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Four-cycle free graphs, height functions, the pivot property and entropy minimality

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Abstract. Fix $d \geq 2$. Given a finite undirected graph $\mathcal{H}$ without self-loops and multiple edges, consider the corresponding 'vertex' shift, $\mathrm{Hom}(\mathbb{Z}^d, \mathcal{H})$, denoted by $X_{\mathcal{H}}$. In this paper, we focus on $\mathcal{H}$ which is 'four-cycle free'. There are two main results of this paper. Firstly, that $X_{\mathcal{H}}$ has the pivot property, meaning that, for all distinct configurations $x, y \in X_{\mathcal{H}}$, which differ only at a finite number of sites, there is a sequence of configurations $x = x^1, x^2, \ldots, x^n = y \in X_{\mathcal{H}}$ for which the successive configurations x^i, x^{i+1} differ exactly at a single site. Secondly, if $\mathcal{H}$ is connected ,then $X_{\mathcal{H}}$ is entropy minimal, meaning that every shift space strictly contained in $X_{\mathcal{H}}$ has strictly smaller entropy. The proofs of these seemingly disparate statements are related by the use of the 'lifts' of the configurations in $X_{\mathcal{H}}$ to the universal cover of $\mathcal{H}$ and the introduction of 'height functions' in this context.

1. *Introduction*

We will always use $\mathcal{H}$ to denote an undirected graph without multiple edges (and by abuse of notation also denote its set of vertices). In this paper, we focus on $\mathcal{H}$ which is four-cycle free: that is, it is finite, it has no self-loops and the four-cycle, denoted by C_4, is not a subgraph of $\mathcal{H}$. Fix $d \geq 2$. The basic object of study is $X_{\mathcal{H}}$, the space of graph homomorphisms from $\mathbb{Z}^d$ to $\mathcal{H}$. Here, by $\mathbb{Z}^d$, we will mean both the group and its standard Cayley graph.

Such a space of configurations $X_{\mathcal{H}}$, referred to as a hom-shift, can be obtained by forbidding certain patterns on edges of $\mathcal{H}^{\mathbb{Z}^d}$. If $\mathcal{H}$ is a finite graph, then $X_{\mathcal{H}}$ is a nearest neighbour shift of finite type. In addition, it is also 'isotropic' and 'symmetric': that is, given vertices $a, b \in \mathcal{H}$, if a is not allowed to sit next to b in $X_{\mathcal{H}}$ for some coordinate direction, then a is not allowed to sit next to b in all coordinate directions. Most of the concepts related to shift spaces are introduced in §2.

Related to a shift space, X is its topological entropy, denoted by $h_{\mathrm{top}}(X)$, which measures the growth rate of the number of patterns allowed in X with the size of the underlying shape (usually rectangular). For a given shift space, its computation is a very

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difficult task (look for instance in [**23**] and the references within). We will focus on a different aspect: as in [**8**], a shift space X is called entropy minimal if, for all shift spaces $Y \subsetneq X$, $h_{\text{top}}(Y) < h_{\text{top}}(X)$. Thus, if a shift space has zero entropy and is entropy minimal, then it is a topologically minimal system.

For $d = 1$, it is well known that all irreducible nearest neighbour shifts of finite type are entropy minimal [**15**, Theorem 4.4.7]. However, not much is known about it in higher dimensions. Shift spaces with a strong mixing property called uniform filling are entropy minimal [**32**]; however, nearest neighbour shifts of finite type with weaker mixing properties like block-gluing may not be entropy minimal [**2**]. If $\mathcal{H}$ is a four-cycle free graph then $X_{\mathcal{H}}$ is not even block-gluing (we do not prove this but it is implied by our results). There has been some recent work [**33**] that describes some conditions which are equivalent to entropy minimality for shifts of finite type. Our first main result is Theorem 2.2 which states that $X_{\mathcal{H}}$ is entropy minimal for all connected four-cycle free graphs $\mathcal{H}$. Our approach to this seemingly combinatorial question will be via thermodynamic formalism, using measures of maximal entropy (more generally, 'adapted' Markov random fields). They are defined and described in §3.

Our second main result is with regard to the pivot property (§4). A shift space X is said to have the pivot property if, for all distinct configurations $x, y \in X$ which differ at a finite number of sites, there exists a sequence $x = x^1, x^2, \ldots, x^n = y \in X$ such that successive configurations x^i, x^{i+1} differ exactly at a single site. We will prove that for four-cycle free graphs $\mathcal{H}$, $X_{\mathcal{H}}$ has the pivot property (Theorem 4.1). Many properties similar to the pivot property have appeared in the literature (often by the name local-move connectedness) [**4**, **20**, **22**, **34**]. For instance, consider the following problem. Let G be a finite undirected graph without multiple edges and self-loops. Given two graph homomorphisms x, y from G to H, can we find graph homomorphisms $x = x^1, x^2, \ldots, x^n = y$ from G to H such that successive configurations x^i, x^{i+1} differ exactly at a single site? Such a problem is called a homomorphism reconfiguration problem. If $\mathcal{H}$ is four-cycle free it was recently shown in [**38**] that the homomorphism reconfiguration problem is solvable in time polynomial in the size of the graph $\mathcal{H}$. Our interest in such problems comes from the connections between the pivot property and the study of Markov and Gibbs cocycles [**7**].

A critical part of our proofs for both Theorems 2.2 and 4.1 depends on the identification of the associated height functions (defined in §7). Some of these ideas come from [**7**]. Let C_n denote the cycle with vertices $0, 1, \ldots, n-1$. If $n \neq 1, 2, 4$, then it was proven that X_{C_n} has the pivot property. Further, [**7**, Lemma 6.7] implied that it is also entropy minimal. We give a brief description of the latter: given a configuration $x \in X_{C_n}$ it was proved that there exists a corresponding height function $h_x \in X_{\mathbb{Z}}$ such that $h_x \bmod n = x$ (for $n = 3$, also look at [**29**]). Further, given any ergodic measure μ (assuming 'adaptedness' cf. §3) on X_{C_n}, it was shown that a well-defined notion of slope (bounded by -1 and 1) exists which measures the average rate of increase of the height for every direction. If the slope is maximal in any direction, it was proven that μ is frozen; otherwise it was shown to be fully supported (frozen means that if $x, y \in \text{supp}(\mu)$ differ only at a finite number of sites, then $x = y$). From standard results in thermodynamic formalism, it follows that X_{C_n} is entropy minimal.

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These ideas do not immediately translate to four-cycle free graphs. For this we will develop a different notion of 'heights'. Fix a connected four-cycle free graph $\mathcal{H}$. For all $u \in \mathcal{H}$ let $D_{\mathcal{H}}(u)$ denote the ball (for the graph distance) of radius one around u. As in algebraic topology, C is a covering space of $\mathcal{H}$ if there is a graph homomorphism (called the covering map) $f : C \longrightarrow \mathcal{H}$ such that for all $u \in \mathcal{H}$, $f^{-1}(D_{\mathcal{H}}(u))$ is a disjoint union of a constant number of subgraphs of C isomorphic to $D_{\mathcal{H}}(u)$ via f. Given a graph $\mathcal{H}$, its universal cover (denoted by $E_{\mathcal{H}}$) is the unique covering space of $\mathcal{H}$ which is a tree (see section 6 and [**1**]). Denote by $\pi : E_{\mathcal{H}} \longrightarrow \mathcal{H}$ the corresponding covering map. π induces a map from $X_{E_{\mathcal{H}}}$ to $X_{\mathcal{H}}$ (also denoted by π). It is not difficult to see that $\pi(X_{E_{\mathcal{H}}}) \subset X_{\mathcal{H}}$; we will prove that the map is surjective and that the preimage of a configuration in $X_{\mathcal{H}}$ is unique once we fix the 'lift' in $X_{E_{\mathcal{H}}}$ at any vertex of $\mathbb{Z}^d$. Thus, for every $x \in X_{\mathcal{H}}$, we can associate $\tilde{x} \in X_{E_{\mathcal{H}}}$ such that $\pi(\tilde{x}) = x$ and a 'height' function $h_x : \mathbb{Z}^d \times \mathbb{Z}^d \longrightarrow \mathbb{Z}$ such that $h_x(\vec{i}, \vec{j})$ is the graph distance between $\tilde{x}_{\vec{i}}$ and $\tilde{x}_{\vec{j}}$. From now on the steps for the proofs of Theorems 2.2 and 4.1 are similar to the steps for the corresponding proofs in [**7**], but the proofs for the individual steps are quite different because, unlike in [**7**], the height functions are not additive but subadditive, meaning

$$h_x(\vec{i}, \vec{j}) \leq h_x(\vec{i}, \vec{k}) + h_x(\vec{k}, \vec{j}).$$

To streamline the proofs, we use graph folding [**21**]; much of this is discussed in §5.

2. *Shifts of finite type and hom-shifts*

In this paper, $\mathcal{H}$ will always denote an undirected graph without multiple edges and single isolated vertices. For such a graph we will denote the adjacency relation by $\sim_{\mathcal{H}}$ and the set of vertices of $\mathcal{H}$ by $\mathcal{H}$ (abusing notation). We identify $\mathbb{Z}^d$ with the set of vertices of the Cayley graph with respect to the standard generators $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_d$: that is, $\vec{i} \sim_{\mathbb{Z}^d} \vec{j}$ if and only if $\|\vec{i} - \vec{j}\|_1 = 1$, where $\|\cdot\|_1$ is the l^1 norm. We drop the subscript in $\sim_{\mathcal{H}}$ when $\mathcal{H} = \mathbb{Z}^d$. Let D_n and B_n denote the $\mathbb{Z}^d$-balls of radius n around $\vec{0}$ in the l^1 and the l^∞ norm, respectively. The graph C_n will denote the n-cycle where the set of vertices is $\{0, 1, 2, \ldots, n-1\}$ and $i \sim_{C_n} j$ if and only if $i \equiv j \pm 1 \bmod n$. The graph K_n will denote the complete graph with n vertices, where the set of vertices is $\{1, 2, \ldots, n\}$ and $i \sim_{K_n} j$ if and only if $i \neq j$.

Let $\mathcal{A}$ be a finite *alphabet* (with the discrete topology) and $\mathcal{A}^{\mathbb{Z}^d}$ be given the product topology, making it compact. We will refer to elements $x \in \mathcal{A}^{\mathbb{Z}^d}$ as *configurations*, denoting them by $(x_{\vec{i}})_{\vec{i} \in \mathbb{Z}^d}$, where $x_{\vec{i}}$ is the value of x at $\vec{i}$. For all $\vec{i} \in \mathbb{Z}^d$ the map $\sigma^{\vec{i}} : \mathcal{A}^{\mathbb{Z}^d} \longrightarrow \mathcal{A}^{\mathbb{Z}^d}$ given by

$$(\sigma^{\vec{i}}(x))_{\vec{j}} := x_{\vec{i}+\vec{j}}$$

is called the *shift map* and defines a $\mathbb{Z}^d$-action on $\mathcal{A}^{\mathbb{Z}^d}$. Closed subsets of $\mathcal{A}^{\mathbb{Z}^d}$ which are invariant under the shift maps are called *shift spaces*. A *sliding block code* from a shift space X to a shift space Y is a continuous map $f : X \longrightarrow Y$ which commutes with the shifts: that is, $\sigma^{\vec{i}} \circ f = f \circ \sigma^{\vec{i}}$ for all $\vec{i} \in \mathbb{Z}^d$. A surjective sliding block code is called a *factor map* and a bijective sliding block code is called a *conjugacy*. We note that a conjugacy defines an equivalence relation; in fact, it has a continuous inverse since it is a continuous bijection between compact sets.

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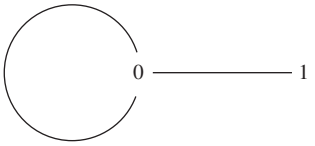


FIGURE 1. Graph for the hard square shift.

There is an alternate description of shift spaces using forbidden patterns: a *pattern* is an element of $\mathcal{A}^F$ for some finite set $F \subset \mathbb{Z}^d$. Given a pattern $a \in \mathcal{A}^F$, we will often denote both the pattern and its corresponding cylinder set by $[a]_F$ or by $[x]_F$ when $x|_F = a$. For any set $F \subset \mathbb{Z}^d$ and $v \in \mathcal{A}$, $x|_F = v$ will mean that $x_{\vec{i}} = v$ for all $\vec{i} \in F$. For a set of (forbidden) patterns $\mathcal{F}$ define

$$X_{\mathcal{F}} := \{x \in \mathcal{A}^{\mathbb{Z}^d} \mid \sigma^{\vec{i}}(x)|_F \notin \mathcal{F} \; \forall \; F \subset \mathbb{Z}^d \text{ and } \vec{i} \in \mathbb{Z}^d\}.$$

It can be proved that a subset $X \subset \mathcal{A}^{\mathbb{Z}^d}$ is a shift space if and only if there exists a set of forbidden patterns $\mathcal{F}$ such that $X_{\mathcal{F}} = X$. Note that given two distinct sets of patterns $\mathcal{F}_1, \mathcal{F}_2$ it is possible that $X_{\mathcal{F}_1} = X_{\mathcal{F}_2}$. A *shift of finite type* is a shift space X such that $X = X_{\mathcal{F}}$ for some finite set $\mathcal{F}$. A *nearest neighbour shift of finite type* is a shift of finite type X such that $X = X_{\mathcal{F}}$ for some set $\mathcal{F}$ consisting of patterns on single edges and vertices of $\mathbb{Z}^d$. It follows, from a simple recoding argument, that any shift of finite type is conjugate to a nearest neighbour shift of finite type [**31**]. In this paper, we will focus on a special class of nearest neighbour shifts of finite type where the forbidden patterns are the same in every 'direction'.

Given a graph $\mathcal{H}$ let

$$X_{\mathcal{H}} := \{x \in \mathcal{H}^{\mathbb{Z}^d} \mid x_{\vec{i}} \sim_{\mathcal{H}} x_{\vec{j}} \; \forall \, \vec{i} \sim \vec{j}\}.$$

Such spaces will be called *hom-shifts*. Note that $X_{\mathcal{H}}$ is the space of graph homomorphisms from $\mathbb{Z}^d$ to $\mathcal{H}$. If $\mathcal{H}$ is finite and

$$\mathcal{F}_{\mathcal{H}} := \{[v, w]_{\vec{0},\vec{e}_j} \mid v \nsim_{\mathcal{H}} w, \, 1 \le j \le d\},$$

then $X_{\mathcal{H}} = X_{\mathcal{F}_{\mathcal{H}}}$. These are exactly the nearest neighbour shifts of finite type with symmetric and isotropic constraints. For example, if the graph $\mathcal{H}$ is given by Figure 1, then $X_{\mathcal{H}}$ is the hard square shift: that is, configurations with alphabet $\{0, 1\}$ such that adjacent symbols cannot both be 1. X_{K_n} is the space of n-colourings of the graph: that is, configurations with alphabet $\{1, 2, \ldots, n\}$, where all adjacent colours are distinct. We note that the properties symmetry and isotropy are not invariant under conjugacy.

$\mathcal{F}$ will always denote a set of patterns and $\mathcal{H}$ will always denote a graph. There will not be any ambiguity in the notations $X_{\mathcal{F}}$, $X_{\mathcal{H}}$.

A graph $\mathcal{H}$ is called *four-cycle free* if it is finite, it has no self-loops and C_4 is not a subgraph of $\mathcal{H}$. For instance, K_4 is not a four-cycle free graph.

The set of *globally allowed patterns* of a shift space X on a set $A \subset \mathbb{Z}^d$ is

$$\mathcal{L}_A(X) := \{a \in \mathcal{A}^A \mid \exists x \in X \text{ such that } x|_A = a\}.$$

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Its *language* is the set of all finite patterns appearing in X: that is

$$\mathcal{L}(X) := \bigcup_{A \subset \mathbb{Z}^d \text{ finite}} \mathcal{L}_A(X).$$

We comment that this is different from the set of locally allowed patterns. Let X be a shift space with a forbidden list $\mathcal{F}$. Given a finite set A, a pattern $a \in \mathcal{A}^A$ is said to be *locally allowed* if no pattern from $\mathcal{F}$ appears in a. In general, it is undecidable for shifts of finite type whether a locally allowed pattern belongs to $\mathcal{L}(X)$ [**27**]; however, it is decidable when X is a hom-shift where it is sufficient to check whether the pattern extends to a locally allowed pattern on B_n, for some n.

