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109.20 Less than equable triangles on the Eisenstein lattice

1. *Introduction*

In this paper, we are interested in triangles that are *perimeter-dominant*, i.e., their perimeter exceeds their area. Recall that a *Heron triangle* is a triangle with integer sides and integer area. By Dolan's Theorem [1], the classification of perimeter-dominant Heronian triangles contains three exceptional examples and four infinite families determined by certain Pell, or Pell-like, equations.

We will obtain an analogous result for triangles with side lengths in $\sqrt{3}\mathbb{N}$ and area in $\frac{\sqrt{3}}{4}\mathbb{N}$. As we will consider in Section 3 of this Note, such triangles can be considered to be the equivalent of Heron triangles on the Eisenstein lattice.

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2. *The Main Theorem*

Theorem: Perimeter-dominant triangles with side lengths in $\sqrt{3}\mathbb{N}$ and area in $\frac{\sqrt{3}}{4}\mathbb{N}$ consist of the equilateral triangle of side length $3\sqrt{3}$ and three infinite families of triangles with side lengths:

(a) $x\sqrt{3}, x\sqrt{3}, \sqrt{3}$, where $1 = 4x^2 - 3y^2$ for some $x, y \in \mathbb{N}$.

(b) $x\sqrt{3}, x\sqrt{3}, 2\sqrt{3}$, where $1 = x^2 - 3y^2$ for some $x, y \in \mathbb{N}$.

(c) $(3x+1)\sqrt{3}, (3x-1)\sqrt{3}, 3\sqrt{3}$, where $1 = 4x^2 - 25y^2$ for some $x, y \in \mathbb{N}$.

Proof: Let a perimeter-dominant triangle T have side lengths $a\sqrt{3}, b\sqrt{3}, c\sqrt{3}$ and area $\Delta = \frac{\sqrt{3}}{4}n$, where $a, b, c, n \in \mathbb{N}$. Heron's formula [2, Chap. 6.7] for the area gives

$$\Delta = \sqrt{s(s - a\sqrt{3})(s - b\sqrt{3})(s - c\sqrt{3})},$$

where $s = \frac{\sqrt{3}}{2}(a + b + c)$ is the semi-perimeter. Thus, as T is perimeter-dominant,

$$\frac{n\sqrt{3}}{4} = \frac{3}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)} < \sqrt{3}(a+b+c),$$

so

$$n^2 = 3(a+b+c)(-a+b+c)(a-b+c)(a+b-c) < 16(a+b+c)^2. \quad (1)$$

Let $p = a + b + c, u = -a + b + c, v = a - b + c, w = a + b - c$. Note that the integers p, u, v, w are all positive and have the same parity. (However, unlike the Heron case, they may be even or odd). Without loss of generality, suppose $u \leqslant v \leqslant w$. Note that $u + v + w = p$, so by (1), $3uvw < 16(u + v + w)$ and hence

$$\frac{1}{uv} + \frac{1}{vw} + \frac{1}{wu} > \frac{3}{16}. \quad (2)$$

Since $u \leqslant v \leqslant w$, (2) gives $\frac{3}{u^2} > \frac{3}{16}$, and hence $u \leqslant 3$. For each of these three values of u, we can use (2) again to get an upper bound on v. Then, using the fact that u, v have the same parity, for each possible pair u, v we can use (2) again, in some cases obtaining an upper bound on w. Table 1 gives the potential values of u, v, w and the corresponding values of $n^2 = 3(u + v + w)uvw$. Certain potential values of u, v, w are eliminated because the resulting values of n^2 cannot be squares.

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u	v	w	n^2	Comment
1	1	w odd	$3w(w+2)$	
1	3	w odd	$9w(w+4)$	Not square mod 8
1	5	w odd	$15w(w+6)$	
1	7	w odd, $7 \leqslant w \leqslant 25$	$21w(w+8)$	Not square mod 8
1	9	w odd, $9 \leqslant w \leqslant 13$	$27w(w+10)$	Not square
1	11	11	$33 \times 11 \times 23$	Not square
2	2	w even	$12w(w+4)$	
2	4	w even, $4 \leqslant w \leqslant 10$	$24w(w+6)$	Not square
2	6	6	$6^3 \times 14$	Not square
3	3	w odd, $3 \leqslant w \leqslant 7$	$27w(w+6)$	Square only for $w = 3$

TABLE 1: Possible values

The side lengths are given by $a = \frac{1}{2}(v+w)$, $b = \frac{1}{2}(u+w)$, $c = \frac{1}{2}(u+v)$. The case $u = v = w = 3$ gives $a = b = c = 3$, which is the equilateral triangle. The remaining three cases from Table 1 are $(u, v) = (1, 1), (1, 5)$ and $(2, 2)$.

