u}}(\bar{\theta}) := \{(q, -\nabla_q b^-_{\bar{\theta}})\colon q \in H^-(\bar{\theta})\},$$

respectively. Let us denote by $\sigma^{\bar{\theta}}_t\colon \tilde{M} \to \tilde{M}$ the integral flow of the vector field $-\nabla b^+_{\bar{\theta}}$. The orbits of this flow are the *Busemann asymptotes* of $\gamma_{\bar{\theta}}$.

We will list some basic properties of horospheres that we shall need (see, for example, [19, 38] for details).

Lemma 2.1. *Let (M, g) be a C^∞ compact connected boundaryless Riemannian manifold without conjugate points. Then the following hold.*

(i) *$b^{\pm}_{\bar{\theta}}$ are C^1 functions for every $\bar{\theta}$.*

(ii) *There exists a constant $K > 0$ depending on the curvature of M such that for given $\theta \in T_1\tilde{M}$, the gradient vector field $\nabla b^{\pm}_{\bar{\theta}}$ is K-Lipschitz and has unit length.*

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Each horosphere is an embedded submanifold of dimension $n-1$ tangent to a Lipschitz plane field.

(iii) *The orbits of $\sigma_t^{\bar\theta}$ are geodesics which are everywhere perpendicular to the horospheres $H^+(\bar\theta)$. In particular, the geodesic $\gamma_{\bar\theta}$ is an orbit of this flow and for every $t \in \mathbb{R}$ we have*

$$\sigma_t^{\bar\theta}(H^+(\bar\theta)) = H^+(\gamma_{\bar\theta}(t)).$$

Lemma 2.1(iii) implies that the horospheres are equidistant, that is, given any point $p \in H^+(\gamma_{\bar\theta}(t))$ for every s we have $d(p, H^+(\gamma_{\bar\theta}(s))) = |t-s|$. Clearly, $H^+(\gamma_{\bar\theta}(t))$ varies continuously with $t \in \mathbb{R}$; however, it is not known whether it varies continuously with $\bar\theta$.

2.2. Visibility, central manifolds and topological dynamics

The universal covering of a compact manifold without conjugate points belongs to a special class of manifolds without conjugate points satisfying the axiom of visibility. This geometric property was introduced by Eberlein and O'Neill [**18**] for manifolds without focal points (or even without conjugate points [**15**]) as a criterion when a manifold behaves as if it had negative curvature.

Definition 2.2. A complete simply connected Riemannian manifold (M, g) is a *visibility manifold* if it has no conjugate points and if for every $\varepsilon > 0$, $p \in M$ there exists $T = T(\varepsilon, p) > 0$ such that for every two unit-speed geodesic rays γ_1, γ_2 with $\gamma_1(0) = p = \gamma_2(0)$, if the distance from p to every point of the geodesic joining $\gamma_1(t)$ to $\gamma_2(s)$ $(0 \le s \le t)$ is larger than T, then the angle formed by $\gamma_1'(0)$ and $\gamma_2'(0)$ is less than ε. When T does not depend on p we say that (M, g) is a *uniform visibility manifold*.

If (M, g) is compact and $(\tilde M, \tilde g)$ is a visibility manifold, then $(\tilde M, \tilde g)$ is a uniform visibility manifold. Moreover, if (M, h) is another Riemannian structure without conjugate points and such that geodesic rays diverge in $\tilde M$, then $(\tilde M, \tilde h)$ is also a visibility manifold (see [**15**, § 5]). Since Green [**23**] proved that in the case of a surface without conjugate points geodesic rays diverge, in particular, we obtain the following conclusion.

Proposition 2.3. *Let (M, g) be a C^∞ compact connected boundaryless Riemannian surface without conjugate points and of genus greater than one. Then $(\tilde M, \tilde g)$ is a visibility manifold.*

Given a point $\bar\theta \in T_1\tilde M$, its *centre stable set* and its *centre unstable set* are defined by

$$\tilde{\mathscr{F}}^{\mathrm{cs}}(\bar\theta) := \bigcup_{t\in\mathbb{R}} \tilde{\mathscr{F}}^{\mathrm{s}}(\phi_t(\bar\theta)) \quad \text{and} \quad \tilde{\mathscr{F}}^{\mathrm{cu}}(\bar\theta) := \bigcup_{t\in\mathbb{R}} \tilde{\mathscr{F}}^{\mathrm{u}}(\phi_t(\bar\theta)),$$

respectively. The images of $\tilde{\mathscr{F}}^{\mathrm{s/u}}(\bar\theta)$ by the natural projection $\bar\pi\colon T_1\tilde M \to T_1M$ are the *stable* and the *unstable leaf* of $\theta = \bar\pi(\bar\theta)$ and denoted by $\mathscr{F}^{\mathrm{s/u}}(\theta)$, respectively. Likewise, the natural projections of $\tilde{\mathscr{F}}^{\mathrm{cs/cu}}(\bar\theta)$ are the *centre stable* and *centre unstable set* of θ and denoted by $\mathscr{F}^{\mathrm{cs/cu}}(\theta)$, respectively.

We list now some important basic properties of the centre stable and centre unstable sets. For item (i) see [**38**, § 6], the proof of item (ii) is analogous to the proof of [**2**, Lemma 2.1].

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Theorem 2.4. *Let (M, g) be a C^∞ compact connected boundaryless Riemannian manifold without conjugate points such that $(\tilde{M}, \tilde{g})$ is a visibility manifold. Then the following assertions hold.*

(i) *The family of sets*

$$\tilde{\mathscr{F}}^{\mathrm{s}} := \bigcup_{\bar{\theta} \in T_1 \tilde{M}} \tilde{\mathscr{F}}^{\mathrm{s}}(\bar{\theta}), \qquad \tilde{\mathscr{F}}^{\mathrm{cs}} := \bigcup_{\bar{\theta} \in T_1 \tilde{M}} \tilde{\mathscr{F}}^{\mathrm{cs}}(\bar{\theta})$$

are collections of C^0 submanifolds. In each union, these submanifolds are either disjoint or coincide.

(ii) *Given $p \in \tilde{M}$, there exists a homeomorphism*

$$\Psi_p \colon \tilde{M} \times \tilde{V}_p \to T_1 \tilde{M}$$

such that

$$\Psi_p(\tilde{M} \times \{(p, v)\}) = \tilde{\mathscr{F}}^{\mathrm{cs}}(p, v).$$

In particular, $\tilde{\mathscr{F}}^{\mathrm{cs}}$ and $\mathscr{F}^{\mathrm{cs}}$ are continuous foliations and the space of leaves of $\mathscr{F}^{\mathrm{cs}}$ is homeomorphic to the vertical fibre $\tilde{V}_p$ for any $p \in \tilde{M}$.

The foliations $\tilde{\mathscr{F}}^{\mathrm{s/cs}}$ and $\tilde{\mathscr{F}}^{\mathrm{u/cu}}$ are called the *stable*, *central stable*, *unstable* and *centre unstable foliations*, respectively. The image of such foliations by the covering map $\bar{\pi} \colon T_1 \tilde{M} \to T_1 M$ form foliations $\mathscr{F}^{\mathrm{s}}$, $\mathscr{F}^{\mathrm{cs}}$, $\mathscr{F}^{\mathrm{u}}$, $\mathscr{F}^{\mathrm{cu}}$ that we call by the same names.

The geodesic flow of a compact manifold whose universal covering is a visibility manifold shares many important dynamical properties with Anosov geodesic flows. The following results are proved by Eberlein [**15**] (see [**17**] for visibility manifolds with non-positive curvature), and extend to visibility universal coverings of compact manifolds without conjugate points, applying Gromov hyperbolic theory [**24**]. In particular, item (3) can be viewed as a sort of coarse local product structure (in general, there is no global product structure for the geodesic flow of the universal covering of compact surfaces without conjugate points and of higher genus).

Recall that a continuous flow $\psi_t \colon X \to X$ is *topologically mixing* if for any two open sets $U, V \subset X$ there exists $T > 0$ such that for $|t| \geq T$, $\psi_t(U) \cap V \neq \emptyset$. Recall that the flow ψ_t is *topologically transitive* if for any open sets U_1 and U_2 there is $t > 0$ such that $\psi_t(U_1) \cap U_2 \neq \emptyset$ or, equivalently, if there exists a dense orbit. Clearly, every topologically mixing flow is topologically transitive. Recall that a foliation is *minimal* if each of its leaves is dense.

Theorem 2.5. *Let (M, g) be a compact Riemannian manifold without conjugate points such that $(\tilde{M}, \tilde{g})$ is a visibility manifold. Then the following assertions hold.*

(i) *The foliations $\mathscr{F}^{\mathrm{s}}$, $\mathscr{F}^{\mathrm{u}}$ are minimal.*

(ii) *The geodesic flow of (M, g) is topologically mixing.*

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(iii) *Every two points* $(p,v),(q,w)\in T_1\tilde{M}$ *with* $(q,w)\neq(p,-v)$ *are* heteroclinically related, *that is, we have*

$$\tilde{\mathscr{F}}^{\mathrm{cs}}(p,v)\cap\tilde{\mathscr{F}}^{\mathrm{cu}}(q,w)\neq\emptyset,\qquad \tilde{\mathscr{F}}^{\mathrm{cs}}(q,w)\cap\tilde{\mathscr{F}}^{\mathrm{cu}}(p,v)\neq\emptyset.$$

Finally, recall that in the case where M is a surface, by a well-known result by Morse [**33**] every minimizing geodesic in $(\tilde{M},\tilde{g})$ is 'shadowed' by a geodesic in the hyperbolic plane. Hence, if (M,g) is a C^∞ compact surface without focal points and of genus greater than one, then there exists $Q>0$ such that each minimizing geodesic in $(\tilde{M},\tilde{g})$ is contained in the Q-tubular neighbourhood of a certain hyperbolic geodesic. In particular, one can immediately conclude the following.

Proposition 2.6. *Let* (M,g) *be a compact Riemannian surface without conjugate points and of genus greater than one. Then there exists* $Q=Q(M)>0$ *such that the Hausdorff distance between any two bi-asymptotic geodesics in* $\tilde{M}$ *is bounded from above by* Q.

Notation

$$M \xleftarrow{\ \pi\ } \sigma_t^{\bar\theta}\colon \tilde{M}\to\tilde{M}, H^{\pm}(\bar\eta)$$

$$\begin{array}{lll}
\theta=(p,v)\in T_1M & \bar\theta=(p,v)\in T_1\tilde{M} & \\
\phi_t\colon T_1M\to T_1M,\mathscr{F}^*(\theta) \xleftarrow{\ \bar\pi\ } & \phi_t\colon T_1\tilde{M}\to T_1\tilde{M},\tilde{\mathscr{F}}^*(\bar\theta) & *=\mathrm{s},\mathrm{u},\mathrm{cs},\mathrm{cu}\\
\downarrow\chi & \downarrow\bar\chi & \\
\psi_t\colon X\to X, W^*([\theta]) \xleftarrow{\ \bar\Pi\ } & \bar\psi_t\colon \bar{X}\to\bar{X},\tilde{W}^*([\bar\theta]) & *=\mathrm{ss},\mathrm{uu},\mathrm{cs},\mathrm{cu}
\end{array}$$

3. The geometry of strips

From now on, we will always assume that (M,g) is a C^∞ compact connected Riemannian surface without focal points and of genus greater than one.

Definition 3.1 (Strip). Given $\bar\theta=(p,v)\in T_1\tilde{M}$, the *strip* $F(\bar\theta)\subset\tilde{M}$ is the maximal subset of geodesics $\gamma_{\bar\eta}\subset\tilde{M}$ such that $\gamma_{\bar\eta}$ is an orbit of both the vector fields $-\nabla b^+_{\bar\theta}$ and $-\nabla b^-_{\bar\theta}$. If the strip $F(\bar\theta)$ consists of a single geodesic, then the point $\bar\theta$ is called *expansive* and the strip is called *trivial*. Otherwise, we call the strip *non-trivial*.

In this section we survey geometric properties of strips. The starting point is the following 'flat strip theorem' (see [**18**, Proposition 5.1], [**38**, Theorem 7.3], and note that it fails for manifolds without conjugate points [**10**]).

Theorem 3.2. *Any two bi-asymptotic geodesics in* $(\tilde{M},\tilde{g})$ *are contained in* a flat strip, *that is, they are contained in a copy of an Euclidean strip* $[a,b]\times\mathbb{R}$ *isometrically and totally geodesically embedded into the universal covering* $\tilde{M}$.

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Definition 3.3. By Theorem 3.2, any strip $F(\bar{\theta})$ is an isometric embedding $f\colon \mathbb{R}\times[a,b]\to\tilde{M}$ into the universal covering. If $[a,b]$ is chosen to be maximal with this property, we call $b-a$ the *width of the strip* $F(\bar{\theta})$.

An immediate consequence of Proposition 2.6 is the following.

Lemma 3.4. *The width of any strip is bounded from above by Q, where Q is the constant in Proposition 2.6.*

We gather in the next lemma the main geometric properties of non-trivial strips (see [**38**]).

Lemma 3.5.

(i) *The intersection of $F(\bar{\theta})$ with the horosphere $H^+(\bar{\theta})$ coincides with*

$$I(\bar{\theta}) := H^+(\bar{\theta})\cap H^-(\bar{\theta}).$$

(ii) *For every $\bar{\theta}\in\tilde{M}$, the set $I(\bar{\theta})$ is a flat convex set, namely, if $p,q\in I(\bar{\theta})$ then the geodesic segment joining p to q also belongs to $I(\bar{\theta})$.*

The following lemma follows from [**9**, Lemma 2.1].

Lemma 3.6. *If the orbit of a point in T_1M by the geodesic flow is dense, then none of its lifts to $T_1\tilde{M}$ has a non-trivial strip.*

In fact, we now have the following much stronger result (in the case of surfaces with non-positive curvature and genus greater than one due to Coudene and Schapira [**12**, Theorem 3.2] and in the general case to Schröder [**43**], where the latter extends to a much more general context).