The topological entropy of the shift space X is the log growth rate of the number of allowed patterns in X: that is,

$$h_{\text{top}}(X) := \lim_{n \longrightarrow \infty} \frac{\log |\mathcal{L}_{B_n}(X)|}{|B_n|}.$$

The existence of the limit follows from subadditivity arguments via the well-known multivariate version of Fekete's Lemma [**5**]. Moreover, the topological entropy is an invariant under conjugacy (for $d = 1$ look at [**15**, Proposition 4.1.9]; the proof extends to higher dimensions). We remark that the computation of this invariant for shifts of finite type in $d > 1$ is a hard problem and very little is known [**23**]. However, there are algorithms to compute approximate upper and lower bounds of the topological entropy of the hom-shifts [**10**, **16**]. Further, if $\mathcal{H}$ is a finite connected graph with at least two edges, then $h_{\text{top}}(X_{\mathcal{H}}) > 0$.

PROPOSITION 2.1. *Let $\mathcal{H}$ be a finite graph with distinct vertices a, b and c such that $a \sim_{\mathcal{H}} b$ and $b \sim_{\mathcal{H}} c$. Then $h_{\text{top}}(X_{\mathcal{H}}) \geq (\log 2)/2$.*

Proof. It is sufficient to see this for a graph $\mathcal{H}$ with exactly three vertices a, b and c such that $a \sim_{\mathcal{H}} b$ and $b \sim_{\mathcal{H}} c$. For such a graph, any configuration in $X_{\mathcal{H}}$ is composed of b on one partite class of $\mathbb{Z}^d$ and a free choice between a and c for vertices on the other partite class. Then

$$|\mathcal{L}_{B_n}(X_{\mathcal{H}})| = 2^{\lfloor (2n+1)^d/2 \rfloor} + 2^{\lceil (2n+1)^d/2 \rceil},$$

proving that $h_{\text{top}}(X_{\mathcal{H}}) = (\log 2)/2$. □

A shift space X is called *entropy minimal* if, for all shift spaces $Y \subsetneq X$, $h_{\text{top}}(X) > h_{\text{top}}(Y)$. In other words, a shift space X is entropy minimal if forbidding any word causes a drop in entropy. From [**26**], we know that every shift space contains an entropy minimal shift space with the same entropy, and also a characterization of same entropy factor maps on entropy minimal shifts of finite type.

One of the main results of this paper is the following theorem.

THEOREM 2.2. *Let $\mathcal{H}$ be a connected four-cycle free graph. Then $X_{\mathcal{H}}$ is entropy minimal.*

For $d = 1$, all irreducible shifts of finite type are entropy minimal [**15**, Theorem 4.4.7]. A necessary condition for the entropy minimality of $X_{\mathcal{H}}$ is that $\mathcal{H}$ has to be connected.

PROPOSITION 2.3. *Suppose $\mathcal{H}$ is a finite graph with connected components $\mathcal{H}_1, \mathcal{H}_2, \ldots \mathcal{H}_r$. Then $h_{\text{top}}(X_{\mathcal{H}}) = \max_{1 \leq i \leq r} h_{\text{top}}(X_{\mathcal{H}_i})$.*

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This follows from the observation that

$$\max_{1\le i\le r} |\mathcal{L}_{B_n}(X_{\mathcal{H}_i})| \le |\mathcal{L}_{B_n}(X_{\mathcal{H}})| = \sum_{i=1}^{r} |\mathcal{L}_{B_n}(X_{\mathcal{H}_i})| \le r \max_{1\le i\le r} |\mathcal{L}_{B_n}(X_{\mathcal{H}_i})|.$$

3. *Thermodynamic formalism*

Here we give a brief introduction of thermodynamic formalism. For more details one can refer to [**28**, **37**].

By μ we will always mean a shift-invariant Borel probability measure on a shift space X. The *support* of μ, denoted by $\mathrm{supp}(\mu)$, is the intersection of all closed sets $Y \subset X$ for which $\mu(Y) = 1$. Note that $\mathrm{supp}(\mu)$ is a shift space as well. The *measure theoretic entropy* is

$$h_\mu := \lim_{i\to\infty} \frac{1}{|D_i|} H_\mu^{D_i},$$

where $H_\mu^{D_i}$ is the Shannon-entropy of μ with respect to the partition of X generated by the cylinder sets on D_i, the definition of which is given by

$$H_\mu^{D_i} := \sum_{a\in\mathcal{L}_{D_i}(X)} -\mu([a]_{D_i}) \log \mu([a]_{D_i}),$$

with the understanding that $0 \log 0 = 0$.

A shift-invariant probability measure μ is a *measure of maximal entropy* of X if the maximum of $\nu \mapsto h_\nu$ over all shift-invariant probability measures on X is obtained at μ. The existence of measures of maximal entropy follows from upper-semi-continuity of the function $\nu \mapsto h_\nu$ with respect to the weak-$*$ topology.

Further, the well-known *variational principle* for topological entropy of $\mathbb{Z}^d$-actions asserts that, if μ is a measure of maximal entropy, $h_{\mathrm{top}}(X) = h_\mu$ whenever X is a $\mathbb{Z}^d$-shift space.

The following is a well-known characterization of entropy minimality (it is used, for instance, in the proof of Theorem 4.1 in [**18**]).

PROPOSITION 3.1. *A shift space X is entropy minimal if and only if every measure of maximal entropy for X is fully supported.*

We understand this by the following. Suppose X is entropy minimal and μ is a measure of maximal entropy for X. Then, by the variational principle for X and $\mathrm{supp}(\mu)$, we get

$$h_{\mathrm{top}}(X) = h_\mu \le h_{\mathrm{top}}(\mathrm{supp}(\mu)) \le h_{\mathrm{top}}(X),$$

proving that $\mathrm{supp}(\mu) = X$. To prove the converse, suppose, for contradiction, that X is not entropy minimal and consider $Y \subsetneq X$ such that $h_{\mathrm{top}}(X) = h_{\mathrm{top}}(Y)$. Then, by the variational principle, there exists a measure μ on Y such that $h_\mu = h_{\mathrm{top}}(X)$. Thus μ is a measure of maximal entropy for X which is not fully supported.

Further is known if X is a nearest neighbour shift of finite type; this brings us to Markov random fields which we introduce next.

Given a set $A \subset \mathbb{Z}^d$ we denote the *r-boundary* of A by $\partial_r A$: that is,

$$\partial_r A = \{w \in \mathbb{Z}^d \setminus A \mid \|w - v\|_1 \le r \text{ for some } v \in A\}.$$

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The *1-boundary* will be referred to as the *boundary* and denoted by ∂A. A *Markov random field* on $\mathcal{A}^{\mathbb{Z}^d}$ is a Borel probability measure μ with the property that, for all finite $A, B \subset \mathbb{Z}^d$ such that $\partial A \subset B \subset A^c$ and $a \in \mathcal{A}^A$, $b \in \mathcal{A}^B$ satisfying $\mu([b]_B) > 0$,

$$\mu([a]_A \mid [b]_B) = \mu([a]_A \mid [b]_{\partial A}).$$

In general, Markov random fields are defined over graphs that are much more general than $\mathbb{Z}^d$. However, we restrict to the $\mathbb{Z}^d$ setting in this paper.

A *uniform Markov random field* is a Markov random field μ such that, in addition,

$$\mu([a]_A \mid [b]_{\partial A}) = \frac{1}{n_{A,b|_{\partial A}}},$$

where $n_{A,b|_{\partial A}} = |\{a \in \mathcal{A}^A \mid \mu([a]_A \cap [b]_{\partial A}) > 0\}|$.

Following [**25**, **30**], we denote by Δ_X the *homoclinic equivalence relation* of a shift space X, which is given by

$$\Delta_X := \{(x, y) \in X \times X \mid x_{\vec{i}} = y_{\vec{i}} \ \forall \text{ but a finite number of } \vec{i} \in \mathbb{Z}^d\}.$$

We say that a measure μ is *adapted* with respect to a shift space X if $\operatorname{supp}(\mu) \subset X$ and

$$x \in \operatorname{supp}(\mu) \Longrightarrow \{y \in X \mid (x, y) \in \Delta_X\} \subset \operatorname{supp}(\mu).$$

To illustrate this definition, let $X \subset \{0, 1\}^{\mathbb{Z}}$ consist of configurations in X in which at most a single 1 appears. X is uniquely ergodic; the delta-measure δ_{0^∞} is the only shift-invariant measure on X. But

$$\operatorname{supp}(\delta_{0^\infty}) = \{0^\infty\} \subsetneq \{y \in X \mid 0^\infty_i = y_i \ \forall \text{ but a finite number of } i \in \mathbb{Z}\} = X,$$

proving that it is not adapted. On the other hand, since the homoclinic relation of $\mathcal{A}^{\mathbb{Z}^d}$ is minimal, meaning that for all $x \in \mathcal{A}^{\mathbb{Z}^d}$

$$\overline{\{y \in \mathcal{A}^{\mathbb{Z}^d} \mid y_{\vec{i}} = x_{\vec{i}} \ \forall \text{ but a finite number of } \vec{i} \in \mathbb{Z}^d\}} = \mathcal{A}^{\mathbb{Z}^d},$$

it follows that a probability measure on $\mathcal{A}^{\mathbb{Z}^d}$ is adapted if and only if it is fully supported.

The relationship between measures of maximal entropy and Markov random fields is established by the following theorem. This is a special case of the Lanford–Ruelle theorem [**14**, **28**].

THEOREM 3.2. *All measures of maximal entropy on a nearest neighbour shift of finite type X are shift-invariant uniform Markov random fields μ adapted to X.*

The converse is also true under further mixing assumptions on the shift space X (called the D-condition). The full strength of these statements is obtained by looking at *equilibrium states* instead of measures of maximal entropy. The measures obtained there are not uniform Markov random fields: rather they are Markov random fields where the conditional probabilities are weighted via an *interaction* giving rise to *Gibbs states*. Uniform Markov random fields are Gibbs states with interaction zero.

We will often restrict our proofs to the ergodic case. We can do so via the following standard facts implied by [**11**, Theorem 14.15] and [**13**, Theorem 4.3.7].

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THEOREM 3.3. *Let μ be a shift-invariant uniform Markov random field adapted to a shift space X. Let its ergodic decomposition be given by a measurable map $x \longrightarrow \mu_x$ on X: that is, $\mu = \int_X \mu_x \, d\mu$. Then, μ-almost everywhere, the measures μ_x are shift-invariant uniform Markov random fields adapted to X such that* $\operatorname{supp}(\mu_x) \subset \operatorname{supp}(\mu)$. *Moreover, $\int h_{\mu_x} \, d\mu(x) = h_\mu$.*

We will prove the following theorem.

THEOREM 3.4. *Let $\mathcal{H}$ be a connected four-cycle free graph. Then every ergodic probability measure adapted to $X_{\mathcal{H}}$ with positive entropy is fully supported.*

This implies Theorem 2.2 by the following. The Lanford–Ruelle theorem implies that every measure of maximal entropy on $X_{\mathcal{H}}$ is a uniform shift-invariant Markov random field adapted to $X_{\mathcal{H}}$. By Proposition 2.1 and the variational principle, we know that these measures have positive entropy. By Theorems 3.3 and 3.4, they are fully supported. Finally, by Proposition 3.1, $X_{\mathcal{H}}$ is entropy minimal.

Alternatively, the conclusion of Theorem 3.4 can be obtained via some strong mixing conditions on the shift space; we will describe one such assumption. A shift space X is called *strongly irreducible* if there exists $g > 0$ such that, for all $x, y \in X$ and $A, B \subset \mathbb{Z}^d$ satisfying $\min_{\vec{i} \in A, \vec{j} \in B} \|\vec{i} - \vec{j}\|_1 \geq g$, there exists $z \in X$ such that $z|_A = x|_A$ and $z|_B = y|_B$. For such a space, the homoclinic relation is minimal, implying the conclusion of Theorem 3.4 and, further, that every probability measure adapted to X is fully supported. Note that this does not prove that X is entropy minimal unless we assume that X is a nearest neighbour shift of finite type. Such an argument is used in the proof of [**18**, Lemma 4.1], which implies that every strongly irreducible shift of finite type is entropy minimal. A more combinatorial approach was used in [**32**] to show that general shift spaces with a weaker mixing property called uniform filling are entropy minimal.

4. *The pivot property*

A *pivot* in a shift space X is a pair of configurations $(x, y) \in X$ such that x and y differ exactly at a single site. A subshift X is said to have *the pivot property* if, for all distinct $(x, y) \in \Delta_X$, there exists a finite sequence of configurations $x^{(1)} = x, x^{(2)}, \ldots, x^{(k)} = y \in X$ such that each $(x^{(i)}, x^{(i+1)})$ is a pivot. In this case, we say that $x^{(1)} = x, x^{(2)}, \ldots, x^{(k)} = y$ is a *chain of pivots* from x to y. Here are some examples of subshifts which have the pivot property.

(1) Any subshift with a trivial homoclinic relation: that is, the homoclinic classes are singletons.
(2) Any subshift with a safe symbol†.
(3) The hom-shifts X_{C_r}. This was proved for $r \neq 4$ in [**7**]; the result for $r = 4$ is a special case of Propositions 5.7.

† A shift space $X \subset \mathcal{A}^{\mathbb{Z}^d}$ has a *safe symbol* $\star$ if, for all $x \in X$ and $A \subset \mathbb{Z}^d$, the configuration $z \in \mathcal{A}^{\mathbb{Z}^d}$ given by

$$z_{\vec{i}} := \begin{cases} x_{\vec{i}} & \text{if } \vec{i} \in A, \\ \star & \text{if } \vec{i} \in A^c, \end{cases}$$

is also an element of X.

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1	2	3	4	5
3	4	5	1	2
5	1	2	3	4
2	3	4	5	1
4	5	1	2	3

FIGURE 2. Frozen pattern.

(4) r-colourings of $\mathbb{Z}^d$ with $r \geq 2d + 2$. (This is well known: look, for instance, in [**7**, §3.2].)
(5) $X_{\mathcal{H}}$ when $\mathcal{H}$ is dismantlable [**4**].

We generalize the class of examples given by (5) in Proposition 5.7. It is not true that all hom-shifts have the pivot property.

The following was observed by Brian Marcus. Recall that K_n denotes the complete graph with n vertices. X_{K_4}, X_{K_5} do not possess the pivot property if the dimension is two. For instance, consider a configuration in X_{K_5} which is obtained by tiling the plane with the pattern given in Figure 2. It is clear that the symbols in the box can be interchanged but no individual symbol can be changed. Therefore X_{K_5} does not have the pivot property. However, both X_{K_4} and X_{K_5} satisfy a more general property as discussed in §9.5.

The following theorem is another main result in this paper.

THEOREM 4.1. *For all four-cycle free graphs $\mathcal{H}$, $X_{\mathcal{H}}$ has the pivot property.*

It is sufficient to prove this theorem for four-cycle free graphs $\mathcal{H}$ which are connected because of the following proposition.

PROPOSITION 4.2. *Let $X_1, X_2, \ldots, X_n$ be shift spaces on disjoint alphabets such that each of them has the pivot property. Then $\bigcup_{i=1}^{n} X_i$ also has the pivot property.*

This is true since $(x, y) \in \Delta_{\cup_{i=1}^{n} X_i}$ implies $(x, y) \in \Delta_{X_i}$ for some $1 \leq i \leq n$.

5. *Folding, entropy minimality and the pivot property*

Given a graph $\mathcal{H}$, we say that a vertex v *folds* into a vertex w if and only if $u \sim_{\mathcal{H}} v$ implies $u \sim_{\mathcal{H}} w$. In this case the graph $\mathcal{H} \setminus \{v\}$ is called a *fold* of $\mathcal{H}$. The folding gives rise to a 'retract' from $\mathcal{H}$ to $\mathcal{H} \setminus \{v\}$, namely the graph homomorphism from $\mathcal{H}$ to $\mathcal{H} \setminus \{v\}$ which is the identity on $\mathcal{H} \setminus \{v\}$ and sends v to w. This was introduced in [**21**] to help characterize cop-win graphs and used in [**4**] to establish many properties which are preserved under 'folding' and 'unfolding'. Given a finite tree $\mathcal{H}$ with more than two vertices, note that a leaf vertex (vertex of degree one) can always be folded to some other vertex of the tree. Thus, starting with $\mathcal{H}$, there exists a sequence of folds resulting in a single edge. In fact, using a similar argument, we can prove the following proposition.

