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ARTICLE

Irregular threefolds with numerically trivial canonical divisor

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Abstract

In this article, we classify irregular threefolds with numerically trivial canonical divisors in positive characteristic. For a threefold, if its Albanese dimension is not maximal, then the Albanese morphism will induce a fibration which either maps to a curve or is fibered by curves. In practice, we treat arbitrary dimensional irregular varieties with either one-dimensional Albanese fiber or one-dimensional Albanese image. We prove that such a variety carries another fibration transversal to its Albanese morphism (a "bi-fibration" structure), which is an analog structure of bielliptic or quasi-bielliptic surfaces. In turn, we give an explicit description of irregular threefolds with trivial canonical divisors.

1. Introduction

Over the field $\mathbb{C}$ of complex numbers, Enriques and Kodaira's classification of surfaces with numerically trivial canonical divisors (K-trivial), has four basic classes: abelian, bielliptic, K3, and Enriques surfaces. This classification is achieved by considering their Albanese morphisms: for a K-trivial surface X, the Albanese morphism $a_X\colon X \to A$, according to $\dim a_X(X) = 2, 1, 0$, is, respectively, an isomorphism, an elliptic fibration to an elliptic curve, or a trivial morphism. When $\dim a_X(X) = 1$, X has another elliptic fibration $g\colon X \to \mathbb{P}^1$, thus X has a so-called bielliptic structure; more precisely, there is an étale cover $A' \to A$ such that $X \times_A A' \cong A' \times F$, where F is a general fiber of a_X. Bielliptic surfaces were fully classified in [2] (see [5, List VI.20]). For higher-dimensional K-trivial varieties, Bogomolov and Beauville ([4], [7] for smooth cases) and Kawamata ([34, Section 8] for varieties with canonical singularities) proved that the Albanese morphism $a_X\colon X \to A$ is a fibration and there exists an étale covering $A' \to A$ such that $X \times_A A' \cong F \times A'$, where F is a general fiber. Especially, for an irregular K-trivial variety X, when $\dim a_X(X) = \dim X$, we have $X \cong A$. When the Albanese image of X has an intermediate dimension, namely, $0 < \dim a_X(X) < \dim X$, we can reduce the study of X to the study of lower-dimensional K-trivial varieties. Here, by an *irregular variety,* we mean a normal projective variety X with *irregularity* $q(X) := \dim \mathrm{Pic}^0(X) > 0$, or equivalently, the Albanese morphism $a_X\colon X \to A$ is not trivial.

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Over a ground field of positive characteristic, Bombieri and Mumford [8], [9] classified K-trivial irregular surfaces X: if $\dim a_X(X) = 2$, then $X = A$; if $\dim a_X(X) = 1$, then a general fiber of the Albanese morphism is either an elliptic curve or a rational curve with a cusp, and X has a bielliptic or a quasi-bielliptic structure accordingly. Note that, in Bombieri and Mumford's classification of (quasi-)bielliptic surfaces, a key step is to show that there is a rational pencil of elliptic curves on X that is transversal to the Albanese morphism of X, that is, X carries a "bi-fibration" structure:

$$\begin{array}{ccc} X & \xrightarrow{g} & Z \cong \mathbb{P}^1. \\ {\scriptstyle a_X}\downarrow & & \\ A & & \end{array}$$

Here, "transversal" means that a general fiber of g is dominant and finite over A under the morphism a_X.

For higher-dimensional K-trivial varieties in positive characteristic, we know that if X is of maximal Albanese dimension, then X is birational to an abelian variety by [27]. In the past decade, a series of progresses have been made in understanding the positivity of the direct image of (pluri)canonical sheaves in positive characteristics ([16], [17], [50], etc.). The application of the powerful positivity engine to the Albanese morphism $a_X : X \to A$ induces some remarkable results for higher-dimensional K-trivial (or more generally $-K$ nef) varieties. A pivotal result by [52] establishes that if X is weakly ordinary (meaning the Frobenius pullback $F^* : H^d(\mathcal{O}_X) \to H^d(\mathcal{O}_X)$ is a σ-linear isomorphism, or equivalently, X is globally F-splitting), then X admits a Beauville–Bogomolov type decomposition. Roughly speaking, there is an isogeny $A' \to A$ such that $X \times_A A' \cong A' \times F$, where F is a general fiber of a_X. Recently, Ejiri and Patakfalvi [20] prove that when $-K$ is nef, under certain conditions on singularities, the Albanese morphism a_X is surjective, and the intermediate variety Y arising from the Stein factorization of a_X is either purely inseparable over A or isomorphic to A. Later, the authors [11] prove that, in case a_X is of relative dimension one, $a_X : X \to A$ is a fibration. Under additional conditions that $\dim a_X(X) = 1$ or that the generic geometric fiber is strongly F-regular, in [18], [20], the authors prove that the fibers of f are isomorphic to each other.

It is worth mentioning that the assumptions of the decomposition theorems above avoid a "bad phenomenon" in positive characteristic: the general fiber of the Albanese morphism might have bad singularities, and sometimes it is non-reduced (see, e.g., [43] or [54, Section 3]). A natural question arises:

Question 1. Let X be an irregular K-trivial variety. Is the Albanese morphism $a_X : X \to A$ a fibration? Does there exist an isogeny $A' \to A$ of abelian varieties such that $X \times_A A' \cong A' \times F$?

In this article, we focus on irregular K-trivial threefolds and treat this "bad phenomenon." Our approach applies to not only threefolds but also arbitrary dimensional X whose Albanese morphism $a_X : X \to A$ satisfies one of the following conditions:

- the Albanese image $a_X(X)$ is of dimension one (Section 4);
- $a_X : X \to a_X(X)$ is of relative dimension one (Section 5).

In both cases, we derive a "bi-fibration" structure

$$\begin{array}{ccc} X & \xrightarrow{g} & Z \\ {\scriptstyle a_X}\downarrow & & \\ A, & & \end{array} \tag{1.1}$$

which is a crucial step to obtain an explicit structure of X.

We explain our strategy as follows. In the first case, $\dim a_X(X) = 1$, we follow the strategy earlier used in [18], [20], [52]: applying the positivity engine to construct a semi-ample divisor D on X, which is relatively ample over A and has $\nu(D) = \dim X - 1$ (Theorem 3.3). In the second case, $f = a_X : X \to A$

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is a fibration with the generic fiber X_η being a curve of arithmetic genus one, thus we treat the following three cases separately:

(C1) X_η is smooth over $k(\eta)$. In this case, $f\colon X \to A$ is an elliptic fibration with $\operatorname{var}(f) = 0$ by [12, Theorem 2.14], so we can apply the Isom functor developed in [52].

If X_η is not smooth over $k(\eta)$, we can show that there is a natural movable divisor which induces the fibration $g\colon X \to Z$ as follows:

(C2) $X_{\bar\eta}$ is reduced but not smooth. Then, $a_X\colon X \to A$ is fibered by quasi-elliptic curves, we prove that the divisor supported on the singular locus Σ of the fibers is movable as required (Theorem 3.7).
(C3) $X_{\bar\eta}$ is not reduced, which means that $a_X\colon X \to A$ is inseparable. Then, the required movable divisor arises from global sections of Ω_A when doing Frobenius base change (Section 2.5). This should be attributed to Ji and Waldron's observation [32] (see [11, Proposition 3.4]).

We see that in cases (C2) and (C3), the "bad phenomenon" that $X_{\bar\eta}$ is singular becomes an advantage. Finally, we apply the two fibrations (1.1) to derive the explicit structure of X.

For threefolds, we have a precise description as follows.

Theorem 1.1 (= Theorem 6.1). *Let k be an algebraically closed field of characteristic $p > 0$. Let X be a normal $\mathbb{Q}$-factorial projective threefold over k with $K_X \equiv 0$. Denote by $a_X\colon X \to A$ the Albanese morphism of X, and assume $\dim A > 0$. Then, the following statements hold:*

(A) *If $\dim a_X(X) = 3$, then $X = A$.*
(B) *If $\dim a_X(X) = 1$, under the condition that:*
  (a) *either X is strongly F-regular and K_X is $\mathbb{Z}_{(p)}$-Cartier;*
  (b) *or X has at most terminal singularities and $p \geq 5$,*
  then a_X is a fibration and there exists an isogeny of elliptic curves $A' \to A$, such that $X \times_A A' \cong A' \times F$, where F is a general fiber of a_X. More precisely, $X \cong A' \times F/H$, where H is a finite group subscheme of A' acting diagonally on $A' \times F$.
(C) *If $\dim a_X(X) = 2$, then a_X is a fibration and X falls into one of the following cases:*
  (C1) *the generic fiber X_η of a_X is smooth. Then, there exists an isogeny of abelian surfaces $A' \to A$, such that $X \times_A A' \cong A' \times E$, where E is an elliptic curve appearing as a general fiber of a_X. More precisely, $X \cong A' \times E/H$, where H is a finite group subscheme of A' acting diagonally on $A' \times E$, with a complete classification as in Section 5.1.2.*
  (C2) *X_η is non-smooth but geometrically reduced. Then, $p = 2$ or 3. Denote by $F_{A/k}\colon A_1 := A^{(-1)} \to A$ the relative Frobenius over k, X_1 the normalization of $X \times_A A_1$ and $f_1\colon X_1 \to A_1$ the induced morphism. Then, $f_1\colon X_1 \to A_1$ is a smooth fibration fibered by rational curves, which falls into one of the following specific cases:*
    (2.1) *In this case, $f_1\colon X_1 \to A_1$ admits a section, and*
      (2.1a) *either $X_1 \cong \mathbb{P}_{A_1}(\mathcal{O}_{A_1} \oplus \mathcal{L})$, where $\mathcal{L}^{\otimes p+1} \cong \mathcal{O}_X$; or*
      (2.1b) *$X_1 \cong \mathbb{P}_{A_1}(\mathcal{E})$, where $\mathcal{E}$ is a unipotent vector bundle of rank two, and there exists an étale cover $\mu\colon A_2 \to A_1$ of degree p^v for some $v \leq 2$ such that $\mu^* F_{A_1/k}^{(2-v)*}\mathcal{E}$ is trivial.*
    (2.2) *In this case, $p = 2$, and there exists a purely inseparable isogeny $A_2 \to A_1$ of degree two, such that $X_2 := X_1 \times_{A_1} A_2$ is a projective bundle over A_1 described as follows:*
      (2.2a) *$X_2 \cong \mathbb{P}_{A_2}(\mathcal{O}_{A_2} \oplus \mathcal{L})$, where $\mathcal{L}^{\otimes 4} \cong \mathcal{O}_{A_2}$; or*
      (2.2b) *$X_2 \cong \mathbb{P}_{A_2}(\mathcal{E})$, where $\mathcal{E}$ is a unipotent vector bundle of rank two, and there exists an étale cover $\mu\colon A_3 \to A_2$ of degree p^v for some $v \leq 2$ such that $\mu^* F_{A_1/k}^{(2-v)*}\mathcal{E}$ is trivial.*
  (C3) *X_η is not geometrically reduced. In this case, we also have $p = 2$ or 3. Let X_1 be the normalization of $(X \times_A A_1)_{red}$, where $A_1 := A^{(-1)} \to A$ is the relative Frobenius. Then, the projection $X_1 \to A_1$ is a smooth morphism, and either:*

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(3.1) $X_1 = A_1 \times \mathbb{P}^1$ *and* $X = A_1 \times \mathbb{P}^1/\mathcal{F}$ *for some smooth rank one foliation* $\mathcal{F}$ *which is described concretely in Section 6.2.2; or*

(3.2) $p = 2$*, and there exists an isogeny of abelian surfaces* $\tau\colon A_2 \to A_1$ *such that* $X_1 \times_{A_1} A_2 \cong A_2 \times \mathbb{P}^1$*, where either:*

(3.2a) $\tau\colon A_2 \to A_1$ *is étale of degree two; or*

(3.2b) $\tau = F_{A_1/k}\colon A_2 := A^{(-2)} \to A_1$ *is the relative Frobenius.*

Remark 1.2. (1) Theorem 1.1 shows that the Albanese fibration $f\colon X \to A$ splits into the product after a sequence of "normalized" Frobenius base changes and an étale base change; say

$$\begin{array}{ccccc} A_2 \times F & \longrightarrow & X_1 & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow f \\ A_2 & \xrightarrow{\mathrm{t}} & A_1 & \xrightarrow{F^n_{A/k}} & A\,, \end{array}$$

where F is the normalization of a general fiber of f, $X_1 = (X \times_A A_1)^{\nu}_{\mathrm{red}}$ and the left square is Cartesian. By Remark 2.3, the same holds if we first do an étale base change followed by Frobenius base changes, namely, we have the following sequence of normalized base changes:

$$\begin{array}{ccccc} B_1 \times F & \longrightarrow & X_B & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow f \\ B_1 & \xrightarrow{F^n_{B/k}} & B & \xrightarrow{\mathrm{t}} & A\,. \end{array}$$

(2) In Case (C), we get a full classification for cases (C1) and (C3.1), and provide examples for the remaining cases (Section 6).

1.1. Effectivity of the pluricanonical map of threefolds

Over the field of complex numbers, for terminal K-trivial threefolds X, Kawamata [35] showed that there exists a positive calculable integer m_0 such that $m_0 K_X \sim 0$; and by [3], [44] and finally [49], the smallest m_0 is $2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$. It is natural to ask the following question.

Question 2. Does there exist a positive integer N such that $NK_X \sim 0$ for all terminal $\mathbb{Q}$-factorial K-trivial threefolds over an algebraically closed field of characteristic $p > 0$?

Applying the structure Theorem 1.1, we can prove the following effectivity result when $\dim a_X(X) = 2$.

Corollary 1.3 (See Section 6.3). *Let X be a terminal $\mathbb{Q}$-factorial threefold such that $K_X \equiv 0$ and $q = 2$, then $(2^4 \cdot 3^3)K_X \sim 0$.*

For the case $\dim a_X(X) = 1$, we may break the effectivity problem into the following two questions.

Question 3. Let $f\colon X \to A$ be a fibration belonging to Case (B) of Theorem 1.1. Denote by F a general fiber of f. Then, there exists an integer $N_F > 0$ such that $N_F K_F \sim 0$.

(1) Is there a uniform bound of N_F? Equivalently, is there a positive integer N such that $NK_F \sim 0$ holds for every fibration $f\colon X \to A$ in Case (B)?

Let H be a finite subgroup scheme of an elliptic curve. Assume that H acts on F as in Case (B). Then, there is a natural group homomorphism $H \to \mathrm{GL}(H^0(F, N_F K_F)) \cong \mathbb{G}_m$; we denote its image by $\overline{H}$, which is a finite group scheme over k.

(2) Is there a uniform bound of the order of $\overline{H}$?

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1.2. Notation and conventions

- By a *variety,* we mean an integral quasi-projective scheme over a field. By a *log pair* (X, Δ), we mean a pair consisting of a variety X and an effective $\mathbb{Q}$-divisor Δ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We denote by $\nu\colon X^\nu \to X$ the normalization morphism of a variety X.
- By a *fibration,* we mean a projective morphism $f\colon X \to Y$ of normal varieties such that $f_*\mathcal{O}_X = \mathcal{O}_Y$.
- Let X be a normal variety and D a Weil divisor on X. Denote by $\mathcal{O}_X(D)$ the reflexive sheaf associated with D. Note that if D_1 and D_2 are Weil divisors, then $\mathcal{O}_X(D_1 + D_2) \cong (\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2))^{\vee\vee}$.
- For a projective morphism $f\colon X \to Y$ of normal varieties, a Weil divisor D on X is called *f-exceptional* if $f(\operatorname{Supp} D) \subset Y$ has codimension ≥ 2, *f-vertical* if $f(\operatorname{Supp} D)$ has codimension ≥ 1, and *f-horizontal* if each irreducible component of D is dominant over Y.
- For a morphism $\sigma\colon Z \to X$ of varieties and a divisor D on X such that the pullback σ^*D is well defined, we often use $D|_Z$ to denote σ^*D for simplicity.
- Let $K = \mathbb{Z}, \mathbb{Q}$, or $\mathbb{R}$. Let D be a K-divisors on X, namely, $D \in N^1(X) \otimes K$. We say that D is effective, with the notation $D \geq 0$, if all coefficients are non-negative. By $D \geq_K 0$, we mean that there exists an effective K-divisor D' such that $D' \sim_K D$. When $K = \mathbb{Z}$, we also denote $D \geq_{\mathbb{Z}} 0$ by $D \geq 0$ for simplicity.
- For every effective integral divisor E on X, the inclusion $\mathcal{O}_X(D) \subseteq \mathcal{O}_X(D + E)$ allows us to regard $H^0(X, \mathcal{O}_X(D))$ as a subspace of $H^0(X, \mathcal{O}_X(D+E))$. This space coincides with $H^0(X, \mathcal{O}_X(D)) \otimes 1_E$, where $1_E \in H^0(X, \mathcal{O}_X(E))$ denotes the section corresponding to the constant function $1 \in K(X)$.

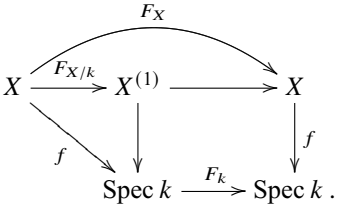
Throughout this article, we let k be an algebraically closed field of characteristic $p > 0$, and unless otherwise mentioned we assume varieties are defined over k.

2. Preparations

In this section, we collect some basic notions and facts which we will use in the sequel.

2.1. Frobenius morphisms

Let $f\colon X \to \operatorname{Spec} k$ be a variety over k. We denote by $F_X\colon X \to X$ the absolute Frobenius morphism of X. Set $X^{(1)} := X \times_{k,F_k} k$. We denote by $F_{X/k}\colon X \to X^{(1)}$ the *relative Frobenius* of X over k, which fits into the following commutative diagram:



Note that since k is perfect, the morphism $X^{(1)} \to X$, though not k-linear, is an isomorphism as schemes. For this reason, we also denote the relative Frobenius by $F_{X/k}\colon X^{(-1)} \to X$.

2.2. Foliations and purely inseparable morphisms

Let Y be a normal variety over k, and denote by $\mathcal{T}_Y := \Omega^\vee_{Y/k}$ the tangent sheaf. A *foliation* on Y is a saturated subsheaf $\mathcal{F} \subseteq \mathcal{T}_Y$, which is p-closed ($\mathcal{F}^p \subseteq \mathcal{F}$) and involutive ($[\mathcal{F}, \mathcal{F}] \subseteq \mathcal{F}$). The subsheaf $\operatorname{Ann} \mathcal{F} \subseteq \mathcal{O}_Y$ is a subring containing $\mathcal{O}_Y^p$, and thus gives a natural morphism $\pi\colon Y \to Y/\mathcal{F} := \operatorname{Spec}(\operatorname{Ann} \mathcal{F})$ over k. By the construction, the relative Frobenius morphism $F_{Y/k}\colon Y \to Y^{(1)}$ factors through $\pi\colon Y \to Y/\mathcal{F}$, thus $\pi\colon Y \to Y/\mathcal{F}$ is a purely inseparable morphism of height one. In fact, there is a one-to-one correspondence ([21] or [51, Proposition 2.9]):

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$$\left\{\begin{array}{l}\text{foliations}\\ \mathcal{F}\subseteq\mathcal{T}_Y\end{array}\right\}\leftrightarrow\left\{\begin{array}{l}\text{finite purely inseparable morphisms } \pi\colon Y\to X\\ \text{over } k \text{ of height one with } X \text{ normal}\end{array}\right\},$$

which is given by

$$\mathcal{F}\ \mapsto\ \pi\colon Y\to Y/\mathcal{F}\quad\text{and}\quad \pi\colon Y\to X\ \mapsto\ \mathcal{F}_{Y/X},$$

where $\mathcal{F}_{Y/X}$ is the subsheaf of $\mathcal{T}_Y$ annihilated by $\mathrm{im}(\pi^*\Omega^1_X\to\Omega^1_Y)$. Recall (see, e.g., [51, Proposition 2.10]) the following formula:

$$\pi^*K_X\sim K_Y-(p-1)\det\mathcal{F}_{X/Y}\sim K_Y+(p-1)\det\Omega^1_{X/Y}. \tag{2.1}$$

If Y is a smooth variety, we call a foliation $\mathcal{F}$ on Y a *smooth foliation* if $\mathcal{F}\subseteq\mathcal{T}_Y$ is a subbundle, namely, both $\mathcal{F}$ and $\mathcal{T}_Y/\mathcal{F}$ are locally free. In this case, by [21, Proposition 2.4] or [42, Proposition 3.1.9], the quotient $Y/\mathcal{F}$ is smooth if and only if $\mathcal{F}$ is a smooth foliation.

