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Geometric realization for substitution tilings

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Abstract. Given an n-dimensional substitution Φ whose associated linear expansion Λ is unimodular and hyperbolic, we use elements of the one-dimensional integer Čech cohomology of the tiling space Ω_Φ to construct a finite-to-one semi-conjugacy $G : \Omega_\Phi \to \mathbb{T}^D$, called a geometric realization, between the substitution induced dynamics and an invariant set of a hyperbolic toral automorphism. If Λ satisfies a Pisot family condition and the rank of the module of generalized return vectors equals the generalized degree of Λ, G is surjective and coincides with the map onto the maximal equicontinuous factor of the $\mathbb{R}^n$-action on Ω_Φ. We are led to formulate a higher-dimensional generalization of the Pisot substitution conjecture: if Λ satisfies the Pisot family condition and the rank of the one-dimensional cohomology of Ω_Φ equals the generalized degree of Λ, then the $\mathbb{R}^n$-action on Ω_Φ has pure discrete spectrum.

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1. *Introduction*

Let Δ be a subset of the n-dimensional Euclidean space $\mathbb{R}^n$, $n \geq 1$. A *tiling* of Δ is a countable collection $T = \{t_j\}_{j\in J}$ of topological closed n-balls in Δ, called *tiles*, that cover Δ and have pairwise disjoint interiors. Consider a finite collection of polyhedra $\mathcal{P} = \{\rho_1, \ldots, \rho_m\}$ in $\mathbb{R}^n$, called *prototiles*. We say that $\mathcal{P}$ generates a tiling $T = \{t_j\}_{j\in J}$ of Δ if each t_i is a translated copy of one of the ρ_j and the t_i meet full face to full face in all dimensions (so each t_i has finitely many faces in each dimension, and if t_i and t_j meet in a point x that is in the relative interior of a face of t_i or t_j, then they meet in that entire face). We denote by $\Omega_{\mathcal{P}}(\Delta)$ the set of all tilings of Δ generated by $\mathcal{P}$, and by $\Omega_{\mathcal{P}}$ the set $\Omega_{\mathcal{P}}(\mathbb{R}^n)$. When $n > 1$, it is not always the case that, for a given family $\mathcal{P}$, the set $\Omega_{\mathcal{P}}$ is not empty. In fact it is well-known that the problem of whether or not $\Omega_{\mathcal{P}}$ is empty is not decidable [**B**]. When $\Omega_{\mathcal{P}} \neq \emptyset$, the group $\mathbb{R}^n$ acts on $\Omega_{\mathcal{P}}$ by translation: for any tiling $T = \{t_j\}_{j\in J}$ in $\Omega_{\mathcal{P}}$, and for any $u \in \mathbb{R}^n$ we define:

$$T - u = \{t_j - u\}_{j\in J}.$$

The set $\Omega_{\mathcal{P}}$ has a natural metric defined as follows. For each $r \geq 0$, let $\bar{B}_r(0)$ denote the closed ball of radius r centered at $0 \in \mathbb{R}^n$ and, for $T \in \Omega_{\mathcal{P}}$, let $B_r[T] := \{t_j \in T : t_j \cap \bar{B}_r(0) \neq \emptyset\}$ be the collection of tiles in T that meet $\bar{B}_r(0)$. Given $T, T' \in \Omega_{\mathcal{P}}$, let A denote the set of $\epsilon \in (0, 1)$ such that there are $u, u' \in \mathbb{R}^n$, with $|u|, |u'| < \epsilon/2$, so that $B_{1/\epsilon}[T - u] = B_{1/\epsilon}[T' - u']$. Then:

$$d(T, T') = \begin{cases} \inf A & \text{if } A \neq \emptyset, \\ 1 & \text{otherwise.} \end{cases}$$

In words, T and T' are close if, up to small translation, they agree exactly in a large neighborhood of the origin. When $\Omega_{\mathcal{P}}$ is equipped with this metric, the translation action is continuous. If $\Omega_{\mathcal{P}}$ has finite local complexity (that is, there are only finitely many patterns of tiles in elements of $\Omega_{\mathcal{P}}$ of any fixed finite radius—see below), $\Omega_{\mathcal{P}}$ is compact and has the structure of a *lamination* whose n-dimensional leaves are the orbits of the translation action and with totally disconnected transverse direction.

Substitutions provide an important method for constructing tilings. Consider a polyhedral family $\mathcal{P} = \{\rho_1, \ldots, \rho_m\}$ of prototiles in $\mathbb{R}^n$ and suppose there is an expanding linear map $\Lambda : \mathbb{R}^n \to \mathbb{R}^n$ (called *inflation*) so that for each $j \in \{1, \ldots, m\}$, there exists a tiling $\mathcal{S}_j$ of $\Lambda(\rho_j)$ generated by $\mathcal{P}$. The collection of tilings $\mathcal{S} = \{\mathcal{S}_1, \ldots, \mathcal{S}_m\}$ is called a *substitution rule*. The *incidence matrix* associated with Λ and $\mathcal{S}$ is the m-by-m matrix $M_{\mathcal{S}} = (m_{i,j})_{i,j}$, where for each $i, j \in \{1, \ldots, m\}$, the entry $m_{i,j}$ is the number of translated copies† of ρ_i in $\mathcal{S}_j$. Recall that a matrix M is *primitive* if there exists $k > 0$ such that all the elements of M^k are positive.

By iterating inflation followed by substitution infinitely many times, one sees that $\Omega_{\mathcal{P}} \neq \emptyset$. On the one hand, the inflation Λ induces a natural continuous map $\mathcal{K}$ from $\Omega_{\mathcal{P}}$

† It is often desirable to mark, or color, prototiles and tiles so that tiles may occupy the same underlying set but nonetheless be distinct. Formally, then, a tile is a pair $\tau = (t, m)$ where t is a closed topological n-ball and m is one of finitely many possible marks. We will stick to the simpler unmarked language in this paper, though all of the results are to be understood in the more general setting.

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to $\Omega_{\Lambda\mathcal{P}}$, where $\Lambda\mathcal{P} = \{\Lambda p \mid p \in \mathcal{P}\}$. On the other hand, the substitution rule $\mathcal{S}$ induces a continuous map $\mathcal{J}$ from $\Omega_{\Lambda\mathcal{P}}$ to $\Omega_{\mathcal{P}}$, which is defined by subdividing the tiles of a tiling in $\Omega_{\lambda\mathcal{P}}$ according to the substitution rule. The composition of these two maps yields a self-map $\Phi := \mathcal{J} \circ \mathcal{K} : \Omega_{\mathcal{P}} \to \Omega_{\mathcal{P}}$ which we call the *substitution map*. We will say that a finite collection $P = \{t_1, \ldots, t_k\}$ of tiles is an *allowed patch* for Φ if there is a prototile ρ_j and a $k \in \mathbb{N}$ so that P is contained in some translate of the patch $\Phi^k(\rho_j)$ obtained by k iterations of inflation and substitution applied to ρ_j. The *tiling space* associated with Φ is the collection

$$\Omega_\Phi := \{T \in \Omega_{\mathcal{P}} : B_r[T] \text{ is an allowed patch for } \Phi \text{ for all } r \geq 0\}.$$

It is clear that Ω_Φ is invariant under both translation and the substitution map.

Throughout this paper we will make the following three assumptions:

- the map Φ is *primitive*, which means that the associated substitution matrix $M_{\mathcal{S}}$ is primitive;
- the tiling space Ω_Φ has *finite local complexity* (FLC), which means that for each $r > 0$ there are, up to translation, only finitely many distinct patches of the form $B_r[T] := \{t \in T : t \cap \bar{B}_r(0) \neq \emptyset\}$, $T \in \Omega_\Phi$; and
- Ω_Φ is *translationally non-periodic* which means that if there exist a tiling T in Ω_Φ and $u \in \mathbb{R}^n$ such that $T - u = T$, then $u = 0$.

Despite the fact that the substitution map is basically a linear inflation, it turns out the laminated structure of the phase space Ω_ϕ induces very rich dynamics. The action of $\mathbb{R}^n$ by translation on Ω_Φ is minimal and uniquely ergodic [**AP**]; the map Φ is a homeomorphism [**S1**] and is ergodic with respect to the unique $\mathbb{R}^n$-invariant measure μ; and the dynamics on Ω_Φ interact by $\Phi(T - v) = \Phi(T) - \Lambda v$.

Our first goal in this paper is to provide a way to understand the dynamical system (Ω_Φ, Φ) in terms of the standard geometric theory of dynamical systems where one studies iteration of maps on compact manifolds. This is what we call *geometric realization*. More precisely, we will show that, under some assumptions on the hyperbolicity of the eigenvalues of the inflation Λ, there exists a finite-to-one continuous map from Ω_Φ to some D-dimensional torus $\mathbb{T}^D$ that factors the dynamics of Φ into those of a linear hyperbolic map. The basic idea is as follows. The one-dimensional cohomology of Ω_Φ supplies a map into the D-torus that, on the level of cohomology, conjugates Φ with a hyperbolic toral automorphism. The technique of global shadowing is then applied to improve the map into an actual dynamical semi-conjugacy of hyperbolic systems. (This technique originated with [**F, Fr**], and is used also in [**BKw**] to embed pseudo-Anosovs into hyperbolic toral automorphisms almost everywhere.)

The second goal is to establish a link between this geometric realization and the traditional Pisot substitution conjecture [**BS**] regarding pure discreteness of the $\mathbb{R}^n$-action. On Ω_Φ we have $\Phi(T - v) = \Phi(T) - \Lambda v$ and a like relation holds between the hyperbolic action on $\mathbb{T}^D$ and a Kronecker action on the u-dimensional leaves of its unstable foliation. Under an assumption on the 'Pisotness' of the inflation Λ, $u = n$ and the semi-conjugacy of hyperbolic systems becomes also a semi-conjugacy of $\mathbb{R}^n$-actions. The coordinate functions of the semi-conjugacy are thus eigenfunctions of $\mathbb{R}^n$-action and their associated eigenvalues constitute a generating set for the discrete part of the spectrum of

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the $\mathbb{R}^n$-action on Ω_Φ. In essence, elements of the first cohomology of Ω_Φ are converted into eigenfunctions of the $\mathbb{R}^n$-action by means of global shadowing.

2. *Main results*

The eigenvalues of the inflation Λ are all algebraic integers ([**KS**], or see Lemma 16 below): let us partition the spectrum, $\mathrm{spec}(\Lambda)$, into families $\mathrm{spec}(\Lambda) = \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_k$ of algebraic conjugacy classes. For each $i \in \{1, \ldots, k\}$, let d_i be the algebraic degree of the elements of $\mathcal{F}_i$, let $m_i := \max_{\lambda \in \mathcal{F}_i} m(\lambda)$, where $m(\lambda)$ is the multiplicity of λ as an eigenvalue of Λ, and let the *degree* of Λ be defined by $D(\Lambda) := \sum_{i=1}^{k} m_i d_i$. We will say that Φ is *unimodular* if each element of $\mathrm{spec}(\Lambda)$ is an algebraic unit, and *hyperbolic* if no element of $\mathrm{spec}(\Lambda)$ has an algebraic conjugate on the unit circle.

If Φ is unimodular, there is a finitely generated subgroup $\mathrm{GR}(\Phi)$ of $\mathbb{R}^n$, the group of *generalized return vectors* (the definition is given below), that has rank $D = D(\mathrm{GR}) \geq D(\Lambda)$ (Corollary 17) and is invariant under Λ. Let A denote the $D \times D$ matrix representing $\Lambda : \mathrm{GR}(\Phi) \to \mathrm{GR}(\Phi)$ with respect to some basis, and let $F_A : \mathbb{T}^D \to \mathbb{T}^D$ be the toral automorphism $x + \mathbb{Z}^D \mapsto Ax + \mathbb{Z}^D$ induced by A. Whenever Φ is hyperbolic, F_A is also hyperbolic.

THEOREM 1. *Suppose that Φ is unimodular and hyperbolic. There is then a finite-to-one continuous map $G : \Omega_\Phi \to \mathbb{T}^D$ so that $G \circ \Phi = F_A \circ G$. Furthermore, G is μ-almost everywhere r-to-one for some $r \in \mathbb{N}$ and G is homologically essential in that $G^* : H^1(\mathbb{T}^D; \mathbb{Z}) \to H^1(\Omega_\Phi; \mathbb{Z})$ is injective.*

Let $\Phi^* : H^1(\Omega_\Phi; \mathbb{Z}) \to H^1(\Omega_\Phi; \mathbb{Z})$ denote the isomorphism induced by Φ on the integer Čech cohomology of Ω_Φ. We will see that there is a Φ^*-invariant subgroup H^1_Λ of $H^1(\Omega_\Phi; \mathbb{Z})$ restricted to which Φ^* is conjugate with the dual isomorphism $\Lambda^* : Hom(\mathrm{GR}(\Phi); \mathbb{Z}) \to Hom(\mathrm{GR}(\Phi); \mathbb{Z})$. It often happens that H^1_Λ is proper (see Example 12 below), in which case it may be possible to lower the r of Theorem 1 by increasing the size of the torus. Let H^1_{hyp} denote the largest subgroup of $H^1(\Omega_\Phi; \mathbb{Z})$ that contains H^1_Λ, is invariant under Φ^*, and restricted to which Φ^* is unimodular and hyperbolic. Let D' be the rank of H^1_{hyp} and let A' be the transpose of a matrix representing $\Phi^* : H^1_{\mathrm{hyp}} \to H^1_{\mathrm{hyp}}$ in some basis.

THEOREM 2. *Suppose that Φ is unimodular and hyperbolic. There is then a finite-to-one, and μ-almost everywhere r'-to-one for some $r' \leq r$, map $G' : \Omega_\Phi \to \mathbb{T}^{D'}$ so that $G' \circ \Phi = F_{A'} \circ G'$. Furthermore, $(G')^* : H^1(\mathbb{T}^{D'}; \mathbb{Z}) \to H^1(\Omega_\Phi; \mathbb{Z})$ is injective with range H^1_{hyp}.*

Theorems 1 and 2 are proved in §6.

CONJECTURE 3. *If Φ is hyperbolic and $H^1_{\mathrm{hyp}} = H^1(\Omega_\Phi; \mathbb{Z})$ (i.e., Φ^* is unimodular and hyperbolic on all of $H^1(\Omega_\Phi; \mathbb{Z})$), then $r' = 1$ (that is, G' is one-to-one almost everywhere).*

A family $\mathcal{F}_i$ of eigenvalues of Λ is a *Pisot family* provided all the elements of $\mathcal{F}_i$ have the same multiplicity as eigenvalues of Λ and, if λ is an algebraic conjugate of the elements

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of $\mathcal{F}_i$ with $|\lambda| \geq 1$, then $\lambda \in \mathcal{F}_i$. We will say that Φ is a *Pisot family substitution* if each $\mathcal{F}_i$ is a Pisot family.

For any group action on a space there is a maximal factor (unique up to conjugacy) on which the group acts equicontinuously. In the setting here, of $\mathbb{R}^n$-actions on tiling spaces Ω, the maximal equicontinuous factor is a Kronecker action on a torus or solenoid. We will denote the maximal equicontinuous factor map by g. It is a consequence of the Halmos–von Neumann theory that the $\mathbb{R}^n$-action on Ω has pure discrete spectrum if, and only if, g is one-to-one almost everywhere [**BK**].

THEOREM 4. *Suppose that Φ is a unimodular Pisot family substitution with linear expansion Λ. If $D(\Lambda) = D(\mathrm{GR}) = D$, then $G : \Omega_\Phi \to \mathbb{T}^D$ is surjective, semi-conjugates the $\mathbb{R}^n$-action on Ω_Φ with a Kronecker action of $\mathbb{R}^n$ on $\mathbb{T}^D$, and the latter action is the maximal equicontinuous factor of the $\mathbb{R}^n$-action on Ω_Φ. That is, we may take $G = g : \Omega_\Phi \to \mathbb{T}^D$.*

CONJECTURE 5. *If Φ is a unimodular Pisot family substitution and* $\mathrm{rank}(H^1(\Omega_\Phi; \mathbb{Z})) = D(\Lambda)$ *then the $\mathbb{R}^n$-action on Ω_Φ has pure discrete spectrum.*

There are counterexamples to Conjecture 5 if the assumption of unimodularity is dropped (see [**BBJS**], where a one-dimensional version of this conjecture is discussed). Our proofs of Theorems 1, 2 and 4 route through a certain non-compact cover of Ω_Φ for which there doesn't appear to be an adequate analog in the non-unimodular case. J. Kwapisz has suggested that corresponding results for non-unimodular hyperbolic substitutions may be obtained using sufficiently large, but compact, covers [**K**].

Theorem 4, and the corollaries below, are proved in §8.