Lemma 3.7. *Any non-trivial strip is periodic.*

As a consequence, we immediately obtain the following.

Corollary 3.8. *For any $\theta\in T_1M$ and any of its lifts $\bar{\theta}\in T_1\tilde{M}$, the set $\tilde{\mathscr{F}}^{\mathrm{cs}}(\bar{\theta})$ contains at most one non-trivial strip. In particular, the set of non-trivial strips is countable and hence the set of expansive points is open and dense in $T_1\tilde{M}$.*

4. Quotient space and the model flow

We now define as a key object in the present paper a relation in T_1M, and it is straightforward to check that it is indeed an equivalence relation. It 'collapses' each strip into a single curve.

Definition 4.1. Two points $\theta,\eta\in T_1M$ are related $\theta\sim\eta$ if, and only if:

- $\eta\in\mathscr{F}^{\mathrm{s}}(\theta)$;

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- $\bar{\theta}$ is any lift of θ and $\bar{\eta}$ a lift of η satisfying $\bar{\eta} \in \tilde{\mathscr{F}}^{\mathrm{s}}(\bar{\theta})$, then the geodesics $\gamma_{\bar{\theta}}$ and $\gamma_{\bar{\eta}}$ are bi-asymptotic.

Given $\theta \in T_1M$, we denote by $[\theta]$ the equivalence class which contains θ. We denote by

$$X := T_1M/_{\sim}$$

the set of all equivalence classes. Consider the quotient map

$$\chi\colon T_1M \to X\colon \theta \mapsto [\theta].$$

Recall that the *quotient topology* associated with a quotient map χ is the topology generated by the sets $U \subset X$ such that $\chi^{-1}(U)$ is an open set of T_1M.

We consider the flow $\psi_t\colon X \to X$ defined by

$$\psi_t([\theta]) := [\phi_t(\theta)].$$

As the geodesic flow preserves the foliation $\mathscr{F}^{\mathrm{s}}$ and asymptoticity, this flow is indeed well defined. Moreover, it is continuous in the quotient topology, and we have the following key fact which immediately follows from the very definition of the flows.

Proposition 4.2. *The geodesic flow $\phi_t\colon T_1M \to T_1M$ and the quotient flow $\psi_t\colon X \to X$ are time-preserving semi-conjugate by means of the quotient map χ.*

The equivalence relation on T_1M with quotient map χ naturally induces an equivalence relation in $T_1\tilde{M}$ with quotient map $\bar{\chi}\colon T_1\tilde{M} \to \bar{X}$. Let us denote by $[\bar{\theta}]$ the corresponding equivalence class of $\bar{\theta} \in T_1\tilde{M}$. Let $\bar{\psi}_t\colon \bar{X} \to \bar{X}$ be the corresponding quotient flow.*

In the present section we will study the topological properties of the quotient space and this flow. The main result of the section is the following.

Theorem 4.3. *Let (M, g) be a compact surface without focal points. Then the quotient space X is a compact topological 3-manifold. In particular, X admits a smooth three-dimensional structure where the quotient flow ψ_t is continuous.*

Theorem 4.3 is not at all obvious and requires a careful analysis of the quotient topology and its relationship with the dynamics of the geodesic flow. The absence of focal points will be crucial in some subtle steps of its proof. The main idea of the proof is to exhibit a special basis for the quotient topology, whose construction will be made in several steps. The proof will be concluded at the end of this section.

4.1. A family of cross-sections and a basis for the quotient topology

As the first step to obtain a basis for the quotient topology we construct a special family of cross-sections for the quotient flow, from which we shall obtain a basis by shifting them by the geodesic flow.

* Notice that the absence of focal points was not needed in the above construction and that it extends to surfaces without conjugate points.

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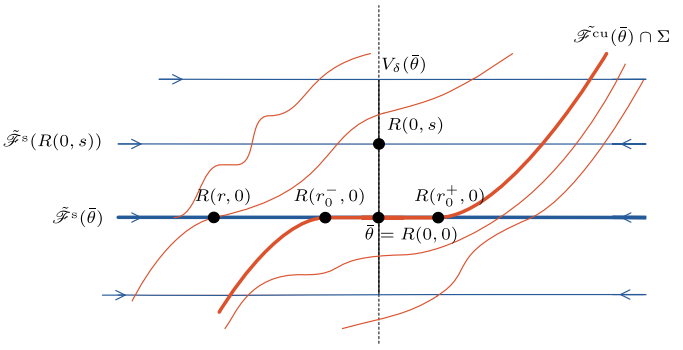


Figure 1. Parametrization of the disk $\Sigma = \Sigma_{\bar\theta}(\varepsilon,\delta)$.

Given $\bar\theta \in T_1\tilde{M}$, let

$$\mathcal{I}(\bar\theta) := \tilde{\mathscr{F}}^{\mathrm{s}}(\bar\theta) \cap \tilde{\mathscr{F}}^{\mathrm{u}}(\bar\theta).$$

Note that the set $\mathcal{I}(\bar\theta)$ is a lift of $I(\bar\theta)$ (defined in Lemma 3.5) to $T_1\tilde{M}$. Moreover, the set $\tilde{\mathscr{F}}^{\mathrm{cs}}(\bar\theta) \cap \tilde{\mathscr{F}}^{\mathrm{cu}}(\bar\theta)$ contains an isometric copy of the strip $F(\bar\theta)$ (which can be trivial or non-trivial).

Let us now choose the local cross-section. Given a point $\bar\theta$, let us construct a continuously embedded closed two-dimensional disc $\Sigma = \Sigma_{\bar\theta}(\varepsilon,\delta) \subset T_1\tilde{M}$ which is transverse to the geodesic flow and which contains $\mathcal{I}(\bar\theta)$. This disk will be foliated by the leaves of $\tilde{\mathscr{F}}^{\mathrm{s}}$. To begin the construction, let $\varepsilon > 0$, $\delta > 0$ be sufficiently small, let $V_\delta(\bar\theta)$ be the δ-tubular neighbourhood of $\bar\theta$ in its vertical fibre and let

$$R\colon (r_0^- - \varepsilon, r_0^+ + \varepsilon) \times (-\delta,\delta) \to T_1\tilde{M}$$

be the homeomorphism with the following properties:

- $R(0,0) = \bar\theta$;
- $R(0,s)$ with $s \in (-\delta,\delta)$ is the arc length parametrization of $V_\delta(\bar\theta)$ in the Sasaki metric;
- $R(r,0)$ is the arc length parametrization of the ε-tubular neighbourhood of $\mathcal{I}(R(0,0)) = \mathcal{I}(\bar\theta)$ in $\tilde{\mathscr{F}}^{\mathrm{s}}(\bar\theta)$, with $R(r_0^-,0)$ and $R(r_0^+,0)$ being the endpoints of $\mathcal{I}(\bar\theta)$;
- for each $s \in (-\delta,\delta)$, $r \mapsto R(r,s)$ is the arc length parametrization of the continuous curve $R(\cdot,s)$ in $\tilde{\mathscr{F}}^{\mathrm{s}}(R(0,s))$.

Since the foliation $\tilde{\mathscr{F}}^{\mathrm{s}}$ is a continuous foliation by Lipschitz curves, by Brouwer's open mapping theorem the image of R is a two-dimensional section that we will denote by $\Sigma_{\bar\theta}(\varepsilon,\delta)$,

$$\Sigma_{\bar\theta}(\varepsilon,\delta) := R((r_0^- - \varepsilon, r_0^+ + \varepsilon) \times (-\delta,\delta))$$

(compare Figure 1). Given $\tau > 0$, let us consider the open neighbourhood of $\bar\theta$ in $T_1\tilde{M}$ of the form

$$B_{\bar\theta}(\varepsilon,\delta,\tau) := \{\phi_t(\Sigma_{\bar\theta}(\varepsilon,\delta))\colon |t| < \tau\} \tag{1}$$

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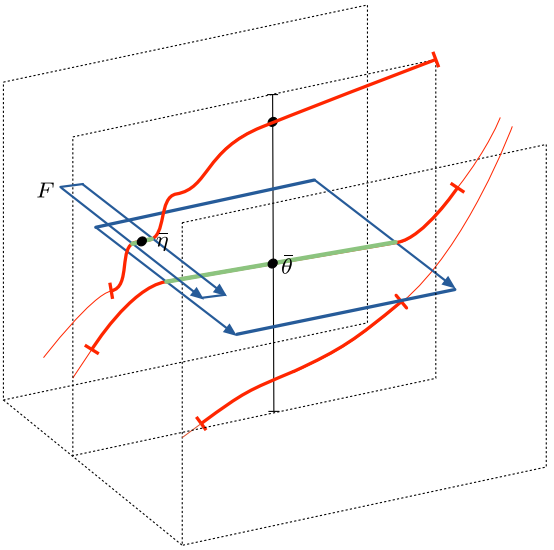


Figure 2. (Colour available online) Flat strips (blue) in $B_{\bar{\theta}}(\varepsilon, \delta, \tau)$ and their projections (green) to $\Sigma_{\bar{\theta}}(\varepsilon, \delta)$.

where ϕ_t is the geodesic flow of $T_1\tilde{M}$. We shall also omit the point $\bar{\theta}$ in the index unless we change it. Clearly, the section depends on the point $\bar{\theta}$ and on the parameters ε and δ; we shall omit this dependence in the notation but keep it in mind. Let

$$\Pi_\Sigma \colon B(\varepsilon, \delta, \tau) \to \Sigma$$

be the projection from $B(\varepsilon, \delta, \tau) := B_{\bar{\theta}}(\varepsilon, \delta, \tau)$ onto $\Sigma = \Sigma_{\bar{\theta}}(\varepsilon, \delta)$ by ϕ_t.

We have the following result (compare Figure 2) essentially saying that $\chi(\Sigma)$ is 'almost' a cross-section for the quotient flow.

Lemma 4.4. *Given a strip F which intersects $B(\varepsilon, \delta, \tau)$, there exists $\bar{\eta} \in \Sigma$ such that $\Pi_\Sigma(F \cap B(\varepsilon, \delta, \tau))$ is a connected component of $\mathcal{I}(\bar{\eta})$. In particular, if $\mathcal{I}(\bar{\eta}) \subset B(\varepsilon, \delta, \tau)$ then $\Pi_\Sigma(F) = \mathcal{I}(\bar{\eta})$ and*

$$F \cap B(\varepsilon, \delta, \tau) = \{\phi_t(\mathcal{I}(\bar{\eta})) \colon |t| < \tau\}.$$

Proof. Observe that, by the construction of Σ, each strip F intersects Σ in a connected component of some $\mathcal{I}(\bar{\eta})$, $\bar{\eta} \in \Sigma$. Moreover, the geodesic flow preserves the sets $\mathcal{I}$, that is, for every $t \in \mathbb{R}$ and for every $\bar{\eta} \in T_1\tilde{M}$ we have $\phi_t(\mathcal{I}(\bar{\eta})) = \mathcal{I}(\phi_t(\bar{\eta}))$. This gives the claim. □

Let us denote

$$\Sigma^+ := \{R(r, s) \colon r \in (r_0^- - \varepsilon, r_0^+ + \varepsilon), s \in (0, \delta)\} \subset \Sigma,$$
$$\Sigma^- := \{R(r, s) \colon r \in (r_0^- - \varepsilon, r_0^+ + \varepsilon), s \in (-\delta, 0)\} \subset \Sigma.$$

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For each point $\bar\eta \in B(\varepsilon,\delta,\tau)$, denote by

$$B^{\mathrm{cs}}_{\bar\eta}(\varepsilon,\delta,\tau) \subset \tilde{\mathscr{F}}^{\mathrm{cs}}(\bar\eta) \cap B(\varepsilon,\delta,\tau), \qquad B^{\mathrm{cu}}_{\bar\eta}(\varepsilon,\delta,\tau) \subset \tilde{\mathscr{F}}^{\mathrm{cu}}(\bar\eta) \cap B(\varepsilon,\delta,\tau)$$

the connected components of the intersections of the central stable and unstable sets of $\bar\eta$ with $B(\varepsilon,\delta,\tau)$ which contain $\bar\eta$, respectively. Given $\bar\eta \in \Sigma$ let

$$\begin{aligned} W^{\mathrm{s}}_{\Sigma}(\bar\eta) &:= \Pi_{\Sigma}(B^{\mathrm{cs}}_{\bar\eta}(\varepsilon,\delta,\tau)) = B^{\mathrm{cs}}_{\bar\eta}(\varepsilon,\delta,\tau) \cap \Sigma_{\bar\theta}(\varepsilon,\delta), \\ W^{\mathrm{u}}_{\Sigma}(\bar\eta) &:= \Pi_{\Sigma}(B^{\mathrm{cu}}_{\bar\eta}(\varepsilon,\delta,\tau)). \end{aligned} \tag{2}$$

Note that, in fact, by the definition of the map R, for every $\bar\eta \in \Sigma$ there exist parameters r, s such that $W^{\mathrm{s}}_{\Sigma}(\bar\eta) = R(r,s)$. However, in general $W^{\mathrm{u}}_{\Sigma}(\bar\eta)$ may not satisfy such a property.

If the section Σ is sufficiently narrow in the vertical direction, being close enough to $\mathcal{I}(\bar\theta)$, then every two different points in Σ are heteroclinically related. Given $\bar\theta$, choose $\delta_0 = \delta_0(\bar\theta) > 0$ small so that $\tilde{\mathscr{F}}^{\mathrm{cu}}(\bar\theta)$ intersects $\tilde{\mathscr{F}}^{\mathrm{s}}(R(0,\pm\delta_0))$. Moreover, for $\delta \in (0,\delta_0)$ there exist $\varepsilon_0 = \varepsilon_0(\bar\theta,\delta) > 0$ such that for every $\varepsilon \in (0,\varepsilon_0)$, every $r \in (r_0^- - \varepsilon, r_0^-) \cup (r_0^+, r_0^+ + \varepsilon)$ and every s with $|s| < \delta$ we have

$$W^{\mathrm{u}}_{\Sigma}(R(r,0)) \cap W^{\mathrm{s}}_{\Sigma}(R(0,s)) \subset \Sigma_{\bar\theta}(\varepsilon,\delta).$$

The basis we will construct is in many respects a 'blow up' of the classical local product neighbourhood in hyperbolic dynamics (see, for example, [**28**, Chapter 6.4]).