2.3. "Pushing down" and pullback foliations

2.3.1. "Pushing-down" foliations along a fibration

Let $f\colon X\to S$ be a fibration of normal varieties over k, and let $\mathcal{F}$ be a foliation on X. We recall the "pushing-down" foliation of $\mathcal{F}$ constructed in [11, Section 3.1.1] as follows. By the results of the previous section, we have the following commutative diagram:

$$\begin{array}{ccccc}X & \xrightarrow{\pi} & \bar{X}=X/\mathcal{F} & \longrightarrow & X^{(1)}\\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} & & \downarrow\\ S & \xrightarrow{\sigma} & \bar{S} & \longrightarrow & S^{(1)},\end{array}$$

where $\bar{f}\colon\bar{X}\to\bar{S}$ arises from the Stein factorization of $\bar{X}\to X^{(1)}\to S^{(1)}$, and hence $\bar{S}$ is obviously between S and $S^{(1)}$. The purely inseparable morphism $\sigma\colon S\to\bar{S}$ corresponds to a foliation $\mathcal{G}$ on S such that $\bar{S}=S/\mathcal{G}$. The following is another characterization of $\mathcal{G}$.

Proposition 2.1 [11, Lemma 3.3]. *Let notation be as above. Assume moreover that S is regular. Let $\mathcal{F}\subseteq\mathcal{T}_X\xrightarrow{\eta}f^*\mathcal{T}_S$ be the natural homomorphisms. Then,*

(1) *the sheaf $\mathcal{G}$ is the minimal foliation on S such that $\eta(\mathcal{F})\subseteq f^*\mathcal{G}$ holds generically;*
(2) *if f is separable, then $\bar{f}$ is separable if and only if $\eta\colon\mathcal{F}\to f^*\mathcal{G}$ is generically surjective.*

2.3.2. Pullback of a foliation

Let $\tau\colon Y\to X$ be a generically finite, separable, and dominant morphism of normal varieties, and let $\mathcal{F}\subset\mathcal{T}_X$ be a foliation on X. We can define the pullback foliation $\mathcal{F}_Y$ on Y as follows. The natural homomorphism $\mathcal{T}_Y\to\tau^*\mathcal{T}_X$ is generically isomorphic. So over some open subset of Y, $\tau^*\mathcal{F}$ can be viewed as a subsheaf of $\mathcal{T}_Y$ under this isomorphism. We define $\mathcal{F}_Y$ to be the saturation of $\tau^*\mathcal{F}$ in $\mathcal{T}_Y$. One can check that $\mathcal{F}_Y$ is a foliation on Y.

Lemma 2.2 [53, Lemma 3.2.1]. *Let $\tau\colon Y\to X$ be a finite étale morphism between normal varieties. Let $\mathcal{F}$ be a foliation on X and let $\mathcal{F}_Y$ be the pullback foliation on Y. There is an étale morphism $\sigma\colon Y/\mathcal{F}_Y\to X/\mathcal{F}$ which give a Cartesian square*

$$\begin{array}{ccc}Y & \longrightarrow & Y/\mathcal{F}_Y\\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\sigma}\\ X & \longrightarrow & X/\mathcal{F}.\end{array}$$

Moreover, if $\tau\colon Y\to X$ is a Galois covering, then so is σ.

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Proof. The first assertion is [53, Lemma 3.2.1]. For the second, assume that G is the group of automorphisms of Y over X. Then, there is a commutative diagram

$$\begin{array}{ccccc} Y/\mathcal{F}_Y & \longrightarrow & Y & \longrightarrow & (Y/\mathcal{F}_Y)^{(1)} \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \sigma^{(1)}} \\ X/\mathcal{F} & \longrightarrow & X & \longrightarrow & (X/\mathcal{F})^{(1)}, \end{array}$$

where the compositions of the horizontal rows are relative Frobenius morphisms over k. Moreover, the left square is Cartesian, that is, $Y/\mathcal{F}_Y \cong X/\mathcal{F} \times_X Y$, thus G acts naturally on $Y/\mathcal{F}_Y$ which makes σ a Galois covering. □

Remark 2.3. Let $f : X \to S$ be a fibration of normal varieties and let $\mathcal{F}$ be a foliation on X. If $\tau : Y \to X$ is obtained by an étale base change, say $Y = X \times_S \tilde{S}$, where $\sigma : \tilde{S} \to S$ is finite étale, then $Y/\mathcal{F}_Y \cong X/\mathcal{F} \times_{S^{(1)}} \tilde{S}^{(1)}$ and the diagram

$$\begin{array}{ccccc} Y & & \xrightarrow{\tilde{\pi}} & & Y/\mathcal{F}_Y \\ & \searrow^{\tau} & & & \vdots \quad \searrow^{\tau'} \\ & & X & \xrightarrow{\pi} & X/\mathcal{F} \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ \tilde{S} & \dashrightarrow & \tilde{S}^{(1)} & & \\ & \searrow^{\sigma} & & \searrow^{\sigma'} & \\ & & S & \xrightarrow{F} & S^{(1)} \end{array} \tag{2.2}$$

is commutative, where $S \to S^{(1)}$ and $\tilde{S} \to \tilde{S}^{(1)}$ are the relative Frobenius morphisms over k and σ' is the natural étale morphism induced by σ.

2.4. A property of fibered varieties under flat base changes

Lemma 2.4. *Let X, S, T be quasi-projective normal varieties over an arbitrary field. Let $f : X \to S$ be a separable fibration and $\sigma : T \to S$ a finite flat morphism. Then, the fiber product $X \times_S T$ is integral, and it is normal if and only if its conductor divisor is zero.*

Proof. Consider the tensor product of the function fields $R := K(X) \otimes_{K(S)} K(T)$. Since f is a fibration, $K(S)$ is algebraically closed in $K(X)$, so $\operatorname{Spec} R$ is irreducible by [25, Proposition 4.3.2]. Since f is separable, $\operatorname{Spec} R$ is reduced by [25, Proposition 4.3.5]. Therefore, $\operatorname{Spec} R$ is integral, which implies that $Y := X \times_S T$ is generically integral (in the sense that it becomes integral restricting to certain open subsets of X, S, T). Since the base change morphism is flat, Y satisfies Serre's condition (S_2) by [41, Corollary of Theorem 23.3]. Consequently, Y satisfies $(S_1) + (R_0)$, and thus it is reduced. Furthermore, since Y is (S_2) and generically irreducible, it is irreducible overall according to Hartshorne's connectedness lemma [58, Tag 0FIV]. Therefore, Y is integral. Since Y is (S_2), it is normal if and only if it satisfies (R_1) condition, which is equivalent to the condition that the conductor divisor is zero. □

2.5. Behavior of the relative canonical divisor under purely inseparable base changes

We shall frequently encounter the following settings:

$$\begin{array}{ccccccc} & & & & \pi & & \\ & & \nu & \curvearrowright & & \curvearrowright & \\ Y & \xrightarrow{\quad} & (X_T)_{\mathrm{red}} & \longrightarrow & X_T & \longrightarrow & X \\ & \searrow^{g} & & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f} \\ & & & & T & \xrightarrow{\tau} & S, \end{array} \tag{2.3}$$

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where X, S, T are normal quasi-projective varieties over k, $f\colon X \to S$ is a fibration and $\tau\colon T \to S$ is a finite purely inseparable morphism of height one. For simplicity, we assume that S and T are regular so that all divisors on them are Cartier. Let us recall the following formulas from [11].

(a) If $X_{K(T)}$ is integral, then, by [11, Proposition 3.5], there exists an effective Weil divisor E and a g-exceptional $\mathbb{Q}$-divisor V on Y such that

$$\pi^* K_{X/S} \sim_{\mathbb{Q}} K_{Y/T} + (p-1)E + V. \tag{2.4}$$

(b) If $S = A$ is an abelian variety, $T = A_1 := A^{(-1)}$ and $\tau\colon A^{(-1)} \to A$ is the relative Frobenius morphism over k, then $\Omega^1_{Y/X}$ is generically globally generated since $\Omega^1_{A_1/A} = \Omega^1_{A_1} \to \Omega^1_{Y/X}$ is generically surjective. As a result, $\det(\Omega^1_{Y/X})$ has global sections (see [11, Proposition 3.4]). By (2.1), we have

$$\pi^* K_X \sim K_Y + (p-1)(E+V), \qquad E \geq 0, V \geq 0, \tag{2.5}$$

where E is a g-horizontal divisor and V is a g-vertical divisor on Y. If moreover, f is separable, then $\omega_{X_{A_1}}$ is locally free restricted on the generic fiber of f_1, and thus E can be chosen such that, on the generic fiber Y_η of g, $(p-1)E$ coincides with the conductor divisor of the normalization of Y_η (see [51, Theorem 1.2]). If f is inseparable, then E contains a nontrivial movable part (see [32, Theorem 1.1] and [11, Proposition 3.4]). Thus,

$$\pi^* K_X \sim K_Y + (p-1)(\mathfrak{M} + \mathfrak{F}), \tag{2.6}$$

where $\mathfrak{M}$ is the movable part, and $\mathfrak{F}$ is the fixed part.

2.6. Numerical dimension

Let X be a normal projective variety over k and D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. The *numerical dimension of* D is

$$\kappa_\sigma(D) = \max\{\ell \in \mathbb{N} \mid \liminf_{m\to\infty}(h^0(X, \mathcal{O}_X(\lfloor mD \rfloor + A))/m^\ell) > 0\},$$

where A is an ample divisor on X and $\kappa_\sigma(D) = -\infty$ if no such ℓ exists. If D is nef, we use $\nu(D)$ to denote the largest natural number $j \geq 0$ such that the cycle class D^j is not numerically trivial. Equivalently, $\nu(D)$ is the largest j such that $(D^j \cdot H^{\dim X - j}) \neq 0$ for some ample Cartier divisor H by [24, Corollary 3.17]. Moreover, for any nef divisor D, we have $\kappa_\sigma(D) = \nu(D)$ by [10, Remark 4.6].

The following lemma is probably well known to experts. In characteristic zero, it is a consequence of [33, Proposition 2.1], whose proof needs resolution of singularities. In characteristic p, we use smooth alteration instead.

Lemma 2.5. *Let X be a normal projective variety, and let D be a nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor with $\nu(D) = \kappa(D) = 1$. Then, D is semi-ample.*

Proof. First, we consider the case when X is smooth. Since $\kappa(D) = 1$, by replacing D with its multiple, we can assume that the linear system $|D|$ contains a movable part. Write that $|D| = |M| + V$, where $|M|$ denotes the movable part and V the fixed part. Since X is smooth, intersections of divisors make sense. By restricting on the intersection of $\dim X - 2$ general hypersurfaces, since M is movable, from $D^2 \equiv M(M+V) + V \cdot D \equiv 0$, we deduce that $M^2 \equiv 0$ and $M \cdot V \equiv 0$. From this, we conclude that:

- the linear system $|M|$ has no base point, hence it induces a fibration $f\colon X \to B$, where B is a smooth projective curve;
- the fixed part V is contained in finitely many closed fibers of f.

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Moreover, since D is nef, we see that V is nef, and V must be like $a_1F_1 + \cdots + a_rF_r$, where F_i are closed fibers of f and $a_i \in \mathbb{Q}^+$. In conclusion, there exists an effective $\mathbb{Q}$-divisor D_B such that $D \sim_{\mathbb{Q}} f^*D_B$. Thus, D is semi-ample.

In general, we can take a smooth alteration $\pi\colon Y \to X$ [14, Theorem 4.1], so that Y is smooth, projective, and dominant over X. We see that π^*D is nef, and $\nu(\pi^*(D)) = \kappa(Y, \pi^*(D)) = \nu(D) = 1$ [10, Remark 4.6]. In the previous paragraph, we have proved that π^*D is semi-ample on Y, which implies that D is semi-ample. □

2.7. Covering theorem

Theorem 2.6 [31, Theorem 10.5]. *Let $f\colon X \to Y$ be a proper surjective morphism between complete normal varieties over an algebraically closed field. If D is a Cartier divisor on Y and E an effective f-exceptional divisor on X, then*

$$\kappa(X, f^*D + E) = \kappa(Y, D).$$

2.8. Adjunction formula and characterization of abelian varieties

Proposition 2.7 [38, Proposition 4.5] and [13, Theorem 4.1]. *Let X be a normal variety and S be a prime Weil divisor of X. Let $S^\nu \to S$ be the normalization. Assume that $K_X + S$ is $\mathbb{Q}$-Cartier. Then,*

(1) *There exists an effective $\mathbb{Q}$-divisor Δ_{S^ν} on S^ν such that*

$$(K_X + S)|_{S^\nu} \sim_{\mathbb{Q}} K_{S^\nu} + \Delta_{S^\nu}.$$

(2) *Let $V \subset S^\nu$ be a prime divisor, then $\operatorname{coeff}_V \Delta_{S^\nu} = 0$ if and only if X, S are both regular at the generic point of V.*
(3) *If the pair (S^ν, Δ_{S^ν}) is strongly F-regular, then S is normal.*

Proposition 2.8 [20, Proposition 3.2], see [11, Proposition 2.9]. *Let X be a normal projective variety of maximal Albanese dimension, then:*

(1) *$K_X \geq_{\mathbb{Q}} 0$, and*
(2) *if $K_X \sim_{\mathbb{Q}} 0$, then X is isomorphic to an abelian variety.*

We give the following useful lemma, which can be proved by the above two propositions.

Lemma 2.9. *Let X be a normal $\mathbb{Q}$-factorial projective variety. Let D be a prime divisor of maximal Albanese dimension, and let $\Delta \geq 0$ be a $\mathbb{Q}$-divisor such that $D \nsubseteq \operatorname{Supp}\Delta$. Then:*

(1) *$(K_X + D + \Delta)|_{D^\nu} \geq_{\mathbb{Q}} 0$;*
(2) *if $(K_X + D + \Delta)|_{D^\nu} \sim_{\mathbb{Q}} 0$, then D is isomorphic to an abelian variety, $\Delta|_D \sim_{\mathbb{Q}} 0$, and X is regular at codimension-one points of D;*
(3) *if $D|_{D^\nu} \geq_{\mathbb{Q}} 0$, and $(K_X + D + aD + \Delta)|_{D^\nu} \sim_{\mathbb{Q}} 0$ for some $a \in \mathbb{Q}_{>0}$, then $D|_{D^\nu} \sim_{\mathbb{Q}} 0$; and as a consequence of (2), D is isomorphic to an abelian variety, $\Delta|_D \sim_{\mathbb{Q}} 0$, and X is regular at codimension-one points of D.*

Proof. By Proposition 2.8(1), we have $K_{D^\nu} \geq_{\mathbb{Q}} 0$. In turn, applying the adjunction formula (Proposition 2.7), we have

$$(K_X + D + \Delta)|_{D^\nu} \sim_{\mathbb{Q}} K_{D^\nu} + \Delta_{D^\nu} + \Delta|_{D^\nu} \geq_{\mathbb{Q}} 0,$$

which is the assertion (1).

Next, assuming $(K_X + D + \Delta)|_{D^\nu} \sim_{\mathbb{Q}} 0$, from the above equation, we see that

$$K_{D^\nu} \sim_{\mathbb{Q}} \Delta_{D^\nu} \sim_{\mathbb{Q}} \Delta|_{D^\nu} \sim_{\mathbb{Q}} 0.$$

Then, the assertion (2) follows immediately from Proposition 2.8(2) and Proposition 2.7(2).

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Finally, assuming $D|_{D^\nu} \geq_{\mathbb{Q}} 0$, applying the adjunction formula (Proposition 2.7) again, we have

$$0 \sim_{\mathbb{Q}} (K_X + D + aD + \Delta)|_{D^\nu} \sim_{\mathbb{Q}} K_{D^\nu} + \Delta_{D^\nu} + aD|_{D^\nu} + \Delta|_{D^\nu} \geq_{\mathbb{Q}} 0.$$

It follows that $K_{D^\nu} \sim_{\mathbb{Q}} \Delta_{D^\nu} \sim_{\mathbb{Q}} D|_{D^\nu} \sim_{\mathbb{Q}} \Delta|_{D^\nu} \sim_{\mathbb{Q}} 0$. Then, the assertion (3) follows by the same argument as above. □

2.9. Some results of elliptic fibrations

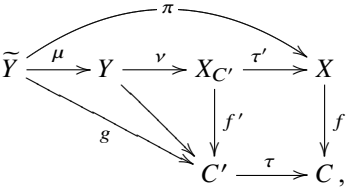
The following result appears as a middle step in the proof of [12, Theorem 1.2].

Theorem 2.10 [12, Claim 3.2 and Remark 3.3]. *Let $f\colon X \to Z$ be an elliptic fibration from a normal variety X onto a smooth variety Z. Then, $\kappa(X, K_{X/Z}) \geq 0$ and $\kappa(X) \geq \max\{\kappa(Z), \mathrm{Var}(f)\}$.*

Remark that in the setting of [12], X is assumed to be smooth, but the proof works when X is normal.

Proposition 2.11. *Let $f\colon X \to C$ be an elliptic fibration from a normal projective surface to a smooth curve. If $K_{X/C} \sim_{\mathbb{Q}} 0$, then there exists a finite flat morphism $\sigma\colon C_1 \to C$ such that $X \times_C C_1 \cong C_1 \times F$, where F is a closed fiber of f.*

Proof. By doing a semi-stable reduction (see [40, Proposition 10.2.33]), we obtain the following commutative diagram:



where C' is a smooth curve, τ is a finite morphism, $X_{C'} := X \times_C C'$ is the fiber product, ν is the normalization, μ is a minimal resolution, $\pi = \tau' \circ \nu \circ \mu$ is the composition, and g is a semi-stable elliptic fibration. We have $K_{Y/C'} = (\tau' \circ \nu)^* K_{X/C} - \mathfrak{C}$, where $\mathfrak{C}$ is the conductor divisor of ν, and $K_{\widetilde{Y}} = \mu^* K_Y - \widetilde{E}$, where $\widetilde{E}$ is an effective μ-exceptional divisor. It follows that

$$K_{\widetilde{Y}/C'} = \pi^* K_{X/C} - \mu^* \mathfrak{C} - \widetilde{E}.$$

We have $K_{\widetilde{Y}/C'} \geq_{\mathbb{Q}} 0$ by Theorem 2.10, in turn, combining with the assumption $K_{X/C} \sim_{\mathbb{Q}} 0$ we can show that $K_{\widetilde{Y}/C'} \sim_{\mathbb{Q}} 0$ and $\mathfrak{C} = \widetilde{E} = 0$. It follows that Y has at most canonical singularities, and that $X_{C'}$ is normal by Lemma 2.4, that is, $Y = X_{C'}$. Since $K_{\widetilde{Y}/C'} \sim_{\mathbb{Q}} 0$, we can apply [12, Theorem 2.14] to obtain a finite morphism $C_1 \to C'$ from a smooth curve C_1 such that $\widetilde{Y} \times_{C'} C_1 \cong C_1 \times F$. In particular, all fibers of g are irreducible, which implies that $\widetilde{Y} \to Y$ is an isomorphism. We complete the proof by taking σ to be the composition $C_1 \to C' \to C$. □

The following result can be derived directly from the proof of [9, Theorem 4].

Proposition 2.12 (cf. [9, Theorem 4]). *Let X be a quasi-projective normal variety which is equipped with a fibration $g\colon X \to Z$ and a morphism $f\colon X \to E$ to an elliptic curve such that $K(E)$ is algebraically closed in $K(X)$. Assume that all the closed fibers C_z of g are elliptic curves and that the induced morphisms $f|_{C_z}\colon C_z \to E$ are finite. Denote by F a general fiber of f. Then, there is an isogeny $\tau\colon E' \to E$ from an elliptic curve E' such that $X \times_E E' \cong E' \times F$. Moreover, we have*

$$X \cong E' \times F/G,$$

where $G := \ker \tau$ acts on $E' \times F$ diagonally: it acts on E' by translation and on F by some injective homomorphism of group functors $\alpha\colon G \to \mathrm{Aut}_F$.