Given a tiling $T = \{\tau_i\}$, a *puncture map* is a function $p : T \to \mathbb{R}^n$ so that $p(\tau_i) \in \tau_i$ and if $\tau_i = \tau_j + v$, then $p(\tau_i) = p(\tau_j) + v$. A set $\Gamma \subset \mathbb{R}^n$ is a *Meyer set* if Γ is relatively dense and $\Gamma - \Gamma$ is uniformly discrete. A tiling T is said to have the Meyer property if for any (and hence all) puncture map(s) p, $p(T)$ is a Meyer set.

COROLLARY 6. *Under the assumptions of Theorem 4:*

- *g is finite-to-one and almost everywhere cr-to-one;*
- *the eigenvalues of the $\mathbb{R}^n$-action on Ω_Φ are relatively dense in $\mathbb{R}^n$;*
- *the tilings $T \in \Omega_\Phi$ have the Meyer property.*

COROLLARY 7. *If Conjecture 3 is true, then so is Conjecture 5.*

A Pisot family substitution is said to be of (m, d)-*Pisot family type* if Λ is diagonalizable over $\mathbb{C}$ and $\mathrm{spec}(\Lambda) = \mathcal{F}_1$ consists of a single family with $m = m_1$ and $d = d_1$. Note in this case that $D(\Lambda) = md$. It is a result of [**BK**] that, if Φ is of (m, d)-Pisot family type, then the maximal equicontinuous factor of the $\mathbb{R}^n$-action on Ω_Φ is a Kronecker action on an md-dimensional torus (or solenoid, in case Φ is not unimodular). Moreover, the factor map $g : \Omega_\Phi \to \mathbb{T}^{md}$ is finite-to-one and almost everywhere cr-to-one for $cr = \min\{\sharp g^{-1}(x) : x \in \mathbb{T}^{md}\}$.

THEOREM 8. *If Φ is unimodular of (m, d)-Pisot family type, then $G : \Omega_\Phi \to \mathbb{T}^D$ is the maximal equicontinuous factor of the $\mathbb{R}^n$-action. That is, $D(\Lambda) = D(\mathrm{GR}) = md$ and $G = g$.*

See §9 for the proof.

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Remark 9. The traditional Pisot substitution conjecture is: if Φ is a one-dimensional substitution with irreducible and unimodular incidence matrix and Pisot inflation factor, then the $\mathbb{R}$-action on Ω_Φ has pure discrete spectrum. There are substitutions that satisfy these hypotheses and not those of Conjecture 5, and vice-versa. In the final section of this paper the conditions in Conjectures 3 and 5 are relaxed a bit to produce conjectures that do generalize the traditional Pisot substitution conjecture. The relaxation involves replacing $H^1(\Omega_\Phi; \mathbb{Z})$ by the potentially smaller $H^1_{\text{ess}}(\Omega_\Phi; \mathbb{Z})$.

Remark 10. Lee and Solomyak [**LS1**] show that if Φ is of (m, d)-Pisot family type, then the eigenvalues of the $\mathbb{R}^n$-action are relatively dense. The difficult part of their proof lies in demonstrating that the return vectors are contained in a $\mathbb{Z}$-module of rank $md = D(\Lambda)$. This is an assumption of Theorem 4 above. But the condition $D(\Lambda) = D(\text{GR})$ is easily verified in practice: it follows from Lemma 16 that if the multiplicity of λ as an eigenvalue of $f_* : H_1(X; \mathbb{Z}) \to H_1(X; \mathbb{Z})$ is the same as its multiplicity as an eigenvalue of Λ, for each $\lambda \in \text{spec}(\Lambda)$, then $D(\Lambda) = D(\text{GR})$. (Here $f : X \to X$ is the map on the collared Anderson–Putnam complex (A–P complex), see below.)

The cohomologies (and *essential* cohomologies—see §10) in the following examples are easily computed by the methods of [**BD1**].

Example 11. Let ϕ be the substitution on letters: $a \mapsto ab'$, $b \mapsto a$, $a' \mapsto a'b$, $b' \mapsto a'$. The corresponding one-dimensional substitution Φ is of (1, 2)-Pisot family type. The group $H^1_{\text{hyp}} = H^1_\Lambda \simeq \mathbb{Z}^2$, so $G' = G = g$. These maps are almost everywhere two-to-one and H^1_{hyp} is proper in $H^1_{\text{ess}}(\Omega_\Phi; \mathbb{Z}) \simeq \mathbb{Z}^4$. (Here $H^1(\Omega_\Phi; \mathbb{Z}) \simeq \mathbb{Z}^7$.)

Example 12. Let ϕ be the substitution on letters: $a \mapsto aba'$, $b \mapsto ab$, $a' \mapsto a'b'a$, $b' \mapsto a'b'$. The corresponding one-dimensional substitution Φ is of (1, 2)-Pisot family type so $G = g : \Omega_\Phi \to \mathbb{T}^2$ and these maps are almost everywhere two-to-one. The group $H^1(\Omega_\Phi; \mathbb{Z}) \simeq \mathbb{Z}^5$, but $H^1_{\text{hyp}} = H^1_{\text{ess}}(\Omega_\Phi; \mathbb{Z}) \simeq \mathbb{Z}^4$ and $G' : \Omega_\Phi \to \mathbb{T}^4$ is almost everywhere one-to-one.

3. *Abelian covers of Ω_Φ and global shadowing*

To simplify notation, let us fix a (primitive, FLC, non-periodic) n-dimensional substitution Φ and let $\Omega := \Omega_\Phi$. Let X be the A–P complex for Φ (see [**AP**]): X is a cell complex with one n-cell $\rho_i \times \{i\}$ for each prototile ρ_i, and these n-cells are glued along faces according to the following scheme. Suppose that ρ_i and ρ_j are prototiles and $u, v \in \mathbb{R}^n$, $T \in \Omega$, are such that $\rho_i + u, \rho_j + v \in T$. Set $\rho_i \times \{i\} \ni (x, i) \sim (y, j) \in \rho_j \times \{j\}$ if $x + u = y + v$ and extend $\sim$ to an equivalence relation on $\bigcup_i \rho_i \times \{i\}$. The A–P complex is the quotient $X := \bigcup_i \rho_i \times \{i\}/\sim$. There is a natural map $p : \Omega \to X$ given by $p(T) = [(x, i)]_\sim$ provided $0 = u + x \in u + \rho_i \in T$ and an induced map $f : X \to X$ with $p \circ \Phi = f \circ p$. The map $\hat{p} : \Omega \to \varprojlim f$ given by $\hat{p}(T) := (p(T), p(\Phi^{-1}(T)), \ldots)$ and the shift homeomorphism $\hat{f} : \varprojlim f \to \varprojlim f$ satisfy:

- $\hat{p} \circ \Phi = \hat{f} \circ p$; and
- $\hat{p}$ is an almost everywhere (with respect to μ on Ω) a one-to-one surjection.

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Moreover, when Φ has the property that it 'forces the border', which can be arranged by replacing the tiles of tilings in Ω by their collared versions, the map $\hat{p}$ is a homeomorphism [**AP**].

Let $\pi : \tilde{X} \to X$ be the universal cover of the A–P complex. The group of deck transformations of $\tilde{X}$ can be identified with the fundamental group of X. We form the *abelian cover* $\pi_{\mathrm{ab}} : \tilde{X}_{\mathrm{ab}} \to X$ by quotienting out the action of the commutator subgroup, C, of the fundamental group of X: $\tilde{X}_{\mathrm{ab}} := \tilde{X}/\sim_{\mathrm{ab}}$ with $\tilde{x} \sim_{\mathrm{ab}} \tilde{y}$ if and only if there is $\gamma \in C$ with $\gamma(\tilde{x}) = \tilde{y}$. Since the map $f_{\sharp}$ induced by f on the fundamental group of X takes C into C, f lifts to $\tilde{f}_{\mathrm{ab}} : \tilde{X}_{\mathrm{ab}} \to \tilde{X}_{\mathrm{ab}}$. The group of deck transformations of $\tilde{X}_{\mathrm{ab}}$ can be identified with the first homology $H_1(X; \mathbb{Z})$. If $h \in H_1(X; \mathbb{Z})$ and $\tilde{x} \in \tilde{X}_{\mathrm{ab}}$ we write $\tilde{x} + h$, or sometimes $h(\tilde{x})$, for the image of $\tilde{x}$ under the deck transformation corresponding to h, and we have $\tilde{f}_{\mathrm{ab}}(\tilde{x} + h) = \tilde{f}(\tilde{x}) + f_*(h)$ with f_* the homomorphism induced on $H_1(X; \mathbb{Z})$ by f.

Suppose that K is any subgroup of $H_1(X; \mathbb{Z})$. There is then a corresponding cover $\pi_K : \tilde{X}_K \to X$ where $\tilde{X}_K$ is the quotient of $\tilde{X}_{\mathrm{ab}}$ by the action of K. The group of deck transformations of $\tilde{X}_K$ is $H_1(X; \mathbb{Z})/K$ and if K is invariant under f_*, f lifts to $\tilde{f}_K : \tilde{X}_K \to \tilde{X}_K$, and $\tilde{f}_K(\tilde{x} + [h]) = \tilde{f}_K(\tilde{x}) + [f_*(h)]$, where $[h]$ denotes the coset $h + K$.

LEMMA 13. *Suppose that the subgroup K of $H_1(X; \mathbb{Z})$ is invariant under f_* and that $\bar{f}_*$ defined by $\bar{f}_*([h]) := [f_*(h)]$ is an isomorphism of $H_1(X; \mathbb{Z})/K$. Then the natural map $\hat{\pi}_K : \varprojlim \tilde{f}_K \to \varprojlim f$, given by $\hat{\pi}_K((\tilde{x})_i) := (\pi_K(\tilde{x}_i))$, is a covering map with group of deck transformations equal to $H_1(X; \mathbb{Z})/K$.*

Proof. The issue is surjectivity. Given $x, y \in X$ with $f(x) = y$, and $\tilde{y} \in \tilde{X}_K$ with $\pi_K(\tilde{y}) = y$, pick $\tilde{x}' \in \tilde{X}_K$ with $\pi_K(\tilde{x}') = x$. Then $\pi_K(\tilde{f}_K(\tilde{x}')) = y$ so there is $h \in H_1(X; \mathbb{Z})$ with $\tilde{f}_K(\tilde{x}') + [h] = \tilde{y}$. Let $\tilde{x} := \tilde{x}' + \bar{f}_*^{-1}([h])$. Then $\tilde{f}_K(\tilde{x}) = \tilde{y}$ and surjectivity of $\hat{\pi}_K$ follows. □

Let us give another description of the covering $\hat{\pi}_K : \varprojlim \tilde{f}_K \to \varprojlim f$. Let $\Omega = \Omega_\Phi$ and let $\tilde{\Omega}_K := \{(T, \tilde{x}) : p(T) = \pi_K(\tilde{x})\} \subset \Omega \times \tilde{X}_K$, with the product topology. Let $\pi_1 : \tilde{\Omega}_K \to \Omega$ and $\pi_2 : \tilde{\Omega}_K \to \tilde{X}_K$ be projections onto first and second factors.

LEMMA 14. *Suppose, as in Lemma 13, that K is invariant under f_* and that $\bar{f}_*$ is an isomorphism. Suppose also that X is the collared A–P complex of Φ. Then $\pi_1 : \tilde{\Omega}_K \to \Omega$ is isomorphic with $\hat{\pi}_K : \varprojlim \tilde{f}_K \to \varprojlim f$.*

Proof. Since X is collared, the map $\hat{p} : \Omega \to \varprojlim f$, $\hat{p}(T) := (p(T), p(\Phi^{-1}(T)), \ldots)$, is a homeomorphism. The continuous map $\tilde{\hat{p}}$ defined by $(T, \tilde{x}) \mapsto (\tilde{x}_i)$, where $\tilde{x}_i$ satisfies $\tilde{x}_0 = \tilde{x}$ and $\pi_K(\tilde{x}_i) = p(\Phi^{-i}(T))$, has continuous inverse $(\tilde{x}_i) \mapsto (\hat{p}^{-1}((\pi_K(\tilde{x}_i))), \tilde{x}_0)$. Moreover, $\hat{\pi}_K \circ \tilde{\hat{p}} = \hat{p} \circ \pi_1$. □

Under the hypotheses of Lemma 14 we have the following.

- We can lift the homeomorphism Φ on Ω to a homeomorphism $\tilde{\Phi}_K : \tilde{\Omega}_K \to \tilde{\Omega}_K$ defined by $\tilde{\Phi}_K((T, \tilde{x})) := (\Phi(T), \tilde{f}_K(\tilde{x}))$. This homeomorphism is conjugated with $\tilde{\hat{f}}_K$, the shift homeomorphism on $\varprojlim \tilde{f}_K$, by $\tilde{\hat{p}}$.

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- We may also lift the $\mathbb{R}^n$-action from Ω to $\tilde{\Omega}_K$ as follows. Given $T \in \Omega$, let $p^T : \mathbb{R}^n \to X$ be defined by $p^T(v) := p(T - v)$. Given $\tilde{x} \in \tilde{X}_K$ with $\pi_K(\tilde{x}) = p(T)$, let $\tilde{p}^T_{\tilde{x}} : \mathbb{R}^n \to \tilde{X}_K$ be the unique lift of p^T satisfying $\tilde{p}^T_{\tilde{x}}(0) = \tilde{x}$. For $\tilde{T} = (T, \tilde{x}) \in \tilde{\Omega}_K$ and $v \in \mathbb{R}^n$ define $\tilde{T} - v := (T - v, \tilde{p}^T_{\tilde{x}}(v))$. Note that $\tilde{\Phi}_K(\tilde{T} - v) = \tilde{\Phi}_K(\tilde{T}) - \Lambda v$.
- Finally, we equip $\tilde{\Omega}_K$ with a metric $\bar{d}$ as follows. First we select a metric $\tilde{d}$ on $\tilde{X}_K$ with two important properties. Suppose that $\{[h_i]\}_{i=1}^N$ is a basis for $H_1(X; \mathbb{Z})/K$. For $[h] = \sum_{i=1}^N b_i[h_i]$, let $|[h]| := \sum_{i=1}^N |b_i|$. We take $\tilde{d}$ so that:
  (1) $\tilde{d}(\tilde{x} + [h], \tilde{y} + [h]) = \tilde{d}(\tilde{x}, \tilde{y})$ for all $\tilde{x}, \tilde{y} \in \tilde{X}_K$, $[h] \in H_1(X; \mathbb{Z})/K$; and
  (2) $\tilde{d}(\tilde{x}, \tilde{y} + [h]) \to \infty$ as $|[h]| \to \infty$ for all $\tilde{x}, \tilde{y} \in \tilde{X}_K$.
  For the metric on $\tilde{\Omega}_K$ we set $\bar{d}((T, \tilde{x}), (S, \tilde{y})) := d(T, S) + \tilde{d}(\tilde{x}, \tilde{y})$.

For $T, S \in \Omega$, we say that T *globally shadows* S (with respect to K), and write $T \sim_{gsK} S$, if there are $\tilde{T} = (T, \tilde{x})$ and $\tilde{S} = (S, \tilde{y})$ in $\tilde{\Omega}_K$ so that $\{\bar{d}(\tilde{\Phi}^k_K(\tilde{T}), \tilde{\Phi}^k_K(\tilde{S}))\}_{k \in \mathbb{Z}}$ is bounded.

4. *Generalized return vectors*

A vector $v \in \mathbb{R}^n$ is called a *return vector* for Φ if there is $T \in \Omega_\Phi$ so that $p(T - v) = p(T)$. Let us call a vector $v \in \mathbb{R}^n$ a *generalized return vector* for Φ if there are $v_i \in \mathbb{R}^n$ and $T_i \in \Omega$, $i = 1, \ldots, k$, with $v = v_1 + \cdots + v_k$, so that $p(T_i - v_i) = p(T_{i+1})$ for $i = 1, \ldots, k - 1$, and $p(T_k - v_k) = p(T_1)$. The collection of generalized return vectors is a subgroup of $\mathbb{R}^n$ which we denote by $\mathrm{GR}(\Phi)$. If Φ is unimodular, one can show that $\mathrm{GR}(\Phi)$ equals the subgroup of $\mathbb{R}^n$ generated by the return vectors.

Any path $\alpha : [0, 1] \to X$ in the A–P complex X for Φ can be 'lifted' to a curve $\tilde{\alpha} : [0, 1] \to \mathbb{R}^n$ (think of unfolding the tiles that α runs through). Let $l(\alpha) := \tilde{\alpha}(1) - \tilde{\alpha}(0)$.