Definition 4.5. Given $\bar\eta, \bar\xi \in \Sigma_{\bar\theta}(\varepsilon,\delta)$, let

$$[\bar\eta,\bar\xi] := W^{\mathrm{s}}_{\Sigma}(\bar\eta) \cap W^{\mathrm{u}}_{\Sigma}(\bar\xi).$$

As a consequence of Corollary 3.8, given $\varepsilon \in (0,\varepsilon_0)$ there exist numbers $\rho^-, \rho^+ \in (0,\varepsilon)$ such that

$$\bar\theta^- = R(r_0^- - \rho^-, 0) \quad \text{and} \quad \bar\theta^+ = R(r_0^+ + \rho^+, 0)$$

are expansive points, and hence the sets $W^{\mathrm{u}}_{\Sigma}(\bar\theta^{\pm})$ are curves which are disjoint from $W^{\mathrm{u}}_{\Sigma}(\bar\theta)$. Moreover, all such curves bound a region in $\Sigma_{\bar\theta}(\varepsilon,\delta)$ which is homeomorphic to a rectangle whose relative interior contains $\mathcal{I}(\bar\theta)$ (compare Figure 3). The set $W^{\mathrm{s}}_{\Sigma}(\bar\theta)$ divides $W^{\mathrm{u}}_{\Sigma}(\bar\theta^-)$ into two parts: one in Σ^+ and one in Σ^-. By Theorem 2.5(iii) we have

$$\bar\eta^- \in W^{\mathrm{u}}_{\Sigma}(\bar\theta^-) \cap \Sigma^+ \quad \text{and} \quad \bar\eta^+ \in W^{\mathrm{u}}_{\Sigma}(\bar\theta^+) \cap \Sigma^-$$

and the intersections $[\bar\eta^-, \bar\theta^+]$ and $[\bar\eta^+, \bar\theta^-]$ are non-empty (though they may be contained in a non-trivial strip), and by the previous remarks we can assume that such intersections are in $\Sigma_{\bar\theta}(\varepsilon,\delta)$. Fixing any such points $\bar\eta^-$ and $\bar\eta^+$, let us denote by $U_{\bar\theta}(\varepsilon,\delta,\bar\theta^-,\bar\theta^+,\bar\eta^-,\bar\eta^+)$ the open two-dimensional region in $\Sigma_{\bar\theta}(\varepsilon,\delta)$ whose boundary is formed by the described pieces of stable and unstable arcs. The region $U_{\bar\theta}(\varepsilon,\delta,\bar\theta^-,\bar\theta^+,\bar\eta^-,\bar\eta^+)$ clearly contains $\mathcal{I}(\bar\theta)$.

Lemma 4.6. *Given $\bar\theta \in T_1\tilde M$, let $\delta_0 = \delta_0(\bar\theta)$, $\delta \in (0,\delta_0)$, $\varepsilon_0 = \varepsilon_0(\bar\theta,\delta)$, $\varepsilon \in (0,\varepsilon_0)$, with expansive points $\bar\theta^- = R(r_0^- - \rho^-, 0)$ and $\bar\theta^+ = R(r_0^+ + \rho^+, 0)$, where $\rho^{\pm} \in$*

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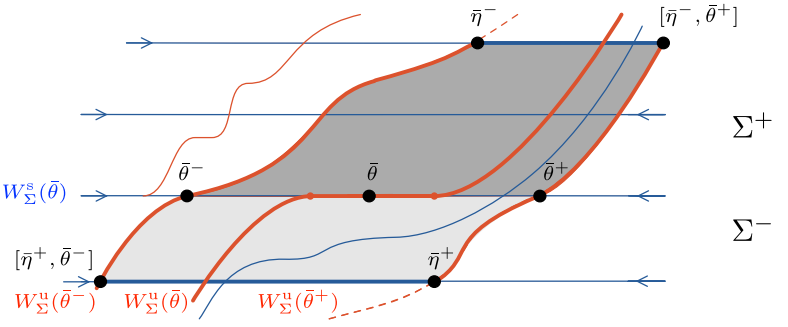


Figure 3. Region defined by expansive points $\bar\theta^\pm$ and $\bar\eta^\pm$, contained in the region $\Sigma_{\bar\theta}(\varepsilon,\delta)$ split into Σ^+ and Σ^- and containing the open set $U_{\bar\theta}(\varepsilon,\delta,\bar\theta^-,\bar\theta^+,\bar\eta^-,\bar\eta^+)$ (shaded region).

$(0,\varepsilon)$, *and points* $\bar\eta^\pm \in W^{\mathrm{u}}_\Sigma(\bar\theta^\pm)\cap\Sigma^\mp$. *Then the above-constructed region* $U = U_{\bar\theta}(\varepsilon,\delta,\bar\theta^-,\bar\theta^+,\bar\eta^-,\bar\eta^+)$ *in the section* $\Sigma_{\bar\theta}(\varepsilon,\delta)$ *has the following properties.*

(i) *We have* $\bar\chi^{-1}(\bar\chi(U)) = U$ *and hence the set* $\bar\chi(U)$ *is an open neighbourhood of* $\bar\chi(\bar\theta)$ *in the quotient topology restricted to* $\bar\chi(\Sigma_{\bar\theta}(\varepsilon,\delta))$.

(ii) *For all positive numbers* t',δ',ε' *we can choose* δ,ε *such that the above-considered region* $U = U_{\bar\theta}(\varepsilon,\delta,\bar\theta^-,\bar\theta^+,\bar\eta^-,\bar\eta^+)$ *for every* t *with* $|t|\le t'$ *satisfies*

$$\phi_t(U) = U_{\phi_t(\bar\theta)}\big(\varepsilon,\delta,\phi_t(\bar\theta^-),\phi_t(\bar\theta^+),\phi_t(\bar\eta^-),\phi_t(\bar\eta^+)\big)\subset\Sigma_{\phi_t(\bar\theta)}(\varepsilon',\delta'),$$

and moreover,

$$\phi_t(U)\subset U_{\phi_t(\bar\theta)}(\varepsilon,\delta',\phi_t(\bar\theta^-),\phi_t(\bar\theta^+),\phi_t(\bar\eta^-),\phi_t(\bar\eta^+)).$$

(iii) *The set* $\bar\chi(U)$ *is a (topological) local cross-section of the quotient flow in* $\bar X$, *that is, there exists an open set containing* $\bar\chi(\bar\theta)$ *such that every orbit of the quotient flow restricted to this set intersects* $\bar\chi(U)$ *at just one point.*

Proof. Given $\bar\xi\in U$, we have $\bar\chi^{-1}(\bar\chi(\bar\xi)) = \mathcal{I}(\bar\xi)\subset\Sigma_{\bar\theta}(\varepsilon,\delta)$. By the construction of $\Sigma_{\bar\theta}(\varepsilon,\delta)$, the curve $\mathcal{I}(\bar\xi)$ meets the boundary C of U if, and only if, $\bar\xi$ is in C already. Because this boundary is made of pieces of centre stable and centre unstable leaves, if a strip through a point $\bar\xi\in\Sigma_{\bar\theta}(\varepsilon,\delta)$ meets C then the whole set $\mathcal{I}(\bar\xi)$ must be included in one of these pieces of centre stable and centre unstable leaves. So we conclude that $\bar\chi^{-1}(\bar\chi(\bar\xi))\in U$ for every $\bar\xi\in U$. Since clearly $\bar\chi^{-1}(\bar\chi(U))\supset U$ we have shown $\bar\chi^{-1}(\bar\chi(U)) = U$.

By the definition of the quotient topology restricted to $\Sigma_{\bar\theta}(\varepsilon,\delta)$, we have that $\bar\chi(U)$ is a relative open neighbourhood of $\bar\chi(\bar\xi)$ in $\bar\chi(\Sigma_{\bar\theta}(\varepsilon,\delta))$, thus proving item (i) in the lemma.

The proof of item (ii) follows from the construction of U. Indeed, all the dynamical objects involved in its construction, that is, stable leaves and heteroclinic intersections, are invariant by the geodesic flow. The constants ε and δ may vary a little, since they are geometric quantifiers of compact pieces of stable leaves which contain strips. The

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size of a strip does not change under the action of the geodesic flow, but the size of a neighbourhood of it changes continuously. From the above statements it is straightforward to conclude item (ii).

Item (ii) follows from construction by the definition of an equivalence relation within any strip. □

Lemma 4.7. *Given $\bar{\theta} \in T_1\tilde{M}$, $\delta_0 = \delta_0(\bar{\theta})$, $\delta \in (0, \delta_0)$, $\varepsilon_0 = \varepsilon_0(\bar{\theta}, \delta)$, $\varepsilon \in (0, \varepsilon_0)$, expansive points $\bar{\theta}^- = R(r_0^- - \rho^-, 0)$, $\bar{\theta}^+ = R(r_0^+ + \rho^+, 0)$ with $\rho^{\pm} \in (0, \varepsilon)$, and points $\bar{\eta}^{\pm} \in W^{\mathrm{u}}_{\Sigma}(\bar{\theta}^{\pm}) \cap \Sigma^{\mp}$, consider the above-constructed region $U = U_{\bar{\theta}}(\varepsilon, \delta, \bar{\theta}^-, \bar{\theta}^+, \bar{\eta}^-, \bar{\eta}^+)$ in the section $\Sigma_{\bar{\theta}}(\varepsilon, \delta)$. Then for $\tau > 0$ small enough the set*

$$A = A_{\bar{\theta}}(\tau, \varepsilon, \delta, \bar{\theta}^-, \bar{\theta}^+, \bar{\eta}^-, \bar{\eta}^+) := \bigcup_{|t|<\tau} \phi_t(U) \tag{3}$$

satisfies $\bar{\chi}^{-1}(\bar{\chi}(A)) = A$.

Hence, the collection of such sets

$$\{\bar{\chi}(A_{\bar{\theta}}(\tau, \varepsilon, \delta, \bar{\theta}^-, \bar{\theta}^+, \bar{\eta}^-, \bar{\eta}^+))\}$$

forms a basis for the quotient topology of $\bar{X}$.

Proof. First observe that A by definition is homeomorphic to $(-\tau, \tau) \times U$. Therefore, Brouwer's open mapping theorem implies that each such set is open in $T_1\tilde{M}$.

Moreover, by Lemma 4.6(ii) we have

$$A = \bigcup_{|t|<\tau} \phi_t(U) = \bigcup_{|t|<\tau} U_{\phi_t(\bar{\theta})}(\varepsilon, \delta, \phi_t(\bar{\theta}^-), \phi_t(\bar{\theta}^+), \phi_t(\bar{\eta}^-), \phi_t(\bar{\eta}^+))$$

for τ small enough. Applying Lemma 4.6(i) to the above union of sets we deduce that $\bar{\chi}^{-1}(\bar{\chi}(A)) = A$ as claimed.

This yields that the family of all such sets $A_{\bar{\theta}}(\tau, \varepsilon, \delta, \bar{\theta}^-, \bar{\theta}^+, \bar{\eta}^-, \bar{\eta}^+)$ specified as in the statement are open sets in the quotient $\bar{X}$ since, by definition, an open set for the quotient topology is any set whose pre-image by the quotient map $\bar{\chi}$ is an open set in $T_1\tilde{M}$. To see that they provide a basis for the quotient topology, observe that the family of open sets $B_{\bar{\theta}}(\varepsilon, \delta, \tau')$ defined in (1) is a basis for the family of open neighbourhoods of the sets $\mathcal{I}(\bar{\theta})$, and that $A_{\bar{\theta}}(\tau, \varepsilon, \delta, \bar{\theta}^-, \bar{\theta}^+, \bar{\eta}^-, \bar{\eta}^+)$ is contained in $B_{\bar{\theta}}(\varepsilon, \delta, \tau')$. So let $\bar{\chi}(\bar{\theta})$ be the equivalence class of $\bar{\theta}$ and V an open neighbourhood of $\bar{\chi}(\bar{\theta})$ in the quotient $\bar{X}$. The set $\bar{\chi}^{-1}(V)$ is an open neighbourhood of $\mathcal{I}(\bar{\theta})$ which contains some $B_{\bar{\theta}}(\varepsilon, \delta, \alpha)$ for some parameters $\varepsilon, \delta, \tau'$. Hence the family of sets $A_{\bar{\theta}}(\tau, \rho, \delta, \bar{\theta}^-, \bar{\theta}^+, \bar{\eta}^-, \bar{\eta}^+)$ provides a basis, too. □

4.2. Smooth manifold structure

Proposition 4.8. *Given $\bar{\theta} \in T_1\tilde{M}$, let $U = U_{\bar{\theta}}(\varepsilon, \delta, \bar{\theta}^-, \bar{\theta}^+, \bar{\eta}^-, \bar{\eta}^+)$ be the local cross-section of the geodesic flow as defined in the previous section and consider the corresponding set $A = A_{\bar{\theta}}(\tau, \varepsilon, \delta, \bar{\theta}^-, \bar{\theta}^+, \bar{\eta}^-, \bar{\eta}^+)$ as defined in (3) for some $\tau > 0$. There exist*

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numbers $a < a'$, $b < b'$ depending on $\bar\theta, \delta, \varepsilon, \bar\theta^-, \bar\theta^+, \bar\eta^-, \bar\eta^+$ and a homeomorphism

$$f\colon (a,a')\times(b,b')\times(-\tau,\tau)\to \bigcup_{|t|<\tau} \bar\psi_t(\bar\chi(U)) = \bar\chi(A)$$

for every $\tau > 0$.