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2.10. Canonical bundle formula of relative dimension one

Let us recall some canonical bundle formulas for fibrations of relative dimension one.

For a fibration $f\colon X \to S$ from X to a variety of maximal Albanese dimension, we collect some useful results from [11].

Theorem 2.13 [11, Theorems 1.3 and 7.3]. *Let (X,Δ) be a normal $\mathbb{Q}$-factorial projective pair. Let $f\colon X \to S$ be a fibration of relative dimension one, where S is a normal variety of maximal Albanese dimension. Let $a_S\colon S \to A$ be the Albanese morphism of S. Assume that the pair $(X_{K(S)}, \Delta_{K(S)})$ is klt and that $K_X + \Delta \sim_{\mathbb{Q}} f^*D$ for some $\mathbb{Q}$-Cartier divisor D on S. Then,*

(1) *If f or a_S is separable, then $\kappa(X, K_X + \Delta) \geq \kappa(S)$.*
(2) *If a_S is inseparable, then $\kappa(X, K_X + \Delta) \geq 0$; furthermore, if $\dim X = 3$, then $\kappa(X, K_X + \Delta) \geq 1$.*

Corollary 2.14 [11, Theorem 8.1]. *Let X be a normal projective $\mathbb{Q}$-factorial variety and Δ an effective $\mathbb{Q}$-divisor on X. Denote by $a_X\colon X \to A_X$ the Albanese morphism of X. Suppose that $-(K_X + \Delta)$ is nef, $X \to a_X(X)$ is of relative dimension one and $(X_{K(A)}, \Delta_{K(A)})$ is klt. Then, $a_X\colon X \to A_X$ is a fibration.*

2.11. Curves of small arithmetic genus

In this section, let K denote an F-finite field of characteristic p. Let X be a normal integral projective K-curve with $H^0(X, \mathcal{O}_X) = K$. Let $p_a(X) = \dim_K H^1(X, \mathcal{O}_X)$ be the arithmetic genus of X. Let $D = \sum_i a_i \mathfrak{p}_i$ be a Cartier divisor on X. Recall that the *degree* of D is defined to be the integer

$$\deg_K D = \sum_i a_i [\kappa(\mathfrak{p}_i) : K],$$

where $\kappa(\mathfrak{p}_i)$ denotes the residue field of $\mathfrak{p}_i$.

If $K_X \equiv 0$, applying Riemann–Roch formula [40, Theorem 7.3.17], we see that $K_X \sim 0$ and $p_a(X) = 1$. Remark that when $p \geq 5$, such a curve X is smooth over K [57]. When $p < 5$, X is possibly geometrically singular. We will focus on the singular behavior of such curves under field extensions.

First, we recall the classification of curves with $p_a = 0$.

Proposition 2.15 [59, Theorem 9.10]. *Let X be regular projective curve over K with $H^0(X, \mathcal{O}_X) = K$ and $p_a(X) = 0$. Then,*

(1) $\deg_K K_X = -2$.
(2) *X is isomorphic to a conic in $\mathbb{P}^2_K$, and $X \cong \mathbb{P}^1_K$ if and only if it has a K-rational point.*
(3) *Either X is a smooth conic or X is geometrically non-reduced. In the latter case, we have $\mathrm{char} K = 2$, and X is isomorphic to the curve defined by a quadric $sx^2 + ty^2 + z^2 = 0$ for some $s, t \in K \setminus K^2$.*

Next, we consider regular curves with $p_a = 1$. We collect some related results here and refer to [11, Section 4] for details.

Proposition 2.16. *Let X be a regular projective curve over K with $H^0(X, \mathcal{O}_X) = K$ and $p_a(X) = 1$. Assume that X is geometrically reduced and non-smooth. Then, there exists a field extension $K \subset L \subseteq K^{1/p}$ such that X_L is singular; and for any such L, the normalization $Y := (X_L)^\nu$ is a smooth curve of genus zero and the following statements hold:*

(1) *The non-smooth locus of X is supported at a closed point $\mathfrak{p}$ and $\kappa(\mathfrak{p})/K$ is purely inseparable of height one with $[\kappa(\mathfrak{p}) : K] \leq p^2$. In particular, there exists a unique point $\mathfrak{q} \in Y$ lying over $\mathfrak{p}$.*
(2) *If $p = 3$, then $Y \cong \mathbb{P}^1_L$, $\pi^*\mathfrak{p} = 3\mathfrak{q}$ (where $\pi\colon Y \to X$ is the induced morphism), $\mathfrak{q}$ is an L-rational point, and $[\kappa(\mathfrak{p}) : K] = 3$.*

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(3) *Assume* $p = 2$.
  (a) *If the point* $\mathfrak{q}$ *is* L*-rational, then* $Y \cong \mathbb{P}^1_L$, $\pi^*\mathfrak{p} = 2\mathfrak{q}$, *and* $[\kappa(\mathfrak{p}) : K] = 2$.
  (b) *If the point* $\mathfrak{q}$ *is not* L*-rational, then* $\deg_L \mathfrak{q} = 2$, *and*

$$\pi^*\mathfrak{p} = \begin{cases} \mathfrak{q}, & \text{if } \deg_K(\mathfrak{p}) = 2; \\ 2\mathfrak{q}, & \text{if } \deg_K(\mathfrak{p}) = 4. \end{cases}$$

Proposition 2.17. *Let* X *be regular projective curve over* K *with* $H^0(X, O_X) = K$ *and* $p_a(X) = 1$. *Assume that* X *is geometrically non-reduced. Then,*

(1) *There exists a height one field extension* $K \subset L \subsetneq K^{1/p}$ *such that* X_L *is integral but not normal, and for its normalization* Y, *we have* $L \subsetneq K' := H^0(Y, O_Y) \subseteq K^{\frac{1}{p}}$. *Note that* $Y \cong (X_{K'})^\nu_{\mathrm{red}}$ *over* K'. *Denote by* $\pi : Y \to X$ *the induced morphism.*
(2) *We have* $0 \sim \pi^*K_X \sim K_Y + (p-1)C$, *where* C *is a Weil divisor such that* $\deg_{K'}(p-1)C = 2$.
(3) *The divisor* C *is supported on either a single point* $\mathfrak{q} \in Y$ *or two points* $\mathfrak{q}_1, \mathfrak{q}_2 \in Y$ *(this happens only when* $p = 2$*). All the possibilities are listed as follows:*
  (i) $p = 3$, X_L *has a unique non-normal point,* $C = \mathfrak{q}$, $Y \cong \mathbb{P}^1_{K'}$, *and either* $\pi^*\mathfrak{p} = \mathfrak{q}$ *or* $\pi^*\mathfrak{p} = 3\mathfrak{q}$.
  (ii) $p = 2$, *and we fall into one of the following cases:*
    (a) $C = 2\mathfrak{q}$, $\mathfrak{q}$ *is a* K'*-rational point of* Y *and* $Y \cong \mathbb{P}^1_{K'}$;
    (b) $C = \mathfrak{q}_1 + \mathfrak{q}_2$, *and* $Y \cong \mathbb{P}^1_{K'}$;
    (c) $C = \mathfrak{q}$, $\kappa(\mathfrak{q})/K'$ *is an extension of degree two, and either*
      (c1) $Y \subset \mathbb{P}^2_K$, *is a smooth conic (possibly* $\mathbb{P}^1_{K'}$*), or*
      (c2) Y *is isomorphic to the curve defined by* $sx^2 + ty^2 + z^2 = 0$ *for some* $s, t \in K' \setminus K'^2$ *such that* $[K'^2(s,t) : K'^2] = 4$.

Remark 2.18. We note that if pdeg $K := [K^{1/p} : K] = 2$, then *case* (c2) *in Proposition 2.17 does not occur*. This follows from Schröer's classification of regular genus-one curves [55]. Indeed, by [55, Theorem 2.3], the p-degree of K is at least $r+1$, where r is the geometric generic embedding dimension (defined as the embedding dimension of the local Artin ring $O_{X,\eta} \otimes_K K^{\mathrm{perf}}$, see [23, Section 1]). If $Y = (X \times_K K^{1/p})^\nu_{\mathrm{red}}$ is not smooth, then, by [55, Theorem 2.3] the second Frobenius base-change $X \times_K K^{1/p^2}$ is isomorphic to some standard model $C^{(i)}_{r,F,\Lambda}$ with $i = 2$ (see [55] for the definition of $C^{(i)}_{r,F,\Lambda}$). Note that $i \leq r$ by Schröer's construction [55, p. 8]. Thus, case (c2) occurs only when pdeg $K \geq r + 1 \geq 3$.

3. Constructions of semi-ample divisors

This section concerns the construction of semi-ample divisors. Let X be a K-trivial variety and denote by $a_X : X \to A$ the Albanese morphism of X. In case $\dim a_X(X) = 1$, we construct a semi-ample divisor which is relatively ample over A; and in case a_X has relative dimension one, we prove a semi-ampleness criterion for divisors with numerical dimension one. We use these results to derive the second fibration $g : X \to Z$ which is transversal to a_X.

3.1. A construction of semi-ample divisors

The construction we present here was recently developed in [18], [20] by use of a powerful positivity engine. Similar approaches were used to treat surfaces [1, Theorem 8.10] and threefolds equipped with fibrations to elliptic curves [61, Section 4.4]. In the settings of [18], [20], a pair (X, Δ) is assumed to be strongly F-regular and the Cartier index of $K_X + \Delta$ is assumed to be indivisible by p (i.e., $K_X + \Delta$ is a $\mathbb{Z}_{(p)}$-Cartier). By refining their argument, we can slightly relax the F-regularity condition and replace the second condition by that X is $\mathbb{Q}$-factorial. Here, we only explain how to modify the argument.

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First, we recall a positivity result due to Ejiri, which is crucial for proving Theorem 3.3, one of the main results of this section.

Theorem 3.1 [19, Corollary 6.5 and Examples 5.7 and 5.8]. *Let $f\colon X \to Y$ be a surjective morphism between normal projective varieties and let $a\colon Y \to A$ be a generically finite morphism to an abelian variety. Let $\Delta \geq 0$ be a $\mathbb{Z}_{(p)}$-divisor on X. Let $V' \subseteq A$ be an open dense subset, $V = a^{-1}V'$ and $U = f^{-1}V$. Assume that:*

- *$(U, \Delta|_U)$ is F-pure;*
- *$K_U + \Delta|_U$ is $\mathbb{Q}$-Cartier and f-ample;*
- *$V \to V'$ is a finite morphism.*

Let H' be a free ample symmetric divisor on A and $H = a^*H'$. Let i be a positive integer such that $i(K_X + \Delta)$ is integral. Then, there exists an integer m_0 such that for any $m \geq m_0$,

(i) *$f_*\mathcal{O}_X(im(K_X + \Delta)) \otimes \mathcal{O}_Y((\dim Y + 1)H)$ is globally generated over V;*
(ii) *$\mathbb{B}_-(f_*\mathcal{O}_X(im(K_X + \Delta))) \subseteq Y \setminus V$ (see [19, Section 4.1] for the definition of $\mathbb{B}_-$).*

We adapt [20, Theorem 7.1] to the following setting, which originally requires that $K_X + \Delta$ is a $\mathbb{Z}_{(p)}$-Cartier divisor and that (X, Δ) is F-pure.

Corollary 3.2. *Let X be a normal $\mathbb{Q}$-factorial projective variety of dimension n and $f\colon X \to Y$ a fibration to a smooth curve Y with $g(Y) \geq 1$. Let $\Delta \geq 0$ be a divisor on X and L be a nef Cartier divisor on X. Assume that (X_η, Δ_η) is strongly F-regular and that $K_X + \Delta + L$ is f-nef. Then, $K_X + \Delta + L$ is nef.*

Proof. We only need to prove that for every ample $\mathbb{Q}$-divisor A on X, $K_X + \Delta + L + A$ is nef. Fix an ample divisor A on X. By [50, Lemma 3.15], we can find an ample $\mathbb{Q}$-divisor $A' > 0$, such that:

(i) $A - A'$ is ample,
(ii) $K_X + \Delta'$ is a $\mathbb{Z}_{(p)}$-Cartier divisor, where $\Delta' = \Delta + A'$, and
(iii) (X_η, Δ'_η) is strongly F-regular.

Here, we remark that in order to apply Theorem 3.1, we only require the pair (X, Δ) to be F-pure over an open subset of Y, which is guaranteed by condition (iii). Therefore, since $K_X + \Delta' + L$ is f-nef, we can apply the same proof of [20, Theorem 7.1] to the pair (X, Δ') and L, which yields that $K_X + \Delta' + L \sim_{\mathbb{Q}} K_X + \Delta + L + A'$ is nef. From this, we conclude that $K_X + \Delta + L$ is nef. □

Theorem 3.3. *Let X be a normal $\mathbb{Q}$-factorial projective variety of dimension n and $f\colon X \to Y$ a fibration to a smooth curve Y with $g(Y) \geq 1$. Let $\Delta \geq 0$ be a divisor on X. Assume that:*

a) *$-(K_X + \Delta)$ is nef;*
b) *the Cartier index of $K_{X_\eta} + \Delta_\eta$ is indivisible by p;*
c) *(X_η, Δ_η) is strongly F-regular.*

Let A be an ample divisor on X. Choose positive integers a, b such that $(aA - bF)^n = a^nA^n - na^{n-1}bA^{n-1}F = 0$, where F is a fiber of f.

Then,

(i) *The divisor $D = aA - bF$ is nef with $v(D) = n - 1$.*
(ii) *$g(Y) = 1$.*
(iii) *For any sufficiently divisible integer $m > 0$, the sheaf $f_*(\mathcal{O}_X(mD))$ is a numerically flat vector bundle, that is, both $f_*(\mathcal{O}_X(mD))$ and its dual $(f_*(\mathcal{O}_X(mD)))^\vee$ are nef. By [48], there exists an isogeny $\pi\colon Z \to Y$ from an elliptic curve Z such that $\pi^* f_*\mathcal{O}_X(mD) \cong \bigoplus_i L_i$, where $L_i \in \mathrm{Pic}^0(Z)$.*
(iv) *Assume moreover that*
    d) *(X, Δ) is strongly F-regular, the Cartier index of $K_X + \Delta$ is indivisible by p and $-(K_X + \Delta)$ is numerically semi-ample.*

Then, there exists some $L \in \mathrm{Pic}^0(Y)$ such that $D + f^*L$ is semi-ample.

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Proof. First, we borrow the argument of [20, Theorem 7.3] to show that the divisor D is nef as follows. Take an ample $\mathbb{Q}$-divisor H on Y. Let us show that $D + f^*H$ is nef. By the construction of D, the divisor $D + f^*H$ is big and f-ample. Take an effective divisor $\Gamma \sim_{\mathbb{Q}} D + f^*H$ and a small rational number $\epsilon > 0$ such that $(X, \Delta' = \Delta + \epsilon\Gamma)$ is strongly F-regular on X_η. Since $-(K_X + \Delta)$ is nef, Corollary 3.2 applies and shows that

$$\epsilon(D + f^*H) \equiv K_X + \Delta' + (-(K_X + \Delta))$$

is a nef divisor. Therefore, we conclude that D is nef, and hence $\nu(D) = n-1$ by $D^n = 0$. This proves (i).

Next, we prove (ii): $g(Y) = 1$. Otherwise, $H = K_Y$ is big. Since D is nef and f-ample and $D - (K_{X/Y} + \Delta) - f^*K_Y = D - (K_X + \Delta)$ is nef, we can apply [60, Theorem 1.5] to show that D is big, which contradicts $\nu(D) = n - 1$.

Having proved $\nu(D) = n - 1$ and $g(Y) = 1$, by the same argument of [20, Theorems 7.4 and 7.5], we can show that $f_*\mathcal{O}_X(mD)$ is numerically flat. This proves (iii).

Finally, we can conclude the assertion (iv) from the proof of [18, Theorem 6.1] under these conditions. □

Remark 3.4. For the assertion (iv) in the theorem above, it seems not easy to drop the Cartier index condition in the proof of [18, Theorem 6.1].

When $\mathrm{char} k \geq 5$, we may run the minimal model program for three-dimensional klt pairs [6], [26], [28]. Taking advantage of this, the arguments of [61, Section 4.4] yield the following theorem.

Theorem 3.5. *Assume $p = \mathrm{char} k \geq 5$. Let (X, Δ) be a normal projective $\mathbb{Q}$-factorial three-dimensional klt pair with $K_X + \Delta \equiv 0$. Let $f : X \to Y$ be a fibration to a curve with $g(Y) \geq 1$. Let A be an ample divisor on X and choose positive integers a, b such that $(aA - bF)^3 = 0$, where F is a fiber of f. Then, there exists some $L \in \mathrm{Pic}^0(Y)$ such that $aA - bF + f^*L$ is semi-ample.*

3.2. A semi-ampleness criterion

If X is a K-trivial variety equipped with a quasi-elliptic fibration $f : X \to A$ to an abelian variety, the non-smooth locus of f provides a divisor. To prove this divisor is semi-ample, we abstract a semi-ampleness criterion from the argument of [61, Section 4.2], which proves a nonvanishing result up to a twist by a numerically trivial line bundle. In [61, Section 4.2], the author treated only threefolds and used results of minimal model program. But our situation is special, we can avoid running MMP in the argument.

We first prove the following nonvanishing result by use of [61, Theorem 3.7] and a similar argument of [61, Section 4.2].

Theorem 3.6. *Let X be a normal projective variety equipped with a surjective morphism $f : X \to A$ to an abelian variety A of dimension d. Let $\Delta \geq 0$ be an effective $\mathbb{Q}$-divisor on X and D a Cartier divisor on X. Assume that:*

(a) *$K_X + \Delta$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor with the Weil index of $K_X + \Delta$ being indivisible by p;*
(b) *the Cartier index of $(K_X + \Delta)|_{X_\eta}$ is indivisible by p;*
(c) *the divisor $D - (K_X + \Delta)$ is nef and relatively ample over A;*
(d) *$r = \dim_{K(\eta)} S^0_\Delta(X_\eta, D|_{X_\eta}) > 0$ (see [60, p. 10] for the definition of S^0_Δ).*

Then,

(1) *$V^0(f_*\mathcal{O}_X(D)) = \{\alpha \in \widehat{A} = \mathrm{Pic}^0(A) \mid h^0(f_*\mathcal{O}_X(D) \otimes \mathcal{P}_\alpha) > 0\} \neq \emptyset$, where $\mathcal{P}$ denotes the Poincaré line bundle over $A \times \widehat{A}$;*
(2) *if $\dim V^0(f_*\mathcal{O}_X(D)) = 0$, then there exists a subsheaf $\mathcal{F} \subseteq f_*\mathcal{O}_X(D)$ of rank r such that $\mathcal{F}|_{X_\eta} = S^0_\Delta(X_\eta, D|_{X_\eta})$, an isogeny $\pi : A_1 \to A$ of abelian varieties, some $P_1, \ldots, P_r \in \mathrm{Pic}^0(A_1)$ and a generically surjective homomorphism*

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$$\beta\colon \bigoplus_i P_i \to \pi^*\mathcal{F}.$$

Proof. By assumption (a), we may fix an integer $g > 0$ such that $(p^g - 1)\Delta$ is integral. For an integer $e > 0$ divisible by g, setting $D_e = (1 - p^e)(K_X + \Delta) + p^e D$, we have the trace map

$$\mathrm{Tr}_D^e : \mathcal{F}^e := f_*(F_{X*}^e \mathcal{O}_X((1 - p^e)(K_X + \Delta)) \otimes \mathcal{O}_X(D)) \cong f_*(F_{X*}^e \mathcal{O}_X(D_e)) \to f_*\mathcal{O}_X(D),$$

and we denote its image by $\mathcal{F}_0^e$. Note that for positive integers $e' > e$ divisible by g, the trace map $\mathrm{Tr}_D^{e'}$ factors through the trace map [61, Section 2.7]

$$\mathrm{Tr}_{D_e}^{e'-e} : \mathcal{F}^{e'} = f_*(F_{X*}^{e'-e}\mathcal{O}_X((D_{e'}))) \to \mathcal{F}^e = f_*\mathcal{O}_X(D_e),$$

which implies that $\mathcal{F}_0^{e'} \subseteq \mathcal{F}_0^e$. Therefore, there exists a positive integer e_0 such that for all $e \geq e_0$ divisible by g, the sheaf $\mathcal{F}_0^e$ has constant rank r, which means that

(c′) *the trace map* $\mathrm{Tr}_D^e : \mathcal{F}^e \to \mathcal{F} := \mathcal{F}_0^{e_0}$ *is generically surjective.*

We shall apply [61, Theorem 3.7], and we need to verify the three conditions required there. We refer to [61, Section 3] for the related notions of Fourier–Mukai transform.