If $u = v = 1$, then w is odd and $n^2 = 3w(w+2)$. So n is divisible by 3. Setting $n = 3y$ and $w = 2x - 1$ gives $1 = 4x^2 - 3y^2$; this is case (a) of the Theorem. Similarly, if $u = 1$, $v = 5$, we have $n^2 = 15w(w+6)$ and setting $n = 45y$ and $w = 6x - 3$ gives $1 = 4x^2 - 15y^2$; this is case (c). Finally, if $u = v = 2$, we have $n^2 = 12w(w+4)$ and setting $n = 12y$ and $w = 2x - 2$ gives $1 = x^2 - 3y^2$, which is case (b).

Remark 1: Table 2 gives the perimeters and areas of the triangles of the Theorem. In cases (a) and (c) of the Theorem, y is necessarily odd. So, from Table 2, it is only in case (b) that the area is an integer multiple of 3.

Case	Relation	Side Lengths / $\sqrt{3}$	Perimeter / $\sqrt{3}$	Area / $\sqrt{3}$
		$(3, 3, 3)$	9	27 / 4
a	$1 = 4x^2 - 3y^2$	$(x, x, 1)$	$2x + 1$	$3y/4$
b	$1 = x^2 - 3y^2$	$(x, x, 2)$	$2x + 2$	$3y$
c	$1 = 4x^2 - 15y^2$	$(3x+1, 3x-1, 3)$	$6x + 3$	$45y/4$

TABLE 2: Perimeters and areas

Remark 2: The three equations of the Theorem are well known. For the equation $1 = 4x^2 - 3y^2$ of case (a), the solutions (x, y) are given by the solutions of the Pell equation $1 = X^2 - 3y^2$ for which X is even, $X = 2x$; see sequence A094347 in [3]. The x-values are given by the recurrence relation $x_n = 14x_{n-1} - x_{n-2}$ with $x_1 = 1, x_2 = 13$. The first five x-values are 1, 13, 181, 2521, 35113.

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For the Pell equation $1 = x^2 - 3y^2$ of case (b), the x-values are given by the recurrence relation $x_n = 4x_{n-1} - x_{n-2}$ with $x_1 = 2$, $x_2 = 7$; see sequence A001075 in [3]. The first five x-values are 2, 7, 26, 97, 362.

For the equation $1 = 4x^2 - 15y^2$ of case (c), the solutions (x, y) are given by the solutions of the Pell equation $1 = X^2 - 15y^2$ for which X is even, $X = 2x$. The x-values are given by the recurrence relation $x_n = 62x_{n-1} - x_{n-2}$ with $x_1 = 2$, $x_2 = 122$; for the sequence $\frac{1}{2}x_n$, see sequence A302329 in [3]. The first five x-values are 2, 122, 7562, 468722, 29053202.

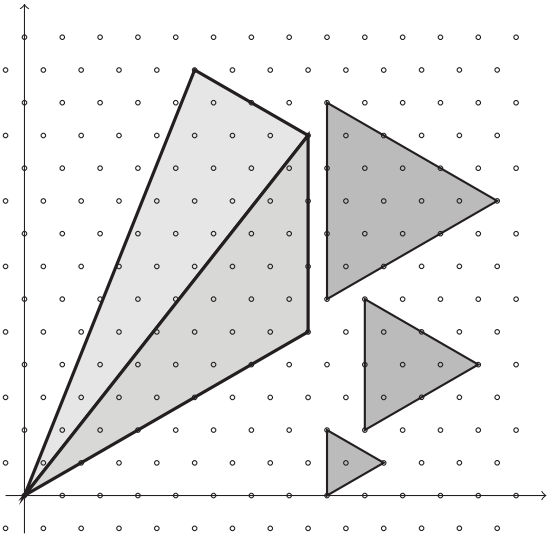


FIGURE 1: Five triangles from the Theorem

3. *The Eisenstein Lattice*

Heron triangles can all be realized on the integer lattice [4], i.e., the lattice of points in the plane with integer coordinates. This is not the case for the triangles studied in this note, since triangles on the integer lattice have integer or half-integer area. However, they can be realised on the *Eisenstein lattice*. The Eisenstein lattice, also known as the triangle lattice or the hexagonal lattice, is the set of vertices of the usual tiling of the plane by equilateral triangles of side length 1. Using complex numbers, the Eisenstein lattice is the additive subgroup of $\mathbb{C}$ generated by the elements 1 and $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. It is shown in [5] that all triangles with side lengths in $\sqrt{3}\mathbb{N}$ and area in $\frac{\sqrt{3}}{4}\mathbb{N}$ can be drawn on the Eisenstein lattice. Figure 1 shows five of the triangles of the Theorem. These are the three equilateral triangles (triangle of side length $3\sqrt{3}$, case (a) with $x = 1$ and case (b) with $x = 2$), the isosceles triangle of case (b) with $x = 7$, and the triangle with sides $(3\sqrt{3}, 5\sqrt{3}, 7\sqrt{3})$ of case (c) with $x = 2$. The results in the present note are

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an extension of those of [6], in which we studied triangles whose vertices lie on the Eisenstein lattice and which are *equable*, i.e., they have equal perimeter and area.

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