PROPOSITION 5.1. *Let $\mathcal{H} \subset \mathcal{H}'$ be trees. Then there is a graph homomorphism $f : \mathcal{H}' \longrightarrow \mathcal{H}$ such that $f|_{\mathcal{H}}$ is the identity map.*

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To show this, first note that if $\mathcal{H} \subsetneq \mathcal{H}'$, then there is a leaf vertex in $\mathcal{H}'$ which is not in $\mathcal{H}$. This leaf vertex can be folded into some other vertex in $\mathcal{H}'$. Thus, by induction on $|\mathcal{H}' \setminus \mathcal{H}|$, we can prove that there is a sequence of folds from $\mathcal{H}'$ to $\mathcal{H}$. Corresponding to this sequence of folds, we obtain a graph homomorphism from $\mathcal{H}'$ to $\mathcal{H}$ which is the identity on $\mathcal{H}$.

Here we consider a related notion for shift spaces. Given a nearest neighbour shift of finite type $X \subset \mathcal{A}^{\mathbb{Z}^d}$, *the neighbourhood* of a symbol $v \in \mathcal{A}$ is given by

$$N_X(v) := \{a \in \mathcal{A}^{\partial\vec{0}} \mid [v]_{\vec{0}} \cap [a]_{\partial\vec{0}} \in \mathcal{L}_{D_1}(X)\},$$

which is the collection of all patterns that can 'surround' v in X. We will say that v *config-folds* into w in X if $N_X(v) \subset N_X(w)$. In such a case, we say that X *config-folds* to $X \cap (\mathcal{A} \setminus \{v\})^{\mathbb{Z}^d}$. Note that $X \cap (\mathcal{A} \setminus \{v\})^{\mathbb{Z}^d}$ is obtained by forbidding v from X and hence it is also a nearest neighbour shift of finite type. Also, if $X = X_{\mathcal{H}}$ for some graph $\mathcal{H}$, then v config-folds into w in $X_{\mathcal{H}}$ if and only if v folds into w in $\mathcal{H}$. Thus, if $\mathcal{H}$ is a tree, then there is a sequence of folds starting at $X_{\mathcal{H}}$ resulting in the two chequerboard configurations with two symbols (the vertices of the edge which $\mathcal{H}$ folds into). This property is weaker than the notion of folding introduced in [**6**].

The main thrust of this property in our context is the following. If v config-folds into w in X then, given any $x \in X$, every appearance of v in x can be replaced by w to obtain another configuration in X. This replacement defines a factor (surjective, continuous and shift-invariant) map $f : X \longrightarrow X \cap (\mathcal{A} \setminus \{v\})^{\mathbb{Z}^d}$ given by

$$(f(x))_{\vec{i}} := \begin{cases} x_{\vec{i}} & \text{if } x_{\vec{i}} \neq v, \\ w & \text{if } x_{\vec{i}} = v. \end{cases}$$

Note that the map f defines a 'retract' from X to $X \cap (\mathcal{A} \setminus \{v\})^{\mathbb{Z}^d}$. Frequently, we will config-fold more than one symbol at once (especially in §8).

Distinct symbols $v_1, v_2, \ldots, v_n$ *config-fold disjointly* into $w_1, w_2, \ldots, w_n$ in X if v_i config-folds into w_i and $v_i \neq w_j$ for all $1 \leq i, j \leq n$. In this case, the symbols $v_1, v_2, \ldots, v_n$ can be replaced by $w_1, w_2, \ldots, w_n$ simultaneously, for all $x \in X$. Suppose $v_1, v_2, \ldots, v_n$ is a maximal set of symbols which can be config-folded disjointly in X. Then $X \cap (\mathcal{A} \setminus \{v_1, v_2, \ldots, v_n\})^{\mathbb{Z}^d}$ is called a *full config-fold* of X.

For example, consider a tree $\mathcal{H} := (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} := \{v_1, v_2, v_3, \ldots, v_{n+1}\}$ and $\mathcal{E} := \{(v_i, v_{n+1}) \mid 1 \leq i \leq n\}$. For all $1 \leq i \leq n$, $\mathcal{V} \setminus \{v_i, v_{n+1}\}$ is a maximal set of symbols which config-folds disjointly in $X_{\mathcal{H}}$ resulting in the chequerboard patterns with the symbols v_i and v_{n+1} for all $1 \leq i \leq n$. Thus the full config-fold of a shift space is not necessarily unique. However, it is unique up to conjugacy.

PROPOSITION 5.2. *The full config-fold of a nearest neighbour shift of finite type is unique up to conjugacy via a change of the alphabet.*

The ideas for the following proof come essentially from the proof of [**4**, Theorem 4.4] and discussions with Professor Brian Marcus.

Proof. Let $X \subset \mathcal{A}^{\mathbb{Z}^d}$ be a nearest neighbour shift of finite type and

$$M := \{v \in \mathcal{A} \mid \forall\, w \in \mathcal{A},\ v \text{ config-folds into } w \Longrightarrow w \text{ config-folds into } v\}.$$

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There is a natural equivalence relation $\equiv$ on M given by $v \equiv w$ if v and w config-fold into each other. Let $A_1, A_2, A_3, \ldots, A_r \subset M$ be the corresponding partition. Clearly, for all distinct $v, w \in M$, v can be config-folded into w if and only if $v, w \in A_i$ for some i. It follows that $A \subset A_i$ can be config-folded disjointly if and only if $\emptyset \neq A \neq A_i$.

Let $v \in \mathcal{A} \setminus M$. We will prove that v config-folds to a symbol in M. By the definition of M there exists $v_1 \in \mathcal{A}$ such that $N_X(v) \subsetneq N_X(v_1)$. If $v_1 \in M$, then we have finished; otherwise, choose $v_2 \in \mathcal{A}$ such that $N_X(v_1) \subsetneq N_X(v_2)$. Continuing this process recursively, we can find a sequence $v = v_0, v_1, v_2, \ldots, v_n$ such that $N_X(v_{i-1}) \subsetneq N_X(v_i)$ for all $1 \leq i \leq n$ and $v_n \in M$. Thus v config-folds into v_n, a symbol in M. Further, if v config-folds to a symbol in A_i, it can config-fold to all the symbols in A_i. Therefore B is a maximal subset of symbols in $\mathcal{A}$ which can be config-folded disjointly if and only if $B = \bigcup_{i=1}^{r} B_i \cup (\mathcal{A} \setminus M)$, where $B_i \subset A_i$ and $|A_i \setminus B_i| = 1$. Let $B' \subset \mathcal{A}$ be another such maximal subset, $\mathcal{A} \setminus B := \{b_1, b_2, \ldots, b_r\}$ and $\mathcal{A} \setminus B' := \{b'_1, b'_2, \ldots, b'_r\}$, where $b_i, b'_i \in A_i$. Then the map

$$f : X \cap (\mathcal{A} \setminus B)^{\mathbb{Z}^d} \longrightarrow X \cap (\mathcal{A} \setminus B')^{\mathbb{Z}^d} \text{ given by } f(x) := y, \text{ where } y_{\vec{i}} = b'_j$$

whenever $x_{\vec{i}} = b_j$ is the required change of alphabet between the two full config-folds of X. □

Let $X \cap (\mathcal{A} \setminus \{v_1, v_2, \ldots, v_n\})^{\mathbb{Z}^d}$ be a *full config-fold* of X where v_i config-folds into w_i for all $1 \leq i \leq n$. Consider $f_X : \mathcal{A} \longrightarrow \mathcal{A} \setminus \{v_1, v_2, \ldots, v_n\}$ given by

$$f_X(v) := \begin{cases} v & \text{if } v \neq v_j \text{ for all } 1 \leq j \leq n, \\ w_j & \text{if } v = v_j \text{ for some } 1 \leq j \leq n. \end{cases}$$

This defines a factor map $f_X : X \longrightarrow X \cap (\mathcal{A} \setminus \{v_1, v_2, \ldots, v_n\})^{\mathbb{Z}^d}$ given by $(f_X(x))_{\vec{i}} := f_X(x_{\vec{i}})$ for all $\vec{i} \in \mathbb{Z}^d$. f_X denotes both the factor map and the map on the alphabet; it should be clear from the context which function is being used.

In many cases we will fix a configuration on a set $A \subset \mathbb{Z}^d$ and apply a config-fold on the rest. Hence we define the map $f_{X,A} : X \longrightarrow X$ given by

$$(f_{X,A}(x))_{\vec{i}} := \begin{cases} x_{\vec{i}} & \text{if } \vec{i} \in A, \\ f_X(x_{\vec{i}}) & \text{otherwise.} \end{cases}$$

The map $f_{X,A}$ can be extended beyond X.

PROPOSITION 5.3. *Let $X \subset Y$ be nearest neighbour shifts of finite type, Z be a full config-fold of X and $y \in Y$ such that, for some $A \subset \mathbb{Z}^d$, $y|_{A^c \cup \partial(A^c)} \in \mathcal{L}_{A^c \cup \partial(A^c)}(X)$. Then the configuration z given by*

$$z_{\vec{i}} := \begin{cases} y_{\vec{i}} & \text{if } \vec{i} \in A, \\ f_X(y_{\vec{i}}) & \text{otherwise,} \end{cases}$$

is an element of Y. Moreover, $z|_{A^c} \in \mathcal{L}_{A^c}(Z)$.

Abusing the notation, in such cases we shall denote the configuration z by $f_{X,A}(y)$.

If A^c is finite, then $f_{X,A}$ changes only a finite number of coordinates. These changes can be applied one-by-one: that is, there is a chain of pivots in Y from y to $f_{X,A}(y)$.

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A nearest neighbour shift of finite type which cannot be config-folded is called a *stiff shift*. We know from [**4**, Theorem 4.4] that all the stiff graphs obtained by a sequence of folds of a given graph are isomorphic. By Proposition 5.2, the corresponding result for nearest neighbour shifts of finite type immediately follows.

PROPOSITION 5.4. *The stiff shift obtained by a sequence of config-folds starting with a nearest neighbour shift of finite type is unique up to conjugacy via a change of the alphabet.*

Starting with a nearest neighbour shift of finite type X, the *fold-radius* of X is the smallest number of full config-folds required to obtain a stiff shift. If $\mathcal{H}$ is a tree, then the fold-radius of $X_{\mathcal{H}}$ is equal to

$$\left\lfloor \frac{\text{diameter}(\mathcal{H})}{2} \right\rfloor.$$

Thus, for every nearest neighbour shift of finite type X, there is a sequence of full config-folds (not necessarily unique) which starts at X and ends at a stiff shift of finite type. Let the fold-radius of X be r and $X = X_0, X_1, X_2, \ldots, X_r$ be a sequence of full config-folds where X_r is stiff. This generates a sequence of maps $f_{X_i} : X_i \longrightarrow X_{i+1}$ for all $0 \leq i \leq r-1$. In many cases, we will fix a pattern on D_n or D_n^c and apply these maps on the rest of the configuration. Consider the maps $I_{X,n} : X \longrightarrow X$ and $O_{X,n} : X \longrightarrow X$ (for $n > r$) given by

$$I_{X,n}(x) := f_{X_{r-1},D_{n+r-1}}(f_{X_{r-2},D_{n+r-2}}(\ldots(f_{X_0,D_n}(x))\ldots)) \text{ (Inward Fixing Map)}$$

and

$$O_{X,n}(x) := f_{X_{r-1},D^c_{n-r+1}}(f_{X_{r-2},D^c_{n-r+2}}(\ldots(f_{X_0,D^c_n}(x))\ldots)) \text{ (Outward Fixing Map).}$$

Similarly, we consider maps which do not fix anything, $F_X : X \longrightarrow X_r$, given by

$$F_X(x) := f_{X_{r-1}}(f_{X_{r-2}}(\ldots(f_{X_0}(x))\ldots)).$$

Note that $D_k \cup \partial D_k = D_{k+1}$ and $D_k^c \cup \partial(D_k^c) = D_{k-1}^c$. This, along with repeated application of Proposition 5.3, implies that the image of $I_{X,n}$ and $O_{X,n}$ lie in X. This also implies the following proposition.

PROPOSITION 5.5. (The onion peeling proposition) *Let $X \subset Y$ be nearest neighbour shifts of finite type, r be the fold-radius of X, Z be a stiff shift obtained by a sequence of config-folds starting with X and y^1, $y^2 \in Y$ such that $y^1|_{D^c_{n-1}} \in \mathcal{L}_{D^c_{n-1}}(X)$ and $y^2|_{D_{n+1}} \in \mathcal{L}_{D_{n+1}}(X)$. Let z^1, $z^2 \in Y$ be given by*

$$z^1 := f_{X_{r-1},D_{n+r-1}}(f_{X_{r-2},D_{n+r-2}}(\ldots(f_{X_0,D_n}(y^1))\ldots)),$$
$$z^2 := f_{X_{r-1},D^c_{n-r+1}}(f_{X_{r-2},D^c_{n-r+2}}(\ldots(f_{X_0,D^c_n}(y^2))\ldots)) \quad \textit{for } n > r.$$

The patterns $z^1|_{D^c_{n+r-1}} \in \mathcal{L}_{D^c_{n+r-1}}(Z)$ and $z^2|_{D_{n-r+1}} \in \mathcal{L}_{D_{n-r+1}}(Z)$. If y^1, $y^2 \in X$ then in addition

$$z^1|_{D^c_{n+r-1}} = F_X(y^1)|_{D^c_{n+r-1}}$$

and

$$z^2|_{D_{n-r+1}} = F_X(y^2)|_{D_{n-r+1}}.$$

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Abusing the notation, in such cases we shall denote the configurations z^1 and z^2 by $I_{X,n}(y^1)$ and $O_{X,n}(y^2)$, respectively. Note that $I_{X,n}(y^1)|_{D_n} = y^1|_{D_n}$ and $O_{X,n}(y^2)|_{D_n^c} = y^2|_{D_n^c}$. Also, $O_{X,n}$ is a composition of maps of the form $f_{X,A}$, where A^c is finite, and there is a chain of pivots in Y from y to $O_{X,n}(y)$. There are two kinds of stiff shifts which will be of interest to us. A configuration $x \in \mathcal{A}^{\mathbb{Z}^d}$ is called *periodic* if there exists $n \in \mathbb{N}$ such that $\sigma^{n\vec{e}_i}(x) = x$ for all $1 \leq i \leq d$. A configuration $x \in X$ is called *frozen* if its homoclinic class is a singleton. This notion coincides with the notion of frozen colouring in [**4**]. A subshift X will be called *frozen* if it consists of frozen configurations: equivalently Δ_X is the diagonal. A measure on X will be called *frozen* if its support is frozen. Note that any shift space consisting just of periodic configurations is frozen. All frozen nearest neighbour shifts of finite type are stiff. Suppose X is a nearest neighbour shift of finite type which is not stiff. Then there is a symbol v which can be config-folded to a symbol w. This means that any appearance of v in a configuration $x \in X$ can be replaced by w. Hence the homoclinic class of x is not a singleton. Therefore X is not frozen.

PROPOSITION 5.6. *Let X be a nearest neighbour shift of finite type such that a sequence of config-folds starting from X results in the orbit of a periodic configuration. Then every shift-invariant probability measure adapted to X is fully supported.*

PROPOSITION 5.7. *Let X be a nearest neighbour shift of finite type such that a sequence of config-folds starting from X results in a frozen shift. Then X has the pivot property.*

Examples.

(1) $X := \{0\}^{\mathbb{Z}^d} \cup \{1\}^{\mathbb{Z}^d}$ is a frozen shift space but not the orbit of a periodic configuration. Clearly, the delta measure $\delta_{\{0\}^{\mathbb{Z}^d}}$ is a shift-invariant probability measure adapted to X but not fully supported. A more non-trivial example of a nearest neighbour shift of finite type which is frozen but not the orbit of a periodic configuration is the set of the Robinson tilings Y [**27**]. There are configurations in Y which have the so-called 'fault lines'; they can occur at most once in a given configuration. Consequently for all shift-invariant probability measures on Y, the probability of seeing a fault line is zero. Thus no shift-invariant probability measure (and hence no adapted shift-invariant probability measure) on Y is fully supported.