With the assumption that $D - (K_X + \Delta)$ is nef and relatively ample over A, the proof of the vanishing condition (C2) in [61, Section 4.2], if substituting the pair (X, B) with (X, Δ) and the divisor $l(K_X + B)$ with D, still works and yields the following.

Claim. If H is an ample line bundle on $\widehat{A}$, then for any $i > 0$ and sufficiently divisible integer $e > 0$,

$$H^i(A, \mathcal{F}^e \otimes \widehat{H}^*) = 0.$$

By induction, we can find integers $e_0 < e_1 < e_2 < \cdots < e_d$ divisible by g, and ample line bundles $H_0, H_1, \ldots, H_{d-1}$ on $\widehat{A}$ such that, if setting

$$\mathcal{F}_0 = \mathcal{F},\ \mathcal{F}_1 = \mathcal{F}^{e_1},\ \mathcal{F}_2 = \mathcal{F}^{e_2}, \ldots,\ \mathcal{F}_d = \mathcal{F}^{e_d},$$

we have

(a′) *for* $0 \leq l \leq d - 1$ *and every* i, *the sheaf* $R^i\Phi_{\mathcal{P}}D_A(\mathcal{F}_l) \otimes H_l$ *is globally generated, and if* $j > 0$, *then* $H^j(\widehat{A}, R^i\Phi_{\mathcal{P}}D_A(\mathcal{F}_l) \otimes H_l) = 0$;
(b′) *for* $0 \leq l < l' \leq d$, *if* $j > 0$, *then* $H^j(A, \mathcal{F}_{l'} \otimes \widehat{H}_l^*) = 0$.

We see that the conditions (a, b, c) of [61, Theorem 3.7] are guaranteed by the above conditions (a′, b′, c′), respectively. Therefore, we can apply [61, Theorem 3.7]:

- by [61, Theorem 3.7(i)] the homomorphism $\alpha_{\mathcal{F}}\colon \mathcal{F}^* \to (-1)_A^* R^0\Psi_{\mathcal{P}}R^0\Phi_{\mathcal{P}}D_A(\mathcal{F})$ is injective, and in turn, applying [61, Proposition 3.4(2)] we can show that

$$V^0(f_*\mathcal{O}_X(D)) = \mathrm{Supp}(-1)_{\widehat{A}}^* R^0\Phi_{\mathcal{P}}D_A(\mathcal{F}) \neq \emptyset;$$

- by [61, Theorem 3.7(ii)], the second assertion of the theorem follows. □

Next, we prove the following semi-ample criterion.

Theorem 3.7. *Let X be a normal $\mathbb{Q}$-factorial projective variety equipped with a fibration $f : X \to A$ of relative dimension one onto an abelian variety. Assume that:*

(a) $K_X \sim_{\mathbb{Q}} 0$;
(b) *there exists an irreducible f-horizontal divisor B on X such that $B|_{B^v} \equiv 0$.*

Then, B is semi-ample, and the associated morphism $g : X \to C$ is a fibration to a curve.

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Proof. Since $B|_{B^\nu} \equiv 0$, B is a nef divisor with numerical dimension $\nu(B) = 1$. To show the semi-ampleness of B, it suffices to find an integer $l > 0$ and a numerically trivial line bundle L such that $h^0(X, lB + L) > 1$. Indeed, granted this, by Lemma 2.5, the divisor $lB + L$ is semi-ample and induces a fibration $g\colon X \to C$ to a curve, then since $lB \equiv lB + L$ and B is irreducible by assumption, we see that a multiple of B coincides with a fiber of g, and hence B is semi-ample.

Since $(K_X + B)|_{B^\nu} \equiv 0$, by Lemma 2.9, B is isomorphic to an abelian variety and thus $B \to A$ is a finite morphism. Note that since X has relative dimension one over A, the divisor B is relatively big over A. Let $\mathbb{E}(B)$ denote the relatively exceptional locus with respect to B, namely, the union of f-exceptional irreducible varieties on which the restriction of B is not f-big.

Claim. The intersection $\mathbb{E}(B) \cap B = \emptyset$, and B is relatively semi-ample over A.

Proof of the claim. Since B is finite over A, if there is an irreducible component Z of $\mathbb{E}(B)$ intersecting B, then $Z \cap B$ is also finite over $f(Z)$, hence, $B|_Z$ is big over $f(Z)$, a contradiction. Therefore, we conclude that $\mathbb{E}(B)$ does not intersect B, and hence B is f-semi-ample by [36, Theorem 0.2]. □

The f-relative semi-ample divisor B induces a birational contraction $\sigma\colon X \to Y$, which is isomorphic both near B and on the generic fiber X_η of f, and there exists a $\mathbb{Q}$-Cartier divisor B_Y on Y such that $B = \sigma^* B_Y$.

Since $B|_{X_\eta}$ is an ample divisor on X_η, we may take a sufficiently divisible integer $l > 0$ such that lB is Cartier and $r := \dim_{K(\eta)} S^0(X_\eta, lB) > 1$. Set $\mathcal{L} = \mathcal{O}_X(lB)$. By construction, we have B_Y is relatively ample over A, and $K_Y \sim_{\mathbb{Q}} 0$. If setting $D = lB_Y$, then $D - K_Y$ is relatively ample over A. Therefore, Theorem 3.6 applies to Y and $D = lB_Y$ and yields that $V^0(f_*\mathcal{L}) \neq \emptyset$.

If $\dim V^0(f_*\mathcal{L}) > 0$, then we can apply the argument Step 1 of the proof of [61, Theorem 4.2] to show that $\kappa(X, B + f^*L) \geq 1$ for some $L \in \mathrm{Pic}^0(A)$, which is sufficient to conclude the proof.

Now, assume $\dim V^0(f_*\mathcal{L}) = 0$. Then, by Theorem 3.6 (2), there exist a subsheaf $\mathcal{F} \subseteq f_*\mathcal{L}$ of rank r, an isogeny $\pi\colon A_1 \to A$ of abelian varieties, some $P_1, \ldots, P_r \in \mathrm{Pic}^0(A_1)$, and a generically surjective homomorphism

$$\beta\colon \bigoplus_i P_i \to \pi^*\mathcal{F}.$$

Applying the covering theorem as in Step 2 of the proof of [61, Theorem 4.2], we show that there exist an integer $m > 0$ and some $L_1, \ldots, L_r \in \mathrm{Pic}^0(A)$ such that:

- $H^0(X, mB + f^*L_i) \neq 0$;
- the sub-linear system of $|(mB)_{K(\eta)}|$ corresponding to the subspace $\sum_i H^0(X, mB + L_i) \otimes_k K(\eta) \subseteq H^0(X_\eta, mB|_{X_\eta})$ defines a non-trivial map.

We may assume each $h^0(X, mB + f^*L_i) = 1$, thus there exists a unique effective divisor $D_i \sim mB + f^*L_i$.

Write $D_i = aB + D'$ such that $B \nsubseteq \mathrm{Supp}\, D'$. We have $D_i|_B \equiv 0$. By $\nu(B) = 1$, we conclude that $\mathrm{Supp}\, B \cap \mathrm{Supp}\, D' = \emptyset$, thus $D'|_B \sim 0$. Moreover, by Lemma 2.9, $B|_B \sim_{\mathbb{Q}} (K_X + B)|_B \geq_{\mathbb{Q}} 0$. Then, it follows that $D_i|_B \sim_{\mathbb{Q}} 0$.

Take $D_1 \neq D_2$. By $D_1 - D_2 \sim f^*(L_1 - L_2)$, we conclude that $f^*(L_1 - L_2)|_B \sim_{\mathbb{Q}} 0$. Since $B \to A$ is dominant, we have $L_1 \sim_{\mathbb{Q}} L_2$, that is, there exists some $N > 0$ such that $NL_1 \sim NL_2$. But then $ND_1 \sim ND_2$. This shows $h^0(X, NmB + NL_1) \geq 2$, which concludes the proof. □

4. Structure theorems of K-trivial irregular varieties with $\dim a_X(X) = 1$

In this section, we treat K-trivial irregular varieties X with $\dim a_X(X) = 1$. The main result is the following.

Theorem 4.1. *Let X be a normal $\mathbb{Q}$-factorial projective variety with $K_X \equiv 0$ and $\dim a_X(X) = 1$, where $a_X\colon X \to A$ is the Albanese morphism of X. Assume moreover that, either:*

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(a) *X is strongly F-regular and the Cartier index of K_X is indivisible by p, or*
(b) $\dim X = 3$, *$p \geq 5$, and X has at worst terminal singularities.*

Then,

(1) *$q(X) = 1$, thus $E := A$ is an elliptic curve and $f = a_X : X \to E$ is a fibration.*
(2) *there exists an isogeny $\pi : \bar{E} \to E$ of elliptic curves such that $X \times_E \bar{E} \cong \bar{E} \times F$, where F is a fiber of f. More precisely, there is a faithful action of $H = \ker(\pi)$ on F, such that*

$$X \cong \bar{E} \times F/H,$$

where H acts diagonally on $\bar{E} \times F$.

Proof. Note that under assumption (b), the generic fiber of f is a regular surface. So in both cases (a) and (b), we can apply Theorem 3.3(i) to show the assertion (1).

Next, we claim that there exists a divisor D such that:

(*) for sufficiently divisible $m > 0$, $f_*\mathcal{O}_X(mD)$ is a numerically flat vector bundle on E, and there exists an isogeny $\tau_1 : E_1 \to E$ such that $\tau_1^* f_*\mathcal{O}_X(mD) \cong \bigoplus_{i=1}^r L_i$ for some $L_i \in \mathrm{Pic}^0(E_1)$;
(**) D is semi-ample and f-ample, and $\nu(D) = \dim X - 1$.

Such a divisor D satisfying the condition $(*)$ exists by Theorem 3.3, and the condition $(**)$ is guaranteed by Theorem 3.3(iv) in case (a) and by Theorem 3.5 in case (b).

Now, let $g : X \to Z$ be the fibration associated with D, where Z is a normal projective variety with $\dim Z = \dim X - 1$. To summarize, we obtain the following "bi-fibration" structure:

$$\begin{array}{ccc} X & \xrightarrow{g} & Z. \\ {\scriptstyle f}\downarrow & & \\ E & & \end{array}$$

Here, for a general closed point $z \in Z$, the fiber X_z of g is a curve which is finite and dominant over E. Since $K_{X_z} \equiv 0$, X_z is in fact an elliptic curve. Moreover, since D is f-ample, we conclude that $g : X \to Z$ is equidimensional and every component of a fiber X_z of g over an arbitrary closed point $z \in Z$ is dominant over E.

In the following, we fix a sufficiently large m such that for any $l > 0$, the natural homomorphism $\eta : S^l(f_*\mathcal{O}_X(mD)) \to f_*\mathcal{O}_X(lmD)$ is surjective, where S^l denotes the lth symmetric power.

Following the approach of [20, Proposition 7.6], we can prove the following result.

Lemma. *There exists an isogeny $\tau : \tilde{E} \to E$ of elliptic curves such that $\tilde{X} := X \times_E \tilde{E} \cong \tilde{E} \times F$.*

Proof of the lemma. First, we prove the following claim.

Claim. There exists an isogeny $\tau : \tilde{E} \to E$ of elliptic curves such that $\tau^* f_*\mathcal{O}_X(mD) \cong \bigoplus^r \mathcal{O}_{\tilde{E}}$, where $r := \mathrm{rank}\, f_*\mathcal{O}_X(mD)$. □

Proof of the claim. By $(*)$, there exists an isogeny $\tau_1 : E_1 \to E$, such that $\tau_1^* f_*\mathcal{O}_X(mD) \cong \bigoplus_{i=1}^r L_i$ for some $L_i \in \mathrm{Pic}^0(E_1)$. Let $X_1 = (X \times_E E_1)^\nu$. Denote by $\pi_1 : X_1 \to X$ and $f_1 : X_1 \to E_1$ the natural projections. Then, $|\pi_1^* mD - f_1^* L_i| \neq \emptyset$. The divisor $\pi_1^* D$, being semi-ample with $\nu(\pi_1^* D) = n-1$, induces a fibration $g_1 : X_1 \to Z_1$. Then, for a general fiber C_1 of g_1, the linear system $|\pi_1^* mD - f_1^* L_i|_{C_1} \neq \emptyset$. We conclude that $f_1^* L_i|_{C_1} \sim 0$, thus each L_i is a torsion point in $\mathrm{Pic}^0(E_1)$. There exists a further isogeny of elliptic curves $\tau_2 : E_2 \to E_1$ such that $\tau_2^* L_i \cong \mathcal{O}_{E_2}$. We may take $\tau : \tilde{E} = E_2 \to E_1 \to E$ to complete the proof of the claim. □

Denote by $\tilde{\pi} : \tilde{X} := X \times_E \tilde{E} \to X$ and $\tilde{f} : \tilde{X} \to \tilde{E}$ the natural projections. Since for each $l \geq 1$, $\eta : S^l(f_*\mathcal{O}_X(mD)) \to f_*\mathcal{O}_X(lmD)$ is surjective, the homomorphism

$$\varphi_l : S^l(\tilde{f}_*\mathcal{O}_{\tilde{X}}(m\tilde{\pi}^* D)) \cong \tau^* S^l(f_*\mathcal{O}_X(mD)) \cong \bigoplus \mathcal{O}_{\tilde{E}} \to \tilde{f}_*\mathcal{O}_{\tilde{X}}(lm\tilde{\pi}^* D) \cong \tau^* f_*\mathcal{O}_X(lmD)$$

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is surjective. Combining this with the numerically flatness of $f_*O_X(lmD)$, $l \geq 1$, we conclude that $\tilde{f}_*O_{\tilde{X}}(lm\tilde{\pi}^*D) \cong \bigoplus O_{\tilde{E}}$, and φ_l is determined by the corresponding homomorphism of the global sections. From this, we conclude that $\tilde{X} = X \times_E \tilde{E} \cong \tilde{E} \times F$.

We regard the morphism $\tau\colon \tilde{E} \to E$ as a morphism of abelian varieties and write $\tilde{H} = \ker(\tau)$. The natural action of $\tilde{H}$ on $\tilde{E}$ induces an action of $\tilde{H}$ on the base change $\tilde{X} = X \times_E \tilde{E}$. To summarize, we have a commutative diagram:

$$\begin{array}{ccccc} & & & & F & \longrightarrow & Z \\ & & & \overset{\tilde{g}}{\nearrow} & & \overset{g}{\nearrow} & \\ \tilde{E} \times F & \underset{\cong}{\xleftarrow{\varphi=(\tilde{f},\tilde{g})}} & \tilde{X} & \xrightarrow{\tilde{\pi}} & X & & \\ & & \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f} & & \\ & & \tilde{E} & \xrightarrow{\tau} & E\,. & & \end{array} \tag{4.1}$$

Through the isomorphism $\varphi\colon \tilde{X} \xrightarrow{\sim} \tilde{E} \times F$, $\tilde{H}$ acts on $\tilde{E} \times F$. Our next step is to show that this action of $\tilde{H}$ on $\tilde{E} \times F$ is diagonal. For this purpose, we consider the second fibration $g\colon X \to Z$. As observed before, for a general closed point $z \in Z$, the fiber X_z of g is an elliptic curve, so $X_z \to E$ is an isogenous of a fixed degree. Let $Z^\circ \subset Z$ be an open subset such that Z° is regular and for each closed point $z \in Z^\circ$, the fiber X_z of g is an elliptic curve. Set $X^\circ := g^{-1}(Z^\circ)$ and $F^\circ := F \cap X^\circ$. By Proposition 2.12, we conclude that:

(***) There exists an isogeny $\tau'\colon E' \to E$ from an elliptic curve E' such that $X^\circ \times_E E' \cong E' \times F^\circ$. Moreover, the induced action of $G := \ker\sigma$ on $X^\circ \times_E E'$ is diagonal on $E' \times F^\circ$, and the following commutative diagram commutes:

$$\begin{array}{ccccccc} & & & \overset{p_2}{\frown} & & F^\circ & \longrightarrow & Z^\circ \cong F^\circ/G \\ & & & & \overset{g_1}{\nearrow} & & \overset{g}{\nearrow} & \\ E' \times F^\circ & \underset{\cong}{\xleftarrow{\quad}} & X_1^\circ := X^\circ \times_E E' & & \xrightarrow{\pi_1} & X^\circ & & \\ & & \downarrow{\scriptstyle f_1} & & & \downarrow{\scriptstyle f} & & \\ & & E' & & \xrightarrow{\tau'} & E\,. & & \end{array}$$

Remark that if we take a further isogeny $\tau''\colon E'' \to E'$ of elliptic curves, then the action of $\ker\tau''$ on the base change $X_1^\circ \times_{E'} E'' \cong E'' \times F^\circ$ induces an action on the product $E'' \times F^\circ$; precisely, $\ker\tau''$ acts on E'' by translation and on F° trivially. Thus, the base change $\tau' \circ \tau''\colon E'' \to E' \to E$ induced a diagonal action of $\ker\tau' \circ \tau''$ on $E'' \times F^\circ$. Therefore, we may choose an isogeny $\tilde{\tau}'\colon \tilde{E}' \to E$ which factors through both $\tau\colon \tilde{E} \to E$ and $\tau'\colon E' \to E$, and obtain the following commutative diagram:

$$\begin{array}{ccccc} & & F^\circ & \longrightarrow & F^\circ/G \\ & \overset{\tilde{g}'_0}{\nearrow} & \cup & \overset{g_0}{\nearrow} & \\ X^\circ \times_E \tilde{E}' & \longrightarrow & X^\circ & & \\ \downarrow & & \downarrow & & \downarrow \\ & \overset{\tilde{g}'}{\nearrow} & F & \dashrightarrow & Z \\ X \times_E \tilde{E}' & \longrightarrow & X & \overset{g}{\nearrow} & \\ \downarrow & & \downarrow & & \\ \tilde{E}' & \xrightarrow{\tilde{\tau}'} & E\,. & & \end{array}$$

Thus, the natural action of $\tilde{H}' = \ker\tilde{\tau}'$ on the base change $X \times_E \tilde{E}' \cong \tilde{E}' \times F$ is compatible with its action on $X^\circ \times_E E_1$ given by (***), which is diagonal. From this, we conclude that $\tilde{H}'$ acts on $\tilde{E}' \times F$

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diagonally. Finally, let K be the kernel of the action of $\tilde{H}'$ on F. Set $H := \tilde{H}'/K$ and $\bar{E} := \tilde{E}'/K$. Then, H acts on F faithfully and the proof is complete. □

5. Structure of K-trivial irregular varieties with $\dim a_X(X) = \dim X - 1$

In this section, we treat K-trivial irregular varieties whose Albanese morphism has relative dimension one. We work in the following setting.

Assumptions 5.1. Let X be a normal projective $\mathbb{Q}$-factorial variety with $K_X \equiv 0$. Let $a_X : X \to A$ be the Albanese morphism of X and assume that $a_X : X \to a_X(X)$ is of relative dimension one. Then, by Theorem 2.13 and Corollary 2.14, we see that $K_X \sim_{\mathbb{Q}} 0$ and $a_X : X \to A$ is a fibration. In the following, we set $f = a_X : X \to A$. We fall into one of the following three cases:

(C1) X_η is smooth over $k(\eta)$, that is, f is an elliptic fibration;
(C2) X_η is non-smooth but geometrically integral, that is, f is a quasi-elliptic fibration;
(C3) X_η is geometrically non-reduced, that is, f is inseparable.

Note that the latter two cases occur only when the characteristic $p = 2$ or 3 (see Section 2.11).

5.1. Case (C1): X_η is smooth

In this case, we give a thorough description of X.

5.1.1. A rough description

We first give a rough description of the structure of X.

Theorem 5.2. *Under Assumption 5.1(C1), there exist an abelian variety A' and an isogeny $\tau : A' \to A$ such that $X \times_A A' \cong A' \times F$, where F is a general fiber. More precisely, there is a faithful action of $G := \ker(\tau)$ on F, such that $X \cong A' \times F/G$, where G acts on $A' \times F$ diagonally.*

Proof. We follow the strategy of [52].