In particular, the quotient spaces $\bar X$ and X are topological 3-manifolds.

Proof. The proof relies essentially on the transitivity of the geodesic flow and the minimality of central foliations. By Theorem 2.5(i), each stable (unstable) leaf is dense in U. In particular, this holds for the central stable (central unstable) leaf of a dense orbit, which by Lemma 3.6 has no non-trivial strips.

Let $\theta^* \in T_1M$ be a point whose orbit is dense in T_1M, and let $\bar\theta^*$ be one of its lifts in $T_1\tilde M$. Suppose that $\bar\theta^* \in U$ and that $d_S(\bar\theta^*, \bar\theta) < \delta$. The lifts of the stable set and the unstable set of the orbit of $\bar\theta^*$ in $T_1\tilde M$ are both dense in $\Sigma = \Sigma_{\bar\theta}(\varepsilon,\delta)$.

Consider the arc length parametrizations

$$R^{\mathrm s}\colon (a,a')\to U \quad \text{with } R^{\mathrm s}(0)=\bar\theta^*$$
$$R^{\mathrm u}\colon (b,b')\to U \quad \text{with } R^{\mathrm u}(0)=\bar\theta^*$$

of the arc of the intersection of the stable leaf $\tilde{\mathscr F}^{\mathrm s}(\bar\theta^*)$ with U and of the arc of the intersection of the centre unstable leaf $\tilde{\mathscr F}^{\mathrm{cu}}(\bar\theta^*)$ with U, respectively. Since we can choose $\bar\theta^*$ as closed to $\bar\theta$ as we wish, we can choose $a'-a$ very close to the length of the connected component of the stable set of $\bar\theta$ in U containing $\bar\theta$. Analogously, $b'-b$ can be chosen very close to the length of the connected component of the unstable set of $\bar\theta$ in U containing $\bar\theta$.

Claim. *For every $r\in(a,a')$ and $s\in(b,b')$, the point $[R^{\mathrm u}(r), R^{\mathrm s}(s)]$ is contained in U. Moreover, the map*

$$h\colon (a,a')\times(b,b')\to \bar\chi(U)\colon (r,s)\mapsto \bar\chi([R^{\mathrm u}(r), R^{\mathrm s}(s)])$$

is a homeomorphism. In particular, $\bar\chi(U)$ is a topological 2-manifold.

Proof. Recall the definition of $W^{\mathrm{s/u}}_\Sigma(\cdot)$ in (2). The set U is foliated by

$$\{W^{\mathrm s}_\Sigma(\bar\eta)\cap U\colon \bar\eta\in U\} \quad \text{and} \quad \{W^{\mathrm u}_\Sigma(\bar\eta)\cap U\colon \bar\eta\in U\}$$

in a trivial way. Both families of sets induce a product foliation of U by embeddings of open segments. Therefore, each of these sets splits its complement in the closure of U into two disjoint open regions (see Figure 4). Let $c^{\mathrm s}(\bar\eta^-)$ denote the curve that is formed by the piece of $W^{\mathrm s}_\Sigma(\bar\eta^-)$ bounded by $\bar\eta^-$ and $[\bar\eta^-,\bar\eta^+]$, and let $c^{\mathrm s}(\bar\eta^+)$ be the curve that is formed by $W^{\mathrm s}_\Sigma(\bar\eta^+)$ bounded by $[\bar\eta^+,\bar\eta^-]$ and $\bar\eta^+$. Analogously, let $c^{\mathrm u}(\bar\eta^-)$ denote the curve that is formed by the piece of $W^{\mathrm u}_\Sigma(\bar\eta^-)$ bounded by $\bar\eta^-$ and $[\bar\eta^+,\bar\eta^-]$, and let $c^{\mathrm u}(\bar\eta^+)$ be the curve that is formed by $W^{\mathrm u}_\Sigma(\bar\eta^+)$ bounded by $\bar\eta^+$ and $[\bar\eta^-,\bar\eta^+]$. Given $\bar\eta\in U$, the set $W^{\mathrm s}_\Sigma(\bar\eta)$ splits the closure of U into two open disjoint regions, one of them containing

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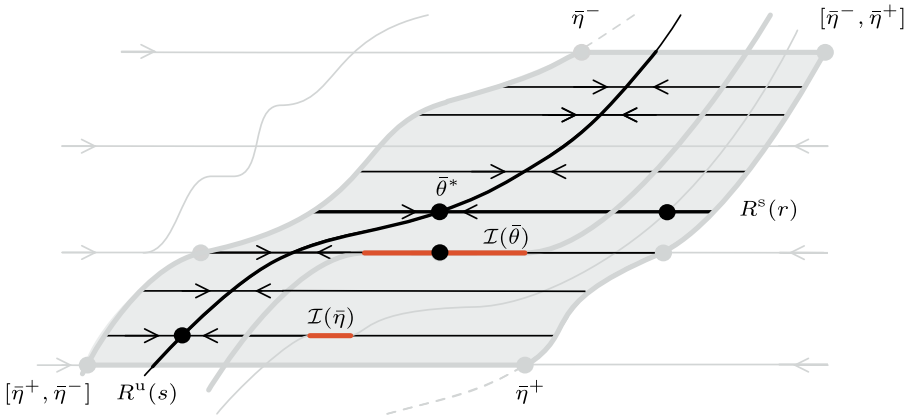


Figure 4. Parametrization based on a recurrent expansive point in U (shaded region). All points in each strip (e.g. $\mathcal{I}(\bar{\eta})$) have *one* common pair (r, s) of parameters.

$c^{\mathrm{s}}(\bar{\eta}^-)$ and the other containing $c^{\mathrm{s}}(\bar{\eta}^+)$. Analogously, each set $W^{\mathrm{u}}_{\Sigma}(\bar{\eta})$ splits the closure of U into two open disjoint regions, one of them containing the curve $c^{\mathrm{u}}(\bar{\eta}^-)$ and the other the curve $c^{\mathrm{u}}(\bar{\eta}^+)$.

For every $r \in (a, a')$ and $s \in (b, b')$, the point $[R^{\mathrm{u}}(r), R^{\mathrm{s}}(s)]$ is contained in U. Indeed, each curve $W^{\mathrm{s}}_{\Sigma}(R^{\mathrm{u}}(r))$ contains points of both $c^{\mathrm{u}}(\bar{\eta}^-)$ and $c^{\mathrm{u}}(\bar{\eta}^+)$, so for every $s \in (b, b')$, by the Jordan curve theorem and the continuity of stable foliation, it has to cross $W^{\mathrm{u}}_{\Sigma}(R^{\mathrm{s}}(s))$. Since by Lemma 4.6 each intersection of the form $W^{\mathrm{s}}_{\Sigma}(\bar{\eta}) \cap W^{\mathrm{u}}_{\Sigma}(\bar{\eta})$ for $\bar{\eta} \in U$ is a class in a strip that is contained in U, we get that $[R^{\mathrm{u}}(r), R^{\mathrm{s}}(s)] \in U$.

Moreover, the parametrizations R^{s} and R^{u} induce continuous parametrizations of their quotients

$$\bar{\chi} \circ R^{\mathrm{s}} \colon (a, a') \to \bar{\chi}(U), \qquad \bar{\chi} \circ R^{\mathrm{u}} \colon (b, b') \to \bar{\chi}(U),$$

because strips of orbits in the centre stable and centre unstable sets of $\bar{\theta}^*$ are trivial. Therefore, the map

$$h \colon (a, a') \times (b, b') \to \bar{\chi}(U),$$
$$h(r, s) := \bar{\chi}([R^{\mathrm{u}}(r), R^{\mathrm{s}}(s)]) = [(\bar{\chi} \circ R^{\mathrm{u}})(r), (\bar{\chi} \circ R^{\mathrm{s}})(s)]$$

defines a homeomorphism from an open rectangle onto $\bar{\chi}(U)$. Indeed, the map h is already a bijection restricted to the dense subset of intersections between the centre stable and centre unstable sets of dense orbits intersecting U. Taking the quotient we have that the map h is continuous and injective in its image, because each set of the form $\bar{\chi}([R^{\mathrm{u}}(r), R^{\mathrm{s}}(s)])$ is just a point in the quotient. By construction, the inverse of h is also continuous. So we have a homeomorphism of a rectangle onto $\bar{\chi}(U)$. By Brouwer's open mapping theorem, the image of h is an open two-dimensional subset, which shows the claim. □

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To conclude the proof of the proposition, we now apply the quotient flow to the section $\bar{\chi}(U)$, which is a topological surface. Indeed, by construction, each set of the form

$$\bigcup_{|t|<\tau} \tilde{\psi}_t(\bar{\chi}(U))$$

is continuously bijective to the product $(a, a') \times (b, b') \times (-\tau, \tau)$. By the claim, this correspondence is a homeomorphism and therefore $\bar{\chi}(U)$ is an open three-dimensional set. This finishes the proof of the proposition. □

Proof of Theorem 4.3. Proposition 4.8 implies that each point in $\bar{X}$ has an open set that is continuously parametrized by an open subset of $\mathbb{R}^3$ which characterizes a topological 3-manifold. Hence, by [**3**, **32**] the space $\bar{X}$ has a smooth structure, which is compatible with the quotient topology. Since the quotient X is locally homeomorphic to $\bar{X}$, the above assertions extend to X. □

Remark 4.9. The fact that the quotient space X is a smooth compact manifold is very important in many respects. Any smooth manifold admits a Riemannian metric, and hence there exists a distance $d\colon X \times X \to \mathbb{R}$ which endows X with the structure of a complete metric space. By the definitions of $\bar{X}$ and X, it is straightforward to see that the map

$$\bar{\Pi}\colon \bar{X} \to X, \qquad \bar{\Pi}(\bar{\chi}(\bar{\theta})) = \chi(\bar{\pi}(\bar{\theta}))$$

is a covering map, where $\bar{\pi}\colon T_1\tilde{M} \to T_1M$ is the natural projection and $\chi\colon T_1M \to X$ and $\bar{\chi}\colon T_1\tilde{M} \to \bar{X}$ are the quotient maps inducing the quotient spaces. The pullback $\bar{d}$ of d to $\bar{X}$ by $\bar{\Pi}$ provides a structure of a complete metric space $(\bar{X}, \bar{d})$ locally isometric to (X, d). The metric d is continuous and hence the quotient flow is continuous with respect to d. We observe that the projection $\bar{\Pi}$ of the basis $\bar{\chi}(A_{(\cdot)}(\cdot))$ defined in Lemma 4.7 naturally defines a basis in X.

Let $\mathrm{Isom}(\bar{X})$ be the group of isometries of $(\bar{X}, \bar{d})$, which contains a representation Γ of the fundamental group $\pi_1(X)$. Notice that for every covering isometry β in $T_1\tilde{M}$ the composition $\bar{\chi} \circ \beta$ induces a deck transformation $\bar{\beta}\colon \bar{X} \to \bar{X}$ that satisfies $\bar{\chi} \circ \beta = \bar{\beta} \circ \bar{\chi}$ and is an element of Γ.

5. The dynamics of the quotient flow

We start this section by recalling some general definitions. Given a general complete continuous flow $\psi_t\colon X \to X$ acting on a complete metric space (X, d), the *strong stable set* $W^{\mathrm{ss}}(x)$ of a point $x \in X$ is the set of points $y \in X$ such that

$$\lim_{t \to +\infty} d(\psi_t(y), \psi_t(x)) = 0.$$

The *strong unstable set* of a point $x \in X$ is defined as the strong stable set of x with respect to ψ_{-t} and denoted by $W^{\mathrm{uu}}(x)$. The *centre stable set* $W^{\mathrm{cs}}(x)$ of a point $x \in X$ is

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the set of points $y \in X$ such that

$$d(\psi_t(y), \psi_t(x)) \leq C$$

for some $C > 0$ and every $t \geq 0$. The *centre unstable set* is defined as the centre stable set of x with respect to ψ_{-t} and denoted by $W^{\mathrm{cu}}(x)$. For $x \in X$ and $\varepsilon > 0$ let

$$W^{\mathrm{cs}}_{\varepsilon}(x) := \{y \in W^{\mathrm{cs}}(x) \colon d(\psi_t(y), \psi_t(x)) \leq \varepsilon \text{ for every } t \geq 0\},$$
$$W^{\mathrm{cu}}_{\varepsilon}(x) := \{y \in W^{\mathrm{cu}}(x) \colon d(\psi_{-t}(y), \psi_{-t}(x)) \leq \varepsilon \text{ for every } t \geq 0\}.$$

The flow ψ_t is said to have *local product structure*[*] if for each sufficiently small $\varepsilon > 0$ there is $\delta > 0$ such that for every $x, y \in X$ with $d(x, y) \leq \delta$ there is a unique $\tau = \tau(x, y)$ with $|\tau| \leq \varepsilon$ satisfying $W^{\mathrm{cs}}_{\varepsilon}(\psi_\tau(x)) \cap W^{\mathrm{cu}}_{\varepsilon}(y) \neq \emptyset$.

In the remainder of this section we consider the quotient space (X, d), the quotient map $\chi \colon T_1M \to X$ and the quotient flow $\psi_t \colon X \to X$ defined in § 4, and we describe the dynamical properties of this flow. The following theorem is the main result of this section.