(2) Let X be a shift space with a safe symbol $\star$. Then any symbol in X can be config-folded into the safe symbol. By config-folding the symbols one-by-one, we obtain a fixed point $\{\star\}^{\mathbb{Z}^d}$. Thus any nearest neighbour shift of finite type with a safe symbol satisfies the hypothesis of both the propositions.

(3) Suppose $\mathcal{H}$ is a graph which folds into a single edge (denoted by $Edge$) or a single vertex v with a loop. Then the shift space $X_{\mathcal{H}}$ can be config-folded to X_{Edge} (which consists of two periodic configurations) or the fixed point $\{v\}^{\mathbb{Z}^d}$, respectively. In the latter case, the graph $\mathcal{H}$ is called *dismantlable* [**21**]. Note that finite non-trivial trees and the graph C_4 fold into an edge. For dismantlable graphs $\mathcal{H}$, [**4**, Theorem 4.1] implies the conclusions of Propositions 5.6 and 5.7 for $X_{\mathcal{H}}$ as well.

Proof of Proposition 5.6. Let μ be a shift-invariant probability measure adapted to X. To prove that $\text{supp}(\mu) = X$ it is sufficient to prove that $\mu([x]_{D_n}) > 0$ for all $n \in \mathbb{N}$ and

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$x \in X$. Let $X_0 = X$, $X_1, X_2, \ldots, X_r$ be a sequence of full config-folds where $X_r := \{\sigma^{\vec{i}_1}(p), \sigma^{\vec{i}_2}(p), \ldots, \sigma^{\vec{i}_{k-1}}(p)\}$ is the orbit of a periodic point. For any two configurations $z, w \in X$ there exists $\vec{i} \in \mathbb{Z}^d$ such that $F_X(z) = F_X(\sigma^{\vec{i}}(w))$. Since μ is shift-invariant, we can choose $y \in \mathrm{supp}(\mu)$ such that $F_X(x) = F_X(y)$. Consider the configurations $I_{X,n}(x)$ and $O_{X,n+2r-1}(y)$. By Proposition 5.5 they satisfy the equations

$$I_{X,n}(x)|_{D^c_{n+r-1}} = F_X(x)|_{D^c_{n+r-1}}$$

and

$$O_{X,n+2r-1}(y)|_{D_{n+r}} = F_X(y)|_{D_{n+r}}.$$

Then $I_{X,n}(x)|_{\partial D_{n+r-1}} = O_{X,n+2r-1}(y)|_{\partial D_{n+r-1}}$. Since X is a nearest neighbour shift of finite type, the configuration z given by

$$\begin{aligned} z|_{D_{n+r}} &:= I_{X,n}(x)|_{D_{n+r}}, \\ z|_{D^c_{n+r-1}} &:= O_{X,n+2r-1}(y)|_{D^c_{n+r-1}} \end{aligned}$$

is an element of X. Moreover,

$$\begin{aligned} z|_{D_n} = I_{X,n}(x)|_{D_n} = x|_{D_n}, \\ z|_{D^c_{n+2r-1}} = O_{X,n+2r-1}(y)|_{D^c_{n+2r-1}} = y|_{D^c_{n+2r-1}}. \end{aligned}$$

Thus $(y, z) \in \Delta_X$. Since μ is adapted, we get that $z \in \mathrm{supp}(\mu)$. Finally,

$$\mu([x]_{D_n}) = \mu([z]_{D_n}) > 0. \qquad \square$$

Note that all the maps being discussed here, f_X, $f_{X,A}$, F_X, $I_{X,n}$ and $O_{X,n}$ are (not necessarily shift-invariant) single block maps: that is, maps f where $(f(x))_{\vec{i}}$ depends only on $x_{\vec{i}}$. Thus, if f is one such map and $x|_A = y|_A$ for some set $A \subset \mathbb{Z}^d$, then $f(x)|_A = f(y)|_A$: they map homoclinic pairs to homoclinic pairs.

Proof of Proposition 5.7. Let $X_0 = X$, $X_1, X_2, \ldots, X_r$ be a sequence of full config-folds where X_r is frozen. Let $(x, y) \in \Delta_X$. Since X_r is frozen, $F_X(x) = F_X(y)$. Suppose $x|_{D^c_n} = y|_{D^c_n}$ for some $n \in \mathbb{N}$. Then $O_{X,n+r-1}(x)|_{D^c_n} = O_{X,n+r-1}(y)|_{D^c_n}$. Also by Proposition 5.5,

$$O_{X,n+r-1}(x)|_{D_n} = F_X(x)|_{D_n} = F_X(y)|_{D_n} = O_{X,n+r-1}(y)|_{D_n}.$$

This proves that $O_{X,n+r-1}(x) = O_{X,n+r-1}(y)$. In fact, it completes the proof since, for all $z \in X$, there exists a chain of pivots in X from z to $O_{X,n+r-1}(z)$. $\square$

6. *Universal covers*

Most cases will not be as simple as in the proofs of Propositions 5.6 and 5.7. We wish to prove the conclusions of these propositions for hom-shifts $X_{\mathcal{H}}$ when $\mathcal{H}$ is a connected four-cycle free graph. Many ideas carry over from the proofs of these results because of the relationship of such graphs with their universal covers; we describe this relationship next. The results in this section are not original: look for instance in [**35**]. We mention them for completeness.

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Let $\mathcal{H}$ be a finite connected graph with no self-loops. We denote by $d_{\mathcal{H}}$ the ordinary graph distance on $\mathcal{H}$ and by $D_{\mathcal{H}}(u)$, the *ball of radius 1* around u. A graph homomorphism $\pi : \mathcal{C} \longrightarrow \mathcal{H}$ is called a *covering map* if, for some $n \in \mathbb{N} \cup \{\infty\}$ and all $u \in \mathcal{H}$, there exist disjoint sets $\{C_i\}_{i=1}^{n} \subset \mathcal{C}$ such that $\pi^{-1}(D_{\mathcal{H}}(u)) = \bigcup_{i=1}^{n} C_i$ and $\pi|_{C_i} : C_i \longrightarrow D_{\mathcal{H}}(u)$ is an isomorphism of the induced subgraphs for $1 \le i \le n$. A *covering space* of a graph $\mathcal{H}$ is a graph $\mathcal{C}$ such that there exists a covering map $\pi : \mathcal{C} \longrightarrow \mathcal{H}$.

A *universal covering space* of $\mathcal{H}$ is a covering space of $\mathcal{H}$ which is a tree. Unique up to graph isomorphism [**35**], these covers can be described in multiple ways. Their standard construction uses non-backtracking walks [**1**]: a *walk* on $\mathcal{H}$ is a sequence of vertices $(v_1, v_2, \ldots, v_n)$ such that $v_i \sim_{\mathcal{H}} v_{i+1}$ for all $1 \le i \le n-1$. The *length* of a walk $p = (v_1, v_2, \ldots, v_n)$ is $|p| = n-1$, the number of edges traversed on that walk. It is called *non-backtracking* if $v_{i-1} \ne v_{i+1}$ for all $2 \le i \le n-1$: that is, successive steps do not traverse the same edge. Choose a vertex $u \in \mathcal{H}$. The vertex set of the universal cover is the set of all non-backtracking walks on $\mathcal{H}$ starting from u; there is an edge between two such walks if one extends the other by a single step. The choice of the starting vertex u is arbitrary; choosing a different vertex gives rise to an isomorphic graph. We denote the universal cover by $E_{\mathcal{H}}$. The covering map $\pi : E_{\mathcal{H}} \longrightarrow \mathcal{H}$ maps a walk to its terminal vertex. Usually, we will denote by $\tilde{u}$, $\tilde{v}$ and $\tilde{w}$ the vertices of $E_{\mathcal{H}}$ such that $\pi(\tilde{u}) = u$, $\pi(\tilde{v}) = v$ and $\pi(\tilde{w}) = w$.

This construction shows that the universal cover of a graph is finite if and only if it is a finite tree. To see this, if the graph has a cycle, then the finite segments of the walk looping around the cycle give us an infinite number of vertices for the universal cover. If the graph is a finite tree, then all walks must terminate at the leaves and their length is bounded by the diameter of the tree. In fact, the universal cover of a tree is itself while the universal cover of a cycle (for instance C_4) is $\mathbb{Z}$ obtained by finite segments of the walks $(1, 2, 3, 0, 1, 2, 3, 0, \ldots)$ and $(1, 0, 3, 2, 1, 0, 3, 2, \ldots)$.

Following the ideas of homotopies in algebraic topology, there is a natural operation on the set of walks: two walks can be joined together if one begins where the other one ends. More formally, given two walks $p = (v_1, v_2, \ldots, v_n)$ and $q = (w_1, w_2, \ldots, w_m)$, where $v_n = w_1$, consider $p \star q = (v_1, v_2, \ldots, v_n, w_2, w_3, \ldots, w_m)$. However, even when p and q are non-backtracking, $p \star q$ need not be non-backtracking. So, instead, we consider the walk $[p \star q]$, which erases the backtracking segments of $p \star q$: that is, if for some $i > 1$, $v_{n-i+1} \ne w_i$ and $v_{n-j+1} = w_j$ for all $1 \le j \le i-1$, then

$$[p \star q] := (v_1, v_2, \ldots, v_{n-i+1}, w_{i-1}, w_i, \ldots, w_m).$$

This operation of erasing the backtracking segments is called *reduction* (look, for instance, in [**35**]). The following proposition is well known [**35**, §4] and shall also be useful in our context.

PROPOSITION 6.1. *Let $\mathcal{H}$ be a finite connected graph without any self-loops. Then for all $\tilde{v}, \tilde{w} \in E_{\mathcal{H}}$ satisfying $\pi(\tilde{v}) = \pi(\tilde{w})$ there exists a graph isomorphism $\phi : E_{\mathcal{H}} \longrightarrow E_{\mathcal{H}}$ such that $\phi(\tilde{v}) = \tilde{w}$ and $\pi \circ \phi = \pi$.*

To see how to construct this isomorphism, consider, as an example, (u), the empty walk on $\mathcal{H}$ and $(v_1, v_2, \ldots, v_n)$, some non-backtracking walk such that $v_1 = v_n = u$. Then the

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map $\phi : E_{\mathcal{H}} \longrightarrow E_{\mathcal{H}}$ given by

$$\phi(\tilde{w}) := [(v_1, v_2, \ldots, v_n) \star \tilde{w}]$$

is a graph isomorphism which maps (u) to $(v_1, v_2, \ldots, v_n)$; its inverse is $\psi : E_{\mathcal{H}} \longrightarrow E_{\mathcal{H}}$ given by

$$\psi(\tilde{w}) := [(v_n, v_{n-1}, \ldots, v_1) \star \tilde{w}].$$

The maps ϕ, π described above give rise to natural maps, also denoted by ϕ and π, where

$$\phi : X_{E_{\mathcal{H}}} \longrightarrow X_{E_{\mathcal{H}}}$$

is given by $\phi(\tilde{x})_{\vec{i}} := \phi(\tilde{x}_{\vec{i}})$ and

$$\pi : X_{E_{\mathcal{H}}} \longrightarrow X_{\mathcal{H}}$$

is given by $\pi(\tilde{x})_{\vec{i}} := \pi(\tilde{x}_{\vec{i}})$ for all $\vec{i} \in \mathbb{Z}^d$, respectively. A *lift* of a configuration $x \in X_{\mathcal{H}}$ is a configuration $\tilde{x} \in X_{E_{\mathcal{H}}}$ such that $\pi \circ \tilde{x} = x$.

Now we shall analyse some consequences of this formalism in our context. More general statements (where $\mathbb{Z}^d$ is replaced by a different graph) are true (under a different hypothesis on $\mathcal{H}$), but we restrict to the four-cycle free condition. We noticed in §5 that if $\mathcal{H}$ is a tree, then $X_{\mathcal{H}}$ satisfies the conclusions of Theorems 3.4 and 4.1. Now we will draw a connection between the four-cycle free condition on $\mathcal{H}$ and the formalism in §5.

PROPOSITION 6.2. (Existence of Lifts) *Let $\mathcal{H}$ be a connected four-cycle free graph. For all $x \in X_{\mathcal{H}}$ there exists $\tilde{x} \in X_{E_{\mathcal{H}}}$ such that $\pi(\tilde{x}) = x$. Moreover, the lift $\tilde{x}$ is unique up to a choice of $\tilde{x}_{\vec{0}}$.*

Proof. We will begin by constructing a sequence of graph homomorphisms $\tilde{x}^n : D_n \longrightarrow E_{\mathcal{H}}$ such that $\pi \circ \tilde{x}^n = x|_{D_n}$ and $\tilde{x}^m|_{D_n} = \tilde{x}^n$ for all $m > n$. Then, by taking the limit of these graph homomorphisms, we obtain a graph homomorphism $\tilde{x} \in X_{E_{\mathcal{H}}}$ such that $\pi \circ \tilde{x} = x$. It will follow that, given $\tilde{x}^0$, the sequence $\tilde{x}^n$ is completely determined proving that the lifting is unique up to a choice of $\tilde{x}_{\vec{0}}$.

The recursion is the following. Let $\tilde{x}^n : D_n \longrightarrow E_{\mathcal{H}}$ be a given graph homomorphism for some $n \in \mathbb{N} \cup \{0\}$ such that $\pi \circ \tilde{x}^n = x|_{D_n}$. For any $\vec{i} \in D_{n+1} \setminus D_n$, choose a vertex $\vec{j} \in D_n$ such that $\vec{j} \sim \vec{i}$. Then $\pi(\tilde{x}^n_{\vec{j}}) = x_{\vec{j}} \sim x_{\vec{i}}$. Since π defines a local isomorphism between $E_{\mathcal{H}}$ and $\mathcal{H}$, there exists a unique vertex $\tilde{v}_{\vec{i}} \sim \tilde{x}^n_{\vec{j}} \in E_{\mathcal{H}}$ such that $\pi(\tilde{v}_{\vec{i}}) = x_{\vec{i}}$. Define $\tilde{x}^{n+1} : D_{n+1} \longrightarrow E_{\mathcal{H}}$ by

$$\tilde{x}^{n+1}_{\vec{i}} := \begin{cases} \tilde{x}^n_{\vec{i}} & \text{if } \vec{i} \in D_n, \\ \tilde{v}_{\vec{i}} & \text{if } \vec{i} \in D_{n+1} \setminus D_n. \end{cases}$$

Then, clearly, $\pi \circ \tilde{x}^{n+1} = x|_{D_{n+1}}$ and $\tilde{x}^{n+1}|_{D_n} = \tilde{x}^n$. Note that the extension $\tilde{x}^{n+1}$ is uniquely defined given $\tilde{x}^n$. We need to prove that this defines a valid graph homomorphism from D_{n+1} to $E_{\mathcal{H}}$. Let $\vec{i} \in D_{n+1} \setminus D_n$ and $\vec{j} \in D_n$ be chosen as described above. Consider, if possible, any $\vec{j}' \neq \vec{j} \in D_n$ such that $\vec{j}' \sim \vec{i}$. To prove that $\tilde{x}^{n+1}$ is a graph homomorphism we need to verify that $\tilde{x}^{n+1}_{\vec{j}'} \sim \tilde{x}^{n+1}_{\vec{i}}$.

Consider $\vec{i}' \in D_n$ such that $\vec{i}' \sim \vec{j}$ and $\vec{j}'$. Then $\vec{i}'$, $\vec{j}$, $\vec{i}$ and $\vec{j}'$ form a four-cycle. Since $\mathcal{H}$ is four-cycle free, either $x_{\vec{i}'} = x_{\vec{i}}$ or $x_{\vec{j}'} = x_{\vec{j}}$.