LEMMA 15. [**BSW**] *If α is a path in the A–P complex X for Φ, the vector $l(\alpha)$ is a well-defined (independent of lift) element of* $\mathrm{GR}(\Phi)$ *and depends only on the homotopy class (relative to endpoints) represented by α. Furthermore, if α is a loop, $l(\alpha)$ depends only on the homology class of α. The resulting function $l : H_1(X; \mathbb{Z}) \to \mathrm{GR}(\Phi)$ is a surjective group homomorphism and $l \circ f_* = \Lambda l$.*

Let K_Λ be the kernel of $l : H_1(X; \mathbb{Z}) \to \mathrm{GR}(\Phi)$. By Lemma 15, K_Λ is invariant under f_*; let $\bar{f}_* : H_1(X; \mathbb{Z})/K_\Lambda \to H_1(X; \mathbb{Z})/K_\Lambda$ be the homomorphism induced by f_*. The group $H_1(X; \mathbb{Z})/K_\Lambda \simeq \mathrm{GR}(\Phi)$ is a finitely generated ($H_1(X; \mathbb{Z})$ is finitely generated) free abelian group ($\mathrm{GR}(\Phi)$ is a subgroup of $\mathbb{R}^n$) so, with the choice of some basis, $\bar{f}_*$ is represented by an integral matrix A. The following lemma is an adaptation of a result in [**KS**].

LEMMA 16. *Let $\mathrm{spec}(\Lambda) = \bigcup_{i=1}^k \mathcal{F}_i$ be the partition of $\mathrm{spec}(\Lambda)$ into families of algebraically conjugate eigenvalues and let m_i be the maximum multiplicity of the elements of $\mathcal{F}_i$. Then λ is an eigenvalue of A of multiplicity m if and only if λ has a conjugate in $\mathcal{F}_i$ for some i. Moreover, $m \geq m_i$.*

Proof. Let $\{[h_1], \ldots, [h_k]\}$ be a basis for the $\mathbb{Z}$-module $H_1(X; \mathbb{Z})/K_\Lambda$ with respect to which $\bar{f}_*$ is represented by A and let $v_i := l(h_i) \in \mathrm{GR}(\Phi)$, $i = 1, \ldots, k$. Let $L : \mathbb{R}^k \to \mathbb{R}^n$ be the linear map that takes the standard basis vector e_i to v_i for each i. Then $LA = \Lambda L$

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and, since the return vectors span $\mathbb{R}^n$ (a consequence of minimality of the $\mathbb{R}^n$-action on Ω), L is surjective. Suppose that λ has a conjugate μ in $\mathcal{F}_i$ for some i and that μ is a real eigenvalue of Λ with multiplicity m_i. There is then a Λ^t-invariant m_i-dimensional subspace $V \subset \mathbb{R}^n$, restricted to which $(\Lambda^t - \mu I)^{m_i}$ is zero. Then $(A^t - \mu I)^{m_i}$ is zero on the m_i-dimensional subspace $L^t V$ of $\mathbb{R}^k$. Since L^t is injective, this means that μ is an eigenvalue of A of multiplicity at least m_i, and thus that λ is an eigenvalue with multiplicity at least m_i of A. We leave it to the reader to reach the same conclusion when λ is complex.

Now suppose that λ is an eigenvalue of A. Let $p(t) = r(t)q(t)$ be the characteristic polynomial of A factored with $r(t), q(t) \in \mathbb{Z}[t]$ so that all roots of $q(t)$, and no roots of $r(t)$, are algebraic conjugates of λ. Then $W := \ker(q(A))$ is a non-trivial subspace of $\mathbb{R}^k$, invariant under A, restricted to which all eigenvalues of A are conjugates of λ. Moreover, since q and A are integer, W has a basis $\{w_1, \ldots, w_d\}$ with $w_i \in \mathbb{Z}^k$ for each i. Let $w_i = \sum_{j=1}^{d} w_{ij} e_j$ with $w_{ij} \in \mathbb{Z}$. Then the elements $[h_i'] \in H_1(X; \mathbb{Z})/K_\Lambda$, $h_i' := \sum_{j=1}^{d} w_{ij} h_i$, span an $\bar{f}_*$-invariant submodule of $H_1(X; \mathbb{Z})/K_\Lambda$ of rank $d > 0$. Since $h_i' \notin K_\Lambda$, $l(h_i') \neq 0$ and it follows that $Lw_i \neq 0$. Thus $L(W)$ is a non-trivial subspace of $\mathbb{R}^n$, invariant under Λ. Applying the argument of the previous paragraph to $A|_W$, $L|_W$ and $\Lambda|_{L(W)}$, we conclude that Λ has an eigenvalue that is also an eigenvalue of $A|_W$, and is hence a conjugate of λ. □

As a corollary to Lemma 16, we have that the rank of the group of generalized return vectors is always at least as large as the degree of the linear inflation.

COROLLARY 17. $D(\mathrm{GR}) \geq D(\Lambda)$.

Question 18. It is a consequence of [**LS1**, Theorem 3.1] that, if Λ is diagonalizable over $\mathbb{C}$ and all of its eigenvalues are algebraic conjugates of the same multiplicity, then $D(\mathrm{GR}) = D(\Lambda)$. Is it ever the case that $D(\mathrm{GR}) > D(\Lambda)$?

We see from Lemma 16 that if Φ is unimodular and hyperbolic, then $\bar{f}_*$ is unimodular and hyperbolic on $H_1(X; \mathbb{Z})/K_\Lambda$. It is sometimes possible to increase the size of the quotient module while retaining the unimodularity and hyperbolicity of the corresponding quotient isomorphism (Example 12). Let T denote the torsion subgroup of $H_1(K; \mathbb{Z})$, let f_*' denote the induced homomorphism $f_*' : H_1(X; \mathbb{Z})/T \to H_1(X; \mathbb{Z})/T$ of the finitely generated free abelian group $H_1(X; \mathbb{Z})/T$, and let A be the matrix representing f_*' in some basis. We may then factor the characteristic polynomial $p(t)$ of A, over $\mathbb{Z}$, as $p(t) = q(t)r(t)$ so that all roots of $q(t)$ are algebraic units and none have modulus 1; and all roots of $r(t)$ are not units, or have a conjugate of modulus 1. Let $F_0 := \mathrm{Ker}(r(f_*'))$ and let $K_{\mathrm{hyp}} := \pi^{-1}(F_0) \subset H_1(X; \mathbb{Z})$, where $\pi : H_1(X; \mathbb{Z}) \to H_1(X; \mathbb{Z})/T$ is the quotient homomorphism. Then $K_{\mathrm{hyp}} \subset K_\Lambda$ is invariant under f_*, is independent of the choice of basis, and is minimal with respect to the property that the induced isomorphism $\bar{f}_* : H_1(X; \mathbb{Z})/K_{\mathrm{hyp}} \to H_1(X; \mathbb{Z})/K_{\mathrm{hyp}}$ is unimodular and hyperbolic.

The group $H^1(X; \mathbb{Z})$ is (naturally) isomorphic with $\mathrm{Hom}(H_1(X; \mathbb{Z}), \mathbb{Z})$ so the Čech cohomology $H^1(\varprojlim f; \mathbb{Z}) = \varinjlim f^* : H^1(X; \mathbb{Z}) \to H^1(X; \mathbb{Z})$ is isomorphic with the direct limit of the dual of f_*. Let $H^1_{\mathrm{hyp}} := \{c \in H^1(X; \mathbb{Z}) : c(h) = 0 \text{ for all } h \in K_{\mathrm{hyp}}\}$. Then $f^*|_{H^1_{\mathrm{hyp}}} : H^1_{\mathrm{hyp}} \to H^1_{\mathrm{hyp}}$ is an isomorphism so that $H^1_{\mathrm{hyp}} \simeq \varinjlim f^*|_{H^1_{\mathrm{hyp}}}$. By means

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of $\hat{p}^* : \varinjlim f^* \to H^1(\Omega_\Phi; \mathbb{Z})$, H^1_{hyp} can be viewed as a subgroup of $H^1(\Omega_\Phi; \mathbb{Z})$; if X is collared (so that $\hat{p}^*$ is an isomorphism), $H^1_{\text{hyp}} \subset H^1(\Omega_\Phi; \mathbb{Z})$ is the largest subgroup of $H^1(\Omega_\Phi; \mathbb{Z})$ that is invariant under Φ^*, contains H^1_Λ, and on which Φ^* is unimodular and hyperbolic.

5. *From substitutions to toral automorphisms*

The first cohomology group $H^1(X; \mathbb{Z})$ is naturally isomorphic with the *Bruschlinski group* consisting of homotopy classes of maps from X to the additive circle group $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ (that is, $\mathbb{T}$ is a $K(\mathbb{Z}, 1)$). An explicit isomorphism is given by $\Theta([\gamma]) = \gamma^*(\mathbf{1})$, where $\mathbf{1}$ is the fundamental class of $H^1(\mathbb{T}; \mathbb{Z})$ and $\gamma^*(\mathbf{1})$ is its pullback to $H^1(X; \mathbb{Z})$. Let us fix a subgroup K of $H_1(X; \mathbb{Z})$, invariant under f_*, and define $H^1_K := \{c \in \text{Hom}(H_1(X; \mathbb{Z}), \mathbb{Z}) : c(h) = 0 \text{ for all } h \in K\} \subset \text{Hom}(H_1(X; \mathbb{Z}), \mathbb{Z}) \simeq H^1(X; \mathbb{Z})$. Let $\{c_1, \ldots, c_N\}$ be a basis for H^1_K and, for each $i = 1, \ldots, N$, choose a map γ_i with $[\gamma_i] := \Theta^{-1}(c_i)$. Then, for each i, $\Theta([\gamma_i]) = (\gamma_i^*)^{-1}(\mathbf{1}) = c_i$ annihilates K.

For the following proposition, let $A = (a_{i,j})$ denote the transpose of the matrix representing the homomorphism induced on H^1_K by f with respect to the basis $\{c_1, \ldots, c_N\}$. Thus $\gamma_j \circ f = \sum_{i=1}^N a_{j,i}\gamma_i$, up to homotopy. We also assume that X is collared so that $\varprojlim f \simeq \Omega_\Phi$ (although the proposition would still be true in general, replacing Ω_Φ by $\varprojlim f$ and making the appropriate adjustments). Let $F_A : \mathbb{T}^N \to \mathbb{T}^N$ be the linear torus map $F_A(x + \mathbb{Z}^N) := Ax + \mathbb{Z}^N$, let $\Gamma : X \to \mathbb{T}^N$ be given by $\Gamma(x) := (\gamma_1(x), \ldots, \gamma_N(x))^t$, and let $G_0 := \Gamma \circ p : \Omega_\Phi \to \mathbb{T}^N$.

PROPOSITION 19. *Suppose that Φ is unimodular and hyperbolic, and that K is a subgroup of $H_1(X; \mathbb{Z})$ that is invariant under f_* and lies between K_{hyp} and K_Λ. Then the map F_A is a hyperbolic toral automorphism and there exists a continuous map $G : \Omega_\Phi \to \mathbb{T}^N$ with the properties:*

(i) $G^* = G_0^* : H^1(\mathbb{T}^N; \mathbb{Z}) \to H^1(\Omega_\Phi; \mathbb{Z})$;

(ii) $G \circ \Phi = F_A \circ G$;

(iii) $G(T) = G(S)$ *if and only if* $T \sim_{gsK} S$.

For the following lemma, let $\tilde{\Omega} \to \Omega$ be a covering map with group of deck transformations H.

LEMMA 20. *Suppose that $G_i : \Omega \to \mathbb{T}^N$, $i = 1, 2$, are continuous maps with lifts $\tilde{G}_i : \tilde{\Omega} \to \mathbb{R}^N$. If $|\tilde{G}_1 - \tilde{G}_2|$ is bounded, then $G_1^* = G_2^* : H^1(\mathbb{T}^N; \mathbb{Z}) \to H^1(\Omega; \mathbb{Z})$.*

Proof. There are homomorphisms $\alpha_i : H \to \mathbb{Z}^N$ so that $\tilde{G}_i(\tilde{x} + h) = \tilde{G}_i(\tilde{x}) + \alpha_i(h)$ for all $h \in H$ and $i = 1, 2$. Clearly, $|\tilde{G}_1 - \tilde{G}_2|$ bounded implies that $\alpha_1 = \alpha_2 =: \alpha$. Suppose that $\gamma : \mathbb{T}^N \to \mathbb{T}$ is continuous. Let $\tilde{\gamma} : \mathbb{R}^N \to \mathbb{R}$ be a lift. Define

$$\tilde{H}(\tilde{x}, t) := t(\tilde{\gamma} \circ \tilde{G}_1(\tilde{x})) + (1 - t)(\tilde{\gamma} \circ \tilde{G}_2(\tilde{x})).$$

Then

$$\tilde{H}(\tilde{x} + h, t) = \cdots = \tilde{H}(\tilde{x}, t) + \alpha(h),$$

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so that $\tilde{H}$ descends to a homotopy from $\gamma \circ G_2$ to $\gamma \circ G_1$. Thus $G_1^*(\gamma^*(\mathbf{1})) = G_2^*(\gamma^*(\mathbf{1}))$. That is, $G_1^* \circ \Theta = G_2^* \circ \Theta$, so $G_1^* = G_2^*$, as $\Theta : [\mathbb{T}^N, \mathbb{T}] \to H^1(\mathbb{T}^N; \mathbb{Z})$ is an isomorphism. □

Proof of Proposition 19. Suppose that α is a loop in $\tilde{X}_K$. Then $\pi_K \circ \alpha$ is a loop that represents a homology class $[\pi_K \circ \alpha]$ in K and $c_i([\pi_K \circ \alpha]) = 0$. That is,

$$(\gamma_i^*)^{-1}(\mathbf{1})(([\pi_K \circ \alpha])) = \mathbf{1}((\gamma_i)_*([\pi_K \circ \alpha])) = \mathbf{1}([\gamma_i \circ \pi_K \circ \alpha]) = 0.$$

This means that the loop $\gamma_i \circ \pi_K \circ \alpha$ in $\mathbb{T}$ is null homotopic and it follows that the map $\gamma_i : X \to \mathbb{T}$ lifts to $\tilde{\gamma}_i : \tilde{X}_K \to \mathbb{R}$. Recall that $\Gamma : X \to \mathbb{T}^N$ is given by $\Gamma(x) := (\gamma_1(x), \ldots, \gamma_N(x))^t$; then

$$\tilde{\Gamma} := (\tilde{\gamma}_1, \ldots, \tilde{\gamma}_N)^t : \tilde{X}_K \to \mathbb{R}^N$$

is a lift of Γ and $\tilde{\Gamma}_0 := \tilde{\Gamma} \circ \pi_0 : \varprojlim \tilde{f}_K \to \mathbb{R}^N$ is a lift of $G_0 = \Gamma \circ p$ (here π_0 is projection onto the zeroth coordinate and we have identified $\tilde{\Omega}_K$ with $\varprojlim \tilde{f}_K$ via Lemma 14).

We know from Lemma 16 that the integer matrix A is unimodular and hyperbolic and thus the linear torus map F_A is a hyperbolic toral automorphism. The homomorphism Γ_* takes K to 0 and the induced homomorphism $\bar{\Gamma}_* : H_1(X; \mathbb{Z})/K \to H_1(\mathbb{T}^N; \mathbb{Z})$ is an isomorphism. Moreover, the choice of A guarantees that $\bar{\Gamma}_* \circ \bar{f}_* = (F_A)_* \circ \bar{\Gamma}_*$. Since $\tilde{\Gamma}$ is a lift of Γ, we have

$$\tilde{\Gamma}(\tilde{x} + [h]) = \tilde{\Gamma}(\tilde{x}) + \bar{\Gamma}_*([h]) \quad \text{for } \tilde{x} \in \tilde{X}_K \text{ and } [h] \in H_1(X; \mathbb{Z})/K.$$

Thus $A\bar{\Gamma}_* = \bar{\Gamma}_* \circ \bar{f}_*$ (we have identified $\mathbb{Z}^N \subset \mathbb{R}^N$ with $H_1(\mathbb{T}^N; \mathbb{Z})$). We see then that

$$A\tilde{\Gamma}_0((\tilde{x}_i) + [h]) = A(\tilde{\Gamma}_0((\tilde{x}_i)) + \bar{\Gamma}_*([h])) = A\tilde{\Gamma}_0((\tilde{x}_i)) + \bar{\Gamma}_* \circ \bar{f}_*([h]),$$

while

$$\tilde{\Gamma}_0 \circ \hat{\tilde{f}}_K((\tilde{x}_i) + [h]) = \tilde{\Gamma}_0(\hat{\tilde{f}}_K((\tilde{x}_i) + \bar{f}_*([h]))) = \tilde{\Gamma}_0 \circ \hat{\tilde{f}}_K((\tilde{x}_i)) + \bar{\Gamma}_* \circ \bar{f}_*([h]).$$

It follows that $|A\tilde{\Gamma}_0 - \tilde{\Gamma}_0 \circ \hat{\tilde{f}}_K|$ is uniformly bounded on $\varprojlim \tilde{f}_K$ (by its bound on a single fundamental domain).