Step 1. Let $d = \dim A$, and let H be a sufficiently ample line bundle on A. For general choices of $H_i \in |H|$, the curve $C := H_1 \cap \cdots \cap H_{d-1}$ is smooth, and $f_C : X_C = X \times_A C \to C$ is flat and smooth over generic point of C. In this step, we aim to show that *there exists a finite flat morphism $C' \to C$ such that $f_{C'} : X_{C'} = X \times_A C' \to C'$ is a trivial fibration.*

Set $Z_i := H_1 \cap \cdots \cap H_i$ for $i = 1, \ldots, d-1$. As $H_1 \in |H|$ is general, we may assume that X_{Z_1} is a prime divisor. Then, applying the adjunction formula, we have

$$0 \sim_{\mathbb{Q}} (K_{X/A})|_{X^\nu_{Z_1}} \sim_{\mathbb{Q}} K_{X^\nu_{Z_1}/Z_1} + \Delta_{X^\nu_{Z_1}}, \tag{5.1}$$

where $\Delta_{X^\nu_{Z_1}} \geq 0$. By Theorem 2.10, we have $K_{X^\nu_{Z_1}/Z_1} \geq_{\mathbb{Q}} 0$. In turn, we deduce that $K_{X^\nu_{Z_1}/Z_1} \sim_{\mathbb{Q}} 0$ and $\Delta_{X^\nu_{Z_1}} = 0$, which implies that X_{Z_1} is normal in codimension one. Inductively, we can prove that this holds for each X_{Z_i}. In particular, for $i = d-1$, the surface X_C is normal in codimension one and $K_{X^\nu_C/C} \sim_{\mathbb{Q}} 0$. Since X is Cohen–Macaulay in codimension two, we can show that X_C is Cohen–Macaulay by induction. Together with that X_C is regular in codimension one, we see that $X_C = (X_C)^\nu$ by Serre's criterion. Finally, since $K_{X_C/C} \sim_{\mathbb{Q}} 0$, we conclude Step 1 by Proposition 2.11.

Therefore, there exists a big open subset A° of A covered by such curves C as above. Over A°, the fibration $f : X \to A$ is smooth, and all the closed fibers are isomorphic to a fixed one $F = f^{-1}(t_0)$ of f for some $t_0 \in A^\circ$. In the following, we equip F with a fixed abelian variety structure.

Step 2. In this step, we aim to show that *there is a finite group scheme G and a G-torsor $I \to A^\circ$ such that $X^\circ \times_{A^\circ} I \to I$ is a trivial fibration, where $X^\circ := f^{-1}(A^\circ)$.*

We follow the proof of [52, Theorem 9.2]. Let L be a very ample line bundle on X, and set $L_0 := L|_F$. We take G to be the automorphism group scheme $\mathrm{Aut}(F, L_0)$ which is finite by [52, Proposition 10.1],

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and I to be the quasi-projective scheme over A° representing the Isom functor $\mathrm{Isom}_{A^\circ}\big((X^\circ, L|_{X^\circ}), (A^\circ \times F, \mathrm{pr}_2^* L_0)\big)$ as constructed in [52, Construction 7.5]. Following the proof of [52, Theorem 9.2], we only need to show that $I \to A^\circ$ is surjective and flat. For each curve $C \subset A^\circ$ as in Step 1, there exists a finite flat morphism $C' \to C$ such that $f_{C'} \colon X_{C'} \to C'$ is a trivial fibration. By the base change property [52, Proposition 7.8] and flattening decomposition [45, Lecture 8], it suffices to verify that $I_{C'} := \mathrm{Isom}_{C'}\big((X_{C'}, L|_{X_{C'}}), (C' \times F, \mathrm{pr}_2^* L_0)\big) \to C'$ is surjective and flat. To verify this, we apply [52, Lemma 8.7], which requires the condition that $-K_{X_{C'}/C'}$ is nef and semi-ample. This condition is satisfied because $f_{C'}$ is a trivial fibration. This finished the proof of this step.

Step 3. We extend the G-torsor $I \to A^\circ$ over A° to a G-torsor $\bar{I} \to A$ over A as follows.

Regard F as an abelian variety with identity 0_F. Since $G = \mathrm{Aut}(F, L_0)$ is a finite group scheme, G is an extension of an étale group scheme by an infinitesimal group scheme. Thus, by a "Purity" theorem [22, Proposition 1.4], the torsor $I \to A^\circ$ can be extended to a torsor $\bar{I} \to A$. Furthermore, according to [47, Proposition], there exist an integer n, a homomorphism $\varphi \colon A[n] \to \mathrm{Aut}(F, L_0)$, and a morphism $\eta \colon A \to \bar{I}$, which is $A[n]$-equivariant with the action of $A[n]$ on $\bar{I}$ induced by φ, leading to the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow[\eta]{A[n]\text{-equivariant}} & \bar{I} \\ & \searrow^{n_A} & \downarrow \\ & & A\,. \end{array}$$

Let $A^{\circ\circ}$ denote the preimage of A° under $n_A \colon A \to A$. By the construction, we obtain an isomorphism $X \times_{A, n_A} A^{\circ\circ} \cong A^{\circ\circ} \times F$.

Step 4. We show that the birational map $\psi \colon Y := X \times_{A, n_A} A \dashrightarrow Y' := F \times A$, which is determined by $X \times_{A, n_A} A^{\circ\circ} \cong A^{\circ\circ} \times F$, is an isomorphism.

Since $f \colon X \to A$ is equidimensional by [52, Theorem 4.1], $Y \to A$ is equidimensional too. Since ψ is an isomorphism in codimension one, Y is regular in codimension one, thus Y is normal by Lemma 2.4.

Let H_Y be an ample Cartier divisor on Y, and denote by $H_{Y'} = \psi_* H_Y$ the strict transform of H_Y. Observe that $H_{Y'}$ is relatively ample over A since each fiber of $Y' \cong F \times A \to A$ is irreducible. If necessary adding the pullback of an ample divisor on A, we may assume that $H_{Y'}$ is ample. Since $\psi \colon Y \dashrightarrow Y'$ is an isomorphism in codimension one, we have a natural ring isomorphism $\bigoplus_{m \geq 0} H^0(Y, mH_Y) \cong \bigoplus_{m \geq 0} H^0(Y', mH_{Y'})$, which implies that $\psi \colon Y \to Y'$ is an isomorphism.

Step 5. Finally, if the action of $A[n]$ on F is not faithful, we may set $H = \mathrm{Ker}(A[n] \to \mathrm{Aut}(F, L_0))$ and $G = A[n]/H$. In turn, we get a faithful action of G on F and an action on $A' = A/H$ such that $X \cong (A \times F)/G \cong (A' \times F)/H$. □

5.1.2. Explicit description of Case (C1)

Based on the analysis in [9, pp. 36–37], we give all the possibilities of X in the context of Theorem 5.2.

Recall that $X \cong (A \times F)/G$, where A is an abelian variety, F is an elliptic curve, and G is a finite group scheme acting diagonally on $A \times F$ via injections $G \hookrightarrow A$ and $\alpha \colon G \hookrightarrow \mathrm{Aut}(F) \cong F \rtimes \mathrm{Aut}(F, 0_F)$. In particular, G is commutative. As observed in [9], the commutativity of G severely constrains its possible structure. In fact, the argument of [9, p. 36] shows that G must have the form

$$\alpha(G) \cong G_0 \times \mathbb{Z}/n\mathbb{Z},$$

where G_0 is a finite subgroup scheme of F, and $\mathbb{Z}/n\mathbb{Z} \subseteq \mathrm{Aut}(F, 0_F)$ is a cyclic subgroup with order $n = 2, 3, 4$, or 6. Moreover, if we denote by $\sigma \in G$ the element corresponding to some generator of $\mathbb{Z}/n\mathbb{Z}$, then $G_0 \subseteq \mathfrak{F}$, where $\mathfrak{F} \subset F$ is the fixed subscheme of σ. There are the following possibilities of the fixed subscheme $\mathfrak{F}$:

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(a) $n = 2$, (so $\sigma = -1_F$), then $\mathfrak{F} \cong \mathrm{Ker}2_F$;
(b) $n = 3$, then $\mathrm{ord}\mathfrak{F} = 3$, so
$\mathfrak{F} \cong \mathbb{Z}/3\mathbb{Z}$, if char $\neq 3$;
$\mathfrak{F} \cong \alpha_3$, if char $= 3$ ($j(F) = 0$ and thus F is supersingular);
(c) $n = 4$, then $\mathrm{ord}\mathfrak{F} = 2$, so
$\mathfrak{F} \cong \mathbb{Z}/2\mathbb{Z}$ if char $\neq 2$;
$\mathfrak{F} \cong \alpha_2$ if char $= 2$ (F is supersingular);
(d) $n = 6$, then $\mathfrak{F} = (e)$.

We can now give a complete list of the possible G:

a1) $X \cong A \times F/(\mathbb{Z}/2\mathbb{Z})$. The action is given by $(x, y) \mapsto (x + a, -y)$ for some $a \in A[2]$. Moreover, A has p-rank ≥ 1 if $p = 2$.

a2) $X \cong A \times F/(\mathbb{Z}/2\mathbb{Z} \cdot \mathbb{Z}/2\mathbb{Z})$. The action is given by $(x, y) \mapsto (x + a, y + b)$ and $(x, y) \mapsto (x + c, -y)$ for some $a \neq c \in A[2]$ and $b \in F[2]$. Moreover, A and F are both ordinary if $p = 2$.

a3) $X \cong A \times F/((\mathbb{Z}/2\mathbb{Z})^2 \cdot \mathbb{Z}/2\mathbb{Z})$. ($p \neq 2$) The factor $\mathbb{Z}/2\mathbb{Z}$ acts as $(x, y) \mapsto (x + c, -y)$ and $(\mathbb{Z}/2\mathbb{Z})^2$ acts by translation on both factors.

a4) $X \cong A \times F/(\mu_2 \cdot \mathbb{Z}/2\mathbb{Z})$. ($p = 2$) The factor $\mathbb{Z}/2\mathbb{Z}$ acts as $(x, y) \mapsto (x + c, -y)$ and μ_2 acts by translation on both factors. Here, A has p-rank ≥ 1 and F is ordinary.

a5) $X \cong A \times F/(\mathbb{Z}/2\mathbb{Z} \cdot \mu_2 \cdot \mathbb{Z}/2\mathbb{Z})$. ($p = 2$) One factor $\mathbb{Z}/2\mathbb{Z}$ acts as $(x, y) \mapsto (x+c, -y)$ and the residue factor $\mathbb{Z}/2\mathbb{Z} \cdot \mu_2$ acts by translation on both factors. Here, A and F are both ordinary.

a6) $X \cong A \times F/(\alpha_2 \cdot \mathbb{Z}/2\mathbb{Z})$. ($p = 2$) The factor $\mathbb{Z}/2\mathbb{Z}$ acts as $(x, y) \mapsto (x + c, -y)$ and α_2 acts by translation on both factors. Here, A has p-rank 1 and F is supersingular.

a7) $X \cong A \times F/(M_2 \cdot \mathbb{Z}/2\mathbb{Z})$, ($p = 2$) where M_2 is the non-split extension of α_p by α_p. The factor $\mathbb{Z}/2\mathbb{Z}$ acts as $(x, y) \mapsto (x+c, -y)$ and M_2 acts by translation on both factors. Here, A has p-rank 1, and F is supersingular.

b1) $X \cong A \times F/(\mathbb{Z}/3\mathbb{Z})$. ($j(F) = 0$) The group $\mathbb{Z}/3\mathbb{Z}$ acts as $(x, y) \mapsto (x + a, \omega y)$, where $a \in A[3]$ and ω is an automorphism of F of order 3.

b2) $X \cong A \times F/(\mathbb{Z}/3\mathbb{Z})^2$. ($j(F) = 0$ and $p \neq 3$) The action is given by $(x, y) \mapsto (x + a, \omega y)$ and $(x, y) \mapsto (x+b, y+c)$ for some $a, b \in A[3]$ and $c \in F[3]$, ω as before and $a \neq b, 2b, \omega c = c$.

b3) $X \cong A \times F/(\alpha_3 \cdot \mathbb{Z}/3\mathbb{Z})$. ($j(F) = 0, p = 3$) The factor $\mathbb{Z}/3\mathbb{Z}$ acts as in b1) and α_3 acts by translation on both factors. Here, A has p-rank 1 and F is supersingular.

c1) $X \cong A \times F/(\mathbb{Z}/4\mathbb{Z})$. ($j(F) = 12^3$) The group $\mathbb{Z}/4\mathbb{Z}$ acts as $(x, y) \mapsto (x+a, iy)$, where $a \in A[4]$ and i is an automorphism of F of order 4.

c2) $X \cong A \times F/(\mathbb{Z}/2\mathbb{Z} \cdot \mathbb{Z}/4\mathbb{Z})$. ($j(F) = 12^3, p \neq 2$) The group $\mathbb{Z}/4\mathbb{Z}$ acts as in c1) and $\mathbb{Z}/2\mathbb{Z}$ acts as $(x, y) \mapsto (x + b, y + c)$ for some $b \in A[2], c \in F[2]$ and $b \neq 2a$, $ic = c$.

c3) $X \cong A \times F/(\alpha_2 \cdot \mathbb{Z}/4\mathbb{Z})$. ($j(F) = 0, p = 2$) The factor $\mathbb{Z}/4\mathbb{Z}$ acts as in c1) and α_2 by translation on both factors. Here, A has p-rank 1 and F is supersingular.

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d) $X \cong A \times F/(\mathbb{Z}/6\mathbb{Z})$. ($j(F) = 0$)	The group $\mathbb{Z}/6\mathbb{Z}$ acts as $(x, y) \mapsto (x + a, -\omega y)$, where $a \in A[6]$ and ω is as in b1).

5.2. *Case (C2): f is a quasi-elliptic fibration*

In this case, we prove the following theorem.

Theorem 5.3. *Let notation and assumptions be as in Assumption 5.1 (C2). Then,*

(i) *The characteristic p is 2 or 3.*
(ii) *X admits another fibration $g\colon X \to \mathbb{P}^1$ that is transversal to f:*

$$\begin{array}{ccc} X & \xrightarrow{g} & \mathbb{P}^1 . \\ {\scriptstyle f}\downarrow & & \\ A & & \end{array}$$

Moreover, a fiber of g is either an abelian variety or a multiple of an abelian variety.

(iii) *Let X_1 be the normalization of $X \times_A A^{(-1)}$ and denote by $f_1\colon X_1 \to A_1 := A^{(-1)}$ the projection. Then, f_1 is a smooth fibration fibered by rational curves, which falls into one of the following cases:*
  (1) *$f_1\colon X_1 \to A_1$ is a projective bundle described as one of the following:*
    (1.a) *$X_1 \cong \mathbb{P}_{A_1}(\mathcal{O}_{A_1} \oplus \mathcal{L})$, where $\mathcal{L}^{\otimes p+1} \cong \mathcal{O}_X$;*
    (1.b) *$X_1 \cong \mathbb{P}_{A_1}(\mathcal{E})$, where $\mathcal{E}$ is a unipotent vector bundle of rank two, and there exists an étale cover $\mu\colon A_2 \to A_1$ of degree p^v ($0 \le v \le d := \dim A$), such that $\mu^* F_{A_1}^{(d-v)*}\mathcal{E}$ is trivial.*
  (2) *$p = 2$, and there exists a purely inseparable isogeny $A_2 \to A_1$ of degree two, such that $X_2 := X_1 \times_{A_1} A_2 \to A_2$ is a projective bundle described as one of the following:*
    (2.a) *$X_2 \cong \mathbb{P}_{A_2}(\mathcal{O}_{A_2} \oplus \mathcal{L})$, where $\mathcal{L}^{\otimes 4} \cong \mathcal{O}_{A_2}$;*
    (2.b) *$X_2 \cong \mathbb{P}_{A_2}(\mathcal{E})$, where $\mathcal{E}$ is a unipotent vector bundle of rank two, and there exists an étale cover $\mu\colon A_3 \to A_2$ of degree p^v ($0 \le v \le d$), such that $\mu^* F_{A_2}^{(d-v)*}\mathcal{E}$ is trivial.*

Proof. The assertion (i) is well known. Let us prove the remaining ones.

Note that $X \times_A A^{\frac{1}{p}}$ is not normal by Proposition 2.16. Let $X_1 := (X \times_A A^{\frac{1}{p}})^\nu$ be the normalization, and let $\pi\colon X_1 \to X$, $f_1\colon X_1 \to A^{\frac{1}{p}}$ be the natural morphisms. We have the following commutative diagram:

$$\begin{array}{ccccc} X_1 & \xrightarrow{\quad\pi\quad} & X \times_A A^{\frac{1}{p}} & \longrightarrow & X \\ & {\scriptstyle f_1}\searrow & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ & & A_1 := A^{\frac{1}{p}} & \xrightarrow{\sigma} & A . \end{array}$$

By results of Section 2.5, we can write that

$$\pi^* K_X \sim K_{X_1} + (p-1)C, \quad \text{with } C \ge 0, \tag{5.2}$$

where C can be chosen such that $C|_{X_{1,\eta}}$ coincides with the conductor of the normalization of the generic fiber of f'. Hereafter, we fix such C. According to Proposition 2.15(1), $\deg_{K(A_1)}(p-1)C = 2$. Write $(p-1)C = H + V$, where H is the f_1-horizontal part and V the vertical part of C. Note that H is irreducible by Proposition 2.16(1), thus $H = nC$, where C is reduced and $n = 1$ or 2. Let D be the reduced divisor supported on $\pi(C)$. We have $\pi^* D = C$ or pC by Proposition 2.16.

Step 1. *We prove that both C and D are semi-ample with numerical dimension one.*

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Since $X_1 \to X$ is a finite purely inseparable morphism, the semi-ampleness of one of the divisors C or D implies the semi-ampleness of the other. We shall show that D is semi-ample with numerical dimension one. By Theorem 3.7, it suffices to verify that $D|_{D^\nu} \sim_{\mathbb{Q}} 0$, which is equivalent to that $C|_{C^\nu} \sim_{\mathbb{Q}} 0$.

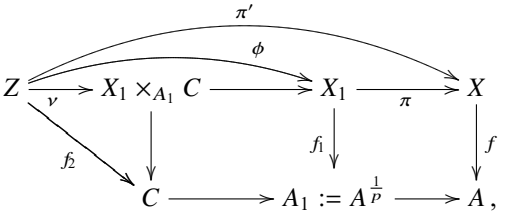
Claim. *Let T be an f_1-horizontal prime divisor of X_1. Then, $T|_{T^\nu} \geq_{\mathbb{Q}} 0$.*

To prove this claim, set $\bar{T} := \pi(T)$. By Lemma 2.9(1), $\bar{T}|_{\bar{T}^\nu} \sim_{\mathbb{Q}} (K_X + \bar{T})|_{\bar{T}^\nu} \geq_{\mathbb{Q}} 0$. Then, since $X_1 \to X$ is finite and purely inseparable, we have $T|_{T^\nu} \geq_{\mathbb{Q}} 0$.

Consequently, since C is f_1-horizontal, we have $C|_{C^\nu} \geq_{\mathbb{Q}} 0$. Thus, applying Lemma 2.9 to

$$0 \sim_{\mathbb{Q}} (K_{X_1} + C)|_{C^\nu} \sim_{\mathbb{Q}} (K_{X_1} + nC + V)|_{C^\nu} \tag{5.3}$$

we see that C is an abelian variety, $V|_{C^\nu} \sim_{\mathbb{Q}} 0$, and moreover, in case $n = 2$, we have $C|_{C^\nu} = C|_C \sim_{\mathbb{Q}} 0$. Thus, we only need to show $C|_C \sim_{\mathbb{Q}} 0$ in case $n = 1$. In this case, we have $p = 2$, and $C \to A_1$ is purely inseparable of degree 2 since $C|_{X_{K(A_1)}}$ is a point which is inseparable of degree 2 over $\operatorname{Spec} K(A_1)$ by Proposition 2.16. By doing the base change $C \to A_1$ and setting $Z = (X_1 \times_{A_1} C)^\nu$, we have the following commutative diagram:

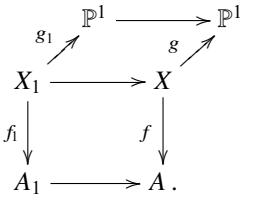


where ϕ, π', f_2 denote the natural morphisms fitting into the above diagram. Since $C \times_{A_1} C$ is non-reduced and $f_1 : X_1 \to A_1$ is smooth over the generic point of A_1, we see that $\phi^* C = 2E$ for some $\mathbb{Z}$-divisor E on Z. Then, we have

$$\pi'^* K_X = K_Z + 2E + V', \tag{5.4}$$

where $V' \geq 0$ is vertical over C. Consequently, $(K_Z + 2E + V')|_{E^\nu} \sim_{\mathbb{Q}} 0$. Since $C|_C \geq_{\mathbb{Q}} 0$, we have $E|_{E^\nu} \geq_{\mathbb{Q}} 0$. It follows from Lemma 2.9 that $E|_{E^\nu} \sim_{\mathbb{Q}} 0$, which is equivalent to that $C|_C \sim_{\mathbb{Q}} 0$.