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Suppose $x_{\vec{i}'} = x_{\vec{i}}$; the other case is similar. Since π is a local isomorphism and $\tilde{x}^{n+1}_{\vec{i}}, \tilde{x}^{n+1}_{\vec{i}'} \sim \tilde{x}^{n+1}_{\vec{j}}$, we get that $\tilde{x}^{n+1}_{\vec{i}} = \tilde{x}^{n+1}_{\vec{i}'}$. But $\vec{i}', \vec{j}' \in D_n$ and $\tilde{x}^{n+1}|_{D_n} = \tilde{x}^n$ is a graph homomorphism. Therefore $\tilde{x}^{n+1}_{\vec{i}} = \tilde{x}^{n+1}_{\vec{i}'} \sim \tilde{x}^{n+1}_{\vec{j}'}$. □

COROLLARY 6.3. *Let $\mathcal{H}$ be a connected four-cycle free graph and $x, y \in X_{\mathcal{H}}$. Consider some lifts $\tilde{x}, \tilde{y} \in X_{E_{\mathcal{H}}}$ such that $\pi(\tilde{x}) = x$ and $\pi(\tilde{y}) = y$. If, for some $\vec{i}_0 \in \mathbb{Z}^d$, $\tilde{x}_{\vec{i}_0} = \tilde{y}_{\vec{i}_0}$, then $\tilde{x} = \tilde{y}$ on the connected subset of*

$$\{\vec{j} \in \mathbb{Z}^d \mid x_{\vec{j}} = y_{\vec{j}}\}$$

which contains $\vec{i}_0$.

Proof. Let D be the connected component of $\{\vec{i} \in \mathbb{Z}^d \mid x_{\vec{i}} = y_{\vec{i}}\}$ and $\tilde{D}$ be the connected component of $\{\vec{i} \in \mathbb{Z}^d \mid \tilde{x}_{\vec{i}} = \tilde{y}_{\vec{i}}\}$ which contain $\vec{i}_0$.

Clearly, $\tilde{D} \subset D$. Suppose $\tilde{D} \neq D$. Since both D and $\tilde{D}$ are non-empty, there exist connected sets $\vec{i} \in D \setminus \tilde{D}$ and $\vec{j} \in \tilde{D}$ such that $\vec{i} \sim \vec{j}$. Then $x_{\vec{i}} = y_{\vec{i}}$, $x_{\vec{j}} = y_{\vec{j}}$ and $\tilde{x}_{\vec{j}} = \tilde{y}_{\vec{j}}$. Since π is a local isomorphism, the lift must satisfy $\tilde{x}_{\vec{i}} = \tilde{y}_{\vec{i}}$, implying that $\vec{i} \in \tilde{D}$. This proves that $D = \tilde{D}$. □

The following corollary says that any two lifts of the same graph homomorphism are 'identical'.

COROLLARY 6.4. *Let $\mathcal{H}$ be a connected four-cycle free graph. Then, for all $\tilde{x}^1, \tilde{x}^2 \in X_{E_{\mathcal{H}}}$ satisfying $\pi(\tilde{x}^1) = \pi(\tilde{x}^2) = x$, there exists an isomorphism $\phi : E_{\mathcal{H}} \longrightarrow E_{\mathcal{H}}$ such that $\phi \circ \tilde{x}^1 = \tilde{x}^2$.*

Proof. By Proposition 6.1, there exists an isomorphism $\phi : E_{\mathcal{H}} \longrightarrow E_{\mathcal{H}}$ such that $\phi(\tilde{x}^1_{\vec{0}}) = \tilde{x}^2_{\vec{0}}$ and $\pi \circ \phi = \pi$. Then $(\phi \circ \tilde{x}^1)_{\vec{0}} = \tilde{x}^2_{\vec{0}}$ and $\pi(\phi \circ \tilde{x}^1) = (\pi \circ \phi)(\tilde{x}^1) = \pi(\tilde{x}^1) = x$. By Proposition 6.2, $\phi \circ \tilde{x}^1 = \tilde{x}^2$. □

At this point, it is worth noting the relationship between the universal cover described here and the universal cover in algebraic topology. Undirected graphs can be identified with one dimensional CW-complexes where the set of vertices correspond to the 0-cells, the edges to the 1-cells of the complex, and the attaching map sends the end-points of the edges to their respective vertices. With this correspondence in mind, the (topological) universal covering space coincides with the (combinatorial) universal covering space described above. Indeed, a one dimensional CW-complex is simply connected if and only if it does not have any loops: that is, the corresponding graph does not have any cycles (it is a tree). The uniqueness, existence and many such facts about the universal covering space follow from purely topological arguments (for instance, look in [**19**, Ch. 13] or [**17**, Ch. 5 and 6]).

7. *Height functions and sub-cocycles*

Existence of lifts, as described in the previous section, enables us to measure the 'rigidity' of configurations. In this section, we define height functions and subsequently the slope

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of configurations, where steepness corresponds to this 'rigidity'. The general method of using height functions is usually attributed to J. H. Conway [**36**].

Fix a connected four-cycle free graph $\mathcal{H}$. Given $x \in X_{\mathcal{H}}$ we can define the corresponding *height function* $h_x : \mathbb{Z}^d \times \mathbb{Z}^d \longrightarrow \mathbb{Z}$ given by $h_x(\vec{i}, \vec{j}) := d_{E_{\mathcal{H}}}(\tilde{x}_{\vec{i}}, \tilde{x}_{\vec{j}})$, where $\tilde{x}$ is a lift of x. It follows, from Corollary 6.4, that h_x is independent of the lift $\tilde{x}$.

Given a finite subset $A \subset \mathbb{Z}^d$ and $x \in X_{\mathcal{H}}$ we define the *range of x on A* as

$$\text{Range}_A(x) := \max_{\vec{j}_1, \vec{j}_2 \in A} h_x(\vec{j}_1, \vec{j}_2).$$

For all $x \in X_{\mathcal{H}}$

$$\text{Range}_A(x) \leq \text{Diameter}(A)$$

and more specifically

$$\text{Range}_{D_n}(x) \leq 2n \tag{7.1}$$

for all $n \in \mathbb{N}$. Since $\tilde{x} \in X_{E_{\mathcal{H}}}$ is a map between bipartite graphs, it preserves the parity of the distance function: that is, if $\vec{i}, \vec{j} \in \mathbb{Z}^d$ and $x \in X_{\mathcal{H}}$, then the parity of $\|\vec{i} - \vec{j}\|_1$ is the same as that of $h_x(\vec{i}, \vec{j})$. As a consequence, it follows that $\text{Range}_{\partial D_n}(x)$ is even for all $x \in X_{\mathcal{H}}$ and $n \in \mathbb{N}$. We note that

$$\text{Range}_A(x) = \text{Diameter}(\text{Image}(\tilde{x}|_A)).$$

The height function h_x is subadditive: that is,

$$h_x(\vec{i}, \vec{j}) \leq h_x(\vec{i}, \vec{k}) + h_x(\vec{k}, \vec{j})$$

for all $x \in X_{\mathcal{H}}$ and $\vec{i}, \vec{j}$ and $\vec{k} \in \mathbb{Z}^d$. This is in contrast with the usual height function (as in [**7**, **24**]) where there is an equality instead of the inequality. This raises some technical difficulties which are partly handled by the subadditive ergodic theorem.

The following terminology is not completely standard. Given a shift space X, a *sub-cocycle* is a measurable map $c : X \times \mathbb{Z}^d \longrightarrow \mathbb{N} \cup \{0\}$ such that for all $\vec{i}, \vec{j} \in \mathbb{Z}^d$

$$c(x, \vec{i} + \vec{j}) \leq c(x, \vec{i}) + c(\sigma^{\vec{i}}(x), \vec{j}).$$

Sub-cocycles arise in a variety of situations (look, for instance, in [**12**]). We are interested in the case $c(x, \vec{i}) = h_x(\vec{0}, \vec{i})$ for all $x \in X_{\mathcal{H}}$ and $\vec{i} \in \mathbb{Z}^d$. The measure of 'rigidity' lies in the asymptotics of this sub-cocycle, the existence of which is provided by the subadditive ergodic theorem. Given a set X, if $f : X \longrightarrow \mathbb{R}$ is a function, then let $f^+ := \max(0, f)$.

THEOREM 7.1. (Subadditive ergodic theorem [**37**, Theorem 10.1]) *Let $(X, \mathcal{B}, \mu)$ be a probability space and let $T : X \longrightarrow X$ be measure preserving. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions $f_n : X \longrightarrow \mathbb{R} \cup \{-\infty\}$ satisfying the conditions:*

(a) $f_1^+ \in L^1(\mu)$*; and*

(b) *for each m, $n \geq 1$, $f_{n+m} \leq f_n + f_m \circ T^n$ μ-almost everywhere.*

Then there exists a measurable function $f : X \longrightarrow \mathbb{R} \cup \{-\infty\}$ such that $f^+ \in L^1(\mu)$, $f \circ T = f$, $\lim_{n\to\infty} (1/n) f_n = f$, μ-almost everywhere and

$$\lim_{n \longrightarrow \infty} \frac{1}{n} \int f_n \, d\mu = \inf_n \frac{1}{n} \int f_n \, d\mu = \int f \, d\mu.$$

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Given a direction $\vec{i} = (i_1, i_2, \ldots, i_d) \in \mathbb{R}^d$, let $\lfloor \vec{i} \rfloor = (\lfloor i_1 \rfloor, \lfloor i_2 \rfloor, \ldots, \lfloor i_d \rfloor)$. We define, for all $x \in X_{\mathcal{H}}$, the *slope of x in the direction $\vec{i}$* as

$$sl_{\vec{i}}(x) := \lim_{n \longrightarrow \infty} \frac{1}{n} h_x(\vec{0}, \lfloor n\vec{i} \rfloor),$$

whenever it exists.

If $\vec{i} \in \mathbb{Z}^d$ we note that the sequence of functions $f_n : X_{\mathcal{H}} \longrightarrow \mathbb{N} \cup \{\vec{0}\}$ given by

$$f_n(x) = h_x(\vec{0}, n\vec{i})$$

satisfies the hypothesis of this theorem for any shift-invariant probability measure on $X_{\mathcal{H}}$: $|f_1| \leq \|\vec{i}\|_1$ and the subadditivity condition in the theorem is just a restatement of the sub-cocycle condition described above: that is, if $T = \sigma^{\vec{i}}$, then

$$f_{n+m}(x) = h_x(\vec{0}, (n+m)\vec{i}) \leq h_x(\vec{0}, n\vec{i}) + h_{\sigma^{n\vec{i}}x}(\vec{0}, m\vec{i}) = f_n(x) + f_m(T^n(x)).$$

The asymptotics of the height functions (or, more generally, the sub-cocycles) are a consequence of the subadditive ergodic theorem as we will describe next. In the following, by an ergodic measure on $X_{\mathcal{H}}$ we mean a probability measure on $X_{\mathcal{H}}$ which is ergodic with respect to the $\mathbb{Z}^d$-shift action on $X_{\mathcal{H}}$.

PROPOSITION 7.2. (Existence of slopes) *Let $\mathcal{H}$ be a connected four-cycle free graph and μ be an ergodic measure on $X_{\mathcal{H}}$. Then for all $\vec{i} \in \mathbb{Z}^d$*

$$sl_{\vec{i}}(x) = \lim_{n \longrightarrow \infty} \frac{1}{n} h_x(\vec{0}, n\vec{i})$$

exists almost everywhere and is independent of x. Moreover, if $\vec{i} = (i_1, i_2 \ldots, i_d)$, then

$$sl_{\vec{i}}(x) \leq \sum_{k=1}^{d} |i_k| sl_{\vec{e}_k}(x).$$

Proof. Fix a direction $\vec{i} \in \mathbb{Z}^d$. Consider the sequence of functions $\{f_n\}_{n=1}^{\infty}$ and the map $T : X_{\mathcal{H}} \longrightarrow X_{\mathcal{H}}$, as described above. By the subadditive ergodic theorem, there exists a function $f : X_{\mathcal{H}} \longrightarrow \mathbb{R} \cup \{-\infty\}$ such that

$$\lim_{n \to \infty} \frac{1}{n} f_n = f \text{ almost everywhere.}$$

Note that $f = sl_{\vec{i}}$. Since, for all $x \in X_{\mathcal{H}}$ and $n \in \mathbb{N}$, $0 \leq f_n \leq n\|\vec{i}\|_1$, $0 \leq f(x) \leq \|\vec{i}\|_1$, whenever it exists. Fix any $\vec{j} \in \mathbb{Z}^d$. Then

$$f_n(\sigma^{\vec{j}}(x)) = h_{\sigma^{\vec{j}}(x)}(\vec{0}, n\vec{i}) = h_x(\vec{j}, n\vec{i} + \vec{j})$$

and hence

$$\begin{aligned} -h_x(\vec{j}, \vec{0}) + h_x(\vec{0}, n\vec{i}) - h_x(n\vec{i}, n\vec{i} + \vec{j}) &\leq f_n(\sigma^{\vec{j}}(x)) \\ &\leq h_x(\vec{j}, \vec{0}) + h_x(\vec{0}, n\vec{i}) + h_x(n\vec{i}, n\vec{i} + \vec{j}) \end{aligned}$$

implying

$$-2\|\vec{j}\|_1 + f_n(x) \leq f_n(\sigma^{\vec{j}}(x)) \leq 2\|\vec{j}\|_1 + f_n(x)$$

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implying

$$f(x) = \lim_{n\longrightarrow\infty} \frac{1}{n} f_n(x) = \lim_{n\longrightarrow\infty} \frac{1}{n} f_n(\sigma^{\vec{j}} x) = f(\sigma^{\vec{j}}(x))$$

almost everywhere. Since μ is ergodic, $sl_{\vec{i}} = f$ is constant almost everywhere. Let $\vec{i}^{(k)} = (i_1, i_2, \ldots, i_k, 0, \ldots, 0) \in \mathbb{Z}^d$. By the subadditive ergodic theorem

$$\begin{aligned} sl_{\vec{i}}(x) = \int sl_{\vec{i}}(x)\, d\mu &= \lim_{n\longrightarrow\infty} \frac{1}{n} \int h_x(\vec{0}, n\vec{i})\, d\mu \\ &\le \sum_{k=1}^{d} \lim_{n\longrightarrow\infty} \frac{1}{n} \int h_{\sigma^{n\vec{i}^{(k-1)}}(x)}(\vec{0}, ni_k\vec{e}_k)\, d\mu \\ &= \sum_{k=1}^{d} \lim_{n\longrightarrow\infty} \frac{1}{n} \int h_x(\vec{0}, ni_k\vec{e}_k)\, d\mu \\ &\le \sum_{k=1}^{d} |i_k| \lim_{n\longrightarrow\infty} \frac{1}{n} \int h_x(\vec{0}, n\vec{e}_k)\, d\mu \\ &= \sum_{k=1}^{d} |i_k| sl_{\vec{e}_k}(x) \end{aligned}$$

almost everywhere. □

COROLLARY 7.3. *Let $\mathcal{H}$ be a connected four-cycle free graph. Suppose μ is an ergodic measure on $X_{\mathcal{H}}$. Then for all $\vec{i} \in \mathbb{R}^d$*

$$sl_{\vec{i}}(x) = \lim_{n\longrightarrow\infty} \frac{1}{n} h_x(\vec{0}, \lfloor n\vec{i} \rfloor)$$

exists almost everywhere and is independent of x. Moreover, if $\vec{i} = (i_1, i_2, \ldots, i_d)$, then

$$sl_{\vec{i}}(x) \le \sum_{k=1}^{d} |i_k| sl_{\vec{e}_k}(x).$$

Proof. Let $\vec{i} \in \mathbb{Q}^d$ and $N \in \mathbb{N}$ such that $N\vec{i} \in \mathbb{Z}^d$. For all $n \in \mathbb{N}$ there exists $k \in \mathbb{N} \cup \{0\}$ and $0 \le m \le N-1$ such that $n = kN + m$. Then for all $x \in X_{\mathcal{H}}$

$$h_x(\vec{0}, kN\vec{i}) - N\|\vec{i}\|_1 \le h_x(\vec{0}, \lfloor n\vec{i} \rfloor) \le h_x(\vec{0}, kN\vec{i}) + N\|\vec{i}\|_1$$

proving

$$sl_{\vec{i}}(x) = \lim_{n\longrightarrow\infty} \frac{1}{n} h_x(\vec{0}, \lfloor n\vec{i} \rfloor) = \frac{1}{N} \lim_{k\longrightarrow\infty} \frac{1}{k} h_x(\vec{0}, kN\vec{i}) = \frac{1}{N} sl_{N\vec{i}}(x)$$

almost everywhere. Since $sl_{N\vec{i}}$ is constant almost everywhere, we have that $sl_{\vec{i}}$ is constant almost everywhere as well. Denote the constant by $c_{\vec{i}}$. Also,

$$sl_{\vec{i}}(x) \le \frac{1}{N} \sum_{l=1}^{d} |Ni_l| sl_{\vec{e}_l}(x) = \sum_{l=1}^{d} |i_l| sl_{\vec{e}_l}(x).$$

Let $X \subset X_{\mathcal{H}}$ be the set of configurations x such that

$$\lim_{n\longrightarrow\infty} \frac{1}{n} h_x(\vec{0}, \lfloor n\vec{i} \rfloor) = c_{\vec{i}}$$

for all $\vec{i} \in \mathbb{Q}^d$. We have proved that $\mu(X) = 1$.