Let E^s and E^u denote the stable and unstable linear subspaces of $\mathbb{R}^N$ under application of A. Then $\mathbb{R}^N = E^s \oplus E^u$, and E^s and E^u are invariant under A. For each $z \in \mathbb{R}^N$, let $z^s \in E^s$ and $z^u \in E^u$ be so that $z = z^s + z^u$. There are C and η, $0 < \eta < 1$, with $||A^k z^s|| \le C\eta^k ||z^s||$ and $||A^{-k} z^u|| \le C\eta^{-k} ||z^u||$ for all $z \in \mathbb{R}^N$ and $k \in \mathbb{N}$. Now, given $(\tilde{x}_i) \in \varprojlim \tilde{f}_K$ and $k \in \mathbb{Z}$, let $y_k := \tilde{\Gamma}_0(\tilde{f}_K^k((\tilde{x}_i)))$. By the above, $b_k := y_{k+1} - Ay_k$ is bounded. For $k \in \mathbb{N}$ we have

$$A^{-k} y_k = y_0 + \sum_{i=1}^{k} A^{-i} b_{i-1} \quad \text{and} \quad A^k y_{-k} = y_0 - \sum_{i=0}^{k-1} A^i b_i.$$

Define $z = z((\tilde{x}_i))$ by $z = z^u + z^s$ where

$$z^u := \lim_{k \to \infty} (A^{-k} y_k)^u = y_0^u + \sum_{i=1}^{\infty} A^{-k} b_{k-1}^u$$

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and

$$z^s := \lim_{k\to\infty}(A^k y_{-k})^s = y_0^s - \sum_{i=0}^{\infty} A^k b^s_{-k}.$$

It is clear that z depends continuously on $(\tilde{x}_i)$, that $z((\tilde{x}_i)+[h]) = z((\tilde{x}_i)) + \bar{\Gamma}_*([h])$, and that $|\tilde{\Gamma}_0((\tilde{x}_i)) - z((\tilde{x}_i))|$ is bounded. Thus the map $\tilde{\Gamma}' : \varprojlim \tilde{f}_K \to \mathbb{R}^N$ given by $\tilde{\Gamma}'((\tilde{x}_i)) := z((\tilde{x}_i))$ is the lift of a continuous map $\Gamma' : \varprojlim f \to \mathbb{T}^N$. Let $\tilde{\hat{p}} : \tilde{\Omega}_K \to \varprojlim \tilde{f}_K$ be the isomorphism of Lemma 14, let $\tilde{G} := \tilde{\Gamma}' \circ \tilde{\hat{p}} : \tilde{\Omega}_K \to \mathbb{R}^N$ and let $G := \Gamma' \circ \hat{p} : \Omega \to \mathbb{T}^N$. By Lemma 20, $G^* = G_0^*$. Furthermore,

$$\begin{aligned}(\tilde{\Gamma}'(\hat{\tilde{f}}_K((\tilde{x}_i))))^u &= (\tilde{\Gamma}'((\tilde{x}_{i+1})))^u = \lim_{k\to\infty}(A^{-k} y_{k+1})^u \\ &= A \lim_{k\to\infty}(A^{-(k+1)} y_{k+1})^u = A(\tilde{\Gamma}'((\tilde{x}_i)))^u.\end{aligned}$$

Similarly,

$$(\tilde{\Gamma}'(\hat{\tilde{f}}_K((\tilde{x}_i))))^s = A(\tilde{\Gamma}'((\tilde{x}_i))))^s,$$

so that $\tilde{\Gamma}' \circ \hat{\tilde{f}}_K = A\tilde{\Gamma}'$, whence $\Gamma' \circ \hat{f} = F_A \circ \Gamma'$. Since $\hat{f} \circ \hat{p} = \hat{p} \circ \Phi$, we have $G \circ \Phi = F_A \circ G$.

It remains to prove (iii). First suppose that $T \sim_{gsK} S$. Let $\tilde{T}$ and $\tilde{S}$, lying over T and S, have the property that $\bar{d}(\tilde{\Phi}_K^k(\tilde{T}), \tilde{\Phi}_K^k(\tilde{S}))$ is bounded for $k \in \mathbb{Z}$. As $\tilde{G}$ is a lift of a continuous function on a compact space, and the metric $\bar{d}$ is equivariant, $\tilde{G}$ is uniformly continuous and hence

$$|\tilde{G}(\tilde{\Phi}_K^k(\tilde{T})) - \tilde{G}(\tilde{\Phi}_K^k(\tilde{S}))| = |A^k(\tilde{G}(\tilde{T}) - \tilde{G}(\tilde{S}))|$$

is also bounded. Since A is hyperbolic, this can only happen if $\tilde{G}(\tilde{T}) = \tilde{G}(\tilde{S})$. Thus $G(T) = G(S)$.

Conversely, if $G(T) = G(S)$, let $\tilde{T}$ and $\tilde{S}$ lie over T and S. There is then $h \in \mathbb{Z}^N = H_1(\mathbb{T}^N; \mathbb{Z})$ so that $\tilde{G}(\tilde{T}) + h = \tilde{G}(\tilde{S})$ and there is $h' \in H_1(X; \mathbb{Z})/K$ so that $\bar{\Gamma}_*(h') = h$. Let $\tilde{T}' := \tilde{T} + h'$. Then $\tilde{T}'$ also lies over T and $\tilde{G}(\tilde{T}') = \tilde{G}(\tilde{S})$. From $\tilde{G} \circ \tilde{\Phi}_K = A\tilde{G}$ it follows that

$$\tilde{G}(\tilde{\Phi}_K^k(\tilde{T}')) = \tilde{G}(\tilde{\Phi}_K^k(\tilde{S})) \quad \text{for all } k \in \mathbb{Z}.$$

Were $\bar{d}(\tilde{\Phi}_K^k(\tilde{T}'), \tilde{\Phi}_K^k(\tilde{S}))$ not bounded, there would be $k_j \in \mathbb{Z}$ and $h_j \in H_1(X; \mathbb{Z})/K$ with $\bar{d}(\tilde{\Phi}_K^{k_j}(\tilde{T}') + h_j, \tilde{\Phi}_K^{k_j}(\tilde{S}))$ bounded and $|h_j| \to \infty$ (see the second of the assumptions on the nature of $\tilde{d}$), hence $|\bar{\Gamma}_*(h_j)| \to \infty$. But (by uniform continuity of $\tilde{G}$ and equivariance of $\bar{d}$),

$$|\tilde{G}(\tilde{\Phi}_K^{k_j}(\tilde{T}') + h_j) - \tilde{G}(\tilde{\Phi}_K^{k_j}(\tilde{S}))| = |\bar{\Gamma}_*(h_j)|$$

is bounded. Thus it must be the case that $T \sim_{gsK} S$. □

6. *Geometric realization*

Proof of Theorems 1 and 2. Let $K = K_\Lambda$ for Theorem 1 or $K = K_{\text{hyp}}$ for Theorem 2 and let A, G, N be as in Proposition 19, with $N = D$ or $N = D'$, depending on whether $K = K_\Lambda$ or $K = K_{\text{hyp}}$. Both $(\mathbb{T}^N, F_A)$ and (Ω, Φ) are Smale spaces—the actions are

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hyperbolic with local product structure (see [**AP**]). A lemma of Putnam [**P**] asserts that a factor map (such as G) between Smale spaces that is injective on unstable manifolds is globally finite-to-one.

Thus we are reduced to proving that G is injective on unstable manifolds. Given $T \in \Omega$, the unstable manifold of T under Φ is the set $W^u(T) := \{T' \in \Omega : d(\Phi^k(T), \Phi^k(T')) \to 0$ as $k \to -\infty\}$. It is easy to see that $W^u(T) = \{T - v : v \in \mathbb{R}^n\}$.

Suppose that $G(T - v) = G(T)$ for some $T \in \Omega$ and $0 \neq v \in \mathbb{R}^n$. According to Proposition 19, $T \sim_{gsK} T - v$. Let us first show that the lifts $\tilde{T} = (T, \tilde{x})$ and $\tilde{T} - v = (T - v, \tilde{p}^T_{\tilde{x}}(v))$ of T and $T - v$ to $\tilde{\Omega}_K$ are such that $\bar{d}(\tilde{\Phi}^k_K(\tilde{T}), \tilde{\Phi}^k_K(\tilde{T} - v)) \to \infty$ as $k \to \infty$. Note that

$$\pi_2(\tilde{\Phi}^k_K(\tilde{T})) = \tilde{p}^{\Phi^k(T)}_{\tilde{f}^k_K(\tilde{x})}(0) \quad \text{and} \quad \pi_2(\tilde{\Phi}^k_K(\tilde{T} - v)) = \tilde{p}^{\Phi^k(T)}_{\tilde{f}^k_K(\tilde{x})}(\Lambda^k v).$$

We may choose return vectors v_k so that $|v_k - \Lambda^k v|$ is bounded and $B_r[\Phi^k(T)] = B_r[\Phi^k(T) - v_k]$ with r twice the maximum diameter of all tiles (recall that $B_r[T]$ denotes the collection of all tiles in T that meet the closed ball centered at 0 with radius r). Then $p(\Phi^k(T)) = p(\Phi^k(T) + v_k)$ in the collared A–P complex X. There is $[h_k] \in H_1(X; \mathbb{Z})/K$ with $l(h_k) = v_k$. Note that there are infinitely many distinct such v_k, and hence infinitely many distinct $[h_k]$ for $k \in \mathbb{N}$. It follows that

$$\tilde{d}(p^{\Phi^k(T)}_{\tilde{f}^k_K(\tilde{x})}(0), p^{\Phi^k(T)}_{\tilde{f}^k_K(\tilde{x})}(0) + [h_k]) = \tilde{d}(p^{\Phi^k(T)}_{\tilde{f}^k_K(\tilde{x})}(0), p^{\Phi^k(T)}_{\tilde{f}^k_K(\tilde{x})}(v_k))$$

is unbounded for $k \in \mathbb{N}$. Hence

$$\begin{aligned}\tilde{d}(p^{\Phi^k(T)}_{\tilde{f}^k_K(\tilde{x})}(0), p^{\Phi^k(T)}_{\tilde{f}^k_K(\tilde{x})}(\Lambda^k v)) &\geq \tilde{d}(p^{\Phi^k(T)}_{\tilde{f}^k_K(\tilde{x})}(0), p^{\Phi^k(T)}_{\tilde{f}^k_K(\tilde{x})}(v_k)) - \tilde{d}(p^{\Phi^k(T)}_{\tilde{f}^k_K(\tilde{x})}(v_k), p^{\Phi^k(T)}_{\tilde{f}^k_K(\tilde{x})}(\Lambda^k v)) \\ &\geq \tilde{d}(p^{\Phi^k(T)}_{\tilde{f}^k_K(\tilde{x})}(0), p^{\Phi^k(T)}_{\tilde{f}^k_K(\tilde{x})}(v_k)) - B,\end{aligned}$$

for some finite B independent of k, is also unbounded for $k \in \mathbb{N}$.

Suppose now $\tilde{T}$ and $\tilde{T}'$ are lifts of T and $T - v$ so that $\bar{d}(\tilde{\Phi}^k_K(\tilde{T}), \tilde{\Phi}^k_K(\tilde{T}'))$ is bounded for $k \in \mathbb{Z}$ and $\tilde{T}' \neq \tilde{T} - v$. Since $\bar{d}(\tilde{\Phi}^k_K(\tilde{T}), \tilde{\Phi}^k_K(\tilde{T} - v)) \to 0$ as $k \to -\infty$, $\bar{d}(\tilde{\Phi}^k_K(\tilde{T}'), \tilde{\Phi}^k_K(\tilde{T} - v))$ must be bounded as $k \to -\infty$. But $\tilde{T}'$ and $\tilde{T} - v$ are not equal and both lie over $T - v$ so there is a non-zero $[h] \in H_1(X; \mathbb{Z})/K$ with $\tilde{T}' + [h] = \tilde{T} - v$. Then the distance between

$$\tilde{\Phi}^k_K(\tilde{T} - v) = \tilde{\Phi}^k_K(\tilde{T}') + \bar{f}^k_*([h]) \quad \text{and} \quad \tilde{\Phi}^k_K(\tilde{T}')$$

is certainly unbounded as $k \to -\infty$ since $\bar{f}^k_*([h])$, by hyperbolicity, takes on infinitely many different values in $H_1(X; \mathbb{Z})/K$. This establishes that G is one-to-one on unstable manifolds and hence G is globally finite-to-one.

That G is μ-almost everywhere r-to-one (or r'-to-one) is a consequence of the ergodicity of Φ with respect to the unique invariant probability measure μ of the $\mathbb{R}^n$-action and the measurability of the function $f : \Omega \to \mathbb{R}$ given by $f(T) := \sharp G^{-1}(G(T))$. □

7. *Non-triviality of the global shadowing relation*

To get some idea of what tilings are $\sim_{gsK}$-related, consider $K = K_\Lambda$.

PROPOSITION 21. *Suppose that Φ is unimodular and hyperbolic and that $T, T' \in \Omega$ are Φ-periodic and share a tile. Then $T \sim_{gsK_\Lambda} T'$.*

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Proof. For simplicity, assume that T and T' are fixed by Φ. Let $K = K_\Lambda$ and suppose that $\tau \in T \cap T'$. Let $v \in \text{int}(\tau)$, so that $B_0[T - v] = B_0[T' - v]$. Choose $\tilde{T} = (T, \tilde{x}) \in \tilde{\Omega}_K$ to be fixed by $\tilde{\Phi}_K$. There is then $\tilde{x}' \in \tilde{X}_K$ so that $\pi_2(\tilde{T}' - v) = \pi_2(\tilde{T} - v)$, where $\tilde{T}' = (T', \tilde{x}')$. Consider the loop α in the A–P complex X:

$$\alpha(t) = \begin{cases} p^T(2t\Lambda v + (1-2t)v) & \text{if } 0 \le t \le 1/2, \\ p^{T'}(2t\Lambda v + (1-2t)v) & \text{if } 1/2 \le t \le 1. \end{cases}$$

Then $l(\alpha) = 0$ so the homology class of α is in K and α lifts to a loop $\tilde{\alpha}$ in $\tilde{X}_K$ with $\tilde{\alpha}(0) = \pi_2(\tilde{T} - v) = \tilde{\alpha}(1)$. The lift $\tilde{\alpha}$ is of the form:

$$\tilde{\alpha}(t) = \begin{cases} \tilde{p}^T_{\tilde{x}}(2t\Lambda v + (1-2t)v) & \text{if } 0 \le t \le 1/2, \\ \tilde{p}^{T'}_{\tilde{y}}(2t\Lambda v + (1-2t)v) & \text{if } 1/2 \le t \le 1, \end{cases}$$

where $\tilde{y}$ is determined by continuity. Since

$$\tilde{\alpha}(1) = \tilde{p}^{T'}_{\tilde{y}}(v) = \tilde{\alpha}(0) = \tilde{p}^T_{\tilde{x}}(v) = \tilde{p}^{T'}_{\tilde{x}'}(v),$$

it must be the case that $\tilde{y} = \tilde{x}'$. Now

$$\begin{aligned} \pi_2(\tilde{T} - v) = \pi_2(\tilde{T}' - v) \Rightarrow \pi_2(\tilde{T} - \Lambda v) &= \pi_2(\tilde{\Phi}_K(\tilde{T} - v)) = \pi_2(\tilde{\Phi}_K(\tilde{T}' - v)) \\ &= \pi_2(\tilde{\Phi}_K(\tilde{T}') - \Lambda v). \end{aligned}$$

We have

$$\pi_2(\tilde{T}' - \Lambda v) = \tilde{p}^{T'}_{\tilde{x}'}(\Lambda v) = \tilde{\alpha}(1/2) = \pi_2(\tilde{T} - \Lambda v) = \pi_2(\tilde{\Phi}_K(\tilde{T}') - \Lambda v)$$

and hence $\tilde{\Phi}_K(\tilde{T}') = \tilde{T}'$. Since $\tilde{T}$ and $\tilde{T}'$ are both fixed by $\tilde{\Phi}_K$, $T \sim_{gsK} T'$. □

It is proved in [**BO**] that, if Φ is an n-dimensional self-similar substitution (that is, $\Lambda = \lambda I$), there are Φ-periodic $T \neq T' \in \Omega$ that agree in a half-space: there is $0 \neq u \in \mathbb{R}^n$ and R so that $B_0[T - v] = B_0[T' - v]$ for all $v \in \mathbb{R}^n$ with $\langle u, v\rangle \ge R$. For such T, T', $\{T, T'\}$ is called an *asymptotic pair*.