Theorem 5.1. *The quotient flow $\psi_t \colon X \to X$ has the following properties.*

(i) *The flow is expansive.*

(ii) *For every $\theta \in T_1M$ the centre stable set (centre unstable set) of $\chi(\theta)$ is the quotient of the centre stable set (centre unstable set) of θ with respect to the geodesic flow.*

(iii) *For every $\theta \in T_1M$ the strong stable set (strong unstable set) of $\chi(\theta) \in X$ is the quotient of $\mathscr{F}^{\mathrm{s}}(\theta)$ (of $\mathscr{F}^{\mathrm{u}}(\theta)$).*

(iv) *If $(p, v) \neq (q, -w)$ then the centre stable set of $[(q, w)]$ intersects the centre unstable set of $[(p, v)]$ at a single orbit of ψ_t. In particular, the flow has a local product structure.*

(v) *The flow is topologically transitive.*

(vi) *Each strong stable set (strong unstable set) is dense.*

(vii) *The flow is topologically mixing.*

We shall prove Theorem 5.1 in several steps and complete its proof at the end of this section. In the forthcoming sections we shall give some interesting applications of Theorem 5.1 and continue exploring the regularity of the quotient space X.

Theorem 2.5 asserts many density properties of dynamical objects associated with the geodesic flow of (M, g). Since in the quotient topology an open set is a set whose pre-image under the quotient map is open in T_1M, all such properties are inherited by the quotient flow ψ_t in a straightforward way.

[*] In other references such as, for example [**5**, **44**], they are sometimes called *canonical coordinates*, which we avoid because of its similarity to geometric objects.

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5.1. Expansiveness

Intuitively, expansiveness is a property we should expect since the quotient collapses strips which are the only 'obstructions' to it. We will need the following basic result of the theory of metric spaces.

Lemma 5.2. *Let Y be a smooth manifold which admits a complete metric space structure (Y, D). Suppose that there exists a sequence of compact sets $(K_n)_{n\geq 1}$ such that:*

(i) *for every $n \geq 1$ there exists an open neighbourhood V_n of K_n contained in K_{n+1};*

(ii) $\bigcup_n K_n = Y$.

Then for every $p \in Y$ and every sequence $(x_n)_{n\geq 1}$ of points $x_n \in K_{n+1} \setminus K_n$, we have that $D(p, x_n) \to +\infty$ if $n \to +\infty$.

Proof. Suppose, by contradiction, that there exists $L > 0$ such that $d(p, x_n) \leq L$ for every $n \geq 1$. Since (Y, D) is complete and the closed ball $\operatorname{cl} B(p, L)$ of radius L centred at p is compact, the sequence $(x_n)_n$ has a subsequence $(x_{n_k})_k$ converging to a point $q \in \operatorname{cl} B(p, L)$. By item (ii), there exists $m \in \mathbb{N}$ such that $q \in K_m$. By item (i), $q \in V_m \subset K_{m+1}$, where V_m is an open set. So there is $k_0 > 0$ such that $x_{n_k} \in V_m$ for every $k \geq k_0$. This contradicts the choice of the sequence $(x_{n_k})_k$, since $x_{n_k} \in K_{n_k+1} \setminus K_{n_k}$ for every k and $V_m \subset K_{m+1} \subset K_k$ for every $k \geq m + 1$. □

Now we show that $\bar{X}$ has a sequence of compact sets K_n satisfying the assumptions of Lemma 5.2.

Lemma 5.3. *Given $\bar{\theta} \in T_1\tilde{M}$, consider the family of balls*

$$\operatorname{cl} B_{d_S}(\bar{\theta}, r) := \{\bar{\eta} \in T_1\tilde{M} \colon d_S(\bar{\theta}, \bar{\eta}) \leq r\},$$

where d_S is the induced Sasaki distance. Let $Q > 0$ be the constant provided by Proposition 2.6. Then the sequence of compact sets $(K_n)_{n\geq 1}$ in the space $\bar{X}$ given by

$$K_n := \chi(\operatorname{cl} B_{d_S}(\bar{\theta}, 3Qn))$$

satisfies the hypotheses of Lemma 5.2.

Proof. By Lemma 3.4, the width of a strip is bounded from above by Q. So the width of the equivalence class of every point in $T_1\tilde{M}$ is bounded from above by Q, since flat strips in $\tilde{M}$ are isometric to strips of bi-asymptotic orbits of the geodesic flow of $T_1\tilde{M}$. Given $\bar{\theta} \in T_1\tilde{M}$, let $C_r \subset T_1\tilde{M}$ be the union of all classes of points in $\operatorname{cl} B_{d_S}(\bar{\theta}, r)$. By the triangular inequality, C_r is a compact connected set whose diameter is at most $r + Q$. Thus, since the interior of C_{r+3Q} contains the open ball of radius $r + 3Q$ centred at $\bar{\theta}$, the set C_{r+3Q} contains C_r. Moreover, there exists a cover $\mathcal{U}_r$ of C_r by open sets taken from the family constructed in Lemma 4.7 which is contained in the interior of C_{r+3Q}. Indeed, we can cover any equivalence class with a set in this family which is arbitrarily close to the class.

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Consider the sets

$$K_n := \chi(C_{3Qn}) = \chi(\operatorname{cl} B_{d_S}(\bar{\theta}, 3Qn))$$

for $n \in \mathbb{N}$. Each such set is compact because it is the continuous image of a compact set. Clearly, $\bigcup_n C_{3Qn}$ covers all of $T_1\tilde{M}$ and hence $\bigcup_n K_n$ covers $\bar{X}$. By the choice of the radius $3Qn$, we have that $C_{3Qn} \subset C_{3(n+1)Q}$ for every $n \in \mathbb{N}$, and hence $K_n \subset K_{n+1}$ for every integer n. This yields item (ii) of Lemma 5.2. Moreover, by the previous paragraph, the open neighbourhood $\chi(\mathcal{U}_{3nQ})$ of $\chi(C_{3Qn}) = K_n$ is in the interior of K_{n+1} for each integer n. This implies item (i) of Lemma 5.2. □

Lemma 5.4. *The flow $\bar{\psi}_t$ is expansive.*

Proof. By contradiction, suppose that that the flow is not expansive, that is, that there exist two points $[\bar{\theta}], [\bar{\eta}] \in \bar{X}$ whose orbits have bounded Hausdorff distance, say, bounded by $L > 0$. Up to some reparametrization we can assume that $\bar{d}([\bar{\theta}], [\bar{\eta}]) \leq L$ and that there is some increasing homeomorphism $\rho\colon \mathbb{R} \to \mathbb{R}$ satisfying $\rho(0) = 0$ such that for every $t \in \mathbb{R}$ we have

$$\bar{d}(\bar{\psi}_t([\bar{\theta}]), \bar{\psi}_{\rho(t)}([\bar{\eta}])) \leq L.$$

We need the following intermediate result.

Claim. *There exists $\bar{L} > 0$ such that for every $t \in \mathbb{R}$ we have*

$$d_S(\phi_t(\bar{\theta}), \phi_{\rho(t)}(\bar{\eta})) \leq \bar{L},$$

where d_S is the Sasaki distance.

Proof. Fix some fundamental domain $\mathcal{D}$ of $T_1\tilde{M}$ containing $\bar{\theta}$, and let D denote its diameter. Given $t \in \mathbb{R}$, let $\beta_t\colon T_1\tilde{M} \to T_1\tilde{M}$ be a covering isometry such that $\beta_t(\phi_t(\bar{\theta})) \in \mathcal{D}$. Hence, for every $t \in \mathbb{R}$ we have

$$d_S(\bar{\theta}, \beta_t(\phi_t(\bar{\theta}))) \leq D.$$

As in Lemma 5.3 and its proof, let $C_r \subset T_1\tilde{M}$ be the union of all classes of points in $\operatorname{cl} B_{d_S}(\bar{\theta}, r)$ and let $K_n := \chi(C_{3Qn})$.

By contradiction, suppose that there exists some sequence $(t_n)_n$ such that the infimum of the Sasaki distance d_S between $\phi_{t_n}(\bar{\theta})$ and $\{\phi_\tau(\bar{\eta})\colon \tau \in \mathbb{R}\}$ is attained at $\tau = \tau_n$ satisfying

$$d_S(\phi_{t_n}(\bar{\theta}), \phi_{\tau_n}(\bar{\eta})) \geq 3Qn + D + Q.$$

This would imply that $\beta_{t_n}(\phi_\tau(\bar{\eta})) \notin C_{3Qn}$ for every $\tau \in \mathbb{R}$ by the definition of C_{3Qn}. Taking the quotient we would get

$$(\bar{\chi} \circ \beta_{t_n})(\phi_\tau(\bar{\eta})) \notin K_n$$

for every $\tau \in \mathbb{R}$ and every $n \geq 1$. By Lemma 5.2 we would get that for each $\tau \in \mathbb{R}$

$$\bar{d}\big(\bar{\chi}(\bar{\theta}), (\bar{\chi} \circ \beta_{t_n})(\phi_\tau(\bar{\eta}))\big) \to \infty$$

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as $n \to \infty$. But on the other hand we would have $(\bar{\chi} \circ \beta_{t_n})(\phi_{t_n}(\bar{\theta})) \in \bar{\chi}(\mathcal{D})$ for every $n \geq 1$, which together would imply that for every $\tau \in \mathbb{R}$

$$\bar{d}((\bar{\chi} \circ \beta_{t_n} \circ \phi_{t_n})(\bar{\theta}), (\bar{\chi} \circ \beta_{t_n} \circ \phi_{\tau})(\bar{\eta})) \to \infty$$

if $n \to \infty$. Recall that by Remark 4.9, each β_t induces a deck transformation $\bar{\beta}_t$ which satisfies $\bar{\chi} \circ \beta_t = \bar{\beta}_t \circ \bar{\chi}$. Hence for every $\tau \in \mathbb{R}$

$$\bar{d}((\bar{\beta}_{t_n} \circ \bar{\chi} \circ \phi_{t_n})(\bar{\theta}), (\bar{\beta}_{t_n} \circ \bar{\chi} \circ \phi_{\tau})(\bar{\eta})) = \bar{d}(\bar{\beta}_{t_n}([\phi_{t_n}(\bar{\theta})]), \bar{\beta}_{t_n}([\phi_{\tau}(\bar{\eta})])) \to \infty$$

Since each $\bar{\beta}_{t_n}$ is an isometry with respect to $\bar{d}$, this together with the definition of the quotient flow $\bar{\psi}$ would imply

$$\sup_{\tau \in \mathbb{R}} \bar{d}([\phi_{t_n}(\bar{\theta})], [\phi_{\tau}(\bar{\eta})]) = \sup_{\tau \in \mathbb{R}} \bar{d}(\bar{\psi}_{t_n}([\bar{\theta}]), \bar{\psi}_{\tau}([\bar{\eta}])) \to \infty.$$

But the latter term is bounded from above by L for all n. This contradiction shows that such a sequence $(t_n)_n$ does not exists. This proves the claim. □

Let us consider the orbits of $\bar{\theta}$ and $\bar{\eta}$ under the flow ϕ_t in $T_1\tilde{M}$. Observe that the claim implies that the strips $F(\bar{\theta})$ and $F(\bar{\eta})$ are within a distance of $\bar{L} + \bar{Q}$, where $\bar{Q}$ depends on the constant in Proposition 2.6. Thus, taking the canonical projection from $T_1\tilde{M}$ onto $\tilde{M}$ we get that the strips of the geodesics $\gamma_{\bar{\theta}}$ and $\gamma_{\bar{\eta}}$ are within a distance $\bar{L} + \bar{Q}$ (recall that the canonical projection is a Riemannian submersion). This can only happen when both geodesics are bi-asymptotic and hence are in the same strip. So $F(\bar{\theta}) = F(\bar{\eta})$ and therefore their quotients in $\bar{X}$ coincide.

This implies that the the quotient flow $\bar{\psi}_t$ is expansive. □

Lemma 5.5. *The flow ψ_t is expansive.*

Proof. Note that by Remark 4.9 $\bar{X}$ covers X. Thus, by compactness of X, there exists $r > 0$ such that the covering map restricted to any ball of radius r is a homeomorphism.

Arguing by contradiction, suppose that ψ_t would not be expansive, that is, suppose that there are two points $[\theta], [\eta] \in X$ with distinct orbits and that there exists a homeomorphism $\rho\colon \mathbb{R} \to \mathbb{R}$ satisfying $\rho(0) = 0$ and such that for every $t \in \mathbb{R}$ we have

$$d\big(\psi_t([\theta]), \psi_{\rho(t)}([\eta])\big) \leq r.$$

Then we could lift these orbits to a pair of orbits in $\bar{X}$ which would stay within a distance r from each other. Therefore, by Lemma 5.4, these orbits would coincide. Thus, the orbits of $[\theta]$ and $[\eta]$ would coincide, which gives a contradiction. □

5.2. Invariant sets and heteroclinic relation

We will consider the following invariant sets of the quotient flows $\bar{\psi}_t$ and ψ_t. Recall the definition of the projection map $\bar{\chi}\colon T_1\tilde{M} \to \bar{X}$ in § 4. Given $\theta \in T_1M$ and one of its lifts

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$\bar\theta \in T_1\tilde M$, let

$$\begin{aligned}
\tilde W^*([\bar\theta]) &:= \bar\chi(\tilde{\mathscr{F}}^*(\bar\theta)), \qquad & W^*([\theta]) &:= \chi(\mathscr{F}^*(\theta)), \qquad & * = \mathrm{cs}, \mathrm{cu},\\
\tilde W^{\mathrm{ss}}([\bar\theta]) &:= \bar\chi(\tilde{\mathscr{F}}^{\mathrm{s}}(\bar\theta)), \qquad & W^{\mathrm{ss}}([\theta]) &:= \chi(\mathscr{F}^{\mathrm{s}}(\theta)),\\
\tilde W^{\mathrm{uu}}([\bar\theta]) &:= \bar\chi(\tilde{\mathscr{F}}^{\mathrm{u}}(\bar\theta)), \qquad & W^{\mathrm{uu}}([\theta]) &:= \chi(\mathscr{F}^{\mathrm{u}}(\theta)).
\end{aligned}$$

In this way, X is the union of the sets $W^{\mathrm{cs}}([\theta])$ and $[\theta] \in X$, which strongly indicates that X should be foliated by the above collection of sets. Analogously, X is the union of the sets $W^*([\theta])$ and $[\theta] \in X$, for $* = \mathrm{ss}, \mathrm{cu}$ and uu, respectively. In this section we will study the dynamical properties of these sets and establish that indeed each set $W^{\mathrm{ss}}([\theta])$ is a strong stable set of the quotient flow as defined above, justifying our notation.