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Fix $x \in X$. Let $\vec{i}, \vec{j} \in \mathbb{R}^d$ such that $\|\vec{i} - \vec{j}\|_1 < \epsilon$. Then

$$\left|\frac{1}{n}h_x(\vec{0}, \lfloor n\vec{i}\rfloor) - \frac{1}{n}h_x(\vec{0}, \lfloor n\vec{j}\rfloor)\right| \leq \frac{1}{n}\|\lfloor n\vec{i}\rfloor - \lfloor n\vec{j}\rfloor\|_1 \leq \epsilon + \frac{2d}{n}.$$

Thus we can approximate $(1/n)h_x(\vec{0}, \lfloor n\vec{i}\rfloor)$ for $\vec{i} \in \mathbb{R}^d$ by $(1/n)h_x(\vec{0}, \lfloor n\vec{j}\rfloor)$ for $\vec{j} \in \mathbb{Q}^d$ to prove that $\lim_{n\longrightarrow\infty}(1/n)h_x(\vec{0}, \lfloor n\vec{i}\rfloor)$ exists for all $\vec{i} \in \mathbb{R}^d$, is independent of $x \in X$ and satisfies

$$sl_{\vec{i}}(x) \leq \sum_{k=1}^{d} |i_k| sl_{\vec{e}_k}(x). \qquad \square$$

The existence of slopes can be generalized from height functions to continuous sub-cocycles. The same proofs work.

PROPOSITION 7.4. *Let $c : X \times \mathbb{Z}^d \longrightarrow \mathbb{R}$ be a continuous sub-cocycle and μ be an ergodic measure on X. Then for all $\vec{i} \in \mathbb{R}^d$*

$$sl^c_{\vec{i}}(x) := \lim_{n\longrightarrow\infty} \frac{1}{n}c(x, \lfloor n\vec{i}\rfloor)$$

exists almost everywhere and is independent of x. Moreover, if $\vec{i} = (i_1, i_2 \ldots, i_d)$, then

$$sl^c_{\vec{i}}(x) \leq \sum_{k=1}^{d} |i_k| sl^c_{\vec{e}_k}(x).$$

Let C_X be the space of continuous sub-cocycles on a shift space X. C_X has a natural vector space structure. Given $c_1, c_2 \in C_X$, $(c_1 + \alpha c_2)$ is also a continuous sub-cocycle on X for all $\alpha \in \mathbb{R}$ where addition and scalar multiplication is point-wise. The following is not hard to prove and follows directly from definition.

PROPOSITION 7.5. *Let X, Y be conjugate shift spaces. Then every conjugacy $f : X \longrightarrow Y$ induces a vector-space isomorphism $f^\star : C_Y \longrightarrow C_X$ given by*

$$f^\star(c)(x, \vec{i}) := c(f(x), \vec{i})$$

for all $c \in C_Y$, $x \in X$ and $\vec{i} \in \mathbb{Z}^d$. Moreover, $sl^c_{\vec{i}}(y) = sl^{f^\star(c)}_{\vec{i}}(f^{-1}(y))$ for all $y \in Y$ and $\vec{i} \in \mathbb{R}^d$ for which the slope $sl^c_{\vec{i}}(y)$ exists.

8. *Proofs of the main theorems*

Proof of Theorem 3.4. If $\mathcal{H}$ is a single edge, then $X_{\mathcal{H}}$ is the orbit of a periodic configuration; the result follows immediately. Suppose this is not the case. The proof loosely follows that of Proposition 5.6 and the ideas from [**33**]. We prove the existence of two kind of configurations in $X_{\mathcal{H}}$: ones which are 'poor' (Lemma 8.1), in the sense that they are frozen and others which are 'universal' (Lemma 8.2), for which the homoclinic class is dense.

Ideas for the following proof were inspired by discussions with Anthony Quas. A similar result in a special case is contained in [**7**, Lemma 6.7].

LEMMA 8.1. *Let $\mathcal{H}$ be a connected four-cycle free graph and μ be an ergodic probability measure on $X_{\mathcal{H}}$ such that $sl_{\vec{e}_k}(x) = 1$ almost everywhere for some $1 \leq k \leq d$. Then μ is frozen and $h_\mu = 0$.*

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Proof. Without loss of generality, assume that $sl_{\vec{e}_1}(x) = 1$ almost everywhere. By the subadditivity of the height function for all $k, n \in \mathbb{N}$ and $x \in X_{\mathcal{H}}$ we know that

$$\frac{1}{kn} h_x(\vec{0}, kn\vec{e}_1) \leq \frac{1}{kn} \sum_{m=0}^{n-1} h_x(km\vec{e}_1, k(m+1)\vec{e}_1) = \frac{1}{n} \sum_{m=0}^{n-1} \frac{1}{k} h_{\sigma^{km\vec{e}_1}(x)}(\vec{0}, k\vec{e}_1) \leq 1.$$

Since $sl_{\vec{e}_1}(x) = 1$ almost everywhere, we get that

$$\lim_{n \longrightarrow \infty} \frac{1}{n} \sum_{m=0}^{n-1} \frac{1}{k} h_{\sigma^{km\vec{e}_1}(x)}(\vec{0}, k\vec{e}_1) = 1$$

almost everywhere. By the ergodic theorem

$$\int \frac{1}{k} h_x(\vec{0}, k\vec{e}_1)\, d\mu = 1.$$

Therefore $h_x(\vec{0}, k\vec{e}_1) = k$ almost everywhere which implies that

$$h_x(\vec{i}, \vec{i} + k\vec{e}_1) = k \tag{8.1}$$

for all $\vec{i} \in \mathbb{Z}^d$ and $k \in \mathbb{N}$ almost everywhere. Let $X \subset \mathrm{supp}(\mu)$ denote the set of such configurations.

For some $n \in \mathbb{N}$ consider two patterns $a, b \in \mathcal{L}_{B_n \cup \partial_2 B_n}(\mathrm{supp}(\mu))$ such that $a|_{\partial_2 B_n} = b|_{\partial_2 B_n}$. We will prove that $a|_{B_n} = b|_{B_n}$. This will verify that μ is frozen, and $|\mathcal{L}_{B_n}(\mathrm{supp}(\mu))| \leq |\mathcal{L}_{\partial_2 B_n}(\mathrm{supp}(\mu))| \leq |\mathcal{A}|^{|\partial_2 B_n|}$, implying that $h_{\mathrm{top}}(\mathrm{supp}(\mu)) = 0$. By the variational principle this implies that $h_\mu = 0$.

Consider $x, y \in X$ such that $x|_{B_n \cup \partial_2 B_n} = a$ and $y|_{B_n \cup \partial_2 B_n} = b$. Noting that $\partial_2 B_n$ is connected, by Corollary 6.3, we can choose lifts $\tilde{x}, \tilde{y} \in X_{E_{\mathcal{H}}}$ such that $\tilde{x}|_{\partial_2 B_n} = \tilde{y}|_{\partial_2 B_n}$. Consider any $\vec{i} \in B_n$ and choose $k \in -\mathbb{N}$ such that $\vec{i} + k\vec{e}_1, \vec{i} + (2n+2+k)\vec{e}_1 \in \partial B_n$. Then, by equation (8.1), $d_{E_{\mathcal{H}}}(\tilde{x}_{\vec{i}+k\vec{e}_1}, \tilde{x}_{\vec{i}+(2n+2+k)\vec{e}_1}) = 2n+2$. But

$$(\tilde{x}_{\vec{i}+k\vec{e}_1}, \tilde{x}_{\vec{i}+(k+1)\vec{e}_1}, \ldots, \tilde{x}_{\vec{i}+(2n+2+k)\vec{e}_1})$$

and

$$(\tilde{y}_{\vec{i}+k\vec{e}_1}, \tilde{y}_{\vec{i}+(k+1)\vec{e}_1}, \ldots, \tilde{y}_{\vec{i}+(2n+2+k)\vec{e}_1})$$

are walks of length $2n+2$ from $\tilde{x}_{\vec{i}+k\vec{e}_1}$ to $\tilde{x}_{\vec{i}+(2n+2+k)\vec{e}_1}$. Since $E_{\mathcal{H}}$ is a tree and the walks are of minimal length, they must be the same. Thus $\tilde{x}|_{B_n} = \tilde{y}|_{B_n}$. Taking the image under the map π we derive that

$$a|_{B_n} = x|_{B_n} = y|_{B_n} = b|_{B_n}.$$

□

This partially justifies the claim that steep slopes lead to greater 'rigidity'. We are left to analyse the case where the slope is submaximal in every direction. As in the proof of [**7**, Proposition 7.1], we will now prove a certain mixing result for the shift space $X_{\mathcal{H}}$.

LEMMA 8.2. *Let $\mathcal{H}$ be a connected four-cycle free graph and $|\mathcal{H}| = r$. Consider any $x \in X_{\mathcal{H}}$ and some $y \in X_{\mathcal{H}}$ satisfying* $\mathrm{Range}_{\partial D_{(d+1)n+3r+k}}(y) \leq 2k$ *for some $n \in \mathbb{N}$.*

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(1) *If either $\mathcal{H}$ is not bipartite or $x_{\vec{0}}$, $y_{\vec{0}}$ are in the same partite class of $\mathcal{H}$, then there exists $z \in X_{\mathcal{H}}$ such that*

$$z_{\vec{i}} = \begin{cases} x_{\vec{i}} & if\ \vec{i} \in D_n, \\ y_{\vec{i}} & if\ \vec{i} \in D^c_{(d+1)n+3r+k}. \end{cases}$$

(2) *If $\mathcal{H}$ is bipartite and $x_{\vec{0}}$, $y_{\vec{0}}$ are in different partite classes of $\mathcal{H}$, then there exists $z \in X_{\mathcal{H}}$ such that*

$$z_{\vec{i}} = \begin{cases} x_{\vec{i}+\vec{e}_1} & if\ \vec{i} \in D_n, \\ y_{\vec{i}} & if\ \vec{i} \in D^c_{(d+1)n+3r+k}. \end{cases}$$

The separation $dn + 3r + k$ between the induced patterns of x and y is not optimal, but sufficient for our purposes.

Proof. We will construct the configuration z only in the case when $\mathcal{H}$ is not bipartite. The construction in the other cases is similar; the differences will be pointed out in the course of the proof.

(1) *Boundary patterns with non-maximal range to monochromatic patterns inside.* Let $\tilde{y}$ be a lift of y and $\mathcal{T}'$ be the image of $\tilde{y}|_{D_{(d+1)n+3r+k+1}}$. Let $\mathcal{T}$ be a minimal subtree of $E_{\mathcal{H}}$ such that

$$\text{Image}(\tilde{y}|_{\partial D_{(d+1)n+3r+k}}) \subset \mathcal{T} \subset \mathcal{T}'.$$

Since $\text{Range}_{\partial D_{(d+1)n+3r+k}}(y) \leq 2k$, $\text{diameter}(\mathcal{T}) \leq 2k$. By Proposition 5.1, there exists a graph homomorphism $f : \mathcal{T}' \longrightarrow \mathcal{T}$ such that $f|_{\mathcal{T}}$ is the identity. Consider the configuration $\tilde{y}^1$ given by

$$\tilde{y}^1_{\vec{i}} = \begin{cases} f(\tilde{y}_{\vec{i}}) & \text{if } \vec{i} \in D_{(d+1)n+3r+k+1}, \\ \tilde{y}_{\vec{i}} & \text{otherwise.} \end{cases}$$

The pattern

$$\tilde{y}^1|_{D_{(d+1)n+3r+k+1}} \in \mathcal{L}_{D_{(d+1)n+3r+k+1}}(X_{\mathcal{T}}) \subset \mathcal{L}_{D_{(d+1)n+3r+k+1}}(X_{E_{\mathcal{H}}}).$$

Moreover, since $f|_{\mathcal{T}}$ is the identity map,

$$\tilde{y}^1|_{D^c_{(d+1)n+3r+k}} = \tilde{y}|_{D^c_{(d+1)n+3r+k}} \in \mathcal{L}_{D^c_{(d+1)n+3r+k}}(X_{E_{\mathcal{H}}}).$$

Since $X_{E_{\mathcal{H}}}$ is given by nearest neighbour constraints, $\tilde{y}^1 \in X_{E_{\mathcal{H}}}$.

Recall that the fold-radius of a nearest neighbour shift of finite type (in our case $X_{\mathcal{T}}$) is the total number of full config-folds required to obtain a stiff shift. Since $\text{diameter}(\mathcal{T}) \leq 2k$, the fold-radius of $X_{\mathcal{T}} \leq k$. Let a stiff shift obtained by a sequence of config-folds starting at $X_{\mathcal{T}}$ be denoted by Z. Since $\mathcal{T}$ folds into a graph consisting of a single edge, Z consists of two chequerboard patterns in the vertices of an edge in $\mathcal{T}$, say $\tilde{v}_1$ and $\tilde{v}_2$. Corresponding to such a sequence of full config-folds, we had defined in §5 the outward fixing map $O_{X_{\mathcal{T}},(d+1)n+3r+k}$. By Proposition 5.5, the configuration $O_{X_{\mathcal{T}},(d+1)n+3r+k}(\tilde{y}^1) \in X_{E_{\mathcal{H}}}$ satisfies

$$O_{X_{\mathcal{T}},(d+1)n+3r+k}(\tilde{y}^1)|_{D_{(d+1)n+3r+1}} \in \mathcal{L}_{D_{(d+1)n+3r+1}}(Z),$$
$$O_{X_{\mathcal{T}},(d+1)n+3r+k}(\tilde{y}^1)|_{D^c_{(d+1)n+3r+k}} = \tilde{y}^1|_{D^c_{(d+1)n+3r+k}} = \tilde{y}|_{D^c_{(d+1)n+3r+k}}.$$

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Note that the pattern $O_{X_{\mathcal{T}},(d+1)n+3r+k}(\tilde{y}^1)|_{\partial D_{(d+1)n+3r}}$ uses a single symbol, say $\tilde{v}_1$. Let $\pi(\tilde{v}_1) = v_1$. Then the configuration $y' = \pi(O_{X_{\mathcal{T}},(d+1)n+3r+k}(\tilde{y}^1)) \in X_{\mathcal{H}}$ satisfies

$$y'|_{\partial D_{(d+1)n+3r}} = v_1,$$
$$y'|_{D^c_{(d+1)n+3r+k}} = y|_{D^c_{(d+1)n+3r+k}}.$$

(2) *Constant extension of an admissible pattern.* Consider some lift $\tilde{x}$ of x. We begin by extending $\tilde{x}|_{B_n}$ to a periodic configuration $\tilde{x}^1 \in X_{E_{\mathcal{H}}}$. Consider the map $f : [-n, 3n] \longrightarrow [-n, n]$ given by

$$f(k) = \begin{cases} k & \text{if } k \in [-n, n], \\ 2n - k & \text{if } k \in [n, 3n]. \end{cases}$$