COROLLARY 22. *Suppose that Φ is self-similar. There are then Φ-periodic $T \neq T' \in \Omega$ and $0 \neq u \in \mathbb{R}^n$ so that $T \sim_{gsK_\Lambda} T'$ and $T - v \sim_{gsK_\Lambda} T' - v$ for all $v \in \mathbb{R}^n$ with $\langle u, v\rangle > 0$.*

Proof. Let T, T' be an asymptotic pair with direction vector u, as above. Then $T \sim_{gsK_\Lambda} T'$ by Proposition 21. There are then lifts $\tilde{T}$, $\tilde{T}'$ that are both fixed by $\tilde{\Phi}^m_K$ for some $m > 0$. Then $\tilde{T} - v$ and $\tilde{T}' - v$ are lifts of $T - v$ and $T' - v$ with the properties: $\tilde{\Phi}^{km}_K(\tilde{T} - v) \to \tilde{T}$ and $\tilde{\Phi}^{km}_K(\tilde{T}' - v) \to \tilde{T}'$ as $k \to -\infty$. Also, if $\langle u, v\rangle > 0$, there is k' so that $\langle u, \lambda^{k'} v\rangle \ge R$, and it follows that $d(\Phi^{mk}(T - v), \Phi^{mk}(T' - v)) \to 0$ as $k \to \infty$. We saw in the proof of Proposition 21 that if $\tilde{T}$ is fixed by $\tilde{\Phi}^m_K$, T' is fixed by Φ^m, and $\pi_2(\tilde{T} - w) = \pi_2(\tilde{T}' - w)$, then $\tilde{T}'$ is also fixed by $\tilde{\Phi}^m_K$. It follows from hyperbolicity of $\tilde{\Phi}_K$ on deck transformations that, conversely, if $\tilde{T}$ and $\tilde{T}'$ are both fixed by $\tilde{\Phi}_K$ and $B_0[T - w] = B_0[T' - w]$ (so that $p^T(w) = p^{T'}(w)$) then $\pi_2(\tilde{T} - w) = \pi_2(\tilde{T}' - w)$. If $\langle u, v\rangle > 0$, then $B_0[T - w] = B_0[T' - w]$ for $w = \lambda^k v$ and $k > k'$. Consequently, for

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such v,

$$\bar{d}(\tilde{\Phi}_K^{mk}(\tilde{T}-v), \tilde{\Phi}_K^{mk}(\tilde{T}'-v)) = d(\Phi^{mk}(T-v), \Phi^{mk}(T'-v)) + \tilde{d}(\pi_2(\tilde{T}-\lambda^{mk}v), \pi_2(\tilde{T}'-\lambda^{mk}v)) \to 0 \quad \text{as } k \to \infty,$$

so $T - v \sim_{gsK_\Lambda} T' - v$. □

8. *Global shadowing and regional proximality for Pisot family substitutions*

A fundamental result of Veech ([**V**], see also [**A**]) asserts that the structure relation of the maximal equicontinuous factor for an abelian group action on a compact metrizable space is given by regional proximality. In the context of the $\mathbb{R}^n$-action on a tiling space Ω, two tilings $T, T' \in \Omega$ are *regionally proximal* provided, for every $\epsilon > 0$, there are $S, S' \in \Omega$ and $v \in \mathbb{R}^n$ so that:

(i) $d(T, S) < \epsilon$;

(ii) $d(T', S') < \epsilon$; and

(iii) $d(S - v, S' - v) < \epsilon$.

If T and T' are regionally proximal, we will write $T \sim_{rp} T'$. Thus $T \sim_{rp} T'$ if and only if $g(T) = g(T')$, where g is the factor map onto the maximal equicontinuous factor of the translation action on Ω.

We thus have two closed equivalence relations on a substitution tiling space Ω_Φ: regional proximality and global shadowing. The aim of this section is to compare these two relations. The following Proposition 24 shows that for unimodular hyperbolic substitutions, global shadowing is stronger than regional proximality. Proposition 25 establishes that if the substitution is Pisot family, and $D = D(\Lambda) = D(\mathrm{GR})$, then the regional proximal and global shadowing relations coincide. From this we will easily deduce Theorem 4, and Corollaries 6 and 7.

First, a lemma about regional proximality.

LEMMA 23. *Suppose that $T, S \in \Omega_\Phi$, $k_i \to \infty$, and $v_i \in \mathbb{R}^n$ are such that $\Phi^{k_i}(T) \to \bar{T} \in \Omega_\Phi$, $\Phi^{k_i}(S) \to \bar{S} \in \Omega_\Phi$, and $d(\Phi^{k_i}(T - v_i), \Phi^{k_i}(S - v_i)) \to 0$. Then $\bar{T} \sim_{rp} \bar{S}$.*

Proof. Let g be the map of Ω_Φ onto the maximal equicontinuous factor $\Omega_\Phi/\sim_{rp}$. If

$$d(\Phi^{k_i}(T - v_i), \Phi^{k_i}(S - v_i)) = d(\Phi^{k_i}(T) - \Lambda^{k_i}v_i, \Phi^{k_i}(S) - \Lambda^{k_i}v_i) \to 0,$$

then

$$d(g(\Phi^{k_i}(T)) - \Lambda^{k_i}v_i, g(\Phi^{k_i}(S)) - \Lambda^{k_i}v_i) \to 0$$

by uniform continuity of g. Then, by equicontinuity of the $\mathbb{R}^n$-action on $\Omega_\Phi/\sim_{rp}$, $d(g(\Phi^{k_i}(T)), g(\Phi^{k_i}(S))) \to 0$. Thus $d(g(\bar{T}), g(\bar{S})) = 0$ and $\bar{T} \sim_{rp} \bar{S}$. □

Given $\tilde{x}, \tilde{x}' \in \tilde{X} = \tilde{X}_{K_\Lambda}$ and a path $\tilde{\gamma}$ in $\tilde{X}$ from $\tilde{x}$ to $\tilde{x}'$, let $l(\tilde{\gamma}) := l(\gamma)$, where γ is the path $\gamma = \pi \circ \tilde{\gamma}$ in X (see Lemma 15). If $\tilde{\gamma}'$ is any other such path, then the concatenation of $\tilde{\gamma}'$ with the reverse of $\tilde{\gamma}$ is a loop in $\tilde{X}$ and hence must project to an element of K_Λ, that is, to a loop of displacement 0. Thus the displacement $l(\tilde{\gamma})$ depends only on $\tilde{x}$ and $\tilde{x}'$, and not on $\tilde{\gamma}$.

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PROPOSITION 24. *Suppose that Φ is unimodular and hyperbolic. Then the global shadowing relation is contained in the regional proximality relation: $T \sim_{gsK_\Lambda} T' \implies T \sim_{rp} T'$.*

Proof. We suppose that the A–P complex X is collared. In this situation, if $S, S' \in \Omega_\Phi$ are such that $p(S) = p(S')$, then $d(\Phi^k(S), \Phi^k(S')) \to 0$ as $k \to \infty$.

Suppose that $\tilde{T} = (T, \tilde{x})$, $\tilde{T}' = (T', \tilde{x}') \in \tilde{\Omega}$ are such that $\bar{d}(\tilde{\Phi}^k(\tilde{T}), \tilde{\Phi}^k(\tilde{T}'))$, $k \in \mathbb{Z}$, is bounded. There are then: a sequence $k_i \to -\infty$; $[h_i]$ in the group $H_1(X; \mathbb{Z})/K_\Lambda$ of deck transformations of $\tilde{X}$; and n-cells $\tilde{\tau}$, $\tilde{\tau}'$ of $\tilde{X}$ so that $\tilde{x}_i \in [h_i](\tilde{\tau})$ and $\tilde{x}_i' \in [h_i](\tilde{\tau}')$ for all i, where $(\tilde{T}_i, \tilde{x}_i) := \tilde{\Phi}^{k_i}(\tilde{T})$ and $(\tilde{T}_i', \tilde{x}_i') := \tilde{\Phi}^{k_i}(\tilde{T}')$. Let $\tilde{\gamma}^i$ be a path in $\tilde{X}$ from $\tilde{x}_i$ to $\tilde{x}_i'$. We may take $\tilde{\gamma}^i = \tilde{\gamma}_1^i * \cdots * \tilde{\gamma}_l^i$ with each $\tilde{\gamma}_j^i$ a lift of a path γ_j^i in X of the form $\gamma_j^i(t) = p(S_j^i - tv_j^i)$, $t \in [0, 1]$, for some $S_j^i \in \Omega_\Phi$ and $v_j^i \in \mathbb{R}^n$ with $S_1^i = T_i$ and $S_l^i = T_i'$. (It is key to this argument that l is constant, independent of i.) We have $p(S_j^i - v_j^i) = p(S_{j+1}^i)$ for $j = 1, \ldots, l-1$ and all i. Passing to a subsequence, we may assume that $\Phi^{|k_i|}(S_j^i + v_1^i + \cdots + v_{j-1}^i) \to \bar{S}_j \in \Omega_\Phi$ as $i \to \infty$ for $j = 2, \ldots, l$. Let $\bar{S}_1 = T$. We conclude from Lemma 23 that $\bar{S}_j \sim_{rp} \bar{S}_{j+1}$ for $j = 1, \ldots, l-1$.

Now, since $\tilde{\Phi}^{|k_i|} \circ \tilde{\gamma}_i$ is a path from $\tilde{x}$ to $\tilde{x}'$, the vector $\Lambda^{|k_i|}(v_1^i + \cdots + v_{l-1}^i) =: v$ is constant (this is the displacement vector $l(\tilde{\Phi}^{|k_i|} \circ \tilde{\gamma}_i)$ of the path $\tilde{\Phi}^{|k_i|} \circ \tilde{\gamma}_i$ in $\tilde{X}$). Thus $\bar{S}_l = T' + v$. We have a path in $\tilde{X}$, call it $\tilde{\gamma}$, from $\tilde{x}$ to $\tilde{x}'$ with displacement v. Then any path in $\tilde{X}$ from $\tilde{f}^k(\tilde{x})$ to $\tilde{f}^k(\tilde{x}')$ has displacement $\Lambda^k v$: since $\bar{d}(\tilde{\Phi}^k(\tilde{T}), \tilde{\Phi}^k(\tilde{T}'))$ is bounded, and $\tilde{\Phi}^k(\tilde{T}) = (\Phi^k(T), \tilde{f}^k(\tilde{x}))$ and $\tilde{\Phi}^k(\tilde{T}') = (\Phi^k(T'), \tilde{f}^k(\tilde{x}'))$, v must be zero—that is, $\bar{S}_l = T'$. Since $\sim_{rp}$ is a translation invariant equivalence relation, $T \sim_{rp} T'$. □

PROPOSITION 25. *Suppose that Φ is a unimodular Pisot family n-dimensional substitution with linear expansion Λ. If $D(\Lambda) = D(\mathrm{GR}) = D$, then the global shadowing relation contains the regional proximal relation: $T \sim_{rp} T' \implies T \sim_{gsK_\Lambda} T'$.*

Proof. Let $f : X \to X$ be the substitution induced map on the collared A–P complex for Φ. Let A represent $f_* : H_1(X; \mathbb{Z})/K_\Lambda \to H_1(X; \mathbb{Z})/K_\Lambda$ in some basis, say $\{[h_1], \ldots, [h_D]\}$, which we fix for the remainder of this proof. Then A is hyperbolic, and, as an isomorphism of $\mathbb{R}^D$, has invariant stable and unstable spaces E^s and E^u with $E^s \oplus E^u = \mathbb{R}^D$. There are $\eta \in (0, 1)$ and C so that $|A^k x| \le C\eta^k |x|$ for $x \in E^s$, $k \in \mathbb{N}$, and $|A^{-k} x| \le C\eta^k |x|$ for $x \in E^u$, $k \in \mathbb{N}$. Let $x = x^s + x^u$ be the decomposition of $x \in \mathbb{R}^D$ into stable and unstable parts. Thus, for each $[h] \in H_1(X; \mathbb{Z})/K_\Lambda$, there are corresponding $[h]^s \in E^s$, $[h]^u \in E^u$.

Given $T \in \Omega_\Phi$ and $\tilde{x} \in \tilde{X} := \tilde{X}_{K_\Lambda}$ with $\pi(\tilde{x}) = p(T)$, we will call the subset $\mathcal{S}(T, \tilde{x}) := \{\tilde{p}_{\tilde{x}}^T(v) : v \in \mathbb{R}^n\}$ of $\tilde{X}$ a *sheet*.

CLAIM 26. *There is $B \in \mathbb{R}$ so that if $\mathcal{S}$ is any sheet in $\tilde{X}$ and $[h] \in H_1(X; \mathbb{Z})/K_\Lambda$ is such that $[h](\mathcal{S}) \cap \mathcal{S}$ contains an n-cell of $\tilde{X}$, then $|[h]^s| \le B$.*

Proof. To prove the claim, let $m \in \mathbb{N}$ and $\alpha < 1$ be so that $|A^{mk} x| \le \alpha^k |x|$ for all $k \in \mathbb{N}$ and $x \in E^s$. We may assume that m is large enough so that, for each tile τ, $\Phi^m(\tau)$ contains, in its interior, a tile of every type. Let B_1 be large enough so that if $\tilde{\tau}$ is any n-cell in $\tilde{X}$, $\tilde{\tau}_1$, $\tilde{\tau}_2$ are two n-cells in $\tilde{f}^{2m}(\tilde{\tau})$ of the same type, and $[h](\tilde{\tau}_1) = \tilde{\tau}_2$, then $|[h]^s| \le B_1$. Now let $T \in \Omega_\Phi$ be fixed by Φ^m and let $\tilde{x} \in \tilde{X}$ be fixed by $\tilde{f}^m$, with $\pi(\tilde{x}) = p(T)$. The sheet $\mathcal{S} := \{\tilde{p}_{\tilde{x}}^T(v) : v \in \mathbb{R}^n\}$ is invariant under $\tilde{f}^m$. We may assume that $\tilde{x}$ is in the

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interior of an n-cell $\tilde{\tau}$ of $\tilde{X}$ so that $\mathcal{S} = \bigcup_{k\in\mathbb{N}} \tilde{\Phi}^{km}(\tilde{\tau})$. Let B_2 be large enough so that $\alpha B_2 + B_1 \le B_2$.

Subclaim: If $[h](\tilde{x}) \in \mathcal{S}$ then $|[h]^s| \le B_2$.

To see that this is the case, let $\mathcal{S}_k := \bigcup_{j=0,\dots,k} \tilde{f}^{mj}(\tilde{\tau})$ for $k = 0, 1, \dots$. We prove the claim, with $\mathcal{S}$ replaced by $\mathcal{S}_k$, by induction on k. If $[h](\tilde{x}) \in \mathcal{S}_0$, then $[h]^s = 0$ and if $[h](\tilde{x}) \in \mathcal{S}_1$, then $|[h]^s| \le B_1 < B_2$. Suppose the claim is true with $\mathcal{S}$ replaced with $\mathcal{S}_k$, $k \ge 1$, and suppose that $[h](\tilde{x}) \in \mathcal{S}_{k+1}$. Let $\tilde{\tau}'$ be the n-cell of $\mathcal{S}_k$ with $[h](\tilde{x}) \in \tilde{f}^{2m}(\tilde{\tau}')$. There is then an n-cell of the same type as $\tilde{\tau}$ in $\tilde{f}^m(\tilde{\tau}') \subset \mathcal{S}_k$. Thus there is $[h_1]$ with $[h_1](\tilde{x}) \in \tilde{f}^m(\tilde{\tau}')$: by hypothesis, $|[h_1]^s| \le B_2$. Then

$$\tilde{f}^m([h_1](\tilde{x})) = [f_*^m(h_1)](\tilde{x}) \in \tilde{f}^{2m}(\tilde{\tau}'),$$

so that $|[h]^s - [f_*^m(h_1)]^s| = |[h - f_*^m(h_1)]^s| \le B_1$. Also, $|[f_*^m(h_1)]^s| \le \alpha|[h_1]^s| \le \alpha B_2$, so that $|[h]^s| \le \alpha B_2 + B_1 \le B_2$, completing the inductive argument and establishing the subclaim.