Lemma 5.6. *Given $D > 0$ there exists $D' > 0$ such that for every $\bar\theta \in T_1\tilde M$, and every pair of points $[\bar\tau], [\bar\eta] \in \tilde{\mathscr{F}}^{\mathrm{s}}(\bar\theta)$ such that $\bar d([\bar\tau], [\bar\eta]) \le D$, we have that*

$$\bar d(\bar\psi_t([\bar\tau]), \bar\psi_t([\bar\eta])) \le D'$$

for every $t \ge 0$.

Proof. Given $\bar\theta \in T_1\bar M$, take $\bar\tau, \bar\eta \in \tilde{\mathscr{F}}^{\mathrm{s}}(\bar\theta)$ as in the hypothesis and note that the distance between their orbits is non-increasing, since we consider a surface (M, g) without focal points.

Claim. *If $K \subset \bar X$ is compact then $\bar\chi^{-1}(K) \subset T_1\tilde M$ is compact.*

Proof. We know that $\bar\chi^{-1}(K)$ is closed by continuity of $\bar\chi$. By Lemma 4.7 the sets $\bar\chi(A)$ for $A = A_{\bar\theta}(\tau, \varepsilon, \delta, \bar\theta^-, \bar\theta^+, \bar\eta^-, \bar\eta^+)$ as in this lemma form a basis for the quotient topology. Here each set of the basis satisfies

$$\bar\chi^{-1}(\bar\chi(A)) = A.$$

Each set A is an open subset with compact closure. Since K is compact, it can be covered by a finite collection $\{\bar\chi(A_i)\}_{i=1,\dots,m}$ of sets in this basis. By Lemma 4.7, this implies that $\bar\chi^{-1}(K)$ is covered by the finite collection A_i of open sets. Since the union of the closures of the sets A_i is a compact set, $\bar\chi^{-1}(K)$ is a closed subset of a compact set. Thus $\bar\chi^{-1}(K)$ is compact. □

Given $D > 0$, let $L = L(D) > 0$ be such that for every pair of points $\bar\theta_1, \bar\theta_2 \in T_1\tilde M$ with $\bar d([\bar\theta_1], [\bar\theta_2]) \le D$ we have $d_S(\bar\theta_1, \bar\theta_2) \le L$. Indeed, such a constant L exists by the above claim, since the pre-image by $\bar\chi$ of the closed ball of radius D centred at $[\bar\theta_1]$ is a compact subset of $T_1\tilde M$.

So for $D := \bar d([\bar\tau], [\bar\eta])$ we have $d_S(\bar\tau, \bar\eta) \le L$ and hence

$$d_S(\phi_t(\bar\tau), \phi_t(\bar\eta)) \le L$$

for every $t \ge 0$. By co-compactness of $T_1\tilde M$, given a fundamental domain $\mathcal{D}$, there exist representatives $\beta_t(\phi_t(\bar\tau)), \beta_t(\phi_t(\bar\eta))$ of $\bar\tau, \bar\eta$ by isometries β_t of the fundamental group of

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T_1M such that

- $\beta_t(\phi_t(\bar{\tau})) \in \mathcal{D}$ for each $t > 0$;
- $d_S(\bar{\tau}, \bar{\eta}) = d_S(\beta_t(\phi_t(\bar{\tau})), \beta_t(\phi_t(\bar{\eta})))$ for every $t > 0$.

Since the image by $\bar{\chi}$ of a compact set is compact, there exists a constant $D' = D'(D)$ such that

$$\bar{d}([\beta_t(\phi_t(\bar{\tau}))], [\beta_t(\phi_t(\bar{\eta}))]) \le D'.$$

But by definition, for each covering isometry β acting on $T_1\tilde{M}$ we have an induced isometry $\bar{\beta}$ of $\bar{d}$ which acts as

$$\bar{\beta}([\bar{\theta}]) = (\bar{\beta} \circ \bar{\chi})(\bar{\eta}) = (\bar{\chi} \circ \beta)(\bar{\eta}) = [\beta(\bar{\theta})].$$

So we get

$$\bar{d}([\beta_t(\phi_t(\bar{\tau}))], [\beta_t(\phi_t(\bar{\eta}))]) = \bar{d}(\bar{\psi}_t([\bar{\tau}]), \bar{\psi}_t([\bar{\eta}])) \le D'$$

for every $t \ge 0$, concluding the proof. □

Lemma 5.7. *The quotient flow $\bar{\psi}_t$ is uniformly contracting on the stable sets $\tilde{W}^{\mathrm{ss}}([\bar{\theta}])$ in the following sense: for every $D, \varepsilon > 0$, there exists $t_0 = t_0(D, \varepsilon) > 0$ such that for every $\bar{\theta} \in T_1\tilde{M}$ and every two points $[\bar{\eta}], [\bar{\xi}] \in \tilde{W}^{\mathrm{ss}}([\bar{\theta}])$ satisfying $\bar{d}([\bar{\eta}], [\bar{\xi}]) \le D$ we have*

$$\bar{d}\big(\bar{\psi}_t([\bar{\xi}]), \bar{\psi}_t([\bar{\eta}])\big) \le \varepsilon$$

for every $t \ge t_0$.

Proof. The proof follows from a standard argument of expansive dynamics which we only sketch. By contradiction, suppose that there exist numbers $D, \varepsilon > 0$ and sequences $([\bar{\eta}_n])_n, ([\bar{\xi}_n])_n \subset \tilde{W}^{\mathrm{ss}}([\bar{\theta}_n])$ such that for every $n \ge 1$

$$\bar{d}([\bar{\eta}_n], [\bar{\xi}_n]) \le D \quad \text{and} \quad \bar{d}(\bar{\psi}_n([\bar{\xi}_n]), \bar{\psi}_n([\bar{\eta}_n])) \ge \varepsilon. \tag{4}$$

By Lemma 5.6 there exists $D' > 0$ such that $\bar{d}(\bar{\psi}_t([\bar{\eta}_n]), \bar{\psi}_t([\bar{\xi}_n])) \le D'$ for every $t \ge 0$.

By co-compactness, there is a fundamental domain $\mathcal{D}$ and we can find a subsequence $(n_k)_k$ and covering isometries $\beta_{n_k} \colon T_1\tilde{M} \to T_1\tilde{M}$ such that for every $k \ge 1$ we have

$$\hat{\eta}_{n_k} := \beta_{n_k}(\bar{\eta}_{n_k}) \in \mathcal{D}, \qquad \hat{\xi}_{n_k} := \beta_{n_k}(\bar{\xi}_{n_k}) \in \mathcal{D},$$

such that these subsequences converge

$$\hat{\eta}_{n_k} \to \bar{\eta}_\infty, \qquad \hat{\xi}_{n_k} \to \bar{\xi}_\infty$$

and

$$\bar{d}(\bar{\psi}_t(\hat{\eta}_{n_k}), \bar{\psi}_t(\hat{\xi}_{n_k})) \le D'$$

for every $t \ge -n_k$. It follows from our assumption that $\bar{\eta}_\infty$ and $\bar{\xi}_\infty$ are distinct points whose orbits are bi-asymptotic. On the other hand, by Lemma 5.5 the flow $\bar{\psi}_t$ is expansive, so these two orbits coincide. But this contradicts the second property of the sequences $[\bar{\xi}_n], [\bar{\eta}_n]$ in (4). This proves the lemma. □

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Another important property of the invariant sets of the quotient flows is the following.

Lemma 5.8. *For each* $* = \mathrm{cs}, \mathrm{ss}, \mathrm{cu}, \mathrm{uu}$, *the set* $\bar{W}^*([\bar{\theta}])$ *varies continuously in* $[\bar{\theta}] \in \bar{X}$, *uniformly on compact sets, with respect to the Hausdorff topology.*

Proof. Recall that, by Theorem 2.4, each family of invariant sets constitutes a continuous foliation by C^1 leaves. So they vary continuously on compact sets with respect to the Hausdorff topology. In fact, in $T_1\tilde{M}$ the invariant foliations are continuous with respect to the C^1 compact open topology, which is stronger than continuity with respect to the Hausdorff topology. The quotient preserves continuity properties, so the same holds for the quotient invariant sets. □

5.3. Further properties

Lemma 5.9. *Given* $\bar{\theta} = (p, v) \in T_1\tilde{M}$, *for every* $\bar{\eta} = (q, w) \in T_1\tilde{M}$ *such that* $(q, -w) \neq (p, v)$, *the set* $\bar{W}^{\mathrm{cs}}([\bar{\eta}]) \cap \bar{W}^{\mathrm{cu}}([\bar{\theta}])$ *consists of a single orbit of* $\bar{\psi}_t$.

Proof. By Theorem 2.5(iii), the intersection $\tilde{\mathscr{F}}^{\mathrm{cs}}(\bar{\eta}) \cap \tilde{\mathscr{F}}^{\mathrm{cu}}(\bar{\theta})$ is non-empty and consists of a strip of bi-asymptotic orbits. The quotient map preserves this intersection. Observing that each strip becomes a single orbit in the quotient space, we deduce the statement. □

Proof of Theorem 5.1. Lemma 5.5 shows item (i). The results in § 5.2 imply items (ii) and (iii). Lemma 5.9 proves item (iv).

Items (v)–(vi) remain to be shown. We will sketch their proofs using the basis of sets $\bar{\chi}(A_{(\cdot)}(\cdot))$ constructed in Lemma 4.7 and their projection to X. As observed at the end of § 4, the projection $\bar{\Pi}\colon \bar{X} \to X$ defines a local homeomorphism and hence the basis $\bar{\chi}(A_{(\cdot)}(\cdot))$ naturally projects to a basis in X.

Item (v) claims the transitivity of the quotient flow. This follows from the transitivity of the geodesic flow of (M, g) (Theorem 2.5(ii)) and the fact that the family of sets $A_{\bar{\theta}}(\cdot)$ established in Lemma 4.7 is a family of open neighbourhoods of sections of strips $\mathcal{I}(\bar{\theta}) \subset T_1\tilde{M}$. Indeed, each dense orbit must intersect any open set, and hence each dense orbit intersects any set A from the basis specified in Lemma 4.7. Hence, the quotient of a dense orbit must intersect any open set $\bar{\chi}(A)$. This proves the density of the quotient of any dense orbit in the quotient topology.

Item (vi) claims the minimality of the quotient of each strong stable set $W^{\mathrm{ss}}([\theta])$ and each strong unstable set $W^{\mathrm{uu}}([\theta])$. By Theorem 2.5(i), the horocycle foliations of T_1M are minimal. The argument applied in the previous paragraph shows that the quotient of any dense set of T_1M is dense in the quotient X.

Item (vii) claims that the quotient flow ψ_t is topologically mixing. This follows from Theorem 2.5(ii) and the fact that the sets from the basis specified in Lemma 4.7 form a family of open neighbourhoods in $T_1\tilde{M}$. Hence, the mixing property is verified in particular in these sets. Taking the quotient, this property is also verified by the quotient flow.

This finishes the proof. □

Proof of Theorem A. The theorem is a consequence of Proposition 4.8 and Theorem 5.1. □

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6. Measures of maximal entropy

In this setting we study the entropy of a flow. We first establish some general results and finally prove Theorem B.

6.1. Expansive flows with local product structure

In this subsection we start by considering a general continuous flow $\psi_t\colon X \to X$ without singular points on a compact metric space X. Of course, we have in mind the quotient flow defined in § 4.

Given positive numbers a and δ we call a pair of sequences $(x_k)_{k=k_0}^{k_1}$ of points $x_k \in X$ and numbers $(\tau_k)_{k=k_0}^{k_1}$ ($k_0 = -\infty$ or $k_1 = \infty$ are permitted) a (δ, a)-*pseudo orbit* for the flow $(\psi_t)_t$ if for every $k = k_0, \dots, k_1 - 1$ we have $\tau_k \geq a$ and $d(\psi_{\tau_k}(x_k), x_{k+1}) < \delta$. Denote $s_0 = 0, s_k = \tau_0 + \cdots + \tau_{k-1}$, and $s_{-k} = \tau_{-k} + \cdots + \tau_{-1}$. A (δ, a)-pseudo orbit $(x_k, \tau_k)_k$ is *ε-traced* by a true orbit $(\psi_t(y))_{t\in\mathbb{R}}$ if there is some increasing homeomorphism $\alpha\colon \mathbb{R} \to \mathbb{R}$ satisfying $\alpha(0) = 0$, such that for every $k = 0, 1, \dots$ for every $t \geq 0$ satisfying $s_k \leq t < s_{k+1}$ we have

$$d(\psi_{\alpha(t)}(y), \psi_{t-s_k}(x_k)) \leq \varepsilon$$

and for every $k = 1, 2, \dots$ and every $t \leq 0$ satisfying $-s_{-k} \leq t \leq -s_{-k+1}$

$$d(\psi_{\alpha(t)}(y), \psi_{t+s_{-k}}(x_{-k})) \leq \varepsilon.$$

The flow $(\psi_t)_t$ is said to have the *pseudo orbit tracing property with respect to time $a > 0$* if for every $\varepsilon > 0$ there is $\delta_0 > 0$ such that every $\delta \in (0, \delta_0)$ for every (δ, a)-pseudo orbit is ε-traced by a true orbit of $(\psi_t)_t$. For $a = 1$ we simply speak of the *pseudo orbit tracing property*.

Recall the definition of local product structure in § 5.