Then we can construct the pattern $\tilde{a} \in \mathcal{L}_{[-n,3n]^d}(X_{E_{\mathcal{H}}})$ given by

$$\tilde{a}_{i_1,i_2,\ldots i_d} = \tilde{x}_{f(i_1),f(i_2),\ldots,f(i_d)}.$$

Given $k, l \in [-n, 3n]$, if $|k - l| = 1$ then $|f(k) - f(l)| = 1$. Thus $\tilde{a}$ is a locally allowed pattern in $X_{E_{\mathcal{H}}}$. Moreover, since $f(-n) = f(3n)$, the pattern $\tilde{a}$ is 'periodic', meaning

$$\tilde{a}_{i_1,i_2,\ldots,i_{k-1},-n,i_{k+1},\ldots,i_d} = \tilde{a}_{i_1,i_2,\ldots,i_{k-1},3n,i_{k+1},\ldots,i_d}$$

for all $i_1, i_2, \ldots, i_d \in [-n, 3n]$. Also, $\tilde{a}|_{B_n} = \tilde{x}|_{B_n}$. Then the configuration $\tilde{x}^1$ obtained by tiling $\mathbb{Z}^d$ with $\tilde{a}|_{[-n,3n-1]^d}$: that is,

$$\tilde{x}^1_{\vec{i}} = \tilde{a}_{(i_1 \bmod 4n,\ i_2 \bmod 4n,\ldots,\ i_d \bmod 4n)-(n,n,\ldots,n)} \quad \text{for all } \vec{i} \in \mathbb{Z}^d$$

is an element of $X_{E_{\mathcal{H}}}$. Moreover, $\tilde{x}^1|_{B_n} = \tilde{a}|_{B_n} = \tilde{x}|_{B_n}$ and $\text{Image}(\tilde{x}^1) = \text{Image}(\tilde{x}|_{B_n})$. Since $\text{diameter}(B_n) = 2dn$, $\text{diameter}(\text{Image}(\tilde{x}^1)) \leq 2dn$. Let $\tilde{\mathcal{T}} = \text{Image}(\tilde{x}^1)$. Then the fold-radius of $X_{\tilde{\mathcal{T}}}$ is less than or equal to dn. Let a stiff shift obtained by a sequence of config-folds starting at $X_{\tilde{\mathcal{T}}}$ be denoted by Z'. Since $\tilde{\mathcal{T}}$ folds into a graph consisting of a single edge, Z' consists of two chequerboard patterns in the vertices of an edge in $\tilde{T}$, say $\tilde{w}_1$ and $\tilde{w}_2$. Then, by Proposition 5.5,

$$I_{X_{\tilde{\mathcal{T}}},n}(\tilde{x}^1)|_{D_n} = \tilde{x}^1|_{D_n} = \tilde{x}|_{D_n},$$
$$I_{X_{\tilde{\mathcal{T}}},n}(\tilde{x}^1)|_{D^c_{(d+1)n-1}} \in \mathcal{L}_{D^c_{(d+1)n-1}}(Z').$$

We note that $I_{X_{\tilde{\mathcal{T}}},n}(\tilde{x}^1)|_{\partial D_{(d+1)n-1}}$ consists of a single symbol, say $\tilde{w}_1$. Let $\pi(\tilde{w}_1) = w_1$. Then the configuration $x' = \pi(I_{X_{\tilde{\mathcal{T}}},n}(\tilde{x}^1)) \in X_{\mathcal{H}}$ satisfies

$$x'|_{D_n} = x|_{D_n}$$

and

$$x'|_{\partial D_{(d+1)n-1}} = w_1.$$

(3) *Patching of an arbitrary pattern inside a configuration with non-maximal range.* We will first prove that there exists a walk on $\mathcal{H}$ from w_1 to v_1, $((w_1 = u_1), u_2, \ldots, (u_{3r+2} = v_1))$. Since the graph is not bipartite, it has a cycle p_1 such that $|p_1| \leq r - 1$ and is odd. Let v' be a vertex in p_1. Then there exist walks p_2 and p_3 from w_1 to v' and from v' to

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v_1, respectively, such that $|p_2|, |p_3| \leq r-1$. Consider any vertex $w' \sim_{\mathcal{H}} v_1$. If $3r+1-|p_2|-|p_3|$ is even, then the walk

$$p_2 \star p_3(\star(v_1, w', v_1))^{(3r+1-|p_2|-|p_3|)/2}$$

and if not, then the walk

$$p_2 \star p_1 \star p_3(\star(v_1, w', v_1))^{(3r+1-|p_1|-|p_2|-|p_3|)/2}$$

is a walk of length $3r+1$ in $\mathcal{H}$ from w_1 to v_1. This is the only place where we use the fact that $\mathcal{H}$ is not bipartite. If it were bipartite, then we would require that x'_0 and y'_0 have to be in the same partite class to construct such a walk.

Given such a walk, the configuration z given by

$$\begin{aligned} z|_{D_{(d+1)n}} &= x'|_{D_{(d+1)n}}, \\ z|_{D^c_{(d+1)n+3r}} &= y'|_{D^c_{(d+1)n+3r}}, \\ z|_{\partial D_{(d+1)n+i-2}} &= u_i \quad \text{for all } 1 \leq i \leq 3r+2 \end{aligned}$$

is an element of $X_{\mathcal{H}}$ for which

$$z|_{D_n} = x'|_{D_n} = x|_{D_n}$$

and

$$z|_{D^c_{(d+1)n+3r+k}} = y'|_{D^c_{(d+1)n+3r+k}} = y|_{D^c_{(d+1)n+3r+k}}. \qquad \square$$

We now return to the proof of Theorem 3.4. Let μ be an ergodic probability measure adapted to $X_{\mathcal{H}}$ with positive entropy.

Suppose $sl_{\vec{e}_i}(x) = \theta_i$ almost everywhere. By Lemma 8.1, $\theta_i < 1$ for all $1 \leq i \leq d$. Let $\theta = \max_i \theta_i$ and $0 < \epsilon < \frac{1}{4}(1-\theta)$. Denote by S^{d-1}, the sphere of radius 1 in $\mathbb{R}^d$ for the l^1 norm. By Corollary 7.3, for all $\vec{v} \in S^{d-1}$

$$\lim_{n \longrightarrow \infty} \frac{1}{n} h_x(\vec{0}, \lfloor n\vec{v} \rfloor) \leq \theta$$

almost everywhere. Since S^{d-1} is compact in $\mathbb{R}^d$ we can choose a finite set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_t\} \subset S^{d-1}$ such that, for all $\vec{v} \in S^{d-1}$, there exists some $1 \leq i \leq t$ satisfying $\|\vec{v}_i - \vec{v}\|_1 < \epsilon$. By Egoroff's theorem [**9**], given ϵ as above, there exists $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$ and $1 \leq i \leq t$

$$\mu(\{x \in X_{\mathcal{H}} \mid h_x(\vec{0}, \lfloor n\vec{v}_i \rfloor) \leq n\theta + n\epsilon \ \forall\ 1 \leq i \leq t\}) > 1 - \epsilon. \tag{8.2}$$

Let $\vec{v} \in \partial D_{n-1}$ and $1 \leq i_0 \leq t$ such that $\|(1/n)\vec{v} - \vec{v}_{i_0}\|_1 < \epsilon$. If, for some $x \in X_{\mathcal{H}}$ and $n \in \mathbb{N}$,

$$h_x(\vec{0}, \lfloor n\vec{v}_{i_0} \rfloor) \leq n\theta + n\epsilon,$$

then

$$h_x(\vec{0}, \lfloor \vec{v} \rfloor) \leq h_x(\vec{0}, \lfloor n\vec{v}_{i_0} \rfloor) + \lceil n\epsilon \rceil \leq n\theta + 2n\epsilon + 1.$$

By inequality (8.2), we get

$$\mu(\{x \in X_{\mathcal{H}} \mid h_x(\vec{0}, \lfloor \vec{v} \rfloor) \leq n\theta + 2n\epsilon + 1 \ \forall\ \vec{v} \in \partial D_{n-1}\}) > 1 - \epsilon$$

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for all $n \geq N_0$. Therefore for all $n \geq N_0$ there exists $x^{(n)} \in \mathrm{supp}(\mu)$ such that

$$\mathrm{Range}_{\partial D_{n-1}}(x^{(n)}) \leq 2n\theta + 4n\epsilon + 2 < 2n(1-\epsilon) + 2.$$

Let $x \in X_{\mathcal{H}}$ and $n_0 \in \mathbb{N}$. It is sufficient to prove that $\mu([x]_{D_{n_0-1}}) > 0$. Suppose $r := |\mathcal{H}|$. Choose $k \in \mathbb{N}$ such that

$$n_0(d+1) + 3r + k + 1 \geq N_0,$$
$$2(n_0(d+1) + 3r + k + 1)(1-\epsilon) + 2 \leq 2k.$$

Then, by Lemma 8.2, there exists $z \in X_{\mathcal{H}}$ such that either

$$z_{\vec{j}} = \begin{cases} x_{\vec{j}} & \text{if } \vec{j} \in D_{n_0}, \\ x^{(n_0(d+1)+3r+k+1)}_{\vec{j}} & \text{if } \vec{j} \in D^c_{n_0(d+1)+3r+k}, \end{cases}$$

or

$$z_{\vec{j}} = \begin{cases} x_{\vec{j}+\vec{e}_1} & \text{if } \vec{j} \in D_{n_0}, \\ x^{(n_0(d+1)+3r+k+1)}_{\vec{j}} & \text{if } \vec{j} \in D^c_{n_0(d+1)+3r+k}. \end{cases}$$

In either case, $(z, x^{(n_0(d+1)+3r+k+1)}) \in \Delta_{X_{\mathcal{H}}}$. Since μ is adapted to $X_{\mathcal{H}}$, $z \in \mathrm{supp}(\mu)$. In the first case we get that $\mu([x]_{D_{n_0-1}}) = \mu([z]_{D_{n_0-1}}) > 0$. In the second case we get that

$$\mu([x]_{D_{n_0-1}}) = \mu(\sigma^{\vec{e}_1}([x]_{D_{n_0-1}})) = \mu([z]_{D_{(n_0-1)}-\vec{e}_1}) > 0.$$

This completes the proof. □

Every shift space conjugate to an entropy minimal shift space is entropy minimal. However, a shift space X that is conjugate to $X_{\mathcal{H}}$, for $\mathcal{H}$ which is connected and four-cycle free, need not even be a hom-shift. By following the proof carefully it is possible to extract a condition for entropy minimality which is conjugacy invariant.

THEOREM 8.3. *Let X be a shift of finite type and c a continuous sub-cocycle on X with the property that $c(\cdot, \vec{i}) \leq \|\vec{i}\|_1$ for all $\vec{i} \in \mathbb{Z}^d$. Suppose every ergodic probability measure μ adapted to X satisfies the following conditions.*

(1) *If $sl^c_{\vec{e}_i}(x) = 1$ almost everywhere for some $1 \leq i \leq d$, then $h_\mu < h_{\mathrm{top}}(X)$.*

(2) *If $sl^c_{\vec{e}_i}(x) < 1$ almost everywhere for all $1 \leq i \leq d$, then $\mathrm{supp}(\mu) = X$.*

Then X is entropy minimal.

By Proposition 3.1 and Theorems 3.2, 3.3 it is sufficient to prove that every ergodic measure of maximal entropy is fully supported. If X is a shift of finite type satisfying the hypothesis of Theorem 8.3, then it is entropy minimal because every ergodic measure of maximal entropy of X is an ergodic probability measure adapted to X: its entropy is either smaller than $h_{\mathrm{top}}(X)$ or it is fully supported. To see why the condition is conjugacy invariant, suppose that $f : X \longrightarrow Y$ is a conjugacy and $c \in C_Y$ satisfies the hypothesis of the theorem. Then by Proposition 7.5, it follows that $f^\star(c) \in C_X$ satisfies the hypothesis as well.

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Proof of Theorem 4.1. By Proposition 4.2, we can assume that $\mathcal{H}$ is connected. Consider some $(x, y) \in \Delta_{X_{\mathcal{H}}}$. By Corollary 6.3 there exist $(\tilde{x}, \tilde{y}) \in \Delta_{X_{E_{\mathcal{H}}}}$ such that $\pi(\tilde{x}) = x$ and $\pi(\tilde{y}) = y$. It is sufficient to prove that there is a chain of pivots from $\tilde{x}$ to $\tilde{y}$. We will proceed by induction on $\sum_{\vec{i}\in\mathbb{Z}^d} d_{E_{\mathcal{H}}}(\tilde{x}_{\vec{i}}, \tilde{y}_{\vec{i}})$. The induction hypothesis (on M) is as follows. If $\sum_{\vec{i}\in\mathbb{Z}^d} d_{E_{\mathcal{H}}}(\tilde{x}_{\vec{i}}, \tilde{y}_{\vec{i}}) = 2M$ then there exists a chain of pivots from $\tilde{x}$ to $\tilde{y}$.

We note that $d_{E_{\mathcal{H}}}(\tilde{x}_{\vec{i}}, \tilde{y}_{\vec{i}})$ is even for all $\vec{i} \in \mathbb{Z}^d$ since there exists $\vec{i}' \in \mathbb{Z}^d$ such that $\tilde{x}_{\vec{i}'} = \tilde{y}_{\vec{i}'}$ and hence $\tilde{x}_{\vec{i}}$ and $\tilde{y}_{\vec{i}}$ are in the same partite class of $E_{\mathcal{H}}$ for all $\vec{i} \in \mathbb{Z}^d$.

The base case ($M = 1$) occurs exactly when $\tilde{x}$ and $\tilde{y}$ differ at a single site; there is nothing to prove in this case. Assume the hypothesis for some $M \in \mathbb{N}$.

Consider $(\tilde{x}, \tilde{y}) \in \Delta_{X_{E_{\mathcal{H}}}}$ such that

$$\sum_{\vec{i}\in\mathbb{Z}^d} d_{E_{\mathcal{H}}}(\tilde{x}_{\vec{i}}, \tilde{y}_{\vec{i}}) = 2M + 2.$$

Let

$$B = \{\vec{j} \in \mathbb{Z}^d \mid \tilde{x}_{\vec{j}} \neq \tilde{y}_{\vec{j}}\}$$

and a vertex $\tilde{v} \in E_{\mathcal{H}}$. Without loss of generality, we can assume that

$$\max_{\vec{i}\in B} d_{E_{\mathcal{H}}}(\tilde{v}, \tilde{x}_{\vec{i}}) \geq \max_{\vec{i}\in B} d_{E_{\mathcal{H}}}(\tilde{v}, \tilde{y}_{\vec{i}}). \tag{8.3}$$

Consider some $\vec{i}_0 \in B$ such that

$$d_{E_{\mathcal{H}}}(\tilde{v}, \tilde{x}_{\vec{i}_0}) = \max_{\vec{i}\in B} d_{E_{\mathcal{H}}}(\tilde{v}, \tilde{x}_{\vec{i}}).$$

Consider the shortest walks $(\tilde{v} = \tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_n = \tilde{x}_{\vec{i}_0})$ from $\tilde{v}$ to $\tilde{x}_{\vec{i}_0}$ and $(\tilde{v} = \tilde{v}'_1, \tilde{v}'_2, \ldots, \tilde{v}'_{n'} = \tilde{y}_{\vec{i}_0})$ from $\tilde{v}$ to $\tilde{y}_{\vec{i}_0}$. By Assumption 8.3, $n' \leq n$. Since these are the shortest walks on a tree, if $\tilde{v}'_k = \tilde{v}_{k'}$ for some $1 \leq k \leq n'$ and $1 \leq k' \leq n$, then $k = k'$ and $\tilde{v}_l = \tilde{v}'_l$ for $1 \leq l \leq k$. Let

$$k_0 = \max\{1 \leq k \leq n' \mid \tilde{v}'_k = \tilde{v}_k\}.$$

Then the shortest walk from $\tilde{x}_{\vec{i}_0}$ to $\tilde{y}_{\vec{i}_0}$ is given by $\tilde{x}_{\vec{i}_0} = \tilde{v}_n, \tilde{v}_{n-1}, \tilde{v}_{n-2}, \ldots, \tilde{v}_{k_0}, \tilde{v}'_{k_0+1}, \ldots, \tilde{v}'_{n'} = \tilde{y}_{\vec{i}_0}$.