For each of the finitely many different types of n-cell in $\tilde{X}$, represented, say, by $\tilde{\tau}_1, \dots, \tilde{\tau}_r$, there is thus a sheet $\mathcal{S}_i = \mathcal{S}(T_i, \tilde{x}_i)$ so that if the n-cell $\tilde{\tau} \subset \mathcal{S}_i$ has the same type as $\tilde{\tau}_i$, and $[h](\tilde{\tau}) \subset \mathcal{S}_i$, then $|[h]^s| \le 2B_2 := B$. Now if $\mathcal{S} = \mathcal{S}(T, \tilde{x})$ is any sheet with $\tilde{\tau} \subset \mathcal{S} \cap [h](\mathcal{S})$, let $\tilde{\tau}_i$ be of the same type as $\tilde{\tau}$. Let Q be a connected finite patch of T containing the tiles τ and $\tau - l([h])$, with $\tilde{p}^T_{\tilde{x}}(\tau) = \tilde{\tau}$, and hence also $\tilde{p}^T_{\tilde{x}}(\tau - l([h])) = [-h](\tilde{\tau})$. There is then v with $Q - v \subset T_i$. Then $[h](\tilde{p}^T_{\tilde{x}_i}(\tau - l[h] - v)) = \tilde{p}^T_{\tilde{x}_i}(\tau - v)$, so $|[h]^s| \le B$. This establishes Claim 26. □

Suppose that $T, T' \in \Omega_\Phi$ are such that $T \sim_{rp} T'$. For each $r = 1, 2, \dots$ there are $S_r, S_r' \in \Omega_\Phi$ and $v_r, w_r \in \mathbb{R}^n$ so that: $B_r[T] = B_r[S_r]$; $B_r[S_r - v_r] = B_r[S_r' - v_r]$; $B_r[T' - w_r] = B_r[S_r']$; and $w_r \to 0$ as $r \to \infty$. Pick a lift $\tilde{T} = (T, \tilde{x}) \in \tilde{\Omega}$, then choose lifts

$$\tilde{S}_r = (S_r, \tilde{x}),\ \tilde{S}_r' = (S_r', \tilde{x}_r') \in \tilde{\Omega}$$

so that $\tilde{p}^{S_r}_{\tilde{x}}(v_r) = \tilde{p}^{S_r'}_{\tilde{x}_r'}(v_r)$.

CLAIM 27. *For sufficiently large r, $\tilde{x}_r' =: \tilde{x}'$ is constant and $w_r = 0$.*

Proof. To see this, let R be large enough so that every R-patch contains a tile of every type. Let $\tilde{\tau}$ be an n-cell of $\tilde{X}$ with $\tilde{x} \in \tilde{\tau} \subset \mathcal{S}(T, \tilde{x})$ and for each $r \ge R$, let $\tilde{\tau}_r$ be an n-cell of $\tilde{X}$ with $\tilde{\tau}_r \subset \mathcal{S}(S_r, \tilde{x}) \cap \mathcal{S}(S_r', \tilde{x}_r')$ of the same type as $\tilde{\tau}$. Pick also n-cells $\tilde{\tau}_r' \subset \mathcal{S}(S_r', \tilde{x}')$, of the same type as $\tilde{\tau}$, in $\tilde{p}^{S_r'}_{\tilde{x}_r'}(B_R(0))$. As $B_r[S_r'] = B_r[T' - w_r]$, we may take $\tilde{\tau}_r' = \tilde{p}^{S_r'}_{\tilde{x}_r'}(\tau' - w_r)$ with $\tau' \in T'$. Let $L : \mathbb{R}^D \to \mathbb{R}^n$ be the linear map given by

$$L((a_1, \dots, a_D)) := \sum_{i=1}^{D} a_i l([h_i]).$$

Then $LA = \Lambda L$ and, since $D = D(\Lambda)$, $L|_{E^u} : E^u \to \mathbb{R}^n$ is an isomorphism. Also, L is one-to-one on $\mathbb{Z}^D$ (recall that $l : H_1(X; \mathbb{Z}) \to \mathrm{GR}$ has kernel K_Λ). Thus $\Gamma := L(\{a \in \mathbb{R}^D : |a^s| \le B\})$, B as in Claim 26, is a regular model set. In particular, Γ is uniformly discrete,

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relatively dense, and has the Meyer property (see [**M**]). Let $[h_r^1](\tilde{\tau}) = \tilde{\tau}_r$ and $[h_r^2](\tilde{\tau}'_r) = \tilde{\tau}_r$. Then $|[h_r^1]^s| \le B$ and $|[h_r^2]^s| \le B$ by Claim 26, so

$$l([h_r^1]),\, l([h_r^2]) \in \Gamma \quad \text{and} \quad \{l([h_r^1]) - l([h_r^2]) : r \ge R\}$$

is uniformly discrete. But $(l([h_{r'}^1]) - l([h_{r'}^2])) - (l([h_r^1]) - l([h_r^2])) = w_r - w_{r'}$, and since $w_r \to 0$, it must be that $w_r = 0$ for all sufficiently large r. Thus, $l([h_r^1 - h_r^2])$ is constant for large r, and since l is injective on $H_1(X; \mathbb{Z})$, $[h_r^1 - h_r^2] := [h]$ is constant for large r. Thus the n-cells $\tilde{\tau}'_r = [h](\tilde{\tau})$ and the patches $B_R[S'_r]$ are constant for all large r and it follows that $\tilde{x}'_r$ is also constant for large r, establishing Claim 27. □

We can deduce more from the above. For large r let

$$\tilde{\gamma}_1(t) := \tilde{p}_{\tilde{x}}^{S_r}(tv_r), \quad \tilde{\gamma}_2(t) := \tilde{p}_{\tilde{x}'}^{S'_r}((1-t)v_r), \quad \text{and} \quad \tilde{\gamma}_3(t) := \tilde{p}_{\tilde{x}'}^{T'}(tw),\ 0 \le t \le 1,$$

where w is such that $\tilde{p}_{\tilde{x}'}^{T'}(w) = [h](\tilde{x})$. Then

$$|l([h])| = |l(\tilde{\gamma}_1 * \tilde{\gamma}_2 * \tilde{\gamma}_3)| = |v_r - v_r + w| \le R.$$

Let

$$\bar{R}_1 := \sup\{\tilde{d}(\tilde{y}, [h']\tilde{y})) : \tilde{y} \in \tilde{X},\, [h'] \in \Gamma,\, |[h']| \le R\}.$$

Then $\bar{R}_1 < \infty$ since Γ is a discrete subset of $\mathbb{R}^n$ and the metric $\tilde{d}$ is equivariant with respect to the deck transformations. Let

$$\bar{R}_2 := \sup\{\tilde{d}(\tilde{p}_{\tilde{y}}^{S}(w),\, \tilde{y}) : |w| \le R,\, (S, \tilde{y}) \in \tilde{\Omega}\};$$

$\bar{R}_2$ is also finite by equivariance. We have: $\tilde{d}(\tilde{x}, \tilde{x}') \le \bar{R}_1 + \bar{R}_2$.

Now let $\tilde{T}' = (T', \tilde{x}')$. For $k \in \mathbb{N}$, it is clear that the sheets

$$\tilde{f}^k(\mathcal{S}(T, \tilde{x})) = \mathcal{S}(\Phi^k(T),\, \tilde{f}^k(\tilde{x})), \quad \tilde{f}^k(\mathcal{S}(S_r, \tilde{x})) = \mathcal{S}(\Phi^k(S_r),\, \tilde{f}^k(\tilde{x})),$$
$$\tilde{f}^k(\mathcal{S}(S'_r, \tilde{x}')) = \mathcal{S}(\Phi^k(S'_r),\, \tilde{f}^k(\tilde{x}')),$$

and

$$\tilde{f}^k(\mathcal{S}(T', \tilde{x}')) = \mathcal{S}(\Phi^k(T'),\, \tilde{f}^k(\tilde{x}'))$$

overlap in the same manner as those sheets with $k = 0$, and the above argument yields $\tilde{d}(\tilde{f}^k(\tilde{x}), \tilde{f}^k(\tilde{x}')) \le \bar{R}_1 + \bar{R}_2$. Now fix $k < 0$. Given r, there is $r' = r'(r)$ big enough so that if $S, S' \in \Omega_\Phi$ satisfy $B_{r'}[S] = B_{r'}[S']$, then $B_r[\Phi^k(S)] = B_r[\Phi^k(S')]$. (This consequence of the invertibility of Φ is referred to as 'recognizability'; see, for example, [**S1**].) For $r > 0$ and $r' = r'(r)$, consider

$$\tilde{\Phi}^k((T, \tilde{x})) = (\Phi^k(T),\, \tilde{y}) \quad \text{and} \quad \tilde{\Phi}^k((S_{r'}, \tilde{x})) = (\Phi^k(S_{r'}),\, \tilde{y}').$$

Since

$$B_0[\Phi^k(S_{r'})] = B_0[\Phi^k(T)], \quad \pi(\tilde{y}) = \pi(\tilde{y}')$$

and there is $[h] \in H_1(X; \mathbb{Z})/K_\Lambda$ so that $[h](\tilde{y}') = \tilde{y}$. Then

$$\tilde{x} = \tilde{f}^k(\tilde{y}') = \tilde{f}^k([h](\tilde{y})) = \bar{f}_*^k([h])(\tilde{f}^k(\tilde{y})) = \bar{f}_*^k([h])(\tilde{x}),$$

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so $[h] = 0$ ($\bar{f}_*$ is invertible on $H_1(X; \mathbb{Z})/K_\Lambda$). Thus $\tilde{y}' = \tilde{y}$. In this way, we see that the sheets determined by $\tilde{\Phi}^k(\tilde{T})$, $\tilde{\Phi}^k(\tilde{S}_{r'})$, $\tilde{\Phi}^k(\tilde{S}'_{r'})$, and $\tilde{\Phi}^k(\tilde{T}')$, for r' large, overlap as above so that

$$\bar{d}(\tilde{\Phi}^k(\tilde{T}),\, \tilde{\Phi}^k(\tilde{T}')) \le d(\Phi^k(T),\, \Phi^k(T')) + \bar{R}_1 + \bar{R}_2.$$

We conclude that $\bar{d}(\tilde{\Phi}^k(\tilde{T}),\, \tilde{\Phi}^k(\tilde{T}'))$, $k \in \mathbb{Z}$, is bounded; that is, $T \sim_{gsK_\Lambda} T'$. □

Proof of Theorem 4. By Proposition 19, $G(T) = G(T')$ if and only if $T \sim_{gsK_\Lambda} T'$, and, by the fundamental result of Veech [**V**], $g(T) = g(T')$ if and only if $T \sim_{rp} T'$. By Propositions 24 and 25, $T \sim_{gsK_\Lambda} T'$ if and only if $T \sim_{rp} T'$. Thus we may identify G with g. □

Proof of Corollary 6. The first statement follows immediately from Theorems 1 and 4 and the characterization of cr in [**BK**]. For the second statement: the eigenvalues form a subgroup of $\mathbb{R}^n$ so are relatively dense if and only if their linear span is all of $\mathbb{R}^n$. Suppose there is a vector $v \neq 0$ perpendicular to the linear span of the eigenvalues E. If $f : \Omega_\Phi \to \mathbb{T}$ is a continuous eigenfunction with eigenvalue β and $T \in \Omega_\Phi$, then $f(T - tv) = \exp(2\pi i \langle \beta, tv \rangle) f(T) = f(T)$ for all $t \in \mathbb{R}$. This means that $g(T - tv) = g(T)$ for all $t \in \mathbb{R}$ (the Halmos–von Neumann theory asserts that the map g is determined by the continuous eigenfunctions—see [**R**] or [**BK**]). But then g is not finite-to-one.

It is shown in [**LS2**] that, in this context, the third statement is equivalent to the second. □

Proof of Corollary 7. The spectrum of the $\mathbb{R}^n$-action on Ω_Φ is pure discrete if and only if g is almost everywhere one-to-one (see, for example, [**BK**]).

9. *Pisot family substitutions*

Proof of Theorem 8. Suppose that Φ is unimodular of (m, d)-Pisot family type. Let X be the collared A–P complex for Φ so that $\varprojlim f$ is identified with Ω via $\hat{p}$. We prove that, for $T, T' \in \Omega$, $g(T) = g(T')$ if and only if $G(T) = G(T')$.

Suppose first that $G(T) = G(T')$. Let $\tilde{X} = \tilde{X}_{K_\Lambda}$ and $\tilde{f} = \tilde{f}_{K_\Lambda} : \tilde{X} \to \tilde{X}$. We will argue that $g : \Omega = \varprojlim f \to \mathbb{T}^{md}$ lifts to $\tilde{g} : \tilde{\Omega}_{K_\Lambda} = \varprojlim \tilde{f} \to \mathbb{R}^{md}$. We may suppose that the origin is in the interior of each prototile. For each $k \in \mathbb{N}$, let X_k denote the A–P complex made of kth order supertiles. That is, the n-faces of X_k are the patches $\Phi^k(\rho_i)$, with face $\Phi^k(\rho_i)$ glued to face $\Phi^k(\rho_j)$ along the $(n-1)$-face $\Lambda^k e$ in X_k if and only if the prototile ρ_i is glued to the prototile ρ_j along the $(n-1)$-face e in X. As each tiling in Ω is uniquely tiled by kth order supertiles, there is a natural map $p^k : \Omega \to X_k$ ($p^k(T) = [v]$ in the face $\Phi^k(\rho_j)$ of X_k if the kth order supertile of T containing the origin is $\Phi^k(\rho_j) - v$). The decomposition of kth order supertiles into $(k-1)$-th order supertiles induces maps $f_k : X_k \to X_{k-1}$ so that $\Omega \simeq \varprojlim f_k$. Furthermore, there are substitution induced maps $f^k : X_k \to X_k$ with $f_k \circ f^k = f^{k-1} \circ f_k$ so that Φ on Ω is conjugated with the homeomorphism induced by (f^k) on $\varprojlim f_k$ by (p^k). In order to lift g we approximate g by maps $g_k \circ p^k$, where the $g_k : X_k \to \mathbb{T}^{md}$ are constructed as follows. For each kth order supertile $\rho^k := \Lambda^k(\rho)$, ρ a prototile for Φ, and $\epsilon_k > 0$, let $\mathcal{N}(\rho^k, \epsilon_k)$ be the ϵ_k-neighborhood of the boundary of ρ^k. Choose (arbitrarily) $T_{\rho^k} \in \Omega$ so that the supertile ρ^k occurs in the decomposition of T_{ρ^k}

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into kth order supertiles (we mean this supertile occurs exactly, with 0 translation). Define g_k on $\rho^k \backslash \mathcal{N}(\rho^k, \epsilon_k)$ by $g_k(p^k(T_{\rho^k} - v)) := g(T_{\rho^k} - v)$ for $v \in \rho^k \backslash \mathcal{N}(\rho^k, \epsilon_k)$. Since we have collared Φ, there is $\delta = \delta(k, \epsilon_k)$ so that if ρ_i^k and ρ_j^k are adjacent faces in X_k, glued along an l-face e, and $e \ni x = p^k(T_{\rho_i^k} - v) = p^k(T_{\rho_j^k} - w)$, with $v \in \rho_i^k$ and $w \in \rho_j^k$, then $d(T_{\rho_i^k} - v, T_{\rho_j^k} - w) < \delta$, and $\delta \to 0$ as $k \to \infty$. Choosing ϵ_k sufficiently small, and using the local convexity of $\mathbb{T}^{md}$, we extend g_k continuously to all of X_k so that, for each i, if $v, w \in \rho_i^k$ are such that $|v - w| < \epsilon_k$, then $d(g_k(p^k(T_{\rho_i^k} - v)), g_k(p^k(T_{\rho_i^k} - w))) < \delta$. Now for $T \in \Omega$, let $T_{\rho_i^k}$ and $v \in \rho_i^k$ be such that $p^k(T) = p^k(T_{\rho_i^k} - v)$. Since we have collared, and $0 \in \operatorname{int}(\rho_i)$, T and $T_{\rho_i^k} - v$ agree in an r_k-ball about the origin, with $r_k \to \infty$ as $k \to \infty$. Thus

$$\begin{aligned} d(g(T), g_k \circ p^k(T)) &= d(g(T), g_k \circ p^k(T_{\rho_i^k} - v)) \le d(g(T), g(T_{\rho_i^k} - v)) \\ &\quad + d(g(T_{\rho_i^k} - v, g_k \circ p^k(T_{\rho_i^k} - v))) \to 0 \quad \text{as } k \to \infty. \end{aligned}$$

That is, $g_k \circ p^k \to g$ uniformly as $k \to \infty$.