In summary, the semi-ample divisor D (resp., C) induces a fibration $g : X \to B$ (resp., $g_1 : X_1 \to B_1$) to a curve. Since general fibers of f and f_1 are rational curves, we have $g(B) = g(B_1) = 0$. In summary, there is a commutative diagram:



Step 2. We prove the following statements:

(a) D is isomorphic to an abelian variety (we have shown this for C in the last step).
(b) X (resp., X_1) is regular at codimension one points of D (resp., C).
(c) $V = 0$.
(d) A general fiber of g (resp., g_1) is an abelian variety, and a special fiber of g (resp., g_1) is a multiple of an abelian variety.

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First, since C and D are irreducible and $C|_{C^\nu} \sim_{\mathbb{Q}} D|_{D^\nu} \sim_{\mathbb{Q}} 0$, we have $(K_X + D)|_{D^\nu} \sim_{\mathbb{Q}} (K_{X_1} + C)|_{C^\nu} \sim_{\mathbb{Q}} 0$, thus the statements (a, b) follow from Lemma 2.9.

To show the last two statements, we denote by G_t (resp., G_t^1) the fiber of g (resp., g_1) over $t \in \mathbb{P}^1$. We first consider the fibration $g_1 \colon X_1 \to \mathbb{P}^1$. Write $G_t^1 = mT + V'$, where T is an f_1-horizontal component and V' is the remaining part with $T \not\subset \operatorname{Supp} V'$. By the claim in Step 1, we have $T|_{T^\nu} \geq_{\mathbb{Q}} 0$. Since C is irreducible, we see that $C \sim_{\mathbb{Q}} rG_t^1$ for some positive rational number r, and it follows that $C|_{T^\nu} \sim_{\mathbb{Q}} rG_t^1|_{T^\nu} \sim_{\mathbb{Q}} 0$. Thus,

$$0 \sim_{\mathbb{Q}} (K_{X_1} + nC + V + G_t^1)|_{T^\nu} \sim_{\mathbb{Q}} (K_{X_1} + mT + V' + V)|_{T^\nu}.$$

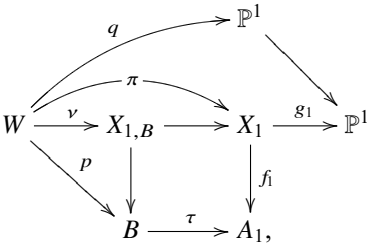
By Lemma 2.9(2), we see that T is an abelian variety, $V|_{T^\nu} = V'|_{T^\nu} = 0$, which implies that $\operatorname{Supp} V' \cap \operatorname{Supp} T = \operatorname{Supp} V \cap \operatorname{Supp} T = \emptyset$. Since the fiber G_t^1 is connected, we have $V' = 0$, and consequently $G_t^1 = mT$ is (a multiple of) an abelian variety. It follows that $\operatorname{Supp} G_t^1 \cap \operatorname{Supp} V = \emptyset$, and thus $V = 0$, which is the statement (c). Moreover, since g_1 is a fibration to a curve, by [1, Corollary 7.3], a general fiber of g is integral, so we obtain the statement (d) for g_1.

Finally, by writing $G_t = mT + V'$ and using $(K_X + G_t)|_{T^\nu} \sim_{\mathbb{Q}} (K_X + mT + V')|_{T^\nu} \sim_{\mathbb{Q}} 0$, Lemma 2.9 implies that G_t is also (a multiple of) an abelian variety. Thus, we obtain the statement (d) for g.

Step 3. *We show that there exists an isogeny $\tau \colon B \to A_1$ of abelian varieties such that $X_1 \times_{A_1} B \cong B \times \mathbb{P}^1$. In turn, $f_1 \colon X_1 \to A_1$ is a smooth morphism.*

Take a general fiber G_t^1 of $g_1 \colon X_1 \to \mathbb{P}^1$ over $t \in \mathbb{P}^1$, which is an abelian variety as established in the previous step. We denote by n the degree $\deg(G_t^1 \to A_1)$, which is independent of $t \in \mathbb{P}^1$. Then, the morphism $\times n \colon B := A_1 \to A_1$ factors through the isogeny $G_t^1 \to A_1$ for a general t (see [46, p. 169, Remark]).

By Lemma 2.4, the fiber product $X_{1,B} := X_1 \times_{A_1} B$ is integral. Let W be the normalization of $X_{1,B}$. We have the following commutative diagram:



where q is the fibration resulting from the Stein factorization of $W \to X_1 \xrightarrow{g_1} \mathbb{P}^1$. By our choice of the base change $B \to A_1$, a general fiber Q_t of q is a birational section of p. Since each fiber of g_1 contains no vertical components over A_1 and ν is finite, each fiber of q contains no vertical components over B. Thus, for each fiber Q_t of q, the morphism $Q_t \to B$ is birational. Denote by $\mathcal{N}$ the conductor divisor of ν. Remark that, since $X_1 \to A_1$ is generically smooth, $\mathcal{N}$ is vertical over B. It follows that $K_W + (p-1)\pi^*C + \mathcal{N} \sim_{\mathbb{Q}} \pi^*(K_{X_1} + (p-1)C) \sim_{\mathbb{Q}} 0$. Thus,

$$(K_W + (p-1)\pi^*C + \mathcal{N} + Q_t)|_{Q_t^\nu} \sim_{\mathbb{Q}} 0.$$

Applying Lemma 2.9, we see that $Q_t = Q_t^\nu$ is an abelian variety and $\mathcal{N}|_{Q_t} \sim_{\mathbb{Q}} 0$. In turn, we conclude that:

- $\mathcal{N} = 0$, thus $X_{1,B}$ is normal by Lemma 2.4;
- for each $t \in \mathbb{P}^1$, the projection $Q_t \to B$ is an isomorphism, hence the morphism $X_{1,B} \cong W \to B \times \mathbb{P}^1$ is an isomorphism by Zariski's main theorem.

Step 4. We consider the case $(p-1)C = 2C$ and prove the statement (iii-1).

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We first show that $(p+1)C|_C \sim 0$. Denote by $\tilde{\pi}\colon C \to D$ the induced finite morphism of abelian varieties and note that $\pi^*D = pC$ by Proposition 2.16. By Step 2 (c), $K_X + D$ is Cartier at codimension-one points of D. Therefore, the adjunction formula gives

$$(K_X + D)|_D^w \sim K_D \sim 0, \tag{5.5}$$

where $|^w$ is the restriction on D by first considering a big open subset $D^\circ \subset D$ over which $K_X + D$ is Cartier and then extending it (see [37, pp. 173–174]). Thus, $\pi^*(K_X + D)|_C = \tilde{\pi}^*((K_X + D)|_D^w) \sim 0$ (here the pullback of π makes sense since π is finite). On the other hand, we have

$$\pi^*(K_X + D)|_C \sim (K_{X_1} + 2C + \pi^*D)|_C \sim (p+1)C|_C.$$

It follows that $(p+1)C|_C \sim 0$.

Next, we equip the smooth morphism $f_1\colon X_1 \to A_1$ with a projective bundle structure over A_1. Note that since $C \to A_1$ is birational and C is isomorphic to an abelian variety, $C \to A_1$ is an isomorphism; this gives a section $s\colon A_1 \to C$ of f_1 which is f_1-ample. Set $\mathcal{E} = f_{1*}\mathcal{O}_{X_1}(C)$. Since each fiber of f_1 is isomorphic to $\mathbb{P}^1$, by exactly the same proof of [29, Proposition V.2.2], we can show that $f_1^*\mathcal{E} \to \mathcal{O}_{X_1}(C)$ is surjective and induces an isomorphism $X_1 \xrightarrow{\sim} \mathbb{P}(\mathcal{E})$. Under this isomorphism, the section $s\colon A_1 \to C$ corresponds to the exact sequence

$$0 \to \mathcal{O}_{A_1} \to \mathcal{E} \to s^*(\mathcal{O}_{X_1}(C)|_C) \to 0. \tag{5.6}$$

Since the divisor $C|_C$ is torsion, $\mathcal{L} := s^*(\mathcal{O}_{X_1}(C)|_C)$ is a torsion line bundle with the same order.

If the above exact sequence (5.6) splits, then $\mathcal{E} \cong \mathcal{O}_{A_1} \oplus \mathcal{L}$, and the torsion order of $\mathcal{L}$ divides $p+1$. If (5.6) does not split, then $\mathrm{Ext}^1(\mathcal{L}, \mathcal{O}_{A_1}) \cong H^1(A_1, \mathcal{L}^{-1}) \neq 0$, which implies that $\mathcal{L} \cong \mathcal{O}_{A_1}$ by [15, Lemma 7.19]. Thus, $\mathcal{E}$ is an extension of $\mathcal{O}_{A_1}$ by $\mathcal{O}_{A_1}$, which corresponds to a nonzero element $\xi \in H^1(A_1, \mathcal{O}_{A_1}) \cong \mathrm{Ext}^1(\mathcal{O}_{A_1}, \mathcal{O}_{A_1})$. With respect to the Frobenius action F^* on $H^1(A_1, \mathcal{O}_{A_1})$, we have a decomposition [46, pp. 143–148])

$$H^1(A_1, \mathcal{O}_{A_1}) = V_n \oplus V_s,$$

where V_n is the nilpotent part and V_s is the semisimple part. Moreover, the semisimple part V_s admits a basis $\alpha_1, \dots, \alpha_r$, where $r = \dim V_s$, such that $F^*(\alpha_i) = \alpha_i$ for $i = 1, \dots, r$. Write $\xi = \xi_n + \xi_s$. Since $\dim V_n = \dim A - r = d - r$, we see that $F^{(d-r)*}\xi_n = 0$. By [39, Satz 1.4], there exist étale covers $\pi_i\colon A_i \to A$ $(i = 1, \dots, r)$ of degree p such that $\pi_i^*\alpha_i = 0$. Define μ as the fiber products $(\cdots((A_1 \times_A A_2) \times_A A_3) \times_A \cdots) \times_A A_r \to A$. Then, $\mu^*\xi_s = 0$. Therefore, $\mu^*(F^{(d-r)*}\xi) = 0$, which is equivalent to $\mu^*(F^{(d-r)*}E)$ being a trivial extension.

Step 5. We consider the case $(p-1)C = C$ and prove the statement (iii-2).

In this case, $p = 2$, and $C \to A_1$ is purely inseparable of degree 2. We do a further base change $A_2 := C \to A_1$. Let $X_2 = X_1 \times_{A_1} A_2$. We obtain the following commutative diagram:

$$\begin{array}{ccccccc} X_2 & \xrightarrow{\pi_2} & X_1 & \xrightarrow{\pi_1} & X & \xrightarrow{g} & \mathbb{P}^1 \\ \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f} & & \\ A_2 & \longrightarrow & A_1 & \longrightarrow & A\,. & & \end{array}$$

We see that $\pi_2^*C = 2C_2$, where C_2 is a section of $X_2 \to A_2$. Similar to Step 4, we have $X_2 \cong \mathbb{P}_{A_2}(\mathcal{E}_2)$, where $\mathcal{E}_2 = f_{2*}\mathcal{O}_{X_2}(C_2)$.

If $\mathcal{E}_2$ does not split, then it is as described as in Step 4 accordingly. Otherwise, $\mathcal{E}_2 \cong \mathcal{O}_{A_1} \oplus \mathcal{L}_2$ and we conclude the proof by determining the torsion order of $C_2|_{C_2}$. By Proposition 2.16, we have $\pi^*D = C$ if $\deg_{K(A)} D = 2$ and $\pi^*D = 2C$ if $\deg_{K(A)} D = 4$. It follows that

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$$\pi^*(K_X + D)|_C \sim (K_{X_1} + C + \pi^* D)|_C \sim \begin{cases} C|_C & \text{if } \deg_{K(A)} D = 2 \\ 2C|_C & \text{if } \deg_{K(A)} D = 4. \end{cases}$$

As in Step 4, we have $\pi^*(K_X + D)|_C \sim 0$, thus $2C|_C \sim 0$. Since $\pi_2^* C = 2C_2$, we see that $4C_2|_{C_2} \sim 0$. □

5.3. *Case (C3): f is an inseparable fibration*

Theorem 5.4. *Let notation and assumptions be as in Assumption 5.1(C3). Then,*

(i) *The characteristic p is* 2 *or* 3.
(ii) *X is equipped with another fibration* $g\colon X \to \mathbb{P}^1$ *transversal to f, whose fibers are abelian varieties or multiples of abelian varieties:*

$$\begin{array}{ccc} X & \xrightarrow{g} & \mathbb{P}^1 . \\ {\scriptstyle f}\downarrow & & \\ A & & \end{array}$$

(iii) *Let* $A_1 = A^{(-1)}$, *and denote* $X_1 = (X \times_A A_1)^\nu_{\mathrm{red}}$. *Then, one of the following happens:*
  (1) $X \cong A_1 \times \mathbb{P}^1/\mathcal{F}$, *where* $\mathcal{F}$ *is a foliation on* $A_1 \times \mathbb{P}^1$ *of height one and* $\operatorname{rank} \mathcal{F} < \dim A$.
  (2) $p = 2$ *and* $X \cong X_1/\mathcal{F}_1$ *for some height one foliation* $\mathcal{F}_1$ *with* $\operatorname{rank} \mathcal{F}_1 < \dim A$, *where* X_1 *is one of the following:*
    (2.a) $X_1 \to A_1$ *is separable and can be trivialized by an étale isogeny* $\tau_2\colon A_2 \to A_1$ *of degree* 2, *namely,* $X_1 \times_{A_1} A_2 \cong A_2 \times \mathbb{P}^1$.
    (2.b) $X_1 \to A_1$ *is separable and can be trivialized by the Frobenius base change* $\tau_2\colon A_2 := (A_1)^{(-1)} \to A_1$, *namely,* $X_1 \times_{A_1} A_2 \cong A_2 \times \mathbb{P}^1$.
    (2.c) $X_1 \to A_1$ *is inseparable, and* $X_1 \cong A_2 \times \mathbb{P}^1/\mathcal{F}_2$, *where* A_2 *is an abelian variety and* $\mathcal{F}_2$ *is a (height-one) foliation with* $\operatorname{rank} \mathcal{F}_2 < \dim A$.

Proof. In this case, $X \times_A A_1$ is non-reduced. Let $X_1 = (X \times_A A_1)^\nu_{\mathrm{red}}$. Denote by $\pi_1\colon X_1 \to X$ the induced morphism, which is a purely inseparable morphism of height one with $\deg \pi_1 < p^{\dim A}$. We have $\pi_1^* K_X \sim K_{X_1} - (p-1) \det \mathcal{F}_{X_1/X}$, where $|- \det \mathcal{F}_{X_1/X}|$ has non-trivial movable part generated by $f_1^* H^0(\Omega_{A^{(-1)}/A})$ (see Section 2.5). We may write that

$$|- \det \mathcal{F}_{X_1/X}| = \mathfrak{M} + \mathfrak{F}, \tag{5.7}$$

where $\mathfrak{M}$ is the movable part and $\mathfrak{F}$ is the fixed part. In turn, we obtain that $\pi_1^* K_X \sim K_{X_1} + (p-1)(\mathfrak{M} + \mathfrak{F})$, where $\deg_{K(A_1)} \mathfrak{M} > 0$, and $\deg_{K(A_1)} (p-1)(\mathfrak{M} + \mathfrak{F}) = 2$ by results of Section 2.11.

By the claim in Step 1 of Theorem 5.3, we have $T|_{T^\nu} \geq_{\mathbb{Q}} 0$ for any f_1-horizontal prime divisor T of X_1. Then, since $\deg_{K(A_1)} (p-1)\mathfrak{F} \leq 1$, no matter $p = 2$ or $p = 3$, we can apply [11, Proposition 5.3] to the pair $(X_1, (p-1)\mathfrak{M} + (p-1)\mathfrak{F})$ and obtain the following:

- $\mathfrak{M}$ is semi-ample with $\nu(\mathfrak{M}) = 1$;
- denote by $g_1\colon X_1 \to \mathbb{P}^1$ the induced fibration by $\mathfrak{M}$. Then, a general fiber of g_1 is an abelian variety and a special fiber is a multiple of an abelian variety.

From this, by a similar argument of Step 2 (c) in Theorem 5.3, we see that there is no f_1-vertical part in $\mathfrak{M} + \mathfrak{F}$.

Moreover, since $X_1 \to X$ is purely inseparable, there exists a fibration $g\colon X \to \mathbb{P}^1$ fitting into the following commutative diagram:

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$$\begin{array}{ccc} & & \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 \\ & {\scriptstyle g_1}\nearrow & & {\scriptstyle g}\nearrow & \\ X_1 & & \longrightarrow & X & \\ {\scriptstyle f_1}\downarrow & & & {\scriptstyle f}\downarrow & \\ A^{(-1)} =: A_1 & & \longrightarrow & A\,. & \end{array}$$

Let G_1 be a general fiber of g_1. Then, $\deg_{K(A_1)} G_1 = 1$ or 2 since $\deg_{K(A_1)} \mathfrak{M} \leq 2$. We distinguish between the following two cases:

(1) $\deg_{K(A_1)} G_1 = 1$;
(2) $\deg_{K(A_1)} G_1 = 2$, which happens only when $p = 2$.

In case (1), the general fiber G_1 of g_1 maps to A_1 isomorphically. Thus, g_1 has no multiple fibers, and it follows that every fiber of g_1 is isomorphic to A_1. Then, the induced morphism $X_1 \xrightarrow{(f_1, g_1)} A_1 \times \mathbb{P}^1$ is an isomorphism by Zariski's main theorem. Therefore, $X \cong A_1 \times \mathbb{P}^1/\mathcal{F}$ for some height one foliation $\mathcal{F}$. By $0 \sim_{\mathbb{Q}} \pi_1^* K_X \sim K_{A_1 \times \mathbb{P}^1} - (p-1) \det \mathcal{F}$, we have

$$\det \mathcal{F} \sim \mathrm{pr}_2^* O_{\mathbb{P}^1}(-2/(p-1)). \tag{5.8}$$

Moreover, since $\deg(X_1 \to X) < \dim A$, we have $\operatorname{rank} \mathcal{F} < \dim A$.

In case (2), we fall into one of the following three subcases:

(2.a) f_1 is separable and $G_1 \to A_1$ is étale.
(2.b) f_1 is separable and $G_1 \to A_1$ is purely inseparable.
(2.c) f_1 is inseparable.

In case (2.a), since the general fibers G_1 of g_1 are étale of degree 2 over A_1, they are isomorphic to each other. Do the étale base change $A_2 := G_1 \to A_1$. Then, $X_2 = X_1 \times_{A_1} A_2$ is normal, and the Stein factorization of $X_2 \to X_1 \xrightarrow{g_1} \mathbb{P}^1$ gives a fibration $g_2 \colon X_2 \to \mathbb{P}^1$. To summarize, we have the following commutative diagram:

$$\begin{array}{ccc} & & \mathbb{P}^1 & \xrightarrow{\delta} & \mathbb{P}^1 \\ & {\scriptstyle g_2}\nearrow & & {\scriptstyle g_1}\nearrow & \\ X_1 \times_{A_1} A_2 =: X_2 & & \xrightarrow{\pi_2} & X_1 & \\ {\scriptstyle f_2}\downarrow & & & {\scriptstyle f_1}\downarrow & \\ G_1 =: A_2 & & \xrightarrow{\tau_2} & A_1\,. & \end{array}$$

Note that the scheme $G_1 \times_{A_1} A_2$ has two disjoint irreducible components G_2', G_2'', that is, $\pi_2^* G_1 \sim G_2' + G_2''$. Thus, δ is a separable morphism of degree two. Therefore, each fiber G_2 of g_2 is mapped birationally to A_2 via f_2. By $(K_{X_2} + G_2)|_{G_2'} \sim_{\mathbb{Q}} 0$, applying Lemma 2.9, we see that G_2 is an abelian variety, hence, $G_2 \to A$ is an isomorphism. It follows that the morphism $X_2 \xrightarrow{(f_2, g_2)} A_2 \times \mathbb{P}^1$ is an isomorphism.