Proposition 6.1 (see [44, Theorem 7.1]). *Every continuous expansive flow without singular points on a compact metric space which has local product structure has the pseudo orbit tracing property.*

We say that the flow $(\psi_t)_t$ has the *periodic orbit specification property*[*] if for every $\varepsilon > 0$ there is a positive number $T = T(\varepsilon)$ such that for any integer $n \geq 2$, any collection of points $x_0, \dots, x_n \in X$, and any sequence of real numbers $t_0 < t_1 < \cdots < t_{n+1}$ satisfying $t_{k+1} - t_k \geq T$ for every $k = 0, \dots, n$, there is a sequence of numbers $r_0, \dots, r_{n+1}$ having the following properties: $r_0 = 0$ and for every $k = 0, \dots, n$

(1) r_{k+1} is determined by $x_0, \dots, x_{k+1}$ and by $t_0, \dots, t_{k+1}$;

(2) $|r_{k+1} - r_k| < \varepsilon$;

[*] We follow the definition in [35] which corrects some results on the uniqueness of an equilibrium state (e.g. measure of maximal entropy) in [20] under stronger hypotheses. Observe that condition (1) is slightly stronger than in usual definitions (for example in [20]). It requires that for the existence of the periodic point in any intermediate construction step of solely considering the subset of points $\{x_0, \dots, x_{k+1}\} \subset \{x_0, \dots, x_n\}$ the numbers r_k are, in fact, defined recursively.

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(3) there is a periodic point $y \in X$ with period τ satisfying

$$|\tau - (t_{k+1} - t_0)| \leq (k+1)\varepsilon$$

and satisfying for every $\ell = 0, \dots, k$ and for every $t \in [t_\ell, t_{\ell+1} - T]$

$$d(\psi_{t+r_\ell}(y), \psi_{t-t_\ell}(x_\ell)) < \varepsilon.$$

Proposition 6.2. *Every continuous expansive topologically mixing flow without singular points on a compact metric space having the pseudo orbit tracing property has the periodic orbit specification property.*

We will need the following auxiliary result (see [44, Proposition 3.2]).

Lemma 6.3. *Let $(\psi_t)_t$ be a continuous expansive flow without singular points on a compact metric space which has local product structure. For every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for every $x, y \in X$, every interval $[T_1, T_2]$ containing 0, and every increasing homeomorphism $\alpha\colon \mathbb{R} \to \mathbb{R}$ satisfying $\alpha(0) = 0$ and $d(\psi_{\alpha(t)}(x), \psi_t(y)) \leq \delta$ for every $t \in [T_1, T_2]$ we have*

$$|\alpha(t) - t| < \varepsilon.$$

Proof of Proposition 6.2. We follow a standard idea of proof (see, e.g. [30]). Let $\varepsilon > 0$. Let $\varepsilon_0 > 0$ be an expansivity constant. Let $\varepsilon_2 := \min\{\varepsilon, \varepsilon_0/2\}$. Let $\delta(\varepsilon)$ be as in Lemma 6.3. Let $\delta_0 = \delta_0(\min\{\delta(\varepsilon), \varepsilon_0/2\})$ be the number given by the pseudo orbit tracing property. Let $\delta \in (0, \min\{\delta_0, \varepsilon_2/2\})$.

Take a covering of X by finitely many open sets U_i satisfying $\operatorname{diam} U_i \leq \delta$. Since the flow is topologically mixing, there exists $T > 0$ such that for every index pair i, j we have

$$\psi_t(U_i) \cap U_j \neq \emptyset \quad \text{for every } |t| \geq T/2. \tag{5}$$

Let $x_0, \dots, x_n \in X$ be a sequence of points and $t_0 < t_1 < \cdots < t_{n+1}$ a sequence of numbers satisfying $t_{k+1} - t_k \geq T$ for every $k = 0, \dots, n$. For every $k = 0, \dots, n$ let

$$y_k := \psi_{t_{k+1} - t_k - T/2}(x_k).$$

For every $k = 0, \dots, n$ there are indices j_k and i_k such that

$$x_k \in U_{j_k}, \qquad y_k \in U_{i_k}.$$

It follows from (5) that for every $k = 0, \dots, n-1$ there is $z_k \in U_{i_k} \subset B(y_k, \delta)$ such that $\psi_{T/2}(z_k) \in U_{j_{k+1}}$, and that for every $k = 0, \dots, n$ there is $w_k \in U_{i_k}$ such that $\psi_{T/2}(w_k) \in U_{j_0}$.

By these choices, for every $k = 1, \dots, n$, consider the pair of finite sequences of points $(\hat{x}_0, \dots, \hat{x}_{2k+3}) := (x_0, z_0, x_1, z_1, \dots x_{k-1}, z_{k-1}, x_k, w_k, x_0)$ and of numbers

$$(\hat{\tau}_1, \dots, \hat{\tau}_{2k+2}) := \left(t_1 - t_0 - \frac{T}{2}, \frac{T}{2}, t_2 - t_1 - \frac{T}{2}, \frac{T}{2}, \dots, t_{k+1} - t_k - \frac{T}{2}, \frac{T}{2}\right)$$

that we concatenate infinitely many times and thus define a periodic $(\delta, T/2)$-pseudo orbit. By our assumption, this pseudo orbit is $\delta(\varepsilon)$-traced by a true orbit $(\psi_t(y))_t$, that

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is, there is a point y and some increasing homeomorphism $\alpha\colon \mathbb{R} \to \mathbb{R}$ satisfying $\alpha(0) = 0$ such that for every k and for every t satisfying $s_k \le t < s_{k+1}$,

$$d(\psi_{\alpha(t)}(y), \psi_t(\hat{x}_k)) \le \min\{\delta(\varepsilon), \varepsilon_0/2\}.$$

Since the flow is expansive, by the choice of ε_0 this bi-infinite tracing orbit $(\psi_t(y))_t$ is uniquely determined. As the pseudo orbit is periodic, the tracing orbit must be closed, that is, we have $\psi_\tau(y) = y$ for some $\tau > 0$. By Lemma 6.3 and the choice of $\delta(\varepsilon)$, as the shadowing orbit $(\psi_{\alpha(t)}(y))_t$ is close to pieces of orbits $(\psi_t(x_k))_t$, the period τ of the tracing periodic orbit must satisfy property (3) in the definition of the periodic orbit specification property. This finishes the proof. □

Propositions 6.1 and 6.2 together imply the following.

Corollary 6.4. *If a continuous expansive topologically mixing flow without singular points on a compact metric space has local product structure, then the flow has the periodic orbit specification property.*

6.2. Entropy

A Borel probability measure is said to be *invariant* under the flow $\psi_t\colon X \to X$ if it is ψ_t-invariant for every $t \in \mathbb{R}$. Let $\mathcal{M}$ denote the set of all flow-invariant Borel probability measures. A set $Z \subset X$ is *invariant* under the flow if $\psi_t(Z) = Z$ for every $t \in \mathbb{R}$. A measure $\nu \in \mathcal{M}$ is said to be *ergodic* if for every invariant set $Z \subset X$ we have either $\nu(Z) = 0$ or $\nu(Z) = 1$.

Given $\nu \in \mathcal{M}$, we denote by $h_\nu(\psi_t)$ the *metric entropy* with respect to the time-t map ψ_t (see [45] for its definition). By Abramov's formula [1], we have

$$h_\nu(\psi_t) = |t| h_\nu(\psi_1) \text{ for every } t.$$

One calls $h_\nu(\psi) := h_\nu(\psi_1)$ the *metric entropy* of ν with respect to the flow $\psi = (\psi_t)_t$.

Given $\varepsilon > 0$, $T > 0$ and $x \in X$, define

$$B(x, \varepsilon, T) := \{y \in X\colon d(\psi_s(y), \psi_s(x)) < \varepsilon \text{ for every } s \in [0, T]\}.$$

Two points x, y are called (T, ε)-*separated* if $y \notin B(x, \varepsilon, T)$. A set $E \subset X$ is (T, ε)-*separated* if every pair of distinct elements in E has this property. Given an invariant compact set $Z \subset X$, denote by $M(T, \varepsilon, Z, \psi)$ the maximal cardinality of any (T, ε)-separated subset of Z. The *topological entropy* of Z with respect to the flow ψ is defined by

$$h(\psi, Z) := \lim_{\varepsilon \to 0} \limsup_{T \to \infty} \frac{1}{T} \log M(T, \varepsilon, Z, \psi). \tag{6}$$

We simply write $h(\psi) = h(\psi, X)$ and call this number the *topological entropy* of the flow ψ. One can also verify Abramov's formula in the case of topological entropy

$$h(\psi_1, Z) = \frac{1}{|t|} h(\psi_t, Z) \text{ for every } t$$

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(see [27] or [4, Proposition 21]) and, together with the variational principle (see [45]), denoting by $\mathcal{M}(\psi_1, Z)$ the set of all ψ_1-invariant measures ν for which $\nu(Z) = 1$, one has

$$h(\psi, Z) = \sup_{\nu \in \mathcal{M}(\psi_1, Z)} h_\nu(\psi_1) = h(\psi_1, Z). \tag{7}$$

In the following we will only work with the entropy of the time-1 map ψ_1. A set of points E is called (n, ε)-*spanning* with respect to ψ_1 if $x, y \in E, x \neq y$, implies $d(\psi_k(x), \psi_k(y)) > \varepsilon$ for some $k \in \{0, \dots, n-1\}$. Given $Z \subset X$ compact, let $N(n, \varepsilon, Z, \psi)$ be the maximal cardinality of a (n, ε)-spanning set $\{x_i\}_i \subset Z$. Recall that

$$h(\psi_1, Z) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(n, \varepsilon, Z, \psi). \tag{8}$$

Consider a continuous flow $\phi_t \colon Y \to Y$ acting on a compact metric space Y being time-preserving semi-conjugate to the continuous flow $\psi_t \colon X \to X$ via $\chi \colon Y \to X$ by $\chi \circ \phi_t = \psi_t \circ \chi$ (ψ_t is also called a *factor* of ϕ_t and ϕ_t is called an *extension* of ψ_t; of course, we have in mind the quotient map introduced in § 4). For $y \in Y$ consider the equivalence class

$$[y] := \{z \in Y \colon \chi(z) = \chi(y)\} = \chi^{-1}(\chi(y)).$$

Note that $[y]$ is compact for every y. Recall that, given an ϕ-invariant compact set $A \subset Y$ (and hence ψ-invariant compact set $\chi(A) \subset X$), by the common property of a factor and by [4, Theorem 17] we have

$$h(\psi_1, \chi(A)) \leq h(\phi_1, A) \leq h(\psi_1, \chi(A)) + \sup_{x \in \chi(A)} h(\phi_1, \chi^{-1}(x)). \tag{9}$$

6.3. Maximal entropy measures

An ergodic measure ν is a *measure of maximal entropy* if it realizes the supremum in (7). Notice that such a measure always exists provided the entropy map $\nu \mapsto h_\nu(\psi_1)$ is upper semi-continuous [45]. This is the case if the flow is smooth [34] or if the map ψ_1 is h-expansive [45]. For any axiom A flow, the measure of maximal entropy is unique [6]. In general, the continuous flows we want to study are not hyperbolic, so we have to rely on a more general result due to Franco [20] (see also Oka [35]).

In what follows let $\psi_t \colon X \to X$ be a continuous expansive flow on a compact metric space X satisfying the periodic orbit specification property. The measure of maximal entropy can be constructed explicitly from the distribution of periodic orbits as follows [5]. Let $\varepsilon > 0$ be an expansivity constant. For every T there is only a finite number of periodic orbits for ψ with minimal period between $T - \varepsilon$ and $T + \varepsilon$; denote this family by $\mathrm{Per}(\psi, T - \varepsilon, T + \varepsilon)$. Denote by $\mathrm{Per}(\psi, T)$ the family of periodic orbits with minimal period less than or equal to T. Denote by $\#$ the cardinality. Consider

$$\hat{\nu}_{\psi,T} := \frac{\sum_\gamma \nu_{\psi,\gamma}}{\#\,\mathrm{Per}(\psi, T - \varepsilon, T + \varepsilon)}$$

with summation taken over all $\gamma \in \mathrm{Per}(\psi, T - \varepsilon, T + \varepsilon)$, and $\nu_{\psi,\gamma}$ denoting the invariant probability measure supported on γ. Note that

$$\#\,\mathrm{Per}(\psi, T - \varepsilon, T + \varepsilon) \leq M(T, \varepsilon, X, \psi) \tag{10}$$

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(compare the proof of [8, Theorem 5]). We also consider

$$\nu_{\psi,T} := \frac{\sum_\gamma \nu_{\psi,\gamma}}{\#\operatorname{Per}(\psi,T)}$$

with summation taken over all $\gamma \in \operatorname{Per}(\psi,T)$.

Proposition 6.5 (Franco [20]). *Let $\psi_t\colon X \to X$ be a continuous expansive flow on a compact metric space satisfying the periodic orbit specification property.*

Then $\hat{\nu}_{\psi,T}$ and $\nu_{\psi,T}$ both converge in the weak$*$ topology to the unique measure of maximal entropy ν_ψ as $T \to \infty$. In particular, for $\varepsilon > 0$ sufficiently small, we have

$$\lim_{T\to\infty} \frac{1}{T} \log \#\operatorname{Per}(\psi, T-\varepsilon, T+\varepsilon) = \lim_{T\to\infty} \frac{1}{T} \log \#\operatorname{Per}(\psi,T) = h(\psi) > 0.$$

Proposition 6.5 and Corollary 6.4 hence imply the following.