With the goal still of lifting g, we show now that $g_k : X_k \to \mathbb{T}^{md}$ lifts to $\tilde{g}_k : \tilde{X}_k \to \mathbb{R}^{md}$. Let E_k be a dual 1-skeleton in X_k. That is, we choose a vertex in the interior of each n-face, a vertex in the (relative) interior of each $(n-1)$-face of X_k, and we make a straight line edge from the vertex interior to every n-face to each of the vertices on the $(n-1)$-subfaces. Let $\tilde{\gamma} : [0, 1] \to \tilde{X}_k$ be a loop in $\tilde{X}_k$. Then the loop $\pi_k \circ \tilde{\gamma}$ in X_k is homotopic to a piecewise affine loop γ lying on E_k. We can write γ as a concatenation $\gamma = c_1 * c_2 * \cdots * c_l$ with each c_i of the form $c_i(t) = p^k(T_{\rho_j^k} - (v_i + t w_i))$, $t_i \le t \le t_{i+1}$, for some $j = j(i)$, v_i, w_i. Let $\eta := g_k \circ \gamma : [0, 1] \to \mathbb{T}^{md}$ and let $\tilde{\eta} : [0, 1] \to \mathbb{R}^{md}$ be a lift of η. Let $\iota : \mathbb{R}^n \to \mathbb{R}^{md}$ be the linear embedding so that $g(T - v) = g(T) - \iota(v)$. We have

$$\begin{aligned} \tilde{\eta}(1) - \tilde{\eta}(0) &= \sum_{i=0}^{l-1} (\tilde{\eta}(t_{i+1}) - \tilde{\eta}(t_i)) \approx \sum_{i=0}^{l-1} (\widetilde{\iota(v_{j(i)})} + t_{i+1}\iota(w_i)) - (\widetilde{\iota(v_{j(i)})} + t_i \iota(w_i)) \\ &= \sum_{i=0}^{l-1} (t_{i+1} - t_i)\iota(w_i), \end{aligned}$$

with the approximation improving as k gets larger. The covering space $\tilde{X}_k$ is the quotient of the abelian cover $(\tilde{X}_k)_{\text{ab}}$ by the action of those deck transformations lying in the kernel K_k of the homomorphism $l_k : H_1(X_k; \mathbb{Z}) \to \mathbb{R}^n$. That $\tilde{\gamma}$ is a loop in $\tilde{X}_k$ means that the homology class $[\gamma]$ lies in K_k. That is,

$$l_k([\gamma]) = \sum_{i=0}^{l-1} (t_{i+1} - t_i) w_i = 0.$$

Thus $\sum_{i=0}^{l-1} (t_{i+1} - t_i)\iota(w_i) = 0$ also. As $\tilde{\eta}(1) - \tilde{\eta}(0) \in \mathbb{Z}^{md}$, it must be that, for sufficiently large k, $\tilde{\eta}(1) - \tilde{\eta}(0) = 0$. The lifting criterion is satisfied: $g_k \circ \tilde{\pi} \circ \tilde{\gamma}$ is homotopic to a loop pushed down from $\mathbb{R}^{md}$ to $\mathbb{T}^{md}$.

Let us record a property of $\tilde{g}_k$ for later use. From the displayed approximation of $\tilde{\eta}(1) - \tilde{\eta}(0)$ above, we see that, for sufficiently large k and any loop γ in X_k, and $\eta := g_k \circ \gamma$, $\tilde{\eta}(1) - \tilde{\eta}(0) = \iota \circ l_k([\gamma])$. It follows that for $\tilde{x} \in \tilde{X}_k$, $[h] \in H_1(X_k; \mathbb{Z})/K_k$,

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and k sufficiently large:

$$\tilde{g}_k(\tilde{x} + [h]) = \tilde{g}_k(\tilde{x}) + \iota \circ l_k(h). \tag{1}$$

In the constructions above we have identified Ω with $\varprojlim f$ (by means of $\hat{p} : T \mapsto (p(\Phi^{-i}(T)))$) and $\varprojlim f$ with $\varprojlim(f_k)$ by means of a rescaling: since $\rho_i^{k+1} = \Lambda\rho_i^k$ and $X = X_0$ there are natural maps $\Lambda^k : X \to X_k$ with $\Lambda^k \circ f = f_k \circ \Lambda^{k+1}$. Then $\varprojlim f$ is identified with $\varprojlim(f_k)$ by $(x_i) \mapsto (\Lambda^i(x_i))$. The spaces $\varprojlim \tilde{f}$ and $\varprojlim(\tilde{f}_k)$ are then identified by first choosing a lift of f to $\tilde{f}$, then choosing lifts $(\tilde{\Lambda}^k)$, $(\tilde{f}_k)$ so that $\tilde{\Lambda}^k \circ \tilde{f} = \tilde{f}_k \circ \widetilde{\Lambda^{k+1}}$. Now choose lifts $\tilde{g}_k : \tilde{X}_k \to \mathbb{R}^{md}$ so that $|\tilde{g}_k \circ \tilde{f}_{k+1}(\tilde{x}_k) - \tilde{g}_{k+1}(\tilde{x}_k)| \to 0$ for some (and hence any) $(\tilde{x}_k) \in \varprojlim(\tilde{f}_k)$ and let $\tilde{g} : \varprojlim \tilde{f} \to \mathbb{R}^{md}$ be defined by $\tilde{g}((\tilde{x}_i)) := \lim_{k\to\infty} \tilde{g}_k \circ \tilde{\Lambda}^k(\tilde{x}_k)$. Convergence of this limit to a lift of g is assured by the convergence of $g_k \circ p^k$ to g.

It is proved in [**BK**] that there is a hyperbolic and unimodular $md \times md$ integer matrix B so that $g \circ \Phi = F_B \circ g$, $F_B(x + \mathbb{Z}^{md}) := Bx + \mathbb{Z}^{md}$ being the corresponding hyperbolic automorphism of $\mathbb{T}^{md}$. It follows that $\tilde{g} \circ \hat{\tilde{f}} = B\tilde{g}$. We have assumed that $G(T) = G(T')$ with the objective of showing that then $g(T) = g(T')$. Let (x_i), $(x_i') \in \varprojlim f$ be such that $\hat{p}(T) = (x_i)$, $\hat{p}(T') = (x_i')$. By Proposition 19, $(x_i) \sim_{gsK_\Lambda} (x_i')$, so there are $(\tilde{x}_i)$, $(\tilde{x}_i') \in \varprojlim \tilde{f}$, living over (x_i), (x_i'), so that $\bar{d}(\hat{\tilde{f}}^k((\tilde{x}_i)), \hat{\tilde{f}}^k((\tilde{x}_i')))$, $k \in \mathbb{Z}$, is bounded. Then

$$|\tilde{g}(\hat{\tilde{f}}^k((\tilde{x}_i))) - \tilde{g}(\hat{\tilde{f}}^k((\tilde{x}_i')))| = |B^k\tilde{g}((\tilde{x}_i)) - B^k\tilde{g}((\tilde{x}_i'))|, \quad k \in \mathbb{Z},$$

is also bounded, and hyperbolicity of B implies that $\tilde{g}((\tilde{x}_i)) = \tilde{g}((\tilde{x}_i'))$. Thus $g(T) = g(T')$.

Now suppose that $g(T) = g(T')$. We wish to show that $T \sim_{gsK_\Lambda} T'$. From [**LS1**, Theorem 3.1] we have (for Φ of (m, d)-Pisot family type) that there are $v_1, \ldots, v_m \in \mathbb{R}^n$ with $\mathrm{GR}(\Phi) \subset \mathbb{Z}[\Lambda]v_1 + \cdots + \mathbb{Z}[\Lambda]v_m$. Thus $D(\mathrm{GR}) \le D(\Lambda)$, and hence $D(\mathrm{GR}) = D(\Lambda)$ since the opposite inequality is always satisfied (Lemma 16). Let K be the kernel of $l : H_1(X; \mathbb{Z}) \to \mathrm{GR}(\Phi)$, let $\pi_K : \tilde{X}_K \to X$ be the corresponding cover of the collared A–P complex X, and let $\tilde{f}_K : \tilde{X}_K \to \tilde{X}_K$ be a lift of f. Let $\pi_1 : \tilde{\Omega} \to \Omega$, with $\tilde{\Omega} := \{(T, \tilde{x}) : p(T) = \pi_K(\tilde{x})\}$ the cover isomorphic with $\hat{\pi}_K : \varprojlim \hat{\tilde{f}}_K \to \varprojlim f$ as in Lemma 14. Let $\bar{d}$ be the metric in $\tilde{\Omega}_K$ given by $\bar{d}((T, \tilde{x}), (T', \tilde{y})) = d(T, T') + \tilde{d}(\tilde{x}, \tilde{y})$. We will show that there are $\tilde{T}$, $\tilde{T}' \in \tilde{\Omega}_K$ lying over T, T' so that $\bar{d}(\tilde{\Phi}^k(\tilde{T}), \tilde{\Phi}^k(\tilde{T}'))$, $k \in \mathbb{Z}$, is bounded.

It is proved in [**BK**] that $g(T) = g(T')$ if and only if T and T' are *strongly regionally proximal*, i.e., for each $k \in \mathbb{N}$ there are S_k, $S_k' \in \Omega$ and $v_k \in \mathbb{R}^n$ so that $B_k[T] = B_k[S_k]$, $B_k[T'] = B_k[S_k']$, and $B_k[S_k - v_k] = B_k[S_k' - v_k]$. Pick $\tilde{T} = (T, \tilde{x}) \in \tilde{\Omega}_K$. Let $\tilde{S}_k := (S_k, \tilde{x}) \in \tilde{\Omega}_K$. There is then $\tilde{S}_k' = (S_k', \tilde{y}_k) \in \tilde{\Omega}_K$ such that $\pi_2(\tilde{S}_k - v_k) = \pi_2(\tilde{S}_k' - v_k)$. Let $\tilde{T}_k' := (T', \tilde{y}_k) \in \tilde{\Omega}_K$.

CLAIM 28. *There are only finitely many distinct $\tilde{y}_k$, $k \in \mathbb{N}$.*

Proof. The proof of Claim 28 will follow from that of Claim 29.

CLAIM 29. *There is R so that if $\tilde{T}$, $\tilde{S} \in \tilde{\Omega}_K$ and $\bar{d}(\tilde{T}, \tilde{S}) < 2\mathrm{diam}(\Omega)$, then $\bar{d}(\tilde{T} - v, \tilde{S} - v) < R$ for every $v \in \mathbb{R}^n$.*

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Proof. To prove Claim 29, let $\tilde{g} : \tilde{\Omega}_K \to \mathbb{R}^{md}$ be a lift of g as constructed above. Under the identification of $\tilde{\Omega}_K$ with $\varprojlim(\tilde{f}_k)$ used above in the construction of $\tilde{g}$, the point $\tilde{T} = (T, \tilde{x}) \in \tilde{\Omega}_K$ corresponds to $(\tilde{x}_0, \tilde{x}_1, \ldots) \in \varprojlim(\tilde{f}_k)$ with: $\tilde{x}_0 = \tilde{x}$; $\tilde{x}_1$ such that $\tilde{f}_1(\tilde{x}_1) = \tilde{x}_0$ and $\pi_1(\tilde{x}_1) = p^1(\Phi^{-1}(T))$; ... (here $\pi_1 : \tilde{X}_1 \to X_1$ is the covering projection). Then, for $[h]$ in the group $H_1(X; \mathbb{Z})/K$ of deck translations of $\tilde{\Omega}_K$, $\tilde{T} + [h]$ (by this we mean the deck translation $[h]$ applied to $\tilde{T}$) corresponds to

$$(\tilde{x} + [h], \ldots, \tilde{x}_k + [((f_1)_* \circ \cdots \circ (f_k)_*)^{-1}(h)], \ldots)$$

(we have used unimodularity to invert $(f_i)_*$ on $H_1(X; \mathbb{Z})/K$). Thus, using equation (1) for k large, we have

$$\begin{aligned}\tilde{g}(T + [h]) &\approx \tilde{g}_k(\tilde{x}_k + [((f_1)_* \circ \cdots \circ (f_k)_*)^{-1}(h)]) \\ &= \tilde{g}_k(\tilde{x}_k) + \iota \circ l_k(((f_1)_* \circ \cdots \circ (f_k)_*)^{-1}(h)).\end{aligned}$$

Using the easily checked fact that $l_k = l_{k-1} \circ (f_k)_*$, we have $l_k(((f_1)_* \circ \cdots \circ (f_k)_*)^{-1}(h)) = l(h)$. Thus $\tilde{g}(\tilde{T} + [h]) \approx \tilde{g}(\tilde{T}) + \iota \circ l(h)$, with an improving approximation as $k \to \infty$. That is,

$$\tilde{g}(\tilde{T} + [h]) = \tilde{g}(\tilde{T}) + \iota \circ l(h).$$

Now $[h] \to l(h)$ is an isomorphism of $H_1(X; \mathbb{Z})/K$ with $\mathrm{GR}(\Phi) \subset \mathbb{R}^n$, and $\iota : \mathbb{R}^n \to \mathbb{R}^{md}$ is a linear embedding, so $\iota \circ l : H_1(X; \mathbb{Z})/K \to \mathbb{Z}^{md}$ is injective. It follows from this and equivariance of $\bar{d}$ that, given C, there is an R so that if $\bar{d}(\tilde{T}, \tilde{S}) > R$ then $|\tilde{g}(\tilde{T}) - \tilde{g}(\tilde{S})| > C$.

From $g(T - v) = g(T) - \iota(v)$ we deduce $\tilde{g}(\tilde{T} - v) = \tilde{g}(\tilde{T}) - \iota(v)$. Since $\tilde{g}$ is a lift and $\bar{d}$ is equivariant, there is C so that $\bar{d}(\tilde{T}, \tilde{S}) < 2\mathrm{diam}(\Omega)$ implies $|\tilde{g}(\tilde{T}) - \tilde{g}(\tilde{S})| < C$. We then have

$$\begin{aligned}\bar{d}(\tilde{T}, \tilde{S}) < 2\mathrm{diam}(\Omega) &\implies |\tilde{g}(\tilde{T}) - \tilde{g}(\tilde{S})| < C \implies |(\tilde{g}(\tilde{T}) - \iota(v)) - (\tilde{g}(\tilde{S}) - \iota(v))| < C \\ &\implies |\tilde{g}(\tilde{T} - v) - \tilde{g}(\tilde{S} - v)| < C \implies \bar{d}(\tilde{T} - v, \tilde{S} - v) < R,\end{aligned}$$

proving Claim 29. □

Now for the proof of Claim 28. We have

$$\bar{d}(\tilde{S}_k, \tilde{S}'_k) = d(S_k, S'_k) + \tilde{d}(\tilde{x}, \tilde{y})$$

and

$$\bar{d}(\tilde{S}_k - v_k, \tilde{S}'_k - v_k) = d(S_k - v_k, S'_k - v_k) + 0 < 2\mathrm{diam}(\Omega).$$

By Claim 29, $\tilde{d}(\tilde{x}, \tilde{y}) < d(S_k, S'_k) + R$ is bounded. The $\tilde{y}_k \in \tilde{X}$ all satisfy $\pi_K(\tilde{y}_k) = p(T')$ and hence there are $[h_k] \in H_1(X; \mathbb{Z})$ so that $\tilde{y}_k = \tilde{y}_0 + [h_k]$. As $\tilde{d}(\tilde{y}_k, \tilde{y}_0)$ is bounded, there can be only finitely many distinct $[h_k]$ and Claim 28 is established. □

Now pick $k_i \to \infty$ with $\tilde{y}_{k_i} =: \tilde{x}'$ constant and let $\tilde{T}' := \tilde{T}'_{k_i}$. To ease notation, we assume $k_i = i$, that is, all $\tilde{y}_k$ are the same. To conclude the proof of Theorem 8, we show that $\bar{d}(\tilde{\Phi}^j(\tilde{T}), \tilde{\Phi}^j(\tilde{T}'))$, $j \in \mathbb{Z}$, is bounded.

First consider $j \geq 0$. We have

$$\pi_2(\tilde{\Phi}^j(\tilde{T})) = \pi_2((\Phi^j(T), \tilde{f}^j_K(\tilde{x}))) = \tilde{f}^j_K(\tilde{x}) = \pi_2(\tilde{\Phi}^j(\tilde{S}_1))$$

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and similarly, $\pi_2(\tilde{\Phi}^j(\tilde{T}')) = \pi_2(\tilde{\Phi}^j(\tilde{S}_1'))$. Also,

$$\pi_2(\tilde{S}_1 - v_1) = \pi_2(\tilde{S}_1' - v_1) \implies \pi_2(\tilde{\Phi}^j(\tilde{S}_1 - v_1)) = \pi_2(\tilde{\Phi}^j(\tilde{S}_1' - v_1)).$$

That is,

$$\pi_2(\tilde{\Phi}^j(\tilde{S}_1) - \Lambda^j v_1) = \pi_2(\tilde{\Phi}^j(\tilde{S}_1') - \Lambda^j v_1),$$

and by Claim 29 we have $\bar{d}(\tilde{\Phi}^j(\tilde{S}_1), \tilde{\Phi}^j(\tilde{S}_1')) < R$. Thus $\tilde{d}(\tilde{f}_K^j(\tilde{x}), \tilde{f}_K^j(\tilde{x}')) < R$, so $\bar{d}(\tilde{\Phi}^j(\tilde{T}), \tilde{\Phi}^j(\tilde{T}')) < R + \operatorname{diam}(\Omega)$ for all $j \geq 0$.