We will prove that, for all $\vec{i} \sim \vec{i}_0$, $\tilde{x}_{\vec{i}} = \tilde{v}_{n-1}$. This is sufficient to complete the proof since then the configuration

$$\tilde{x}^{(1)}_{\vec{j}} = \begin{cases} \tilde{x}_{\vec{j}} & \text{if } \vec{j} \neq \vec{i}_0, \\ \tilde{v}_{n-2} & \text{if } \vec{j} = \vec{i}_0, \end{cases}$$

is an element of $X_{E_{\mathcal{H}}}$, $(\tilde{x}, \tilde{x}^{(1)})$ is a pivot and

$$n + n' - 2k_0 - 2 = d_{E_{\mathcal{H}}}(\tilde{x}^{(1)}_{\vec{i}_0}, \tilde{y}_{\vec{i}_0}) < d_{E_{\mathcal{H}}}(\tilde{x}_{\vec{i}_0}, \tilde{y}_{\vec{i}_0}) = n + n' - 2k_0,$$

giving us a pair $(\tilde{x}^{(1)}, \tilde{y})$ such that

$$\sum_{\vec{i}\in\mathbb{Z}^d} d_{E_{\mathcal{H}}}(\tilde{x}^{(1)}_{\vec{i}}, \tilde{y}_{\vec{i}}) = \sum_{\vec{i}\in\mathbb{Z}^d} d_{E_{\mathcal{H}}}(\tilde{x}_{\vec{i}}, \tilde{y}_{\vec{i}}) - 2 = 2M,$$

to which the induction hypothesis applies. There are two possible cases.

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(1) If $\vec{i} \in B$, then $d_{E_{\mathcal{H}}}(\tilde{v}, \tilde{x}_{\vec{i}}) = d_{E_{\mathcal{H}}}(\tilde{v}, \tilde{x}_{\vec{i}_0}) - 1$ and $\tilde{x}_{\vec{i}} \sim_{E_{\mathcal{H}}} \tilde{x}_{\vec{i}_0}$. Since $E_{\mathcal{H}}$ is a tree, $\tilde{x}_{\vec{i}} = \tilde{v}_{n-1}$.

(2) If $\vec{i} \notin B$, then $\tilde{x}_{\vec{i}} = \tilde{y}_{\vec{i}}$ and we get that $d_{E_{\mathcal{H}}}(\tilde{x}_{\vec{i}_0}, \tilde{y}_{\vec{i}_0}) = 2$. Since $\tilde{x}_{\vec{i}} \sim_{E_{\mathcal{H}}} \tilde{x}_{\vec{i}_0}$, the shortest walk joining $\tilde{v}$ and $\tilde{x}_{\vec{i}}$ must either be $\tilde{v} = \tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_{n-1} = \tilde{x}_{\vec{i}}$ or $\tilde{v} = \tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_n = \tilde{x}_{\vec{i}_0}, \tilde{v}_{n+1} = \tilde{x}_{\vec{i}}$. We want to prove that the former is true. Suppose not. Since $\tilde{y}_{\vec{i}_0} \sim_{E_{\mathcal{H}}} \tilde{x}_{\vec{i}}$ and $\vec{i}_0 \in B$, the shortest walk from $\tilde{v}$ to $\tilde{y}_{\vec{i}_0}$ is $\tilde{v} = \tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_n = \tilde{x}_{\vec{i}_0}, \tilde{v}_{n+1} = \tilde{x}_{\vec{i}}, \tilde{v}_{n+2} = \tilde{y}_{\vec{i}_0}$. This contradicts Assumption 8.3 and completes the proof. □

9. *Further directions*

9.1. *Getting rid of the four-cycle free condition.* In the context of the results in this paper, the four-cycle free condition seems *a priori* artificial; we feel that, in many cases, it is a mere artefact of the proof. To the author, getting rid of this condition is an important and interesting topic for future research. Here we will illustrate what goes wrong when we try to apply our proofs to the simplest possible example with four-cycles: that is, C_4.

We have shown (Example 3) that X_{C_4} satisfies the hypothesis of Propositions 5.6 and 5.7 and thus it also satisfies the conclusions of Theorems 3.4 and 4.1. The proofs of Theorems 3.4 and 4.1, however, rely critically on the existence of lifts to the universal cover (that is, Proposition 6.2). However, the conclusion of this proposition does not hold for X_{C_4}. The universal cover of C_4 is $\mathbb{Z}$ and the corresponding covering map $\pi : \mathbb{Z} \longrightarrow C_4$ is given by $\pi(i) = i \bmod 4$. By the second remark following [**7**, Theorem 4.1], it follows that the induced map $\pi : X_{\mathbb{Z}} \longrightarrow X_{C_4}$ is not surjective, so disproving the conclusion of Proposition 6.2 for X_{C_4}.

9.2. *Identification of hom-shifts.*

Question 1. Given a shift space X, are there some nice decidable conditions which imply that X is conjugate to a hom-shift?

Being conjugate to a hom-shift lays many restrictions on the shift space, for instance, on its periodic configurations. Consider a conjugacy $f : X \longrightarrow X_{\mathcal{H}}$ where $\mathcal{H}$ is a finite undirected graph. Let $Z \subset X_{\mathcal{H}}$ be the set of configurations invariant under $\{\sigma^{2\vec{e}_i}\}_{i=1}^{d}$. Then there is a bijection between Z and $\mathcal{L}_A(X_{\mathcal{H}})$, where A is the rectangular shape

$$A := \left\{ \sum_{i=1}^{d} \delta_i \vec{e}_i \;\middle|\; \delta_i \in \{0, 1\} \right\},$$

because every pattern in $\mathcal{L}_A(X_{\mathcal{H}})$ extends to a unique configuration in Z. More generally, given a graph $\mathcal{H}$, it is not hard to compute the number of periodic configurations for a specific finite-index subgroup of $\mathbb{Z}^d$. Moreover, periodic points are dense in these shift spaces and there are algorithms to compute approximate upper and lower bounds of their entropy [**10**, **16**]. Hence the same has to hold for the shift space X as well. We are not familiar with nice, decidable conditions which imply that a shift space is conjugate to a hom-shift.

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9.3. *Hom-shifts and strong irreducibility.*

Question 2. Which hom-shifts are strongly irreducible?

We know of two such conditions.

(1) [**4**] If $\mathcal{H}$ is a finite graph which folds into $\mathcal{H}'$, then $X_{\mathcal{H}}$ is strongly irreducible if and only if $X_{\mathcal{H}'}$ is strongly irreducible. This reduces the problem to graphs $\mathcal{H}$ which are stiff. For instance if $\mathcal{H}$ is dismantlable, then $X_{\mathcal{H}}$ is strongly irreducible.

(2) [**3**] If $X_{\mathcal{H}}$ is single site fillable. A shift space $X_{\mathcal{F}} \subset \mathcal{A}^{\mathbb{Z}^d}$ is said to be *single site fillable* if, for all patterns $a \in \mathcal{A}^{\partial\{\vec{0}\}}$, there exists a locally allowed pattern in $X_{\mathcal{F}}$, $b \in \mathcal{A}^{D_1}$ such that $b|_{\partial\{\vec{0}\}} = a$. In the case $X_{\mathcal{F}} = X_{\mathcal{H}}$ for some graph $\mathcal{H}$, then it is single site fillable if and only if, given vertices $v_1, v_2, \ldots, v_{2d} \in \mathcal{H}$, there exists a vertex $v \in \mathcal{H}$ adjacent to all of them.

It follows that X_{K_5} is single site fillable and hence strongly irreducible for $d = 2$. In fact, strong irreducibility has been proved in [**3**] for shifts of finite type with a weaker mixing condition called TSSM. This does not cover all possible examples. For instance, it was proved in [**3**] that X_{K_4} is strongly irreducible for $d = 2$ even though it is not TSSM and K_4 is stiff. We do not know if it is possible to verify whether a given hom-shift is TSSM.

9.4. *Hom-shifts and entropy minimality.*

Question 3. Given a finite connected graph $\mathcal{H}$, when is $X_{\mathcal{H}}$ entropy minimal?

We have provided some examples in the paper.

(1) When $\mathcal{H}$ can be folded to a single vertex with a loop or a single edge (Proposition 5.6).

(2) When $\mathcal{H}$ is four-cycle free (Theorem 2.2).

Again, this does not provide the full picture. For instance, X_{K_4} is strongly irreducible when $d = 2$ and hence entropy minimal even though K_4 is stiff and not four-cycle free. A possible approach might be via identifying the right sub-cocycle and Theorem 8.3.

Conjecture. Let $d = 2$ and $\mathcal{H}$ be a finite connected graph. Then $X_{\mathcal{H}}$ is entropy minimal.

9.5. *Hom-shifts and the pivot property.* We have given a list of examples of graphs $\mathcal{H}$ for which the shift space $X_{\mathcal{H}}$ has the pivot property in §4. In this paper we have provided two further sets of examples.

(1) When $\mathcal{H}$ can be folded to a single vertex with a loop or a single edge (Proposition 5.7).

(2) When $\mathcal{H}$ is four-cycle free (Theorem 4.1).

We saw in §4 that X_{K_4} and X_{K_5} do not have the pivot property when $d = 2$. However, they do satisfy a weaker property which we will describe next.

A shift space X is said to have the *generalized pivot property* if there is an $r \in \mathbb{N}$ such that, for all $(x, y) \in \Delta_X$, there exists a chain $x^1 = x, x^2, x^3, \ldots, y = x^n \in X$ such that x^i and x^{i+1} differ at most on some translate of D_r.

It can be shown that any nearest neighbour shift of finite type $X \subset \mathcal{A}^{\mathbb{Z}}$ has the generalized pivot property. In higher dimensions this is not true without any hypothesis (look, for instance, in [**7**, §9]). It is not hard to prove that any single site fillable nearest

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neighbour shift of finite type has the generalized pivot property. This can be generalized further: in [**3**] it is proven that every shift space satisfying TSSM has the generalized pivot property.

Question 4. For which graphs $\mathcal{H}$ does $X_{\mathcal{H}}$ satisfy the pivot property? What about the generalized pivot property?

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References

[**1**] D. Angluin. Local and global properties in networks of processors (extended abstract). *Proc. 12th Ann. ACM Symp. on Theory of Computing (Los Angeles, CA, USA, 28–30 April 1980)*. ACM, New York, 1980, pp. 82–93.
[**2**] M. Boyle, R. Pavlov and M. Schraudner. Multidimensional sofic shifts without separation and their factors. *Trans. Amer. Math. Soc.* **362**(9) (2010), 4617–4653.
[**3**] R. Briceño. The topological strong spatial mixing property and new conditions for pressure approximation. *Preprint*, 2014, arXiv:org/abs/1411.2289.
[**4**] G. R. Brightwell and P. Winkler. Gibbs measures and dismantlable graphs. *J. Combin. Theory Ser. B* **78**(1) (2000), 141–166.
[**5**] S. Capobianco. Multidimensional cellular automata and generalization of Fekete's lemma. *Discrete Math. Theor. Comput. Sci.* **10**(3) (2008), 95–104.
[**6**] N. Chandgotia. Generalisation of the Hammersley–Clifford theorem on bipartite graphs. *Preprint*, 2014, arXiv:org/abs/1406.1849.
[**7**] N. Chandgotia and T. Meyerovitch. Markov random fields, Markov cocycles and the 3-colored chessboard. *Preprint*, 2013, arXiv:org/abs/1305.0808.
[**8**] E. M. Coven and J. Smítal. Entropy-minimality. *Acta Math. Univ. Comenian. (N.S.)* **62**(1) (1993), 117–121.
[**9**] G. B. Folland. *Real Analysis: Modern Techniques and their Applications (Pure and Applied Mathematics (New York))*, 2nd edn. Wiley-Interscience, New York, 1999.
[**10**] S. Friedland. On the entropy of $\mathbf{Z}^d$ subshifts of finite type. *Linear Algebra Appl.* **252** (1997), 199–220.
[**11**] H.-O. Georgii. *Gibbs Measures and Phase Transitions (de Gruyter Studies in Mathematics, 9)*. Walter de Gruyter & Co., Berlin, 1988.
[**12**] J. M. Hammersley and D. J. A. Welsh. First-passage percolation, subadditive processes, stochastic networks, and generalized renewal theory. *Proc. Int. Res. Semin. (Statistics Laboratory, University of California)*. Springer, New York, 1965, pp. 61–110.
[**13**] G. Keller. *Equilibrium States in Ergodic Theory (London Mathematical Society Student Texts, 42)*. Cambridge University Press, Cambridge, 1998.

https://doi.org/10.1017/etds.2015.88 Published online by Cambridge University Press

[**14**] O. E. Lanford III and D. Ruelle. Observables at infinity and states with short range correlations in statistical mechanics. *Comm. Math. Phys.* **13** (1969), 194–215.
[**15**] D. Lind and B. Marcus. *An Introduction to Symbolic Dynamics and Coding*. Cambridge University Press, Cambridge, 1995, reprinted 1999.
[**16**] E. Louidor and B. H. Marcus. Improved lower bounds on capacities of symmetric 2D constraints using Rayleigh quotients. *IEEE Trans. Inform. Theory* **56**(4) (2010), 1624–1639.
[**17**] W. S. Massey. *Algebraic Topology: an Introduction (Graduate Texts in Mathematics, 56)*. Springer, New York, 1977, reprint of the 1967 edition.
[**18**] R. Meester and J. E. Steif. Higher-dimensional subshifts of finite type, factor maps and measures of maximal entropy. *Pacific J. Math.* **200**(2) (2001), 497–510.
[**19**] J. R. Munkres. *Topology: A First Course*. Prentice-Hall, Englewood Cliffs, NJ, 1975.
[**20**] F. Nakano, H. Ono and T. Sadahiro. Local move connectedness of domino tilings with diagonal impurities. *Discrete Math.* **310**(13–14) (2010), 1918–1931.
[**21**] R. Nowakowski and P. Winkler. Vertex-to-vertex pursuit in a graph. *Discrete Math.* **43**(2–3) (1983), 235–239.
[**22**] E. Ordentlich and R. M. Roth. When data must satisfy constraints upon writing. *Proc. IEEE Int. Symp. on Information Theory (Honolulu, Hawaii, June 2014)*. IEEE, Piscataway, NJ, 2014, pp. 2257–2261.
[**23**] R. Pavlov. Approximating the hard square entropy constant with probabilistic methods. *Ann. Probab.* **40**(6) (2012), 2362–2399.
[**24**] R. Peled. High-dimensional Lipschitz functions are typically flat. *Ann. Probab.* to appear. *Preprint*, 2010, arXiv:org/abs/1005.4636.
[**25**] K. Petersen and K. Schmidt. Symmetric Gibbs measures. *Trans. Amer. Math. Soc.* **349**(7) (1997), 2775–2811.
[**26**] A. N. Quas and P. B. Trow. Subshifts of multi-dimensional shifts of finite type. *Ergod. Th. & Dynam. Sys.* **20**(3) (2000), 859–874.
[**27**] R. M. Robinson. Undecidability and nonperiodicity for tilings of the plane. *Invent. Math.* **12** (1971), 177–209.
[**28**] D. Ruelle. *Thermodynamic Formalism: The Mathematical Structures of Equilibrium Statistical Mechanics*, 2nd edn. Cambridge University Press, Cambridge, 2004.
[**29**] K. Schmidt. The cohomology of higher-dimensional shifts of finite type. *Pacific J. Math.* **170**(1) (1995), 237–269.
[**30**] K. Schmidt. Invariant cocycles, random tilings and the super-K and strong Markov properties. *Trans. Amer. Math. Soc.* **349**(7) (1997), 2813–2825.
[**31**] K. Schmidt. Tilings, fundamental cocycles and fundamental groups of symbolic $\mathbb{Z}^d$-actions. *Ergod. Th. & Dynam. Sys.* **18**(6) (1998), 1473–1525.
[**32**] M. Schraudner. Projectional entropy and the electrical wire shift. *Discrete Contin. Dyn. Syst.* **26**(1) (2010), 333–346.
[**33**] M. Schraudner and S. Lightwood. Entropy minimality of Zd shifts of finite type and the family of wire shifts. Work in progress.
[**34**] S. Sheffield. Ribbon tilings and multidimensional height functions. *Trans. Amer. Math. Soc.* **354**(12) (2002), 4789–4813 (electronic).
[**35**] J. R. Stallings. Topology of finite graphs. *Invent. Math.* **71**(3) (1983), 551–565.
[**36**] W. P. Thurston. Conway's tiling groups. *Amer. Math. Monthly* **97**(8) (1990), 757–773.
[**37**] P. Walters. *An Introduction to Ergodic Theory (Graduate Texts in Mathematics, 79)*. Springer, New York, 1982.
[**38**] M. Wrochna. Homomorphism reconfiguration via homotopy. *32nd Int. Symp. on Theoretical Aspects of Computer Science (STACS, 2015) (Leibniz International Proceedings in Informatics, 30)*. Eds. E. W. Mayr and N. Portier. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, Dagstuhl, Germany, 2015, pp. 730–742.

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