We use a similar argument for case (2.b). Take $\tau_2 \colon A_2 := A_1 \xrightarrow{F} A_1$ to be the Frobenius morphism, which factors through $G_1 \to A_1$ for a general fiber G_1 of g_1 [46, p. 169, Remark]. Let $\nu \colon X_2 \to X_1 \times_{A_1} A_2$ be the normalization morphism. Since f_1 is generically smooth, the projection $f_2 \colon X_2 \to A_2$ is in fact a fibration. Notice that $G_1 \times_{A_1} A_2$ is non-reduced. Then, by $\deg_{K(A_2)} G_1 \times_{A_1} A_2 = 2$, we conclude $\pi_2^* G_1 = 2G_2$ for some integral divisor G_2 in X_2. Let $g_2 \colon X_2 \to \mathbb{P}^1$ be the induced fibration from the Stein factorization of $X_2 \to X_1 \xrightarrow{g_1} \mathbb{P}^1$. Then, a general fiber G_2 is mapped to A_2 birationally via f_2. Let $\mathcal{N}$ be the conductor divisor of ν, which is vertical over A_2 since the generic fiber of f_1

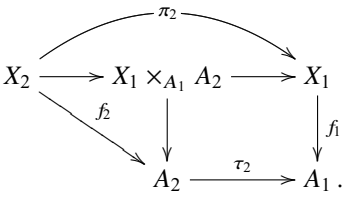
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is smooth over $K(A_1)$. Now fix a fiber G_2^0, and take a horizontal irreducible component G_2', which is unique since $\deg_{K(A_2)} G_2^0 = 1$ and write that $G_2^0 = G_2' + G_2''$. Then, by

$$0 \sim_{\mathbb{Q}} (\pi_2^* K_{X_1} + G_2)|_{(G_2')^\nu} \sim_{\mathbb{Q}} (K_{X_2} + G_2' + G_2'' + \mathcal{N})|_{(G_2')^\nu},$$

and applying Lemma 2.9, we see that $G_2 \to A_2$ is an isomorphism and $G_2''|_{(G_2')^\nu} = \mathcal{N}|_{(G_2')^\nu} = 0$. It follows that $\operatorname{Supp}\mathcal{N} \cap \operatorname{Supp} G_2^0 = \emptyset$ and $G_2'' = 0$ since G_2^0 is connected. In turn, we conclude that $\mathcal{N} = 0$. As before, we show that the morphism $X_2 \xrightarrow{(f_2, g_2)} A_2 \times \mathbb{P}^1$ is an isomorphism.

Finally, we deal with case (2.c). Take $\tau_2 \colon A_2 := A_1 \xrightarrow{F} A_1$ to be the Frobenius morphism, and let $X_2 := (X_1 \times_{A_1} A_2)^\nu_{\mathrm{red}}$. We have the following diagram:



Since f_1 is inseparable, there exists a moving linear system $\mathfrak{N}$ and $V_2 \geq 0$ on X_2 such that

$$K_{X_2} + \mathfrak{N} + V_2 \sim \pi_2^* K_{X_1}. \tag{5.9}$$

We have that $K_{X_2} + \mathfrak{N} + V_2 + \pi_2^*\mathfrak{M} \sim_{\mathbb{Q}} \pi_2^*\pi_1^* K_X \sim_{\mathbb{Q}} 0$. By $\deg_{K(A_2)}(K_{X_2}) = -2$, we see that $\deg_{K(A_2)}(\mathfrak{N}) = \deg_{K(A_2)}(\pi_2^*\mathfrak{M}) = 1$. Fix a divisor $N_0 \in \mathfrak{N}$, and take an irreducible f_2-horizontal component N' and write that $N_0 = N' + N''$. Arguing as in case (2.b), we can prove that $N_0 = N'$ and $V_2 = 0$. In turn, we can prove that $X_2 \cong A_2 \times \mathbb{P}^1$. □

6. Irregular *K*-trivial threefolds

In this section, we focus on irregular K-trivial threefolds.

6.1. Structure theorem

Theorem 6.1. *Let X be a normal $\mathbb{Q}$-factorial projective threefold with $K_X \equiv 0$. Denote the Albanese morphism of X by $a_X \colon X \to A$. Assume $\dim a_X(X) > 0$. Then, X can be described as follows:*

(A) *If $\dim a_X(X) = 3$, then X is an abelian variety.*

(B) *If $\dim a_X(X) = 1$, under the condition that:*
- *either (i) X is strongly F-regular and K_X is $\mathbb{Z}_{(p)}$-Cartier;*
- *or (ii) X has at most terminal singularities and $p \geq 5$,*

then the Albanese morphism a_X is a fibration and there exists an isogeny of elliptic curves $A' \to A$, such that $X \times_A A' \cong A' \times F$, where F is a general fiber of a_X. More precisely, $X \cong A' \times F/H$, where H is a finite group subscheme of A' acting diagonally on $A' \times F$.

(C) *If $\dim a_X(X) = 2$, then a_X is a fibration and one of the following holds:*

(C1) *a_X is a smooth fibration and there exists an isogeny of abelian surfaces $A' \to A$, such that $X \times_A A' \cong A' \times E$, where E is an elliptic curve appearing as a general fiber of a_X. More precisely, $X \cong A' \times E/H$, where H is a finite group subscheme of A' acting diagonally on $A' \times E$, with a full classification as in Section 5.1.2.*

(C2) *$p = 2$ or 3, and a_X is a quasi-elliptic fibration. Set X_1 to be the normalization of $X \times_A A^{(-1)}$ and $f_1 \colon X_1 \to A_1 := A^{(-1)}$ the induced morphism. Then, $f_1 \colon X_1 \to A_1$ is a smooth fibration fibered by rational curves, which falls into one of the following two specific cases:*

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(2.1) *In the first case, $f_1 : X_1 \to A_1$ has a section, and*
  (2.1a) *either $X_1 \cong \mathbb{P}_{A_1}(\mathcal{O}_{A_1} \oplus \mathcal{L})$, where $\mathcal{L}^{\otimes p+1} \cong \mathcal{O}_X$; or*
  (2.1b) *$X_1 \cong \mathbb{P}_{A_1}(\mathcal{E})$, where $\mathcal{E}$ is a unipotent vector bundle of rank two, and there exists an étale cover $\mu : A_2 \to A_1$ of degree p^v for some $v \leq 2$ such that $\mu^* F_{A_1/k}^{(2-v)*} \mathcal{E}$ is trivial.*

(2.2) *In the second case, $p = 2$, and there exists a purely inseparable isogeny $A_2 \to A_1$ of degree two, such that $X_2 := X_1 \times_{A_1} A_2$ is a projective bundle over A_1 described as follows:*
  (2.2a) *$X_2 \cong \mathbb{P}_{A_2}(\mathcal{O}_{A_2} \oplus \mathcal{L})$, where $\mathcal{L}^{\otimes 4} \cong \mathcal{O}_{A_2}$; or*
  (2.2b) *$X_2 \cong \mathbb{P}_{A_2}(\mathcal{E})$, where $\mathcal{E}$ is a unipotent vector bundle of rank two, and there exists an étale cover $\mu : A_3 \to A_2$ of degree p^v for some $v \leq 2$ such that $\mu^* F_{A_1/k}^{(2-v)*} \mathcal{E}$ is trivial.*

(C3) *$p = 2$ or 3, and a_X is inseparable. Set $X_1 = (X \times_A A_1(:= A^{(-1)}))^{\nu}_{\mathrm{red}}$. Then, the projection $X_1 \to A_1$ is a smooth morphism, and*
  (3.1) *either $X_1 = A_1 \times \mathbb{P}^1$ and $X = A_1 \times \mathbb{P}^1/\mathcal{F}$ for some smooth rank one foliation $\mathcal{F}$, which can be described concretely (see Section 6.2.2); or*
  (3.2) *$p = 2$, and there exists an isogeny of abelian surfaces $\tau : A_2 \to A_1$ such that $X_1 \times_{A_1} A_2 \cong A_2 \times \mathbb{P}^1$, where either:*
    (3.2a) *$\tau : A_2 \to A_1$ is an étale of degree two; or*
    (3.2b) *$\tau = F_{A_1/k} : A_2 := A_1^{(-1)} \to A_1$ is the relative Frobenius.*

Proof. When $\dim a_X(X) = 3$, we can apply Proposition 2.8 and show that $X \cong A$. This is Case (A).

When $\dim a_X(X) = 1$, we apply Theorem 4.1 and show that $q(X) = 1$ and X is described as in Case (B).

At last, we consider the case $\dim a_X(X) = 2$. When $a_X : X \to A$ is separable, we can apply Theorems 5.2 and 5.3 to obtain cases (C1) and (C2), respectively. When $a_X : X \to A$ is inseparable, we can apply Theorem 5.4 to obtain (C3). Note that as $\dim A = 2$, by Remark 2.18, we see that the generic fiber of $X_1 \to A_1$ is geometrically normal, and thus Case (2.c) in Theorem 5.4 does not occur here. □

6.2. *Concrete description and examples*

In this section, we first describe concretely the foliation in case (C3.1), then give examples of cases (C3.2) and (C2).

6.2.1. Notation

(1) On the projective line $\mathbb{P}^1$, we fix an affine open cover $\mathbb{P}^1 = \mathbb{A}^1_{(t)} \cup \mathbb{A}^1_{(s)}$, with $s = 1/t$. We identify the line bundle $\mathcal{O}_{\mathbb{P}^1}(i)$ as

$$\mathcal{O}_{\mathbb{P}^1}(i)|_{\mathbb{A}^1_{(t)}} = \mathcal{O}_{\mathbb{A}^1_{(t)}} \cdot 1, \quad \mathcal{O}_{\mathbb{P}^1}(i)|_{\mathbb{A}^1_{(s)}} = \mathcal{O}_{\mathbb{A}^1_{(s)}} \cdot \frac{1}{s^i};$$

and the twisted tangent bundle $\mathcal{T}_{\mathbb{P}^1}(i)$ as

$$\mathcal{T}_{\mathbb{P}^1}(i)|_{\mathbb{A}^1_{(t)}} = \mathcal{O}_{\mathbb{A}^1_{(t)}} \cdot \partial_t, \quad \mathcal{T}_{\mathbb{P}^1}(i)|_{\mathbb{A}^1_{(s)}} = \mathcal{O}_{\mathbb{A}^1_{(s)}} \cdot \frac{1}{s^i}\partial_s, \quad \text{with } \partial_t = s^2\partial_s.$$

(2) Let A be an abelian surface. We can choose a basis α, β of Lie algebra Lie$A$ which falls into one of the following cases:

(i) $\alpha^p = \beta^p = 0$ (superspecial);
(ii) $\alpha^p = \beta$ and $\beta^p = 0$ (supersingular, not superspecial);
(iii) $\alpha^p = \alpha$ and $\beta^p = 0$ (p-rank one);
(iv) $\alpha^p = \alpha$ and $\beta^p = \beta$ (ordinary).

Then, $\mathcal{T}_A = \mathcal{O}_A \cdot \alpha \oplus \mathcal{O}_A \cdot \beta$.

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6.2.2. A concrete description of foliation in Case (C3.1)
In this case, $X \cong A \times \mathbb{P}^1/\mathcal{F}$, where $\mathcal{F} \subset \mathcal{T}_{A\times\mathbb{P}^1}$ is a rank one foliation. We shall give a concrete description of $\mathcal{F}$.

As $\mathcal{F}$ is reflexive and of rank one on a factorial variety, it is locally free [30, Proposition 1.9]. By Equation (5.8), we have $\mathcal{F} \cong \det\mathcal{F} \cong \mathrm{pr}_2^* O_{\mathbb{P}^1}(-i)$, where $i = 1$ if $p = 3$ and $i = 2$ if $p = 2$, therefore, the inclusion $\mathcal{F} \subset \mathcal{T}_{A\times\mathbb{P}^1}$ is determined by a non-zero element (unique up to scaling)

$$\alpha_{\mathcal{F}} \in \mathrm{Hom}(\mathrm{pr}_2^* O_{\mathbb{P}^1}(-i), \mathcal{T}_{A\times\mathbb{P}^1}) \cong H^0(A \times \mathbb{P}^1, \mathcal{T}_A(i) \oplus \mathrm{pr}_2^*\mathcal{T}_{\mathbb{P}^1}(i)), \tag{6.1}$$

where $\mathcal{T}_A(i) := \mathrm{pr}_1^*\mathcal{T}_A \otimes \mathrm{pr}_2^* O_{\mathbb{P}^1}(i)$. Using the basis of $O_{\mathbb{P}^1}(i)$, $\mathcal{T}_{\mathbb{P}^1}(i)$, and $\mathcal{T}_A$ given in Section 6.2.1, the element $\alpha_{\mathcal{F}}$, over the subset $A \times \mathbb{A}^1_{(t)}$, can be written as

$$D = u\alpha + v\beta + w\partial_t, \tag{6.2}$$

where $u, v, w \in k[t]$ with $\deg u, \deg v \le i$, $\deg w \le i + 2$.

Since $\mathcal{F}$ is saturated, we have $\gcd(u, v, w) = 1$, thus the ideal $(u, v, w) = k[t]$. From this, we conclude that on $A \times \mathbb{A}^1_{(t)}$, the quotient $\mathcal{T}_{A\times\mathbb{P}^1}/\mathcal{F}$ is also locally free, thus the foliation $\mathcal{F}$ is smooth. For the same reason, $\mathcal{F}$ is smooth on $A \times \mathbb{A}^1_{(s)}$. To summarize, we have the following proposition.

Proposition 6.2. *The foliation $\mathcal{F}$ is smooth, and thus X is a smooth variety.*

6.2.2.1. The case $\boldsymbol{p = 3}$. In this case, $\deg u(t) \le 1$, $\deg v(t) \le 1$, $\deg w(t) \le 3$, and we can write

$$D = (a_1 t + a_0)\alpha + (b_1 t + b_0)\beta + (c_3 t^3 + c_2 t^2 + c_1 t + c_0)\partial_t.$$

The rank one subsheaf $\mathcal{F}$ is a foliation if and only if $\mathcal{F}^p \subseteq \mathcal{F}$, which is equivalent to the condition $D^p = \lambda D$ for some $\lambda \in k[t]$. By Proposition 2.1, the Albanese morphism $a_X : X \to A_X$ is inseparable if and only if

$$\Delta := a_1 b_0 - a_0 b_1 \neq 0.$$

Therefore, we can characterize the foliation $\mathcal{F}$ as the following:

(♣) $\Delta \neq 0$, and $D^p = \lambda D$ for some $\lambda \in k[t]$.

By a direct calculation, we have

$$D^3 = \big(u(t)\alpha + v(t)\beta + w(t)\partial_t\big)^3 = u^3\alpha^3 + v^3\beta^3 + ww'u'\alpha + ww'v'\beta + (ww'^2 + w^2w'')\partial_t.$$

Then, we can translate the condition (♣) into the following in each of the cases (i)–(iv) of Lie$A$ in Section 6.2.1(2):

(i) $\Delta \neq 0$ and $w' = 0$.
(ii) Invalid.
(iii) $\Delta \neq 0$, $c_1c_3 - c_2^2 = \dfrac{-b_0a_1^3}{\Delta}$, $c_2c_3 = \dfrac{b_1a_1^3}{\Delta}$, $c_0c_2 - c_1^2 = \dfrac{b_1a_0^3}{\Delta}$, $c_0c_1 = \dfrac{-b_0a_0^3}{\Delta}$.
(iv) $\Delta \neq 0$, $c_1c_3 - c_2^2 = \dfrac{a_0b_1^3 - b_0a_1^3}{\Delta}$, $c_2c_3 = \dfrac{a_1^3b_1 - a_1b_1^3}{\Delta}$, $c_0c_2 - c_1^2 = \dfrac{a_0^3b_1 - b_0^3a_1}{\Delta}$, $c_0c_1 = \dfrac{a_0b_0^3 - b_0a_0^3}{\Delta}$.

6.2.2.2. The case $\boldsymbol{p = 2}$. In this case, $\deg u(t) \le 2$, $\deg v(t) \le 2$, $\deg w(t) \le 4$, and we can write

$$D = (a_2 t^2 + a_1 t + a_0)\alpha + (b_2 t^2 + b_1 t + b_0)\beta + (c_4 t^4 + c_3 t^3 + c_2 t^2 + c_1 t + c_0)\partial_t,$$

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which satisfies the following conditions:

$$(\spadesuit)\quad \begin{cases} D^p = \lambda D \text{ for some } \lambda \in k[t]; \\ \text{one of } \Delta_{01} := a_0b_1 + a_1b_0, \Delta_{12} := a_1b_2 + a_2b_1, \text{ and } \Delta_{02} := a_0b_2 + a_2b_0 \text{ is nonzero;} \\ \gcd(u, v, w) = 1 \text{ and } (a_2, b_2, c_4) \neq 0. \end{cases}$$

The conditions (♠) can be translated into the following in each of the cases (i)–(iv) of Lie$A$ in Section 6.2.1(2):

(i) either (1) $w = 0$, $(\Delta_{01}, \Delta_{12}, \Delta_{02}) \neq 0$, $\gcd(u, v) = 1 \neq 0$, and $(a_2, b_2) \neq 0$ or (2) $u' = v' = w' = 0$ and $\Delta_{02} \neq 0$.

(ii) $\gcd(u, v, w) = 1$, $(a_2, b_2) \neq 0$, $(\Delta_{01}, \Delta_{12}) \neq 0$, and

	c_0	c_1	c_2	c_3	c_4
$\Delta_{01} \neq 0, \Delta_{12} \neq 0$	$\frac{a_0^3}{\Delta_{01}}$	$\frac{a_0^2a_1}{\Delta_{01}}$	$\frac{a_0a_1^2}{\Delta_{01}} = \frac{a_2a_1^2}{\Delta_{12}}$	$\frac{a_1a_2^2}{\Delta_{12}}$	$\frac{a_2^3}{\Delta_{12}}$
$b_1 \neq 0, a_1 = 0, a_0$ or $a_2 = 0$	$\frac{a_0^2}{b_1}$	0	0	0	$\frac{a_2^2}{b_1}$

By symmetric reason, we omit the case $\Delta_{01} = 0$ and $\Delta_{12} \neq 0$ in the following.