For $j < 0$ we will need the following claim.

CLAIM 30. *Given R_1 there is R_2 so that if $\tilde{T}$, $\tilde{S} \in \tilde{\Omega}_K$ satisfy $\pi_2(\tilde{\Phi}(\tilde{T}) - v) = \pi_2(\tilde{\Phi}(\tilde{S}) - v)$ for all $v \in B_{R_2}(0)$, then $\pi_2(\tilde{T} - v) = \pi_2(\tilde{S} - v)$ for all $v \in B_{R_1}(0)$.*

Proof. Let R_1 be given. It is a consequence of invertibility of Φ that there is an R_2 so that if T, $S \in \Omega$ satisfy $B_{R_2}[\Phi(T)] = B_{R_2}[\Phi(S)]$, then $B_{R_1}[T] = B_{R_1}[S]$. (This is 'recognizability'.) Now let $\tilde{T} = (T, \tilde{x})$ and $\tilde{S} = (S, \tilde{y})$. Then $\tilde{T} - v = (T - v, \tilde{p}_{\tilde{x}}^T(v))$ and $\tilde{S} - v = (S - v, \tilde{p}_{\tilde{y}}^S(v))$. Now suppose that $\pi_2(\tilde{\Phi}(\tilde{T}) - v) = \pi_2(\tilde{\Phi}(\tilde{S}) - v)$ for all $|v| < R_2$. This means that $\tilde{f}_K(\tilde{p}_{\tilde{x}}^T(\Lambda^{-1}v)) = \tilde{f}_K(\tilde{p}_{\tilde{y}}^S(\Lambda^{-1}v))$, which is to say $\tilde{p}_{\tilde{f}_K(\tilde{x})}^{\Phi(T)}(v) = \tilde{p}_{\tilde{f}_K(\tilde{y})}^{\Phi(S)}(v)$, for all such v. From this we have $\tilde{f}_K(\tilde{x}) = \tilde{f}_K(\tilde{y})$ and $p(\Phi(T) - v) = p(\Phi(S) - v)$ for $|v| < R_2$. So $B_{R_2}[\Phi(T)] = B_{R_2}[\Phi(S)]$ so that $B_{R_1}[T] = B_{R_1}[S]$. In particular, $p(T) = p(S)$ and $\tilde{y} = \tilde{x} + [h]$ for some $[h] \in H_1(X; \mathbb{Z})/K$. Then $\tilde{f}_K(\tilde{x}) = \tilde{f}_K(\tilde{y}) = \tilde{f}_K(\tilde{x} + [h]) = \tilde{f}_K(\tilde{x}) + [f_*(h)] \implies [f_*(h)] = 0$. Since f_* is invertible on $H_1(X; \mathbb{Z})/K$ (unimodularity), $[h] = 0$ and $\tilde{x} = \tilde{y}$. Together with $B_{R_1}[T] = B_{R_1}[S]$, this last implies that $\tilde{p}_{\tilde{x}}^T(v) = \tilde{p}_{\tilde{y}}^S(v)$, that is, $\pi_2(\tilde{T} - v) = \pi_2(\tilde{S} - v)$, for all $v \in B_{R_1}(0)$. □

To finish the proof of the theorem, let $R_2 = R_2(R_1)$ be as in Claim 30 and for $m \in \mathbb{N}$ let $R_2^m := R_2(R_2(\cdots(R_2(1))\cdots))$, iterated m times. Pick $j < 0$ and take $k > R_2^{|j|}$. Then $\pi_2(\tilde{T} - v) = \pi_2(\tilde{S}_k) - v$ for $|v| < k$ implies that $\pi_2(\tilde{\Phi}^{-1}(\tilde{T}) - v) = \pi_2(\tilde{\Phi}^{-1}(\tilde{S}_k) - v)$ for $|v| < R_2^{|j|-1}$, which implies that …, which gives $\pi_2(\tilde{\Phi}^j(\tilde{T}) - v) = \pi_2(\tilde{\Phi}^j(\tilde{S}_k) - v)$ for $|v| < 1$. Similarly, for this k, we have $\pi_2(\Phi^j(\tilde{S}_k - v_k) - v) = \pi_2(\Phi^j(\tilde{S}_k' - v_k) - v)$ and $\pi_2(\Phi^j(\tilde{S}_k' - v)) = \pi_2(\Phi^j(\tilde{T}') - v)$ for $|v| < 1$. Using Claim 29 (as in the $j > 0$ case above) we conclude that $\bar{d}(\tilde{\Phi}^j(\tilde{T}), \tilde{\Phi}^j(\tilde{T}')) < \operatorname{diam}(\Omega) + R$. Thus $G(T) = G(T')$. □

10. *Connections with the traditional Pisot substitution conjecture*

The traditional Pisot substitution conjecture is as follows.

CONJECTURE 31. *If Φ is a one-dimensional substitution with unimodular and irreducible incidence matrix M (that is, the characteristic polynomial of M is irreducible over $\mathbb{Q}$) and with inflation λ a Pisot number, then the $\mathbb{R}$-action on Ω_Φ has pure discrete spectrum.*

We will see below that this is a special case of a slight strengthening of Conjecture 5.

Given a substitution Ψ with A–P complex X (not necessarily collared), substitution induced map $f : X \to X$, and natural semi-conjugacy $p : \Omega_\Psi \to \varprojlim f$, as defined previously, let

$$H_f^*(\Omega_\Psi) := p^*(H^*(\varprojlim f; \mathbb{Z})).$$

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Let us say that a substitution Ψ is *compatible* with a substitution Φ if there is a homeomorphism $h_\Psi : \Omega_\Phi \to \Omega_\Psi$ that conjugates some positive powers of the substitution homeomorphisms Φ and Ψ. We define the *essential cohomology* of Ω_Φ to be

$$H^*_{\text{ess}}(\Omega_\Phi; \mathbb{Z}) := \bigcap_\Psi h^*_\Psi(H^*_f(\Omega_\Psi)),$$

the intersection being over all Ψ compatible with Φ.

CONJECTURE 32. *If Φ is unimodular and hyperbolic, and $H^1_{\text{hyp}}(\Omega_\Phi; \mathbb{Z}) = H^1_{\text{ess}}(\Omega_\Phi; \mathbb{Z})$, then $G' : \Omega_\Phi \to \mathbb{T}^{D'}$ is almost everywhere one-to-one.*

CONJECTURE 33. *If Φ is a unimodular Pisot family substitution and* $\text{rank}(H^1_{\text{ess}}(\Omega_\Phi; \mathbb{Z})) = D(\Lambda)$ *then the $\mathbb{R}^n$-action on Ω_Φ has pure discrete spectrum.*

Example 34. Consider the one-dimensional substitution Φ generated by the substitution on letters: $1 \mapsto 12221111$; $2 \mapsto 21112$. The incidence matrix is unimodular and irreducible and the inflation λ is the fourth power of the golden mean, a Pisot number. Thus Φ satisfies the hypotheses of the traditional Pisot substitution conjecture (and, in fact, the spectrum is pure discrete as the traditional Pisot conjecture is correct if the incidence matrix is 2×2 (see [**HS**])). But $D(\Lambda) = \deg(\lambda) = 2$ while $\text{rank}(H^1(\Omega_\Phi; \mathbb{Z})) = 3$, so the hypotheses of Conjecture 5 are not met. (The cohomology is easily computed by the techniques of [**BD1**].) On the other hand, the A–P complex for Φ is a wedge of two circles; it follows that $H^1_{\text{ess}}(\Omega_\Phi; \mathbb{Z}) = H^1_f(\Omega_\Phi)$ has rank two, and the hypotheses of Conjecture 33 are satisfied.

PROPOSITION 35. *Suppose Φ is a one-dimensional substitution with unimodular and irreducible $d \times d$ incidence matrix M and whose inflation λ is a Pisot number. Then* $\text{rank}(H^1_{\text{ess}}(\Omega_\Phi; \mathbb{Z})) = d$.

Hence any substitution satisfying the hypotheses of the traditional Pisot substitution conjecture also satisfies the hypotheses of Conjecture 33.

LEMMA 36. *Suppose that Φ is an n-dimensional substitution. Let $f : X \to X$ be the substitution induced map on the A–P complex for Φ and let $p : \Omega_\Phi \to X$ be the usual map, inducing $\hat{p} : \Omega_\Phi \to \varprojlim f$. Then the map g, from Ω_Φ onto the maximal equicontinuous factor, factors through $\varprojlim f$ via $\hat{p}$.*

Proof. Let $f_c : X_c \to X_c$ be the substitution induced map of the collared A–P complex for Φ, let $\pi : X_c \to X$ be the map that forgets collars, and let $p_c : \Omega_\Phi \to X_c$ be as usual. It suffices to show that if $(x_i), (x'_i) \in \varprojlim f_c$ are such that $\hat{\pi}((x_i)) = \hat{\pi}((x'_i))$ then the tilings $T = {\hat{p}_c}^{-1}((x_i))$ and $T' = {\hat{p}_c}^{-1}((x'_i))$ are regionally proximal. Suppose then that $\hat{\pi}((x_i)) = \hat{\pi}((x'_i))$. If x_i, and hence x'_i, are in the interior of an n-cell for infinitely many (hence all) i, let $i_j \to \infty$, ρ, and v_j be such that $B_0[\Phi^{-i_j}(T)] = \rho - v_j = B_0[\Phi^{-i_j}(T')]$. Then the patches $\Phi^{i_j}(\rho) - \Lambda^{i_j}$ are subsets of both T and T'. That is, T and T' share patches of arbitrarily large internal diameter, and hence T and T' are proximal.

If, on the other hand, x_i and x'_i are in the $n-1$ skeleton of X_c for all i, $\pi(x_i) = \pi(x'_i) =: y_i$ is in the $n-1$ skeleton of X for all i. From the definition of the equivalence relation that specifies the way in which the prototiles are glued along their boundaries to form X,

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there are, for each $i \in \mathbb{N}$, $T_1^i, \ldots, T_{k_i}^i \in \Omega_\Phi$ and tiles $\tau_j^i \neq \tau_{j+1}^i \in T_j^i$ so that $0 \in \tau_j^i \cap \tau_{j+1}^i$ for all i, j, $T_1^i = \Phi^{-i}(T)$, and $T_{k_i}^i = \Phi^{-i}(T')$. The k_i are bounded and we may select $i_l \to \infty$ so that $k = k_{i_l}$ is constant and the collection $\{\tau_j^{i_l} : j = 1, \ldots, k\}$ is constant up to translation. Using compactness of Ω_Φ we may pass to a subsequence $I_{l_s} \to \infty$ so that $\Phi^{i_{l_s}}(T_j^{i_{l_s}})$ converges, say to T_j, for $j = 1, \ldots, k$. As above, T_j and T_{j+1} share patches of arbitrarily large internal diameter, and are hence proximal, for each j. As regional proximality is an equivalence relation and contains the proximality relation, $T = T_1$ and $T' = T_k$ are regionally proximal. □

Suppose that Φ is a one-dimensional (primitive) substitution whose inflation is a degree d Pisot unit. Let X_c be the collared A–P complex for Φ, with vertex set S_c and substitution induced map $f : (X_c, S_c) \to (X_c, S_c)$. In an appropriate basis, $f^* : H^1(X_c, S_c; \mathbb{Z}) \to H^1(X_c, S_c; \mathbb{Z})$ is represented by the transpose, M^t, of the incidence matrix for Φ. There is then a unique f^*-invariant subgroup $\mathcal{P}$ of $H^1(X_c, S_c; \mathbb{Z})$ of rank d with the properties that $f^*|_\mathcal{P}$ is represented by an integer matrix with eigenvalue λ, and $\mathcal{P}$ is maximal in the sense that $H^1(X_c, S_c; \mathbb{Z})/\mathcal{P}$ is torsion free. By taking the direct limit by f^* of the short exact sequence for the pair

$$0 \to \tilde{H}^0(S_c, \mathbb{Z}) \to H^1(X_c, S_c; \mathbb{Z}) \to H^1(X_c; \mathbb{Z}) \to 0$$

and noting that all eigenvalues of (a matrix representing) $f^* : \tilde{H}^0(S_c; \mathbb{Z}) \to \tilde{H}^0(S_c; \mathbb{Z})$ are 0 or roots of unity, we see that $H^1(\Omega_\Phi; \mathbb{Z}) = \varinjlim f_* : H^1(X_c; \mathbb{Z}) \to H^1(X_c; \mathbb{Z})$ also contains a unique subgroup, invariant under Φ^*, with the properties of $\mathcal{P}$. We will call this subgroup the *Pisot subgroup* of $H^1(\Omega_\Phi; \mathbb{Z})$.

Proof of Proposition 35. Let Φ be as in the proposition, let $f : X \to X$ be the substitution induced map on its A–P complex, and let S be the vertex set of X. The homomorphism $f^* : H^1(X, S; \mathbb{Z}) \to H^1(X, S; \mathbb{Z})$ is represented by the transpose M^t of the incidence matrix (in the basis dual to the prototile edges of X). As M is invertible over $\mathbb{Z}$, the direct limit of this homomorphism is $\mathbb{Z}^d$. Moreover, the subgroup $\delta^*(H^0(S; \mathbb{Z})) \subset H^1(X, S; \mathbb{Z})$ is f^*-invariant, so, by irreducibility of M, this subgroup must be trivial. From the exact sequence for the pair we see that $f^* : H^1(X, S; \mathbb{Z}) \to H^1(X; \mathbb{Z})$ is conjugate with $f^* : H^1(X; \mathbb{Z}) \to H^1(X; \mathbb{Z})$, and hence $H^1(\varprojlim f; \mathbb{Z})$ is also $\mathbb{Z}^d$.

Let X_c denote the collared A–P complex for Φ with vertex set S_c and map $f_c : (X_c, S_c) \to (X_c, S_c)$ and let $p_c : \Omega_\Phi \to X_c$ and $p : \Omega_\Phi \to X$ be the standard maps onto the A–P complexes. Let $\pi : (X_c, S_c) \to (X, S)$ be the map that forgets collars. Then $\pi^* : H^1(X, S; \mathbb{Z}) \to H^1(X_c, S_c; \mathbb{Z})$ is injective and it follows that the image of $H^1(X; \mathbb{Z})$ in $H^1(X_c; \mathbb{Z})$ under π^* is an f_c^*-invariant copy of $\mathbb{Z}^d$ on which f_c^* acts like M^t. Thus $H_f^1(\Omega_\Phi)$ has rank d and $\operatorname{rank}(H_{\text{ess}}^1(\Omega_\Phi; \mathbb{Z}))$ is at most d.

Suppose that Ψ is a one-dimensional substitution with inflation η such that Ψ^n is conjugate with Φ^m, by means of h_Ψ, for some $n, m \in \mathbb{N}$. It is easily checked that the topological entropies of the homeomorphisms Ψ and Φ are $\log(\eta)$ and $\log(\lambda)$, hence $\eta^m = \lambda^n$, as conjugacies preserve entropy. Thus η^m is a Pisot unit, also of degree d (as M^n is irreducible). Now, by Lemma 36, if $f : X \to X$ is the substitution induced map on the A–P complex for Ψ, and $p : \Omega_\Psi \to X$ is the usual map, the map $g : \Omega_\Psi \to \mathbb{T}^d$,

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onto the maximal equicontinuous factor, factors through $\varprojlim f$ via $\hat{p}$. Thus $H^1_f(\Omega_\Psi) = \hat{p}^*(H^1(\varprojlim f; \mathbb{Z})) \supset g^*(H^1(\mathbb{T}^d; \mathbb{Z}))$. By Theorems 1 and 8, g^* conjugates $F_A^*: H^1(\mathbb{T}^d; \mathbb{Z}) \to H^1(\mathbb{T}^d; \mathbb{Z})$ with Ψ^* restricted to the image of g^*. The matrix A is a companion matrix for η^m (see the proof of Theorem 1 or [**BK**]). It follows that $H^1_f(\Omega_\Psi)$ contains the Pisot subgroup for Ψ^* and hence $h^*_\Psi(H^1(\Omega_\Psi))$ contains the Pisot subgroup for Φ. Thus $\operatorname{rank}(H^1_{\mathrm{ess}}(\Omega_\Phi; \mathbb{Z}))$ is at least d. □

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