Corollary 6.6. *If a continuous expansive topologically mixing flow without singular points on a compact metric space has local product structure, then this flow has a unique (hence ergodic) measure of maximal entropy and $\hat{\nu}_{\psi,T}$ and $\nu_{\psi,T}$ both converge in the weak$*$ topology to this measure as $T \to \infty$.*

Consider a continuous flow $\phi_t\colon Y \to Y$ acting on a compact metric space Y and being time-preserving semi-conjugate to the continuous flow $\psi_t\colon X \to X$ via a semi-conjugacy $\chi\colon Y \to X$. If x is a periodic point for ψ with period ℓ and $\gamma = \{\psi_t(x)\colon 0 \le t < \ell\}$ is its orbit, then $[\beta] = \chi^{-1}(\gamma)$ is compact and ϕ-invariant and has period ℓ. Hence we can pick a probability measure $\mu_{\phi,[\beta]}$ supported on $[\beta]$ which is invariant with respect to the flow $\phi = (\phi_t)_t$. We define

$$\mu_{\phi,T} := \frac{\sum_{[\beta]} \mu_{\phi,[\beta]}}{\#\operatorname{Per}(\phi,T)},$$

with summation taken over all $[\beta] \in \operatorname{Per}(\phi,T)$. The probability measure $\mu_{\phi,T}$ is ϕ-invariant and, in particular, ϕ_1-invariant.

We can now formulate [11, Theorem 1.5] in our setting.

Theorem 6.7. *Let $\psi_t\colon X \to X$ be a continuous expansive flow on a compact metric space satisfying the periodic orbit specification property and denote by ν_ψ its unique (hence ergodic) Borel probability measure of maximal entropy. Let $\phi_t\colon Y \to Y$ be a continuous flow time-preserving semi-conjugate to ψ_t through some continuous surjective map $\chi\colon Y \to X$ and assume that the following conditions are satisfied:*

(i) *$h(\phi_1, [y]) = 0$ for every $y \in Y$;*

(ii) *$\nu_\psi(\{\chi(y)\colon [y] = \{y\}\}) = 1$.*

Then $\mu_{\phi,T}$ converges in the weak$*$ topology as $T \to \infty$ to the unique ergodic Borel probability measure of maximal entropy for $\phi = (\phi_t)_t$.

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6.4. Lyapunov exponents

We continue to assume that (M, g) is a compact surface without focal points. Lyapunov exponents and associated vector bundles provided by the Oseledec multiplicative ergodic theorem are in this setting conveniently described by orthogonal stable and unstable Jacobi fields (see [21] for more details). A Jacobi field J along a geodesic γ_θ, $\theta \in TM$, with $\gamma'_\theta(0) = \theta$ is uniquely determined by its initial conditions $J(0), J'(0) \in T_pM$. The set of *orthogonal Jacobi fields* is the one such that $J(0)$ and $J'(0)$ are orthogonal to v and are exactly those such that for every $t \in \mathbb{R}$, $J(t)$ is normal to $\gamma'_\theta(t)$. Given a vector $\xi \in T_\theta T_1 M$, we denote by J_ξ the Jacobi field such that $J_\xi(0)$ and $J'_\xi(0)$ are the horizontal and the vertical component of ξ, respectively. Recall that with this notation the linearized geodesic flow acts such that for every $t \in \mathbb{R}$, $J_\xi(t)$ and $J'_\xi(t)$ are the horizontal and the vertical component of $D\phi_t(\xi)$, respectively. An (orthogonal) Jacobi field J is called *stable* (respectively, *unstable*) if $\|J(t)\|$ is uniformly bounded for all $t \geq 0$ (respectively, uniformly bounded for all $t \leq 0$). We denote by J^{s} (respectively, J^{u}) the set of stable (respectively, unstable) orthogonal Jacobi fields and consider the subspaces

$$E^{\mathrm{s}}_\theta := \{\xi \in T_\theta T_1 M \colon J_\xi \in J^{\mathrm{s}}\}, \qquad E^{\mathrm{u}}_\theta := \{\xi \in T_\theta T_1 M \colon J_\xi \in J^{\mathrm{u}}\}.$$

In our setting, each such subspace is one-dimensional. The distributions $E^{\mathrm{s}} \colon \theta \in T_1M \mapsto E^{\mathrm{s}}_\theta \in T_\theta T_1 M$ and $E^{\mathrm{u}} \colon \theta \in T_1M \mapsto E^{\mathrm{u}}_\theta \in T_\theta T_1 M$ are invariant under the linearized geodesic flow and continuous. The bundles E^{s}_θ and E^{u}_θ may not be linearly independent. They generate the space of orthogonal Jacobi fields along a geodesic if, and only if, the curvature does not vanish everywhere along this geodesic. In fact, the geodesic flow is an Anosov flow precisely if, and only if, these bundles are linearly independent at *every* point of the unit tangent bundle [16].

We denote by $\mathcal{R}$ the subset of vectors $\theta \in T_1M$ such that E^{s}_θ and E^{u}_θ are linearly independent; this set is open and invariant under $(\phi_t)_t$ (a vector in $\mathcal{R}$ is also said to have *rank* 1). The complementary set $\mathcal{H} \subset T_1M$ is closed and invariant. Based on the quotient map χ we introduced in § 4, for the corresponding equivalence classes we have

$$\mathcal{R} \subset \mathcal{R}_0 := \{\theta \in T_1M \colon [\theta] = \{\theta\}\}. \tag{11}$$

Let $\lambda(\theta)$ denote the *Lyapunov exponent* at θ associated with the linearization of the geodesic flow and its action on the subbundle E^{u}

$$\lambda(\theta) := \lim_{t \to \pm\infty} \frac{1}{t} \log \|D\phi_t|_{E^{\mathrm{u}}_\theta}\|,$$

whenever both limits exist and are equal, where $\|\cdot\|$ denotes the norm induced by the Sasaki metric on TM. The Lyapunov exponent $\lambda(\theta)$ is well defined for any so-called *Lyapunov regular* vector θ. By the above facts, in our setting, $\lambda(\theta)$ can in fact be computed as the Birkhoff average of a continuous function which is the instantaneous rate of expansion caused by the derivative of the geodesic flow acting on the subspace E^{u}_θ. It vanishes on $\mathcal{H}$ because unstable Jacobi fields are covariantly constant along geodesics tangent to vectors in $\mathcal{H}$.

The set of Lyapunov regular vectors is of full measure with respect to any invariant Borel probability measure. It coincides at almost every point with the non-negative Lyapunov exponent provided by the Oseledec decomposition. Ruelle's inequality [40] asserts

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that for all ϕ-invariant probability measures μ we have

$$h_\mu(\phi_1) \le \lambda(\mu) := \int_{T_1M} \lambda(\theta)\,\mathrm{d}\mu(\theta). \tag{12}$$

Let $\widetilde{m}$ be the Liouville measure m restricted to $\mathcal{R}$ and normalized to obtain a probability measure. The ergodicity of $\widetilde{m}$ was proved in [**38**]. In all known examples, $\widetilde{m}$ coincides with the Liouville measure, but this has not been proved in general. Since $\widetilde{m}$ is absolutely continuous and has positive density throughout $\mathcal{R}$ we have $h_{\widetilde{m}}(\phi_1) = \lambda(\widetilde{m}) > 0$, by Pesin's formula [**39**]. Thus, the variational principle for topological entropy (7) implies

$$h(\phi_1) \ge h_{\widetilde{m}}(\phi_1) = \lambda(\widetilde{m}) > 0.$$

The following is an immediate consequence of Ruelle's inequality (12) and the variational principle for entropy (7) (recalling that $\mathcal{H}$ is closed and invariant).

Lemma 6.8. *We have $h_\mu(\phi_1) = 0$ for every invariant probability measure μ satisfying $\mu(\mathcal{H}) = 1$. We have $h(\phi_1, \mathcal{H}) = 0$.*

6.5. Small entropy on strips

We now return to our setting in the rest of the paper. In this section let $\phi_t\colon T_1M \to T_1M$ be the geodesic flow of a C^∞ compact connected boundaryless surface (M, g) without focal points and of genus greater than one. By Proposition 4.2, $\phi = (\phi_t)_t$ is time-preserving semi-conjugate to the quotient flow $\psi = (\psi_t)_t$ from Definition 4.1 through the quotient map $\chi\colon T_1M \to X$. We will need the following lemma.

Lemma 6.9. *$h(\phi_1, [\theta]) = 0$ for any $\theta \in T_1M$.*

Proof. The result is trivial for every θ for which $[\theta] = \{\theta\}$.

Assume that $[\theta]$ contains a point $\eta = (q, w) \neq \theta$. Then $\chi(\eta) = \chi(\theta)$ and by Definition 4.1 we have $\eta \in \mathscr{F}^{\mathrm{s}}(\theta)$. If $\tilde{\mathscr{F}}^{\mathrm{ss}}(\bar{\theta})$ is any lift of $\mathscr{F}^{\mathrm{s}}(\theta)$ in $T_1\tilde{M}$ and $\bar{\eta} \in \tilde{\mathscr{F}}^{\mathrm{ss}}(\bar{\theta})$ a lift of η then the geodesics $\gamma_{\bar{\theta}}, \gamma_{\bar{\eta}}$ are bi-asymptotic and by Theorem 3.2 bound a non-trivial strip in $\tilde{M}$ and stay in constant distance for every $t \in \mathbb{R}$. Hence, for every $\delta > 0$ a $(1, \delta)$-spanning set of $[\theta]$ is (n, δ)-spanning for every n. Hence, in particular, $N(n, \delta, [\theta], \phi) \le N(1, \delta, [\theta], \phi)$ and thus $h(\phi_1, [\theta]) = 0$ by (8). □

Now Lemma 6.9 and (9) together imply the following.

Lemma 6.10. *For any compact invariant set $Z \subset T_1M$ we have $h(\phi_1, Z) = h(\psi_1, \chi(Z))$.*

By Theorem 5.1 and Corollary 6.6, the flow $\psi = (\psi_t)_t$ has a unique (hence ergodic) measure ν_ψ of maximal (positive) entropy.

Lemma 6.11. *We have $\nu_\psi(\{\chi(\theta)\colon [\theta] = \{\theta\}\}) = 1$.*

Proof. With (11) we have $\{\chi(\theta)\colon [\theta] = \{\theta\}\} = \chi(\mathcal{R}_0)$. Note that $\mathcal{R}_0$ and its complementary set $\mathcal{H} \setminus \mathcal{R}_0$ are ϕ-invariant and hence $\chi(\mathcal{R}_0)$ and $\chi(\mathcal{H} \setminus \mathcal{R}_0)$ are ψ-invariant.

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By contradiction, suppose that $\nu_\psi(\chi(\mathcal{R}_0)) < 1$. Hence, by ergodicity, $\nu_\psi(\chi(\mathcal{H} \setminus \mathcal{R}_0)) = 1$, which implies $\nu_\psi(\chi(\mathcal{H})) = 1$. Since $\mathcal{H}$ is compact and invariant, by the variational principle (7) we then have $h(\psi_1, \chi(\mathcal{H})) > 0$. Hence, by Lemma 6.10 we obtain $h(\phi_1, \mathcal{H}) > 0$. This is, however, in contradiction with Lemma 6.8. □

We are now ready to give the proof of Theorem B.

Proof of Theorem B. Apply Theorem 6.7 using Lemmas 6.9 and 6.11. □

For completeness, we formulate the following result. For compact rank one surfaces (in fact, for rank one manifolds of any dimension) this is contained in [**29**, Corollary 6.2]. Based on the semi-conjugacy, using the above introduced notation, for $T > 0$ the set $\mathrm{Per}_{\mathcal{R}}(T) := \mathrm{Per}(\phi, T) \cap \mathcal{R}$ is the set of primitive periodic orbits γ in $\mathcal{R}$ of period $\ell(\gamma) \leq T$.

Corollary 6.12. *Let $\phi_t \colon T_1M \to T_1M$ be the geodesic flow of a C^∞ compact connected boundaryless surface (M, g) without focal points and of genus greater than one. We have $0 = h(\phi_1, \mathcal{H}) < h(\phi_1)$ and*

$$\lim_{T\to\infty} \frac{1}{T} \log \# \mathrm{Per}_{\mathcal{R}}(T) = h(\phi_1).$$

Proof. By Lemma 6.8 we have $h(\phi_1, \mathcal{H}) = 0$. By Proposition 6.5 we have

$$\lim_{T\to\infty} \log \# \mathrm{Per}(\psi, T) = h(\psi) > 0.$$

Hence, by Lemma 6.10 applied to T_1M and (7) we have $h(\phi_1) = h(\psi_1) = h(\psi) > 0$.

Since $h(\psi, \chi(\mathcal{H})) = 0$ by the above and by Lemma 6.10, by definition (6) of the entropy on $\chi(\mathcal{H})$, for any $\delta \in (0, h(\psi))$ there exists $\varepsilon > 0$ sufficiently small (and smaller than an expansivity constant for ψ) and $T_0 = T(\varepsilon)$ sufficiently large, such that for every $T \geq T_0$ we have

$$M(T, \varepsilon, \chi(\mathcal{H}), \psi) \leq \mathrm{e}^{T\delta}.$$

By (10), we conclude

$$\mathrm{Per}(\psi, T - \varepsilon, T + \varepsilon) \cap \chi(\mathcal{H}) \leq M(T, \varepsilon, \chi(\mathcal{H}), \psi) \leq \mathrm{e}^{T\delta}.$$

Since by Proposition 6.5 we have

$$\lim_{T\to\infty} \frac{1}{T} \log \# \mathrm{Per}(\psi, T - \varepsilon, T + \varepsilon) = h(\psi) > 0,$$

the above implies

$$\lim_{T\to\infty} \frac{1}{T} \log \# \mathrm{Per}(\psi, T - \varepsilon, T + \varepsilon) \cap \chi(\mathcal{R}) = h(\psi) > 0$$

from which we conclude

$$\lim_{T\to\infty} \frac{1}{T} \log \# \mathrm{Per}(\psi, T) \cap \chi(\mathcal{R}) = h(\psi).$$

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Since χ is a homeomorphism on $\mathcal{R}$ which preserves parametrization, this and the above immediately imply

$$\lim_{T\to\infty} \frac{1}{T} \log \# \operatorname{Per}_{\mathcal{R}}(T) = h(\psi) = h(\phi_1).$$

This proves the corollary. □

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