(iii) $\gcd(u, v, w) = 1$, $(a_2, b_2) \neq 0$, $(\Delta_{01}, \Delta_{12}) \neq 0$, and

	c_0	c_1	c_2	c_3	c_4
$\Delta_{01}\neq0, \Delta_{12}\neq0,$ $b_1a_0\Delta_{02}\Delta_{12}+a_2b_1\Delta_{02}\Delta_{01}+a_1b_1\Delta_{01}\Delta_{12}=0$	$\frac{b_0a_0^2}{\Delta_{01}}$	$\frac{b_1a_0^2}{\Delta_{01}}$	$\frac{b_0a_0a_2}{\Delta_{01}} + \frac{a_0a_2b_2}{\Delta_{12}} + a_1$	$\frac{b_1a_2^2}{\Delta_{12}}$	$\frac{b_2a_2^2}{\Delta_{12}}$
$\Delta_{01}\neq0, \Delta_{12}=0,$ $a_1b_1=a_2b_1=a_2b_2=0$			$\frac{b_0a_1^2}{\Delta_{01}} + \frac{b_1a_0^2\Delta_{02}}{\Delta_{01}^2}$	0	$\frac{b_0a_2^2}{\Delta_{01}}$

(iv) $\gcd(u, v, w) = 1$, $(a_2, b_2) \neq 0$, $(\Delta_{01}, \Delta_{12}) \neq 0$, $(a_1b_0^2 + b_1a_0^2)\Delta_{12}^2 + (a_1b_2^2 + b_1a_2^2)\Delta_{01}^2 + (b_1a_1^2 + a_1b_1^2)\Delta_{01}\Delta_{12} = 0$, and

	c_0	c_1	c_2	c_3	c_4
$\Delta_{01}\neq0, \Delta_{12}\neq0, a_1\neq0$			$\frac{a_2b_0^2+a_2a_0b_0}{\Delta_{01}} + \frac{a_0b_2^2+a_0a_2b_2}{\Delta_{12}} + a_1$	$\frac{a_1b_2^2+b_1a_2^2}{\Delta_{12}}$	$\frac{a_2b_2^2+b_2a_2^2}{\Delta_{12}}$
$\Delta_{01}\neq0, \Delta_{12}\neq0, b_1\neq0$	$\frac{a_0b_0^2+b_0a_0^2}{\Delta_{01}}$	$\frac{a_1b_0^2+b_1a_0^2}{\Delta_{01}}$	$\frac{b_0a_0b_2+a_0^2b_2}{\Delta_{01}} + \frac{b_0b_2a_2+b_0a_2^2}{\Delta_{12}} + b_1$		
$\Delta_{01}\neq0, \Delta_{12}=0$			$\frac{(a_1b_0^2+b_1a_0^2)\Delta_{02}+(a_0b_1^2+b_0a_1^2)\Delta_{01}}{\Delta_{01}^2}$	0	$\frac{(a_0b_2^2+b_0a_2^2)}{\Delta_{01}}$

Remark 6.3. For each valid case, it is easy to give examples. For instance, if $p = 2$, $\alpha^2 = \alpha$, and $\beta^2 = \beta$ (so we are in case (iv)), then $\left(\begin{smallmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{smallmatrix}\right)$ is a solution of (♠) which corresponds to the following vector field:

$$D = \alpha + (t + t^2)\beta + (t + t^4)\partial_t.$$

Some special cases have appeared in the literature (see [43], [54, Section 3], and [52, Example 14.1]).

6.2.3. Examples of Case (C3.2)

Example 6.4 (of Case (C3.2a)). We aim to find a free action of $G = \mathbb{Z}/2\mathbb{Z}$ on $A \times \mathbb{P}^1$ and a foliation $\mathcal{F}$ on $X_0 := A \times \mathbb{P}^1$ in a compatible way such that $\mathcal{F}$ descents to a foliation $\mathcal{F}_1$ on $A \times \mathbb{P}^1/G$ and the quotient $X := (A \times \mathbb{P}^1/G)/\mathcal{F}_1$ is an example of Case (C3.2a).

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Let $p = 2$, and let A be an ordinary abelian surface with $\alpha^p = \alpha$ and $\beta^p = \beta$ for some bases $\alpha, \beta \in \mathrm{Lie}A$. Let $G = \mathbb{Z}/2$ be the cyclic group of order 2 with a generator σ. Let $P_0 \in A$ be a nontrivial 2-torsion point. Then, we can define an action of G on $A \times \mathbb{P}^1$ diagonally by

$$\sigma(P) = P + P_0 \text{ for each } P \in A, \text{ and } \sigma(t) = t + 1 \text{ for each } [t : 1] \in \mathbb{P}^1.$$

Let $X_1 := A \times \mathbb{P}^1/G$. Since the action of G is free, $\pi_2 \colon A \times \mathbb{P}^1 \to X_1$ is étale. Let $\mathcal{F}$ be the foliation on $A \times \mathbb{P}^1$ generated by

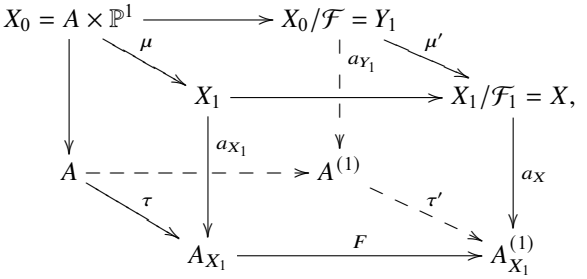
$$\alpha + (t^2 + t)\beta + (t^4 + t)\partial_t.$$

Let $Y_1 = (A \times \mathbb{P}^1)/\mathcal{F}$. Since $K_{A\times\mathbb{P}^1} = \mathrm{pr}_2^* K_{\mathbb{P}^1}$ and $\det \mathcal{F} \cong \mathrm{pr}_2^* O_{\mathbb{P}^1}(-2)$, we have $K_{Y_1} \equiv 0$.

The foliation $\mathcal{F}$ is invariant under the action of σ, thus it descends to a foliation on X_1, denoted by $\mathcal{F}_1$. Moreover, the action of σ descends to an action σ_1 on $Y_1 = X/\mathcal{F}$ (Lemma 2.2). Let

$$X := X_1/\mathcal{F}_1 \cong Y_1/\langle\sigma_1\rangle.$$

By Remark 2.3, we have the following commutative diagram:



where $\tau \colon A \to A_{X_1} = A/\langle\sigma\rangle$ and τ' are the étale morphisms, the left and the right squares are Cartesian and thus μ, μ' are étale. As stated, we have $K_{Y_1} \equiv 0$, thus $K_X \equiv 0$. Moreover, a_{Y_1} is inseparable, so is a_X.

Example 6.5 (of Case (C3.2b)). In this example, we construct a threefold X by two consecutive quotients of $A \times \mathbb{P}^1$ by certain foliations:

$$\begin{array}{ccccc} X_0 = A \times \mathbb{P}^1 & \xrightarrow{\pi_0} & X_1 = X_0/\mathcal{F}_0 & \xrightarrow{\pi_1} & X = X_2 = X_1/\mathcal{F}_1 \\ \downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 \\ A & \xrightarrow{\tau_0} & A_1 & \xrightarrow{\tau_1} & A_1^{(1)}, \end{array} \tag{6.3}$$

where τ_0 is purely inseparable of degree p, τ_1 is the relative Frobenius over k, and f_i $(i = 0, 1, 2)$ are the Albanese morphisms, respectively. We work over a base field k of characteristic 2.

Let E and E' be elliptic curves with the following defining equations:

$$\begin{aligned} E &: y^2 + y = x^3, \\ E' &: y'^2 + x'y' = x'^3 + 1. \end{aligned} \tag{6.4}$$

Note that E is supersingular which admits a invariant vector field $\alpha \in H^0(E, \mathcal{T}_E)$ such that $\alpha^p = 0$; and E' is ordinary which admits an invariant vector field $\beta \in H^0(E', \mathcal{T}_{E'})$ such that $\beta^p = \beta$. Concretely, we can take α to be the dual of $dx \in H^0(E, \Omega^1_E)$ such that $\alpha(dx) = \alpha(x) = 1$ and $\alpha(y) = x^2$; and β the dual of $dx'/x' \in H^0(E, \Omega^1_E)$, so $\beta(x') = x'$, $\beta(y') = y' + x'^2$ (see [56, Section III.1]).

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Let $A = E \times E'$. In the following, we usually describe vector fields on the given affine pieces of E and E' by (6.4). The reader can check that the vector fields involved extend to the whole variety.

With notation of Section 6.2.1, let $X_0 = A \times \mathbb{P}^1$ and $Y_0 = E \times \mathbb{P}^1$ and let $D_0 = \alpha + \partial_t \in H^0(Y_0, \mathcal{T}_{Y_0})$. We may also regard D_0 as an element of $H^0(X_0, \mathcal{T}_{X_0}) \cong H^0(Y_0, \mathcal{T}_{Y_0}) \oplus H^0(E', \mathcal{T}_{E'})$. Then, $D_0^p = 0$, and D_0 extends to a global vector field on $X_0 = A \times \mathbb{P}^1$, which is non-zero everywhere. Let $\mathcal{F}_0$ and $\mathcal{G}_0$ be the foliations generated by D_0 on X_0 and Y_0, respectively. Let $Y_1 = Y_0/\mathcal{G}_0$ and $X_1 = X_0/\mathcal{F}_0 \cong Y_0 \times E'$, which are both nonsingular. By construction, X_1 is equipped with two fibrations $f_1 \colon X_1 \to A_1 = E^{(1)} \times E'$ and $g_1 \colon X_1 \to B = \mathbb{P}^1$, which fit into the following commutative diagrams:

$$\begin{array}{ccc} X_0 = A \times \mathbb{P}^1 & \xrightarrow{\pi_0} & X_1 = X_0/\mathcal{F}_0 \\ \downarrow{\scriptstyle f_0 = \mathrm{pr}_1} & & \downarrow{\scriptstyle f_1} \\ A & \xrightarrow{\tau_0} & A_1 \end{array} \qquad \begin{array}{ccc} X_0 & \longrightarrow & X_1 \\ \downarrow{\scriptstyle g_0 = \mathrm{pr}_2} & & \downarrow{\scriptstyle g_1} \\ \mathbb{P}^1 & \xrightarrow{F} & B = (\mathbb{P}^1)^{(1)}. \end{array}$$

Here, we have an open covering $(\mathbb{P}^1)^{(1)} = \mathbb{A}^1_{(t^p)} \cup \mathbb{A}^1_{(s^p)}$. Taking the corresponding inverse images of g_0, g_1, we obtain the coverings

$$X_0 = A \times \mathbb{P}^1 = (X_{0,t} := A \times \mathbb{A}^1_{(s)}) \cup (X_{0,s} := A \times \mathbb{A}^1_{(s)}), \quad X_1 = X_{1,t^p} \cup X_{1,s^p} \text{ and } Y_1 = Y_{1,t^p} \cup Y_{1,s^p}.$$

We may identify

$$\Gamma(Y_{1,t^p}, \mathcal{O}_{Y_{1,t^p}}) = R = k[x_1 = x^2, y_1 = y^2, t_1 = t^2, u = x + t]/(y_1^2 + y_1 - x_1^3, u^2 - (x_1 + t_1)),$$

then $\Omega_{R/k} = R \cdot dt_1 \bigoplus R \cdot du$. Let $\alpha_1 \in H^0(Y_{1,t^p}, \mathcal{T}_{Y_{1,t^p}})$ be determined by $\alpha_1(dt_1) = \alpha_1(dx_1) = t_1, \alpha_1(du) = 0$. On the other piece Y_{1,s^p}, we have

$$\Gamma(Y_{1,s^p}, \mathcal{O}_{Y_{1,s^p}}) \cong k[x_1 = x^2, y_1 = y^2, s_1 = s^2, v = s^2 x + s]/(y_1^2 + y_1 - x_1^3, v^2 - (s_1^2 x_1 + s_1)),$$
$$\Omega_{R'/k} = R' \cdot dx_1 \oplus R' dv, \text{ and } \alpha_1(x_1) = 1/s_1, \alpha_1(v) = \alpha_1(s_1 u) = v.$$

We abuse notation $\alpha_1 \in H^0(X_1, \mathcal{T}_{X_0}) \cong H^0(Y_1, \mathcal{T}_{Y_1}) \oplus H^0(E', \mathcal{T}_{E'})$ for the lifting of $\alpha_1 \in H^0(Y_1, \mathcal{T}_{Y_1})$ and β_1 for the lifting of $\beta \in H^0(E', \mathcal{T}_{E'})$.

Set $D_1 = \alpha_1 + \beta_1 \in H^0(X_1, \mathcal{T}_{X_1})$. Clearly, $D_1^2 = D_1$, thus D_1 determines a rank one foliation $\mathcal{F}_1$ on X_1. More precisely, on $X_{1,t}$, applying Jacobi criterion, we see that (u, t_1, x') form a local coordinate for X_{1,t^p} and

$$D_1 = t_1 \frac{\partial}{\partial t_1} + x' \frac{\partial}{\partial x'}.$$

Since the vector field D_1 has no zeros on $X_{1,t}$, we have $\mathcal{F}_1|_{X_{1,t^p}} = \mathcal{O}_{X_{1,t^p}} \cdot D_1$. While on $X_{1,s}$, the set of functions $\{v, x_1, x'\}$ forms a local coordinate and correspondingly

$$D_1 = \frac{1}{s_1}\left(\frac{\partial}{\partial x_1} + s_1 v \frac{\partial}{\partial v} + s_1 x' \frac{\partial}{\partial x'}\right).$$

We see that $\mathcal{F}_1|_{X_{1,s^p}} = \mathcal{O}_{X_{1,s^p}} \cdot s_1 D_1$. Thus, $\mathcal{F}_1 \sim g_1^* \mathcal{O}_{\mathbb{P}^1}(-1)$.

Let $X = X_1/\mathcal{F}_1$. By Proposition 2.1, the Albanese morphism of X is $f \colon X \to A_1^{(1)}$, and therefore the diagram (6.3) holds. Finally, X is as required since

$$\pi_0^* \pi_1^* K_X \sim \pi_0^*(K_{X_1} - (p-1) \det \mathcal{F}_1) \sim K_{X_0} + \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}^1}(-1) \sim 0.$$

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6.2.4. Examples of case (C2): a_X being a quasi-elliptic fibration

Example 6.6 (of (C2.1a)). We give this example to show that, in the structure theorem 5.3, the second fibration $g\colon X \to \mathbb{P}^1$ is not necessarily isotrivial. Recall that, in this case, the general fiber C of the Albanese fibration $f\colon X \to A$ is a rational curve with a cusp. Therefore, there does not exist an isogeny $A' \to A$ and a diagonal group action of G on $A' \times C$ such that $X \cong A' \times C/G$.

Assume $p = 2$. Let E be a supersingular elliptic curve with $\alpha \in \mathrm{Lie}E$ a nonzero vector field. Let E' be a copy of E over which we use β to denote the same vector field α. Let $A = E \times E'$. Then, $\{\alpha, \beta\}$ form a basis of $\mathrm{Lie}A$. Let $\mathcal{F}$ be the rank two foliation on $A \times B(:= \mathbb{P}^1)$, which is generated by

$$\begin{cases} t^2\alpha + \beta, \\ \beta + \partial_t, \end{cases}$$

on $A \times \mathbb{A}^1_{(t)}$ and by $\alpha + s^2\beta, \beta + s^2\partial_s$ on $A \times \mathbb{A}^1_{(s)}$. We see that $\det \mathcal{F} = \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}^1}(-2)$ on $A \times B$. Let $X = A \times B/\mathcal{F}$. Then, we have $K_X \sim_{\mathbb{Q}} 0$. Thus, the generic fiber of $a_X\colon X \to A^{(1)}$ is of arithmetic genus one, so it is geometrically non-normal; and by Proposition 2.1, a_X is separable, thus a_X forms a quasi-elliptic fibration. Since $t \notin \mathrm{Ann}(\mathcal{F})$, we have a fibration $g\colon X \to B' = B^{(1)} \cong \mathbb{P}^1$. For $b = [t^2 : 1] \in B'$, the fiber X_b of g is isomorphic to $A/(t^2\alpha + \beta)$. In particular, two general closed fibers of g are not isomorphic to each other, which implies that $g\colon X \to \mathbb{P}^1$ is not an isotrivial fibration.

Example 6.7 (of (C2.1b)). Assume $p = 2$. Let A be an ordinary abelian surface. We take a basis α, β of $\mathrm{Lie}A$ such that $\alpha^2 = \alpha$ and $\beta^2 = \beta$. Let $G = \langle\sigma\rangle \cong \mathbb{Z}/2$ be the cyclic group of order 2 which acts on $A \times \mathbb{P}^1$ as in Example 6.4:

$$\sigma(P) = P + P_0 \text{ for each } P \in A, \text{ and } \sigma(t) = t + 1 \text{ for each } [t : 1] \in \mathbb{P}^1,$$

where $P_0 \in A$ is a nontrivial 2-torsion point. Let $\mathcal{F}$ be the saturated subsheaf of $\mathcal{T}_{A\times\mathbb{P}^1}$ generated by

$$\begin{cases} \alpha + (t^4 + t)\partial_t, \\ \alpha + \beta. \end{cases}$$

The sheaf $\mathcal{F}$ is p-closed so it is a foliation. We have $\det \mathcal{F} \cong \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}^1}(-2)$. Also, $\mathcal{F}$ is invariant under the action of σ, thus it descends to a foliation $\mathcal{G}$ on $Y = (A \times \mathbb{P}^1)/\langle\sigma\rangle$; or equivalently, the action of σ descends to an action on $(A \times \mathbb{P}^1)/\mathcal{F}$ (Lemma 2.2). Let $X = Y/\mathcal{G} \cong (A \times \mathbb{P}^1)/\mathcal{F}/\langle\sigma\rangle$. Then, $K_X \equiv 0$. Finally, by Proposition 2.1, $a_X\colon X \to A^{(1)}$ is separable, thus a_X forms a quasi-elliptic fibration; for which the non-smooth locus of a_X is located at $t = \infty$. This gives an example of Case (C2.1b).

6.3. Effectivity of the pluricanonical maps of threefolds in case $q = 2$

Proof of Corollary 1.3. We discuss separately cases (C1)–(C3) according to Theorem 6.1.

(C1) By the classification of Section 5.1.2, the torsion order of K_X is exactly the same as [9, p. 37]; namely,

$$\begin{aligned} \text{the torsion order of } K_X &= 2, 3, 4, 6 \text{ in cases a), b), c), d) for } p \neq 2, 3 \\ &= 1, 3, 1, 3 \text{ in cases a), b), c), d) for } p = 2 \\ &= 2, 1, 4, 2 \text{ in cases a), b), c), d) for } p = 3. \end{aligned}$$

(C2) We shall use the following statement.

Lemma 6.8. *Let $\pi\colon Y \to X$ be a finite dominant morphism of degree d between normal varieties, and let L be a Weil divisor on X. Assume $H^0(Y, \pi^*L) \neq 0$. Then,*

(1) *We have $H^0(X, dL) \neq 0$; in particular, if $\pi^*L \sim 0$, then $dL \sim 0$.*
(2) *If π is purely inseparable of height r, then $H^0(X, p^rL) \neq 0$.*

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Proof. (1) On a normal variety, the global sections of a reflexive sheaf are the same as the sections over a big open subset. Thus, by restricting on the regular locus, we may assume that X and Y are regular. Then, given a section $s \in H^0(Y, \pi^* L)$, taking its norm gives a section of $H^0(X, dL)$.

(2) If π is purely inseparable of height r, then the rth Frobenius of X factors through π. So the assertion follows. □

We only show how to treat case (2.2b), because the other cases are similar and easier. We have the following commutative diagram:

$$\begin{array}{ccccccc} A_3 \times \mathbb{P}^1 \cong X_3 & \xrightarrow{\pi_3} & X_1 = (X \times_A A_1)^\nu & \xrightarrow{\pi_1} & X \\ \downarrow & & \downarrow & & \downarrow \\ A_3 & \longrightarrow & A_1 = A^{(-1)} & \xrightarrow{F} & A, \end{array}$$

where $A_3 \to A_1$ is a composition of three morphisms

$$A_3 \xrightarrow[\text{degree } p^r]{\text{étale of}} A_2 \xrightarrow{F^{2-r}} A_2 \xrightarrow[\text{of degree 2}]{\text{inseparable}} A_1, \quad r = 0, 1, \text{ or } 2.$$

Thus, π_3 is a composition of three morphisms in a similar way. Therefore, $X_3 \to X$ is a composition of an étale morphism of degree p^r and a purely inseparable morphism of height $4 - r$.

Since

$$\pi_3^* \pi_1^* K_X = \pi_3^*(K_{X_1} + C) = \pi_3^*(K_{X_1/A_1} + C) = K_{X_3/A_3} + \pi_3^* C \sim 0,$$

we conclude that $2^4 K_X \sim 0$ by Lemma 6.8.

We list the results as follows, leaving the detailed computation of other cases to the reader:

$$\text{the torsion order of } K_X \text{ is a divisor of } \begin{cases} (p+1)p & \text{in case (2.1a) for } p = 2 \text{ or } 3, \\ p^3 & \text{in case (2.1b) for } p = 2 \text{ or } 3, \\ 4 \cdot 2^2 & \text{in case (2.2a) for } p = 2, \\ 2^4 & \text{in case (2.2b) for } p = 2. \end{cases}$$

(C3) Since only purely inseparable and étale base change are involved in Theorem 5.4, we see that $3K_X \sim 0$ if $p = 3$ and $4K_X \sim 0$ if $p = 2$.

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